Research Article

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Inequalities between height and deviation of polynomials

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Abstract: In this paper, for polynomials with real coefficients P, Q satisfying $|P(x)| \le |Q(x)|$ for each x in a real interval I, we prove the bound $L(P) \le cL(Q)$ between the lengths of P and Q with a constant c, which is exponential in the degree d of P. An example showing that the constant c in this bound should be at least exponential in d is also given. Similar inequalities are obtained for the heights of P and Q when the interval I is infinite and P, Q are both of degree d. In the proofs and in the constructions of examples, we use some translations of Chebyshev polynomials.

Keywords: height of a polynomial, Chebyshev polynomials

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1 Introduction

Throughout, for a polynomial

$$f(x) = a_0 + a_1x + \cdots + a_dx^d \in \mathbb{C}[x], \quad a_d \neq 0,$$

of degree d by $H(f) := \max_{0\le k\le d}|a_k|$ and $L(f) := \sum_{k=0}^{d}|a_k|$ we denote its *height* and its *length*, respectively. It is clear that

$$H(f) \le \left(\sum_{k=0}^{d}|a_k|^2\right)^{1/2} \le \max_{|z|=1}|f(z)| \le L(f) \le (d+1)H(f) \tag{1}$$

(see, e.g., [1]) and

$$|f(x)| \le L(f)\max\{1, |x|\}^d \tag{2}$$

for any $x \in \mathbb{C}$. In particular, by (1), for polynomials $f, g \in \mathbb{C}[x]$ the inequality for their maximums on the unit circle

$$\max_{|z|=1}|f(z)| \le \max_{|z|=1}|g(z)| \tag{3}$$

implies the inequality for their heights

$$H(f) \le (\deg g + 1)H(g).$$

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See also [2–11] for some other inequalities between various heights of polynomials and their factors, not only $H(f)$, $L(f)$ and $\max_{|z|=1}|f(z)|$ but also their *house* $\overline{|\alpha|} := \max_{\alpha:f(\alpha)=0} |\alpha|$, *Mahler's measure* $M(f) := |a_d| \prod_{\alpha:f(\alpha)=0} \max\{1, |\alpha|\}$, etc.

In this paper, we will consider the case when in (3) the unit circle is replaced by a real (finite or infinite) interval. The following questions related to his research in the study of solutions of singular parabolic differential equations were recently asked by my colleague Prof. V. Mackevičius.

Find an explicit constant $\xi(d)$ for which the inequality

$$H(P) \le \xi(d)H(Q) \tag{4}$$

holds for any $P, Q \in \mathbb{R}[x]$ of the degree d satisfying

$$|P(x)| \le |Q(x)| \quad \text{for } x \ge 0. \tag{5}$$

He is also interested in an explicit constant $\eta(d)$ for which the inequality

$$H(P) \le \eta(d)H(Q) \tag{6}$$

holds for any $P, Q \in \mathbb{R}[x]$ of the degree d satisfying

$$|P(x)| \le |Q(|x|)| \quad \text{for } x \in \mathbb{R}. \tag{7}$$

Note that for $d = 1$ we have $P(x) = a_0 + a_1x$ and $Q(x) = b_0 + b_1x$. Inserting $x = 0$ into (5) or (7), we get $|a_0| \le |b_0|$. Similarly, inserting $x = r$ and letting $r \to \infty$, in both cases, (5) and (7), we obtain $|a_1| \le |b_1|$. Hence,

$$\xi(1) = \eta(1) = 1 \tag{8}$$

are the best possible constants in (4) and (6) for $d = 1$.

Likewise, for $d = 2$, assuming that $P(x) = a_0 + a_1x + a_2x^2$ and $Q(x) = b_0 + b_1x + b_2x^2$, we find that $|a_0| \le |b_0|$ and $|a_2| \le |b_2|$. Thus, $|a_0|, |a_2| \le H(Q)$. Inserting $x = 1$ into (5) we deduce $|a_0 + a_1 + a_2| \le 3H(Q)$. Thus, $|a_1| \le 3H(Q) + |a_0| + |a_2| \le 5H(Q)$. Consequently, $H(P) \le 5H(Q)$ for any quadratic polynomials P, Q satisfying (5). This, combined with the example $P(x) = 1 - 5x + x^2$, $Q(x) = 1 + x + x^2$ satisfying (5) and $H(P) = 5H(Q)$, shows that

$$\xi(2) = 5 \tag{9}$$

is the best possible constant in (4) for $d = 2$.

A simple example of polynomials $P(x) = (x - 1)^d$ and $Q(x) = 1 + x^d$ satisfying (5) shows that the constant $\xi(d)$ should be at least exponential in d. Indeed, by Stirling's formula (see, e.g., [12]), for each $n \in \mathbb{N}$ one has

$$n! = \sqrt{2\pi n}\,(n/e)^n e^{\theta_n}, \tag{10}$$

where $\frac{1}{12n+1} < \theta_n < \frac{1}{12n}$. Hence, $H(P)/H(Q) = \binom{d}{\lfloor d/2 \rfloor} \sim 2^{d+1/2}/\sqrt{\pi d}$ as $d \to \infty$.

First, we will show the following lower bounds with better exponents:

Theorem 1. *The constants $\xi(d)$ and $\eta(d)$ for which* (4) *and* (6) *hold under corresponding assumptions* (5) *and* