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Time-periodic Poiseuille-type solution with minimally regular flow rate*

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Abstract. The nonstationary Navier–Stokes equations are studied in the infinite cylinder $\Pi = \{x = (x', x_n) \in \mathbb{R}^n\colon x' \in \sigma \subset \mathbb{R}^{n-1}, -\infty < x_n < \infty, n = 2, 3\}$ under the additional condition of the prescribed time-periodic flow rate (flux) $F(t)$. It is assumed that the flow rate F belongs to the space $L^2(0, 2\pi)$, only. The time-periodic Poiseuille solution has the form $\mathbf{u}(x, t) = (0, \dots, 0, U(x', t))$, $p(x, t) = -q(t)x_n + p_0(t)$, where $(U(x', t), q(t))$ is a solution of an inverse problem for the time-periodic heat equation with a specific over-determination condition. The existence and uniqueness of a solution to this problem is proved.

Keywords: Navier–Stokes equations, cylindrical domain, time-periodic Poiseuille-type solution, inverse problem, minimal regularity.

1 Introduction

Mathematical modelling is very useful in many practical applications. For example, in medicine, it can be helpful by choosing the optimal strategy of medical treatment. In such modelling, very important issues are multiscale mathematical models of the blood circulation in a network of vessels. The full 3D computations are nowadays very time consuming and may be applied only for small parts of the blood circulation system. Therefore, a new trend is related to the creation of hybrid dimension models that combine the 1D reduction in the regular zones (mostly in straight vessels) with 3D zooms in small zones of singular behaviour. This method of asymptotic partial decomposition of a domain was proposed by Panasenko (see [12]) and developed in [13–16]. It mathematically justifies a size of zoomed areas and prescribes asymptotically exact junction conditions. These hybrid-dimension models require significantly smaller computational resources.

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The 1D Poiseuille-type flows in straight vessels play a very important role in hybrid-dimension models.

The steady-state Poiseuille flow in an infinite straight pipe $\Pi = \{x = (x', x_n) \in \mathbb{R}^n\colon x' \in \sigma \subset \mathbb{R}^{n-1}, -\infty < x_n < \infty, n = 2, 3\}$ of constant cross-section σ was invented by Jean Louis Poiseuille in 1841 (see [2,11,24]). The Poiseuille flow is described by the fact that the associated velocity field has only one nonzero component $u(x')$ directed along the x_n-axis of Π, which depends only on $x' \in \sigma$, and the pressure function $p = p(x_n)$ is linear. The Poiseuille-type solutions can be also defined in the nonstationary case (see [7, 17–21, 23, 26]). Moreover, in [14] the behaviour of the nonstationary Poiseuille flow was studied in a thin cylinder (with the cross-section of radius ε), and the asymptotics of it as $\varepsilon \to 0$ was found.

In the time-periodic case, such flow is usually called Womersley's flow (see [28]). The time-periodic Poiseuille-type solutions were studied in [3] and [8]. The time periodic solutions for the full Navier–Stokes problem were considered in many papers (see, e.g., [4–6]). Notice that the time-periodic case is very important because of applications to hemodynamics.

In mentioned above papers the Poiseuille-type solutions were studied in the case when data is sufficiently regular. However, in real applications, one usually does not have data defined by smooth functions, and it is important to study the case of minimal regularity of data. The nonstationary Poiseuille-type solution with a prescribed initial condition and given flow rate $F(t)$ belonging to $L^2(0,T)$ was studied in [22], where a new class of weak solutions was introduced, and the unique existence of the solution in such class was proved. The goal of the present paper is to extend the result obtained in [22] to the case of time-periodic Poiseuille-type solutions.

Let us consider the time-periodic Navier–Stokes problem describing the motion of a viscous incompressible fluid in the infinite cylinder Π:

$$\begin{aligned} &\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0,\\ &\operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Pi} = 0, \qquad \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure function, and $\nu > 0$ is the constant kinematic viscosity of the fluid.

We look for the solution $\mathbf{u}$ of (1) in Π satisfying the additional condition of prescribed flow rate (flux) $F(t)$:

$$\int\limits_\sigma u_n(x,t)\,\mathrm{d}x' = F(t),$$

where $F(0) = F(2\pi)$.[1]

The solution $(\mathbf{u}(x,t), p(x,t))$ of problem (1) has the following expression:

$$\mathbf{u}(x,t) = \bigl(0,\dots,0,U_n(x',t)\bigr), \qquad p(x,t) = -q(t)x_n + p_0(t) \tag{2}$$

[1]Without loss of generality, we suppose that the period is equal to 2π.

with an arbitrary function $p_0(t)$. Putting (2) into (1), we obtain the following problem on the cross-section σ:

$$\begin{aligned} &U_t(x',t) - \nu\Delta' U(x',t) = q(t), \\ &U(x',t)\big|_{\partial\sigma} = 0, \qquad U(x',0) = U(x',2\pi), \end{aligned} \tag{3}$$

where $U(x',t) = U_n(x',t)$ and $q(t)$ are unknown functions, Δ' is the Laplace operator with respect to x'.

The Poiseuille flow can be uniquely determined either prescribing the pressure drop $q(t)$ or the flow rate $F(t)$. In the first case the problem is reduced to the standard time-periodic problem for the heat equation for unknown velocity $U = U(x',t)$ with time-periodic forcing $q(t)$. Problems of such type are well studied (see, e.g., [9]). However, in the real word applications the pressure is unknown, and only the flow rate (flux) of the fluid is given. Therefore, it is necessary to prescribe the additional condition

$$\int\limits_{\sigma} U(x',t)\,\mathrm{d}x' = F(t), \quad F(0) = F(2\pi). \tag{4}$$

In this case the solution of problem (3), (4) is a pair of functions $(U(x',t), q(t))$, and one has to solve for $U(x',t)$ and $q(t)$ more complicated *inverse* parabolic problem: *for given $F(t)$, to find a pair of functions $(U(x',t), q(t))$ solving problem* (3) *with $U(x',t)$ satisfying the flux condition* (4).

Thus, in the second case the relation between $q(t)$ and $F(t)$ depends on the solution of the inverse problem (in the stationary case the flux F and the pressure gradient q are proportional, and the problem remains very simple). The solvability of the time-periodic problem with the assumption that the flux $F(t)$ is from the Sobolev space $W^{1,2}(0,2\pi)$ was proved by Beirão da Veiga [3], and in [8] an elementary relationship between the pressure drop $q(t)$ and the flux $F(t)$ was found. However, in applications and numerical computations, the data usually is not regular. Therefore, in this paper, we study problem (3), (4) assuming only that $F \in L^2(0,2\pi)$.

Problem (3), (4) can be reduced to the case when all involved functions have zero mean values. Let us denote by $\bar{H} = 1/(2\pi)\int_0^{2\pi} H(t)\,\mathrm{d}t$ the mean value of a function H. Let $(\bar{U}, \bar{q})$ be a solution of the following problem on σ (the stationary Poiseuille solution corresponding to the flux $\bar{F}$):

$$-\nu\Delta'\bar{U}(x') = \bar{q}, \qquad \bar{U}(x')|_{\partial\sigma} = 0, \qquad \int\limits_{\sigma} \bar{U}(x')\,\mathrm{d}x' = \bar{F}. \tag{5}$$

The solution $\bar{U}(x')$ of (5) can be represented in the form $\bar{U}(x') = (\bar{F}/\kappa_0)U_0(x')$, where

$$-\nu\Delta' U_0(x') = 1, \qquad U_0(x')|_{\partial\sigma} = 0, \tag{6}$$

and

$$\bar{q} = \frac{\bar{F}}{\kappa_0}, \qquad \kappa_0 = \int\limits_{\sigma} U_0(x')\,\mathrm{d}x' = \nu\int\limits_{\sigma} \left|\nabla' U_0(x')\right|^2 \mathrm{d}x' > 0. \tag{7}$$

Let us represent the solution (U,q) in the form

$$U(x',t) = V(x',t) + \bar{U}(x'), \qquad q(t) = s(t) + \bar{q}.$$

Then, obviously, $\bar{V}(x') = 0$, $\bar{s} = 0$, and (V,s) is the solution of the problem

$$\begin{aligned} & V_t(x',t) - \nu\Delta' V(x',t) = s(t), \\ & V(x',t)|_{\partial\sigma} = 0, \qquad V(x',0) = V(x',2\pi), \\ & \int\limits_{\sigma} V(x',t)\,\mathrm{d}x' = \widetilde{F}(t), \end{aligned} \tag{8}$$

where $\widetilde{F}(t) = F(t) - \bar{F}$, $\bar{\widetilde{F}} = 0$. So, without loss of generality, we assume $F(t) = \widetilde{F}(t)$, that is $\bar{F} = 0$.

Below, we deal with a weak solution of problem (3), (4). The reasoning about the reduction to the case of functions with zero mean values remains valid for weak solutions as well, and we will study only this case.

The rest of the paper is organized in the following way. In Section 2, function spaces are defined and the main result is formulated. In Section 3 the Galerkin approximations of the solution are constructed, and in Section 4 a priori estimates for these approximations are proved. In Section 5 the main result of the paper, that is the existence and the uniqueness of the solution, is proved.

2 Notation and formulation of main result

2.1 Function spaces

Below, we will use the following notation. If G is the domain in $\mathbb{R}^n$, $C^\infty(G)$ means, as usual, the set of all infinitely differentiable functions in G, and $C_0^\infty(G)$ is the subset of functions from $C^\infty(G)$ with compact supports in G. We use the usual (see [1,9]) notation for Lebesgue and Sobolev spaces: $L^2(G)$, $W^{l,2}(G)$, $l \geqslant 0$, and $\mathring{W}^{1,2}(G)$. The norm of an element u in the function space V is denoted by $\|u\|_V$. $L^2(0,T;V)$ is the Bochner space of functions u such that $u(\cdot,t) \in V$ for almost all $t \in [0,T]$, and the norm

$$\|u\|_{L^2(0,T;V)} = \left(\int\limits_0^T \|u(\cdot,t)\|_V^2 \,\mathrm{d}t \right)^{1/2}$$

is finite.

Let us consider the set of smooth periodic functions $C_\wp^\infty(0,2\pi) = \{h \in C^\infty(\mathbb{R})\colon h(t) = h(t+2\pi)\ \forall t \in \mathbb{R}\}$ defined on the interval $[0,2\pi]$. Let $L^2(0,2\pi)$ be a Lebesgue space on the interval $(0,2\pi)$. We extend the functions from $L^2(0,2\pi)$ to the whole line $\mathbb{R}$ by putting $f(t+2\pi) = f(t)$ for any t. To emphasize that functions are periodically extended to $\mathbb{R}$, we use the notation $L_\wp^2(0,2\pi)$. Let $L_\sharp^2(0,2\pi) = \{h \in L_\wp^2(0,2\pi)\colon \int_0^{2\pi} h(t)\,\mathrm{d}t = 0\}$. Clearly, $L_\sharp^2(0,2\pi)$ is a closure of $C_\sharp^\infty(0,2\pi) = \{h \in C_\wp^\infty(0,2\pi)\colon \int_0^{2\pi} h(t)\,\mathrm{d}t = 0\}$ in

$L^2(0,2\pi)$-norm, and it is a proper subspace of $L^2_\wp(0,2\pi)$. Let $W^{1,2}_\wp(0,2\pi)$ be the closure of the set $C^\infty_\wp(0,2\pi)$ in $W^{1,2}$-norm. Since function f from $W^{1,2}_\wp(0,2\pi)$ coincides with a continuous function on a set whose complement is of measure zero, we may assume that $f(0) = f(2\pi)$. Let $W^{-1,2}_\wp(0,2\pi)$ be dual of $W^{1,2}_\wp(0,2\pi)$, i.e., $W^{-1,2}_\wp(0,2\pi) = (W^{1,2}_\wp(0,2\pi))^*$.

For any function $f \in L^2_\wp(0,2\pi)$, denote by $S_f(t)$ its primitive:

$$S_f(t) = -\int_t^{t_0+2\pi} f(\tau)\,\mathrm{d}\tau, \quad \text{where } t_0 \in [0,2\pi),\ t \in [t_0,\, t_0+2\pi]. \tag{9}$$

Clearly, $\mathrm{d}S_f(t)/\mathrm{d}t = f(t)$, $S_f(t_0+2\pi) = 0$.

If $f \in L^2_\sharp(0,2\pi)$, then $S_f(t_0) = -\int_{t_0}^{t_0+2\pi} f(\tau)\,\mathrm{d}\tau = -\int_0^{2\pi} f(t)\,\mathrm{d}t = 0$. Moreover,

$$\begin{aligned}\int_0^{2\pi} \left|S_f(t)\right|^2 \mathrm{d}t &\leqslant 2\pi \int_0^{2\pi}\int_t^{t_0+2\pi} \left|f(\tau)\right|^2 \mathrm{d}\tau\,\mathrm{d}t \leqslant 4\pi^2 \int_{t_0}^{t_0+2\pi} \left|f(\tau)\right|^2 \mathrm{d}\tau \\ &= 4\pi^2 \int_0^{2\pi} \left|f(\tau)\right|^2 \mathrm{d}\tau,\end{aligned}$$

and $S_f(t)$ is a periodic function:

$$\begin{aligned}S_f(t+2\pi) &= -\int_{t+2\pi}^{t_0+2\pi} f(\tau)\,\mathrm{d}\tau = -\int_t^{t_0} f(\tau)\,\mathrm{d}\tau = -\int_t^{t_0+2\pi} f(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t_0+2\pi} f(\tau)\,\mathrm{d}\tau \\ &= S_f(t) - S_f(t_0) = S_f(t).\end{aligned}$$

Thus, $S_f \in L^2_\wp(0,2\pi)$. Note that functions $S_f(t)$ defined by (9) with various t_0 differ from each other by a constant.

Note that the L^2-limit of a sequence $\{S_{f_n}\} \subset C^\infty_\wp(0,2\pi)$ is not necessary a primitive of some function from $L^2_\sharp(0,2\pi)$. We will prove that an element $h \in W^{-1,2}_\wp(0,2\pi)$ possesses a primitive in the distributional sense.

Lemma 1. *Any functional $h \in W^{-1,2}_\wp(0,2\pi)$ can be represented in the form*

$$\langle h,\eta\rangle = \int_0^{2\pi} H(t)\eta'(t)\,\mathrm{d}t \quad \forall \eta \in W^{1,2}_\wp(0,2\pi) \tag{10}$$

with the uniquely determined $H \in L^2_\sharp(0,2\pi)$.

Proof. Obviously, the functional given by formula (10) obeys the estimate

$$\left|\langle h,\eta\rangle\right| \leqslant \|H\|_{L^2_\sharp(0,2\pi)} \|\eta\|_{W^{1,2}_\wp(0,2\pi)},$$

and hence, $h \in W^{-1,2}_\wp(0,2\pi)$.

Let us take an arbitrary functional $h \in W_{\wp}^{-1,2}(0,2\pi)$ and show that it can be represented in the form (10). Consider the operator $\partial : \eta \in W_{\wp}^{1,2}(0,2\pi) \mapsto \eta' \in L_{\sharp}^2(0,2\pi)$ (due to periodicity, $\int_0^{2\pi} \eta'(t)\,\mathrm{d}t = \eta(2\pi) - \eta(0) = 0$). Since for any $\varphi \in L_{\sharp}^2(0,2\pi)$, the equality $\varphi = \eta'$ holds with

$$\eta(t) = -\int\limits_t^{2\pi} \varphi(\tau)\,\mathrm{d}\tau \in W_{\wp}^{1,2}(0,2\pi), \tag{11}$$

we have $\mathcal{R}(\partial) = L_{\sharp}^2(0,2\pi)$, and the operator ∂ is an isomorphism from $W_{\wp}^{1,2}(0,2\pi)$ to $L_{\sharp}^2(0,2\pi)$, where the bounded operator $\partial^{-1} : L_{\sharp}^2(0,2\pi) \mapsto W_{\wp}^{1,2}(0,2\pi)$ is given by (11).

For $\varphi \in L_{\sharp}^2(0,2\pi)$, define the functional $M(\varphi) = \langle h, \partial^{-1}\varphi\rangle$. Clearly,

$$\bigl|M(\varphi)\bigr| \leqslant c\|h\|_{W_{\wp}^{-1,2}(0,2\pi)} \bigl\|\partial^{-1}\varphi\bigr\|_{W_{\wp}^{1,2}(0,2\pi)} \leqslant c\|h\|_{W_{\wp}^{-1,2}(0,2\pi)} \|\varphi\|_{L_{\sharp}^2(0,2\pi)}.$$

Hence, there exists a uniquely defined $H \in L_{\sharp}^2(0,2\pi)$ such that

$$M(\varphi) = \int\limits_0^{2\pi} H(\tau)\varphi(\tau)\,\mathrm{d}\tau \quad \forall \varphi \in L_{\sharp}^2(0,2\pi).$$

Thus,

$$\langle h, \eta\rangle = \bigl\langle h, \partial^{-1}\varphi\bigr\rangle = \int\limits_0^{2\pi} H(\tau)\varphi(\tau)\,\mathrm{d}\tau = \int\limits_0^{2\pi} H(\tau)\eta'(\tau)\,\mathrm{d}\tau \quad \forall \eta \in W_{\wp}^{1,2}(0,2\pi),$$

and $\langle h, \eta\rangle$ is represented in the form (10). □

Remark 1. Note that if the functional h can be represented in the form $\langle h, \eta\rangle = \int_0^{2\pi} H(t)\eta(t)\,\mathrm{d}t$ with $H \in L_{\sharp}^2(0,2\pi)$ and arbitrary $\eta \in W_{\wp}^{1,2}(0,2\pi)$, then $H(t) = -\int_t^{2\pi} h(\tau)\,\mathrm{d}\tau$. Therefore, also in the case of a general functional h, for the distributional primitive H, we use the notation $H(t) = S_h(t)$.

2.2 Formulation of main result

Definition of a weak solution

Let $F \in L_{\sharp}^2(0,2\pi)$. By a *weak solution* of problem (8) we understand a pair (V, s) such that $V \in L_{\sharp}^2(0,2\pi; L^2(\sigma))$,[2] $S_V \in L_{\wp}^2(0,2\pi; \mathring{W}^{1,2}(\sigma))$,[3] $s \in W_{\wp}^{-1,2}(0,2\pi)$, $V(x',t)$ satisfies the flux condition

$$\int\limits_\sigma V(x',t)\mathrm{d}x' = F(t), \tag{12}$$

[2]Since $V \in L_{\sharp}^2(0,2\pi; L^2(\sigma))$, S_V is a primitive of V, i.e., $(S_V)_t = V$, and we have the following inclusions: $S_V \in L_{\wp}^2(0,2\pi; L^2(\sigma))$, $(S_V)_t \in L_{\sharp}^2(0,2\pi; L^2(\sigma))$.

[3]The condition that $V|_{\partial\sigma} = 0$ is understood in the usual sense of traces [1]. If $S_V \in L^2(0,2\pi; \mathring{W}^{1,2}(\sigma))$, then $S_V = -\int_t^{2\pi} V(\cdot,\tau)\,\mathrm{d}\tau \in \mathring{W}^{1,2}(\sigma)$ for a.a. $t \in (0,2\pi)$ and $\int_t^{2\pi} V(x',\tau)\,\mathrm{d}\tau|_{\partial\sigma} = 0$ in the sense of traces for such t. But then also $V(x',t)_{\partial\sigma} = 0$ in the sense of traces for a.a. $t \in (0,2\pi)$.

and the pair (V,s) satisfies the integral identity

$$\int_0^{2\pi}\int_\sigma V(x',t)\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t + \nu\int_0^{2\pi}\int_\sigma \nabla' S_V(x',t)\cdot\nabla'\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t = \int_0^{2\pi} S_s(t)\int_\sigma \eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t \tag{13}$$

for any test function $\eta \in L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma))$ such that $\eta_t \in L^2_\sharp(0,2\pi;W^{1,2}(\sigma))$.

For a regular solution (V,s), taking into account that $\nabla V = (\nabla S_V)_t$, $s = S_s'$, identity (13) can be easily obtained multiplying equation $(8)_1$ by η, integrating over σ and over the interval $(0,2\pi)$ and integrating by parts with respect to x' and t. On the other hand, by uniqueness of such weak solution (V,s) (see Theorem 1 below) it follows that for $F \in W^{1,2}_\sharp(0,2\pi)$, the solution (V,s) coincides with the regular one, that is $V \in L^2_\sharp(0,2\pi;\mathring{W}^{1,2}(\sigma)\cap W^{2,2}(\sigma))$, $V_t \in L^2_\sharp(0,2\pi;L^2(\sigma))$, $s \in L^2_\sharp(0,2\pi)$. Thus, the proposed definition is an extension of the concept of weak solutions.

Theorem 1. *Let $F \in L^2_\sharp(0,2\pi)$. Then problem* (8) *admits a unique weak solution* (V,s). *Then there holds the estimate*

$$\int_0^{2\pi}\int_\sigma \left|V(x',t)\right|^2\mathrm{d}x'\,\mathrm{d}t + \int_0^{2\pi}\int_\sigma \left|\nabla' S_V(x',t)\right|^2\mathrm{d}x'\,\mathrm{d}t + \int_0^{2\pi}\left|S_s(\tau)\right|^2\mathrm{d}\tau \leqslant c\int_0^{2\pi}\left|F(\tau)\right|^2\mathrm{d}\tau, \tag{14}$$

where the constant c depends only on σ.

Remark 2. Since $\int_0^{2\pi}\eta'(t)\,\mathrm{d}t = 0$ for $\eta \in W^{1,2}_\wp(0,2\pi)$ and all primitives of the function $V(x',t)$ differ from each other by a function independent of t, the integral identity (13) remains valid for any primitive function S_V, and we can assume, for example, that S_V is taken to be zero at the point $t = 2\pi$.

Theorem 1 will be proved applying some version of Galerkin approximations (see Sections 3 and 4). Notice that in order to get estimates of approximate solutions $(V_\alpha^{(N)}(x',t), s_\alpha^{(N)}(t))$, we have used primitive functions defined over the integrals $\int_t^{t_*+2\pi}$ with specially chosen points $t_* = t(\alpha,N) \in [0,2\pi)$ and $t \in (t_*, t_*+2\pi)$. Thus, estimate (14) is valid only for the primitive function S_V obtained as a limit of the sequence $\{S_{V_\alpha^{(N)}}\}$.

3 Construction of Galerkin approximations

Let $u_k(x') \in \mathring{W}^{1,2}(\sigma)$ and λ_k be eigenfunctions and eigenvalues of the Laplace operator:

$$-\nu\Delta' u_k(x') = \lambda_k u_k(x'), \qquad u_k(x')|_{\partial\sigma} = 0.$$

Note that $\lambda_k > 0$ and $\lim_{k\to\infty} \lambda_k = \infty$. The eigenfunctions $u_k(x')$ are orthogonal in $L^2(\sigma)$, and we assume that $u_k(x')$ are normalized in $L^2(\sigma)$. Then

$$\nu \int_\sigma \left|\nabla' u_k(x')\right|^2 \mathrm{d}x' = \lambda_k, \quad \int_\sigma \nabla' u_k(x') \cdot \nabla' u_l(x')\,\mathrm{d}x' = 0, \quad k \neq l.$$

Moreover, $\{u_k(x')\}$ is a basis in $L^2(\sigma)$ and $\mathring{W}^{1,2}(\sigma)$.

We look for an approximate solution of problem (8) in the form

$$V^{(N)}(x',t) = \sum_{k=1}^{N} w_k^{(N)}(t) u_k(x'). \tag{15}$$

The coefficients $w_k^{(N)}(t)$ and the function $s^{(N)}(t)$ are obtained by solving the following linear problems:

$$\begin{aligned}
&\int_\sigma V_t^{(N)}(x',t) u_k(x')\mathrm{d}x' + \nu \int_\sigma \nabla' V^{(N)}(x',t) \cdot \nabla' u_k(x')\mathrm{d}x' \\
&\quad = s^{(N)}(t) \int_\sigma u_k(x')\mathrm{d}x', \quad k = 1,2,\ldots,N, \\
&w_k^{(N)}(0) = w_k^{(N)}(2\pi), \quad k = 1,\ldots,N, \\
&\int_\sigma V^{(N)}(x',t)\,\mathrm{d}x' = F(t),
\end{aligned} \tag{16}$$

which, in virtue of the orthonormality of functions $u_k(x')$, are equivalent to ordinary differential equations for the functions $w_k^{(N)}(t)$:

$$\begin{aligned}
&\left(w_k^{(N)}(t)\right)' + \lambda_k w_k^{(N)}(t) = \beta_k s^{(N)}(t), \quad t \in (0, 2\pi), \\
&w_k^{(N)}(0) = w_k^{(N)}(2\pi),
\end{aligned} \tag{17}$$

where $\beta_k = \int_\sigma u_k(x')\mathrm{d}x'$. Note that $\sum_{k=1}^\infty \beta_k u_k(x') = 1$ and $\sum_{k=1}^\infty \beta_k^2 = |\sigma|$.

The solution of problem (17) has the form

$$w_k^{(N)}(t) = \beta_k \int_0^{2\pi} G_k(t,\tau) s^{(N)}(\tau)\,\mathrm{d}\tau, \tag{18}$$

where

$$G_k(t,\tau) = \begin{cases} \frac{\mathrm{e}^{-\lambda_k(t-\tau)}}{1-\mathrm{e}^{-2\pi\lambda_k}}, & 0 \leqslant \tau \leqslant t \leqslant 2\pi, \\ \frac{\mathrm{e}^{-\lambda_k(t-\tau+2\pi)}}{1-\mathrm{e}^{-2\pi\lambda_k}}, & 0 \leqslant t \leqslant \tau \leqslant 2\pi. \end{cases}$$

It is easy to see that $w_k^{(N)}(0) = w_k^{(N)}(2\pi)$.

Substituting expression (18) into (15), we obtain

$$V^{(N)}(x',t)=\sum_{k=1}^{N}\beta_k\int_0^{2\pi}G_k(t,\tau)s^{(N)}(\tau)\,\mathrm{d}\tau\,u_k(x').$$

Now the flux condition yields

$$\begin{aligned}F(t)&=\int_\sigma V^{(N)}(x',t)\mathrm{d}x'=\sum_{k=1}^{N}\beta_k\int_0^{2\pi}G_k(t,\tau)s^{(N)}(\tau)\,\mathrm{d}\tau\int_\sigma u_k(x')\,\mathrm{d}x'\\&=\sum_{k=1}^{N}\beta_k^2\int_0^{2\pi}G_k(t,\tau)s^{(N)}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Thus, for the function $s^{(N)}$, we derived Fredholm integral equation of the first kind:

$$\int_0^{2\pi}\sum_{k=1}^{N}\beta_k^2G_k(t,\tau)s^{(N)}(\tau)\,\mathrm{d}\tau=F(t).\tag{19}$$

It is well known (see, e.g., [10, 25]) that such equations, in general, are ill-posed in L^2 setting. In order to regularize equation (19), we consider the following Fredholm integral equation of the second kind:

$$\alpha s_\alpha^{(N)}(t)+\int_0^{2\pi}\sum_{k=1}^{N}\beta_k^2G_k(t,\tau)s_\alpha^{(N)}(\tau)=F(t),\tag{20}$$

where α later will tend to 0, i.e., instead of problem (16), (19), we study the regularized problem

$$\begin{aligned}&\int_\sigma (V_\alpha^{(N)})_t(x',t)u_k(x')\,\mathrm{d}x'+\nu\int_\sigma\nabla'V_\alpha^{(N)}(x',t)\cdot\nabla'u_k(x')\,\mathrm{d}x'\\&\qquad=s_\alpha^{(N)}(t)\int_\sigma u_k(x')\,\mathrm{d}x',\quad k=1,2,\dots,N,\\&V_\alpha^{(N)}(x',0)=V_\alpha^{(N)}(x',2\pi),\\&\alpha s_\alpha^{(N)}(t)+\int_0^{2\pi}\sum_{k=1}^{N}\beta_k^2G_k(t,\tau)s_\alpha^{(N)}(\tau)\,\mathrm{d}\tau=F(t),\end{aligned}\tag{21}$$

where

$$V_\alpha^{(N)}(x',t)=\sum_{k=1}^{N}w_{k,\alpha}^{(N)}(t)u_k(x'),\qquad w_{k,\alpha}^{(N)}(t)=\beta_k\int_0^{2\pi}G_k(t,\tau)s_\alpha^{(N)}(\tau)\,\mathrm{d}\tau.$$

Lemma 2. *Let* $F \in L^2_\sharp(0,2\pi)$. *Then equation* (20) *admits a unique solution* $s^{(N)}_\alpha \in L^2_\sharp(0,2\pi)$.

Proof. First, we show that equation (20) is well defined in the space $L^2_\sharp(0,2\pi)$. Obviously, if F is periodic and $s^{(N)}_\alpha$ is the solution of (20), then $s^{(N)}_\alpha$ also is a periodic function. Assume that mean value $\bar F = 0$. Then

$$\alpha \bar s^{(N)}_\alpha + \overline{\int_0^{2\pi} \sum_{k=1}^N \beta_k^2 s^{(N)}_\alpha(\tau) \int_0^{2\pi} G_k(t,\tau)\,\mathrm{d}t\,\mathrm{d}\tau} = 0. \tag{22}$$

Since

$$\int_0^{2\pi} G_k(t,\tau)\,\mathrm{d}t = \frac{1}{1-\mathrm{e}^{-2\pi\lambda_k}}\left(\int_\tau^{2\pi} \mathrm{e}^{-\lambda_k(t-\tau)}\,\mathrm{d}t + \int_0^\tau \mathrm{e}^{-\lambda_k(t-\tau+2\pi)}\,\mathrm{d}t\right) = \frac{1}{\lambda_k},$$

the second term in (22) is equal to $\sum_{k=1}^N (\beta_k^2/\lambda_k)\bar s^{(N)}_\alpha$, and from (22) it follows that

$$\left(\alpha + \sum_{k=1}^N \frac{\beta_k^2}{\lambda_k}\right)\bar s^{(N)}_\alpha = 0,$$

and thus, $\bar s^{(N)}_\alpha = 0$.

From this it follows that the mean value $\bar V^{(N)}_\alpha(x')$ of $V^{(N)}_\alpha(x',t)$ also vanishes: $\bar V^{(N)}_\alpha(x') = 0$.

It is well known that Fredholm integral equations of the second kind satisfy the Fredholm alternative (e.g., [27]). So, it is enough to prove the uniqueness of the solution to (20). Let $F(t) = 0$. By construction,

$$\int_0^{2\pi} \sum_{k=1}^N \beta_k^2 G_k(t,\tau) s^{(N)}_\alpha(\tau)\,\mathrm{d}\tau = \int_\sigma V^{(N)}_\alpha(x',t)\,\mathrm{d}x',$$

and the homogeneous equation (20) gives

$$\int_\sigma V^{(N)}_\alpha(x',t)\,\mathrm{d}x' = -\alpha s^{(N)}_\alpha(t).$$

Multiplying $(21)_1$ by $w^{(N)}_{k,\alpha}(t)$, summing by k from 1 to N and integrating over the interval $(0,2\pi)$ yield

$$\begin{aligned}
&\int_0^{2\pi}\int_\sigma \big(V^{(N)}_\alpha\big)_t(x',t) V^{(N)}_\alpha(x',t)\,\mathrm{d}x'\,\mathrm{d}t + \nu \int_0^{2\pi}\int_\sigma \big|\nabla' V^{(N)}_\alpha(x',t)\big|^2\,\mathrm{d}x'\,\mathrm{d}t \\
&\quad = \int_0^{2\pi} s^{(N)}_\alpha(t) \int_\sigma V^{(N)}_\alpha(x',\tau)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}t = -\alpha \int_0^{2\pi} \big|s^{(N)}_\alpha(t)\big|^2\,\mathrm{d}t.
\end{aligned}$$

Integrating by parts with respect to t and taking into account the time-periodicity of $V_\alpha^{(N)}(x',t)$, we obtain

$$\nu\int_0^{2\pi}\int_\sigma \left|\nabla' V_\alpha^{(N)}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t+\alpha\int_0^{2\pi}\left|s_\alpha^{(N)}(t)\right|^2 \mathrm{d}t=0.$$

Thus, $s_\alpha^{(N)}(t)=0$ for a.a. $t\in[0,2\pi]$, and the lemma is proved. □

4 A priori estimates of Galerkin approximations

Let the pair $(V_\alpha^{(N)}(x',t), s_\alpha^{(N)}(t))$ be the solution of problem (21) and $U_0(x')$ be the solution of problem (6). Consider the integral $\int_\sigma V_\alpha^{(N)}(x',t)U_0(x')\,\mathrm{d}x'$. Since the mean value $\bar V_\alpha^{(N)}(x')=0$, we have

$$\int_0^{2\pi}\int_\sigma V_\alpha^{(N)}(x',t)U_0(x')\,\mathrm{d}x'\,\mathrm{d}t=\int_\sigma U_0(x')\left(\int_0^{2\pi} V_\alpha^{(N)}(x',t)\,\mathrm{d}t\right)\mathrm{d}x'=0.$$

Therefore, there exists $t_*=t(\alpha,N)$ such that $\int_\sigma V_\alpha^{(N)}(x',t_*)\,U_0(x')\,\mathrm{d}x'=0$.[4] By periodicity we also have $\int_\sigma V_\alpha^{(N)}(x',\,t_*+2\pi)\,U_0(x')\,\mathrm{d}x'=0$.

Let $f\in L^2_\sharp(0,2\pi)$. For $t\in[t_*,\,t_*+2\pi]$, define the notation $S_f^*(t)=-\int_t^{t_*+2\pi} f(\tau)\,\mathrm{d}\tau$. Since the mean value of f vanishes, we have $S_f^*(t_*+2\pi)=S_f^*(t_*)=0$. Moreover, $\mathrm{d}S_f^*(t)/\mathrm{d}t=f(t)$.

Lemma 3. *The following estimate*

$$\begin{aligned}&\int_{t_*}^{t_*+2\pi}\int_\sigma\left|V_\alpha^{(N)}(x',t)\right|^2\mathrm{d}x'\,\mathrm{d}t+\int_{t_*}^{t_*+2\pi}\int_\sigma\left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2\mathrm{d}x'\,\mathrm{d}t+\int_{t_*}^{t_*+2\pi}\left|S^*_{s_\alpha^{(N)}}(\tau)\right|^2\mathrm{d}\tau\\&\quad\leqslant c\int_{t_*}^{t_*+2\pi}\left|F(\tau)\right|^2\mathrm{d}\tau\end{aligned}\tag{23}$$

holds with a constant c independent of α and N.

Proof. Define $\gamma_{k,\alpha}^{(N)}(t)=\int_t^{2\pi+t_*} w_{k,\alpha}^{(N)}(\tau)\,\mathrm{d}\tau$, $t_*\leqslant t\leqslant 2\pi+t_*$. Multiplying $(21)_1$ by $\gamma_{k,\alpha}^{(N)}(t)$, summing the obtained relation from $k=1$ to $k=N$ and integrating them over

[4] The point $t(\alpha,N)$ depends on α and N, but in this section, we denote it just t_* in order to simplify the notation.

the interval $(t_*,\, t_*+2\pi)$, we obtain

$$\begin{aligned}
&\int\limits_{t_*}^{t_*+2\pi}\int\limits_{\sigma}\bigl(V_\alpha^{(N)}\bigr)_t(x',t)\int\limits_{t}^{t_*+2\pi}V_\alpha^{(N)}(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t\\
&\quad+\nu\int\limits_{t_*}^{t_*+2\pi}\int\limits_{\sigma}\nabla'V_\alpha^{(N)}(x',t)\int\limits_{t}^{t_*+2\pi}\nabla'V_\alpha^{(N)}(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t\\
&\quad=\int\limits_{t_*}^{t_*+2\pi}s_\alpha^{(N)}(t)\int\limits_{t}^{t_*+2\pi}\int\limits_{\sigma}V_\alpha^{(N)}(x',\tau)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}t. \qquad (24)
\end{aligned}$$

Using the relation

$$\int\limits_{\sigma}V_\alpha^{(N)}(x',t)\mathrm{d}x'=\sum_{k=1}^{N}\beta_k^2\int\limits_{0}^{2\pi}G_k(t,\tau)s_\alpha^{(N)}(\tau)\,\mathrm{d}\tau$$

and $(21)_3$, we derive

$$\int\limits_{\sigma}V_\alpha^{(N)}(x',t)\,\mathrm{d}x'=F(t)-\alpha s_\alpha^{(N)}(t).$$

Therefore, (24) can be written as

$$\begin{aligned}
&\int\limits_{t_*}^{t_*+2\pi}\int\limits_{\sigma}\bigl(V_\alpha^{(N)}\bigr)_t(x',t)\int\limits_{t}^{t_*+2\pi}V_\alpha^{(N)}(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t\\
&\quad+\nu\int\limits_{t_*}^{t_*+2\pi}\int\limits_{\sigma}\nabla'V_\alpha^{(N)}(x',t)\int\limits_{t}^{t_*+2\pi}\nabla'V_\alpha^{(N)}(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t\\
&\quad=\int\limits_{t_*}^{t_*+2\pi}s_\alpha^{(N)}(t)\int\limits_{t}^{t_*+2\pi}\bigl(F(\tau)-\alpha s_\alpha^{(N)}(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}t. \qquad (25)
\end{aligned}$$

Then

$$\begin{aligned}
&\int\limits_{t_*}^{t_*+2\pi}s_\alpha^{(N)}(t)\int\limits_{t}^{t_*+2\pi}\bigl(F(\tau)-\alpha s_\alpha^{(N)}(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}t\\
&\quad=\int\limits_{t_*}^{t_*+2\pi}\frac{\mathrm{d}S^*_{s_\alpha^{(N)}}(t)}{\mathrm{d}t}\int\limits_{t}^{t_*+2\pi}\bigl(F(\tau)-\alpha s_\alpha^{(N)}(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}t\\
&\quad=\int\limits_{t_*}^{t_*+2\pi}S^*_{s_\alpha^{(N)}}(t)F(t)\,\mathrm{d}t-\alpha\int\limits_{t_*}^{t_*+2\pi}S^*_{s_\alpha^{(N)}}(t)s_\alpha^{(N)}(t)\,\mathrm{d}t
\end{aligned}$$

$$= \int\limits_{t_*}^{t_*+2\pi} S^*_{s^{(N)}_\alpha}(t)F(t)\,\mathrm{d}t - \alpha \int\limits_{t_*}^{t_*+2\pi} S^*_{s^{(N)}_\alpha}(t)\frac{\mathrm{d}S^*_{s^{(N)}_\alpha}(t)}{\mathrm{d}t}\,\mathrm{d}t$$
$$= \int\limits_{t_*}^{t_*+2\pi} S^*_{s^{(N)}_\alpha}(t)F(t)\,\mathrm{d}t.$$

Calculating similarly two integrals J_1 and J_2 on the left-hand side of (25), we derive that

$$J_1 = \int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma \left|V^{(N)}_\alpha(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t, \qquad J_2 = 0.$$

Hence, relation (25) takes the form

$$\int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma \left|V^{(N)}_\alpha(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t = \int\limits_{t_*}^{t_*+2\pi} S^*_{s^{(N)}_\alpha}(t)F(t)\,\mathrm{d}t. \tag{26}$$

Now multiply $(21)_1$ by $(t-t_*)\gamma^{(N)}_{k,\alpha}(t)$, sum from $k=1$ till $k=N$ and integrate over the interval $(t_*,\, t_*+2\pi)$:

$$\begin{aligned}
&\int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma (t-t_*)\big(V^{(N)}_\alpha\big)_t(x',t) \int\limits_t^{t_*+2\pi} V^{(N)}_\alpha(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t \\
&\quad + \nu \int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma (t-t_*)\nabla' V^{(N)}_\alpha(x',t) \int\limits_t^{t_*+2\pi} \nabla' V^{(N)}_\alpha(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}x'\,\mathrm{d}t \\
&= \int\limits_{t_*}^{t_*+2\pi} (t-t_*)s^{(N)}_\alpha(t) \int\limits_t^{t_*+2\pi} \big(F(\tau) - \alpha s^{(N)}_\alpha(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned} \tag{27}$$

Evaluating each of the three integrals in (27) and having in mind that mean values of participating functions are zero, we obtain

$$\begin{aligned}
&\int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma (t-t_*)\left|V^{(N)}_\alpha(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t + \frac{\nu}{2} \int\limits_{t_*}^{t_*+2\pi} \int\limits_\sigma \left|\nabla' S^*_{V^{(N)}_\alpha}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t \\
&= \int\limits_{t_*}^{t_*+2\pi} (t-t_*)S^*_{s^{(N)}_\alpha}(t)F(t)\,\mathrm{d}t - \frac{\alpha}{2} \int\limits_{t_*}^{t_*+2\pi} \left|S^*_{s^{(N)}_\alpha}(t)\right|^2 \mathrm{d}t \\
&\quad - \int\limits_{t_*}^{t_*+2\pi} S^*_{s^{(N)}_\alpha}(t) \int\limits_t^{t_*+2\pi} F(\tau)\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned} \tag{28}$$

From (26) and (28) it follows that

$$\int_{t_*}^{t_*+2\pi}\int_\sigma \left|V_\alpha^{(N)}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t \leqslant \varepsilon \int_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(t)\right|^2 \mathrm{d}t + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 \mathrm{d}t \qquad (29)$$

and

$$\begin{aligned}
&\frac{\nu}{2}\int_{t_*}^{t_*+2\pi}\int_\sigma \left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t \\
&\leqslant \left|\int_{t_*}^{t_*+2\pi} (t-t_*) S^*_{s_\alpha^{(N)}}(t) F(t)\,\mathrm{d}t\right| + \left|\int_{t_*}^{t_*+2\pi} S^*_{s_\alpha^{(N)}}(t) \int_t^{t_*+2\pi} F(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\right| \\
&\leqslant \varepsilon\left(4\pi^2+1\right)\int_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(t)\right|^2 \mathrm{d}t + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 \mathrm{d}t + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} \left|\int_t^{t_*+2\pi} F(\tau)\,\mathrm{d}\tau\right|^2 \mathrm{d}t \\
&\leqslant \left(4\pi^2+1\right)\left(\varepsilon\int_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(t)\right|^2 \mathrm{d}t + \frac{1}{2\varepsilon}\int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 \mathrm{d}t\right). \qquad (30)
\end{aligned}$$

Let us estimate the integral $\int_{t_*}^{t_*+2\pi} |S^*_{s_\alpha^{(N)}}(t)|^2\,\mathrm{d}t$. Let $U_0 \in \mathring{W}^{1,2}(\sigma)$ be a solution of problem (6). Remind that the flux of U_0 is nonzero, $\kappa_0 = \int_\sigma U_0(x')\,\mathrm{d}x' > 0$ (see (7)). Since $\{u_k(x')\}$ is a basis in $\mathring{W}^{1,2}(\sigma)$, U_0 can be expressed as a Fourier series in $\mathring{W}^{1,2}(\sigma)$:

$$U_0(x') = \sum_{k=1}^\infty a_k u_k(x'), \quad a_k \in \mathbb{R}.$$

Let us multiply relations $(21)_1$ by a_k and sum them over k. This gives

$$\begin{aligned}
&\int_\sigma \left(V_\alpha^{(N)}\right)_t(x',t)\,U_0(x')\,\mathrm{d}x' + \nu\int_\sigma \nabla' V_\alpha^{(N)}(x',t)\cdot\nabla' U_0(x')\,\mathrm{d}x' \\
&\quad = s_\alpha^{(N)}(t)\int_\sigma U_0(x')\,\mathrm{d}x' = s_\alpha^{(N)}(t)\kappa_0. \qquad (31)
\end{aligned}$$

On the other hand, multiplying (6) by $V_\alpha^{(N)}(x',t)$ and integrating by parts in σ, we obtain

$$\nu\int_\sigma \nabla' U_0(x')\cdot\nabla' V_\alpha^{(N)}(x',t)\,\mathrm{d}x' = \int_\sigma V_\alpha^{(N)}(x',t)\,\mathrm{d}x' = F(t) - \alpha s_\alpha^{(N)}(t). \qquad (32)$$

Substituting (32) into (31) yields

$$\int_\sigma \left(V_\alpha^{(N)}\right)_t(x',t)\,U_0(x')\,\mathrm{d}x' + F(t) - \alpha s_\alpha^{(N)}(t) = s_\alpha^{(N)}(t)\kappa_0,$$

i.e.,

$$(\kappa_0+\alpha)s_\alpha^{(N)}(t)=\int\limits_\sigma \bigl(V_\alpha^{(N)}\bigr)_t(x',t)\,U_0(x')\,\mathrm{d}x'+F(t). \tag{33}$$

Integrating (33) with respect to t from τ to $t_*+2\pi$, we obtain

$$\begin{aligned}(\kappa_0+\alpha)\int\limits_\tau^{t_*+2\pi} s_\alpha^{(N)}(t)\,\mathrm{d}t&=-(\kappa_0+\alpha)S^*_{s_\alpha^{(N)}}(\tau)\\&=-\int\limits_\sigma V_\alpha^{(N)}(x',\tau)U_0(x')\,\mathrm{d}x'+\int\limits_\tau^{t_*+2\pi}F(t)\,\mathrm{d}t.\end{aligned} \tag{34}$$

Here we have used the choice of the point t_*, that is

$$\int\limits_\sigma V_\alpha^{(N)}(x',t_*)\,U_0(x')\,\mathrm{d}x'=\int\limits_\sigma V_\alpha^{(N)}(x',\,t_*+2\pi)\,U_0(x')\,\mathrm{d}x'=0,$$

and hence,

$$\int\limits_\tau^{t_*+2\pi}\int\limits_\sigma \bigl(V_\alpha^{(N)}\bigr)_t(x',t)\,U_0(x')\,\mathrm{d}x'\,\mathrm{d}t=-\int\limits_\sigma V_\alpha^{(N)}(x',\tau)\,U_0(x')\,\mathrm{d}x'.$$

From (34) it follows that

$$\begin{aligned}&(\kappa_0+\alpha)^2\int\limits_{t_*}^{t_*+2\pi}\bigl|S^*_{s_\alpha^{(N)}}(\tau)\bigr|^2\,\mathrm{d}\tau\\&\leqslant 2\int\limits_{t_*}^{t_*+2\pi}\Bigl(\int\limits_\sigma V_\alpha^{(N)}(x',\tau)U_0(x')\,\mathrm{d}x'\Bigr)^2\mathrm{d}\tau+2\int\limits_{t_*}^{t_*+2\pi}\Bigl(\int\limits_\tau^{t_*+2\pi}F(t)\,\mathrm{d}t\Bigr)^2\mathrm{d}\tau\\&\leqslant 2\int\limits_{t_*}^{t_*+2\pi}\int\limits_\sigma\bigl|V_\alpha^{(N)}(x',\tau)\bigr|^2\,\mathrm{d}x'\int\limits_\sigma\bigl|U_0(x')\bigr|^2\,\mathrm{d}x'\,\mathrm{d}\tau+2\int\limits_{t_*}^{t_*+2\pi}\bigl|S_F^*(\tau)\bigr|^2\,\mathrm{d}\tau\\&\leqslant c\int\limits_{t_*}^{t_*+2\pi}\int\limits_\sigma\bigl|V_\alpha^{(N)}(x',\tau)\bigr|^2\mathrm{d}x'\int\limits_\sigma\bigl|\nabla' U_0(x')\bigr|^2\,\mathrm{d}x'\,\mathrm{d}\tau+2\int\limits_{t_*}^{t_*+2\pi}\bigl|S_F^*(\tau)\bigr|^2\,\mathrm{d}\tau\\&\leqslant c\Biggl(\int\limits_{t_*}^{t_*+2\pi}\int\limits_\sigma\bigl|V_\alpha^{(N)}(x',\tau)\bigr|^2\,\mathrm{d}x'\,\mathrm{d}\tau+\int\limits_{t_*}^{t_*+2\pi}\bigl|F(\tau)\bigr|^2\,\mathrm{d}\tau\Biggr).\end{aligned} \tag{35}$$

Substituting (35) into (29) yields

$$\int\limits_{t_*}^{t_*+2\pi}\int\limits_\sigma\bigl|V_\alpha^{(N)}(x',t)\bigr|^2\,\mathrm{d}x'\,\mathrm{d}t\leqslant c\varepsilon\int\limits_{t_*}^{t_*+2\pi}\int\limits_\sigma\bigl|V_\alpha^{(N)}(x',t)\bigr|^2\,\mathrm{d}x'\,\mathrm{d}t+c_\varepsilon\int\limits_{t_*}^{t_*+2\pi}\bigl|F(t)\bigr|^2\,\mathrm{d}t,$$

and choosing ε sufficiently small, we obtain

$$\int_{t_*}^{t_*+2\pi}\int_\sigma \left|V_\alpha^{(N)}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t \leqslant c \int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 \mathrm{d}t. \tag{36}$$

Estimates (36) and (35) give

$$\int_{t_*}^{t_*+2\pi} \left|S^*_{s_\alpha^{(N)}}(\tau)\right|^2 \mathrm{d}\tau \leqslant c \int_{t_*}^{t_*+2\pi} \left|F(\tau)\right|^2 \mathrm{d}\tau. \tag{37}$$

Finally, from (30) and (37) it follows that

$$\int_{t_*}^{t_*+2\pi}\int_\sigma \left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t \leqslant c \int_{t_*}^{t_*+2\pi} \left|F(t)\right|^2 \mathrm{d}t. \tag{38}$$

The constants in (36)–(38) are independent of α and N. □

5 Convergence of Galerkin approximations. Proof of Theorem 1

5.1 Proof of existence

The constructed approximate solutions $(V_\alpha^{(N)}(x',t), s_\alpha^{(N)}(t))$ satisfy equalities $(21)_1$. Multiplying these relations by arbitrary functions $d_k(t) \in L^2_\wp(0,2\pi)$ such that $d_k'(t) \in L^2_\sharp(0,2\pi)$, summing over k from $k=1$ to $k=M, M \leqslant N$, integrating with respect to t and then integrating by parts, we obtain the integral identity

$$\int_0^{2\pi}\int_\sigma V_\alpha^{(N)}(x',t)\,\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t + \nu\int_0^{2\pi}\int_\sigma \nabla' S^*_{V_\alpha^{(N)}}(x',t)\cdot\nabla'\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t$$
$$= \int_0^{2\pi} S^*_{s_\alpha^{(N)}}(\tau)\int_\sigma \eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t \tag{39}$$

for test functions η having the form $\eta(x',t) = \sum_{k=1}^M d_k(t)u_k(x')$.

Recall that $(V_\alpha^{(N)}(x',t), s_\alpha^{(N)}(t))$ obey a priori estimate (23) with a constant c independent of α and N.

Since all functions in (23) are 2π-periodic, inequality (23) is equivalent to

$$\int_0^{2\pi}\int_\sigma \left|V_\alpha^{(N)}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t + \int_0^{2\pi}\int_\sigma \left|\nabla' S^*_{V_\alpha^{(N)}}(x',t)\right|^2 \mathrm{d}x'\,\mathrm{d}t + \int_0^{2\pi} \left|S^*_{s_\alpha^{(N)}}(\tau)\right|^2 \mathrm{d}\tau$$
$$\leqslant c\int_0^{2\pi} \left|F(\tau)\right|^2 \mathrm{d}\tau. \tag{40}$$

Let us fix N and choose a subsequences $\{\alpha_l\}$ and $\{(V_{\alpha_l}^{(N)}(x',t), s_{\alpha_l}^{(N)}(t))\}$ such that $\lim_{l\to\infty}\alpha_l = 0$, $\{V_{\alpha_l}^{(N)}\}$ converges weakly in $L^2_\sharp(0,2\pi; L^2(\sigma))$ to some $V^{(N)}$, $\{S^*_{V_{\alpha_l}^{(N)}}\}$ converges weakly in $L^2_\wp(0,2\pi; \mathring{W}^{1,2}(\sigma))$ to $S_{V^{(N)}}$[5], while $\{s_{\alpha_l}^{(N)}\}$ converges weakly in $W_\wp^{-1,2}(0,2\pi)$ to $s^{(N)}$. The last convergence means that

$$\lim_{l\to\infty}\int_0^{2\pi} S^*_{s_{\alpha_l}^{(N)}}(t)\,\eta'(t)\,\mathrm{d}t = \int_0^{2\pi} S_{s^{(N)}}(t)\,\eta'(t)\,\mathrm{d}t = \langle s^{(N)},\eta\rangle \quad \forall\eta\in W_\wp^{1,2}(0,2\pi),$$

where $s^{(N)}\in W_\wp^{-1,2}(0,2\pi)$, and $S_{s^{(N)}}\in L^2_\sharp(0,2\pi)$ is a primitive of $s^{(N)}$ in the distributional sense.

Obviously, for the limit functions $V^{(N)}$ and $S_{s^{(N)}}$, estimate (40) remains valid with a constant c independent of N. In (39), taking $\alpha=\alpha_l$ and passing to the limit as $\alpha_l\to 0$, we get

$$\begin{aligned}\int_0^{2\pi}\int_\sigma V^{(N)}(x',t)\,\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t + \nu\int_0^{2\pi}\int_\sigma \nabla' S_{V^{(N)}}(x',t)\cdot\nabla'\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t \\ = \int_0^{2\pi} S_{s^{(N)}}(\tau)\int_\sigma \eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t. \qquad (41)\end{aligned}$$

Let us show that $V^{(N)}(x',t)$ satisfy the flux condition

$$\int_\sigma V^{(N)}(x',t)\,\mathrm{d}x' = F(t). \qquad (42)$$

Integrating equation $(21)_3$ for $\alpha=\alpha_l$ from t to 2π yields

$$\alpha_l S_{s_{\alpha_l}^{(N)}}(t) + \int_t^{2\pi}\int_\sigma V_{\alpha_l}^{(N)}(x',\tau)\,\mathrm{d}x'\,\mathrm{d}\tau = S_F(t). \qquad (43)$$

Obviously, the sequence $\{\varphi_l^{(N)}(\tau) = \int_\sigma V_{\alpha_l}^{(N)}(x',\tau)\mathrm{d}x'\}$ is bounded in $L^2(0,2\pi)$. So we may assume, without loss of generality, that $\{\varphi_l^{(N)}\}$ is weakly convergent to $\varphi^{(N)}$ in $L^2(0,2\pi)$. Then the sequence of primitives $S_{\varphi_l^{(N)}}(t) = -\int_t^{2\pi}\varphi_l^{(N)}(\tau)\,\mathrm{d}\tau \to S_{\varphi^{(N)}}(t)$ for all $t\in[0,2\pi]$, and hence, $\|S_{\varphi_l^{(N)}} - S_{\varphi^{(N)}}\|_{L^2(0,2\pi)} \to 0$ as $l\to\infty$ ($\alpha_l\to 0$). From (43) we have

$$\|S_{\varphi_l^{(N)}} - S_F\|_{L^2(0,2\pi)} = \alpha_l\|S_{s_{\alpha_l}^{(N)}}\|_{L^2(0,2\pi)} \leqslant c\alpha_l \to 0 \quad \text{as } l\to\infty.$$

[5]Recall that for $U\in L^2_\sharp(0,T;L^2(\sigma))$, the function S_U coincides with the primitive of U.

Therefore,

$$\int_t^{2\pi}\int_\sigma V^{(N)}(x',\tau)\,\mathrm{d}x'\,\mathrm{d}\tau=\int_t^{2\pi}F(\tau)\,\mathrm{d}\tau\quad\text{for a.a. } t\in[0,2\pi],$$

and differentiating this equality with respect to t, we get (42).

Now we choose a subsequence $\{(V^{(N_k)}(x',t),s^{(N_k)}(t))\}$ such that $\{V^{(N_k)}\}$ converges weakly in $L^2_\sharp(0,2\pi;L^2(\sigma))$ to some V, $\{S_{V^{(N_k)}}\}$ converges weakly in $L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma))$ to S_V, and $\{s^{(N_k)}\}$ converges weakly in $W^{-1,2}_\wp(0,2\pi)$ to s. In (41), passing to the limit over the subsequence as $N_k\to+\infty$ yields

$$\begin{aligned}&\int_0^{2\pi}\int_\sigma V(x',t)\,\eta_t(x',t)\mathrm{d}x'\,\mathrm{d}t+\nu\int_0^{2\pi}\int_\sigma\nabla' S_V(x',t)\cdot\nabla'\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t\\&\quad=\int_0^{2\pi}S_s(\tau)\int_\sigma\eta_t(x',t)\,\mathrm{d}x'\,\mathrm{d}t.\end{aligned}\tag{44}$$

Exactly as above, we can prove that $V(x',t)$ satisfies the flux condition (12). However, the integral identity (44) is proved, up to now, only for test functions η, which can be represented as the sums: $\eta(x',t)=\sum_{k=1}^M d_k(t)u_k(x')$ with $d_k(t)\in L^2_\wp(0,2\pi)$ such that $d'_k(t)\in L^2_\sharp(0,2\pi)$. After we have passed to the limit in (41) as $N_l\to+\infty$, the subscript M in these sums can be arbitrary large natural number. Such sums are dense in the space $\mathcal{V}=\{\eta\colon\ \eta\in L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma)),\ \eta_t\in L^2_\sharp(0,2\pi;L^2(\sigma))\}$. This can be proved exactly in the same way as it is done in the book [9] for the case of an initial boundary value problem. Thus, they also are dense in the subspace $\mathcal{V}_1=\{\eta\colon\ \eta\in L^2_\wp(0,2\pi;\mathring{W}^{1,2}(\sigma)),\ \eta_t\in L^2_\sharp(0,2\pi;W^{1,2}(\sigma))\}\subset\mathcal{V}$, and therefore, (41) remains valid for all $\eta\in\mathcal{V}_1$. This proves that $(V(x',t),s(t))$ satisfies the integral identity (13), and thus, it is a weak solution of problem (8). Estimate (14) for $(V(x',t),s(t))$ follows from estimate (40) for the approximate solutions.

5.2 Proof of uniqueness

Assume that $F(t)=0$. Since $\eta_t\in L^2_\sharp(0,2\pi;W^{1,2}(\sigma))$, we take

$$\begin{aligned}\eta(x',t)&=\int_t^{2\pi}\big(S_V(x',\tau)-\bar S_V(x')\big)\,\mathrm{d}\tau\\&=\int_t^{2\pi}\Bigg(-\int_\tau^{2\pi}V(x',\mu)\,\mathrm{d}\mu+\frac{1}{2\pi}\int_0^{2\pi}\int_t^{2\pi}V(x',\mu)\,\mathrm{d}\mu\,\mathrm{d}t\Bigg)\,\mathrm{d}\tau.\end{aligned}$$

Obviously, $\eta \in \mathcal{V}_1$. Putting this η into identity (13), we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_\sigma V(x',t)\big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t \\
&\qquad + \nu\int_0^{2\pi}\int_\sigma \nabla' S_V(x',t)\cdot\nabla'\big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t \\
&\quad = \int_0^{2\pi} S_s(t)\int_\sigma \big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t. \qquad (45)
\end{aligned}$$

Since

$$\int_\sigma V(x',t)\,\mathrm{d}x' = F(t) = 0,$$

by Fubini's theorem and the time-periodicity of the function $V(x',t)$ we have

$$\begin{aligned}
&\int_\sigma \big(S_V(x',t)-\bar S_V(x')\big)\mathrm{d}x' \\
&\quad = \int_\sigma\bigg(-\int_t^{2\pi} V(x',\tau)\,\mathrm{d}\tau\bigg)\mathrm{d}x' + \frac{1}{2\pi}\int_\sigma\bigg(\int_0^{2\pi}\int_t^{2\pi} V(x',\tau)\,\mathrm{d}\tau\,\mathrm{d}t\bigg)\mathrm{d}x' \\
&\quad = -\int_t^{2\pi}\bigg(\int_\sigma V(x',\tau)\,\mathrm{d}x'\bigg)\,\mathrm{d}\tau + \frac{1}{2\pi}\int_0^{2\pi}\int_t^{2\pi}\bigg(\int_\sigma V(x',\tau)\,\mathrm{d}x'\bigg)\,\mathrm{d}\tau\,\mathrm{d}t = 0;
\end{aligned}$$

$$\begin{aligned}
&\int_0^{2\pi}\int_\sigma V(x',t)\big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t \\
&\quad = \frac{1}{2}\int_0^{2\pi}\frac{d}{\mathrm{d}t}\int_\sigma \big|S_V(x',t)\big|^2\,\mathrm{d}t - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_\sigma S_V(x',t)\bar S_V(x')\,\mathrm{d}x\,\mathrm{d}t = 0;
\end{aligned}$$

$$\begin{aligned}
&\int_0^{2\pi}\int_\sigma \nabla' S_V(x',t)\cdot\nabla'\big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t \\
&\quad = \int_0^{2\pi}\int_\sigma \big|\nabla'\big(S_V(x',t)-\bar S_V(x')\big)\big|^2\,\mathrm{d}x'\,\mathrm{d}t \\
&\qquad + \int_0^{2\pi}\int_\sigma \nabla'\bar S_V(x')\cdot\big(S_V(x',t)-\bar S_V(x')\big)\,\mathrm{d}x'\,\mathrm{d}t
\end{aligned}$$

$$= \int_0^{2\pi} \int_\sigma \left| \nabla' \big(S_V(x', t) - \bar{S}_V(x') \big) \right|^2 \mathrm{d}x' \, \mathrm{d}t$$

$$- \int_\sigma \nabla' \bar{S}_V(x') \cdot \nabla' \left(\int_0^{2\pi} \big(S_V(x', t) - \bar{S}_V(x') \big) \right) \mathrm{d}t \, \mathrm{d}x'$$

$$= \int_0^{2\pi} \int_\sigma \left| \nabla' \big(S_V(x', t) - \bar{S}_V(x') \big) \right|^2 \mathrm{d}x' \, \mathrm{d}t.$$

Therefore, from (45) it follows that

$$\int_0^{2\pi} \int_\sigma \left| \nabla' \big(S_V(x', t) - \bar{S}_V(x') \big) \right|^2 \mathrm{d}x' \, \mathrm{d}t = 0.$$

Then $\nabla'(S_V(x', t) - \bar{S}_V(x')) = 0$, and hence, $S_V(x', t) - \bar{S}_V(x') = m(t)$. Integrating over σ, we get

$$|\sigma| m(t) = - \int_\sigma \int_t^{2\pi} V(x', \tau) \, \mathrm{d}\tau \, \mathrm{d}x' + \frac{1}{2\pi} \int_\sigma \int_0^{2\pi} \int_\tau^{2\pi} V(x', \mu) \, \mathrm{d}\mu \, \mathrm{d}\tau \, \mathrm{d}x'$$

$$= - \int_t^{2\pi} \left(\int_\sigma V(x', \tau) \, \mathrm{d}x' \right) \mathrm{d}\tau + \frac{1}{2\pi} \int_0^{2\pi} \int_\tau^{2\pi} \left(\int_\sigma V(x', \mu) \mathrm{d}x' \right) \mathrm{d}\mu \, \mathrm{d}\tau = 0.$$

Thus, $S_V(x', t) = \bar{S}_V(x')$, and since $S_V(x', 0) = S_V(x', 2\pi) = 0$, we conclude that $S_V(x', t) = 0$ for a.a. $(x', t) \in \sigma \times [0, 2\pi]$, that is $\int_t^{2\pi} V(x', \tau) \, \mathrm{d}\tau = 0$ for a.a. x' and t. This implies $V(x', t) = 0$.

From identity (13) it follows that

$$\int_0^{2\pi} S_s(t) \int_\sigma \eta_t(x', t) \, \mathrm{d}x' \, \mathrm{d}t = 0 \quad \forall \eta \in \mathcal{V}_1. \tag{46}$$

In (46), we take $\eta = b(t) U_0(x')$, where $b \in W_\wp^{1,2}(0, 2\pi)$ is arbitrary, and $U_0(x')$ is the solution of problem (6). Recall that $\int_\sigma U_0(x') \, \mathrm{d}x' = \kappa_0 \neq 0$. Then (46) takes the form

$$\int_0^{2\pi} S_s(t) \int_\sigma \eta_t(x', t) \, \mathrm{d}x' \, \mathrm{d}t = \kappa_0 \int_0^{2\pi} S_s(t) b'(t) \, \mathrm{d}t = 0 \quad \forall b \in W_\wp^{1,2}(0, 2\pi).$$

Thus, $S_s(t) = \mathrm{const}$. Since $S_s \in L^2_\sharp(0, 2\pi)$, i.e., the mean value of $S_s(t)$ is equal to zero, we get that $S_s(t) = 0$. Therefore, the functional $s = 0$.

References

1. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, San Francisco, London, 1975.
2. G.K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, Cambridge, 2002.
3. H. Beirão da Veiga, Time-periodic solutions of the Navier–Stokes equations in an unbounded cylindrical domains. Leray's problem for periodic flows, *Arch. Ration. Mech. Anal.*, **178**:301–325, 2005, `https://doi.org/10.1007/s00205-005-0376-3`.
4. G.P. Galdi, On the problem of steady bifurcation of a falling sphere in a Navier–Stokes liquid, *J. Math. Phys.*, **61**(8), 2020, `https://doi.org/10.1063/5.0011248`.
5. G.P. Galdi, On the self-propulsion of a rigid body in a viscous liquid by time-periodic boundary data, *J. Math. Fluid Mech.*, **22**(61), 2020, `https://doi.org/10.1007/s00021-020-00537-z`.
6. G.P. Galdi, T. Hishida, Attainability of time–periodic flow of a viscous liquid past an oscillating body, *J. Evol. Equ.*, 2021, `https://doi.org/10.1007/s00028-020-00661-3`.
7. G.P. Galdi, K. Pileckas, A. Silvestre, On the unsteady Poiseuille flow in a pipe, *Z. Angew. Math. Phys.*, **58**:994–1007, 2007, `https://doi.org/10.1007/s00033-006-6114-3`.
8. G.P. Galdi, A.M. Robertson, The relation between flow rate and axial pressure gradient for time-periodic Poiseuille flow in a pipe, *J. Math. Fluid Mech.*, **7**:215–223, 2005, `https://doi.org/10.1007/s00021-005-0154-x`.
9. O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer, New York, 1985.
10. P.K. Lamm, *Surveys on Solution Methods for Inverse Problems*, Springer, Vien, 2000.
11. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Oxford, New York, 1987.
12. G. Panasenko, Asymptotic expansion of the solution of Navier–Stokes equation in a tube structure, *C. R. Acad. Sci., Paris, Sér. II, Fasc. b, Méc. Phys. Astron.*, **326**(12):867–872, 1998, `https://doi.org/10.1016/S1251-8069(99)80041-6`.
13. G. Panasenko, *Multi-Scale Modelling for Structures and Composites*, Springer, Dordrecht, 2005.
14. G. Panasenko, K. Pileckas, Asymptotic analysis of the nonsteady viscous flow with a given flow rate in a thin pipe, *Appl. Anal.*, **91**(3):559–574, 2012, `https://doi.org/10.1080/00036811.2010.549483`.
15. G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier–Stokes equations in a tube structure. I: The case without boundary layer-in-time, *Nonlinear Anal., Theory Methods Appl.*, **122**:125–168, 2015, `https://doi.org/10.1016/j.na.2015.03.008`.
16. G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier–Stokes equations in a tube structure. II: General case, *Nonlinear Anal., Theory Methods Appl.*, **125**:582–607, 2015, `https://doi.org/10.1016/j.na.2015.05.018`.
17. K. Pileckas, On the behavior of the nonstationary Poiseuille solution as $t \to \infty$, *Sib. Math. J.*, **46**:707–716, 2005, `https://doi.org/10.1007/s11202-005-0071-5`.
18. K. Pileckas, Existence of solutions with the prescribed flux of the Navier–Stokes system in an infinite cylinder, *J. Math. Fluid Mech.*, **8**:542–563, 2006, `https://doi.org/10.1007/s00021-005-0187-1`.

19. K. Pileckas, The Navier–Stokes system in domains with cylindrical outlets to infinity. Leray's problem, in S. Friedlander, D. Serre (Eds.), *Handbook of Mathematical Fluid Dynamics, Vol. 4*, North-Holland, 2007, pp. 445–647, https://doi.org/10.1016/S1874-5792(07)80012-7.
20. K. Pileckas, Solvability in weighted spaces of the three-dimensional Navier–Stokes problem in domains with cylindrical outlets to infinity, *Topol. Methods Nonlinear Anal.*, **29**(2):333–360, 2007.
21. K. Pileckas, Global solvability in $W_2^{2,1}$-weighted spaces of the two-dimensional Navier–Stokes problem in domains with strip-like outlets to infinity, *J. Math. Fluid Mech.*, **10**(2):272–309, 2008, https://doi.org/10.1007/s00021-006-0232-8.
22. K. Pileckas, R. Čiegis, Existence of nonstationary Poiseuille-type solutions under minimal regularity assumptions, *Z. Angew. Math. Phys.*, **71**(192), 2020, https://doi.org/10.1007/s00033-020-01422-5.
23. K. Pileckas, V. Keblikas, Existence of nonstationary Poiseuille solution, *Sib. Math. J.*, **46**: 514–526, 2005, https://doi.org/10.1007/s11202-005-0053-7.
24. J.L.M. Poiseuille, *Recherches Expérimentales sur le Mouvement des Liquides dans les Tubes de Très-Petits Diamètres*, Imprimerie Royale, Paris, 1844.
25. W.W. Schmaedeke, Approximate solutions for the Volterra equations of the first kind, *J. Math. Anal. Appl.*, **23**(3):604–613, 1968, https://doi.org/10.1016/0022-247X(68)90140-6.
26. S.S. Sritharan, On the acceleration of viscous fluid through an unbounded channel, *J. Math. Anal. Appl.*, **168**(1):255–283, 1992, https://doi.org/10.1016/0022-247X(92)90204-Q.
27. F.G. Tricomi, *Integral Equations*, Intersience, New York, 1957.
28. J.R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known, *J. Physiol.*, **127**(3):553–563, 1955, https://doi.org/10.1113/jphysiol.1955.sp005276.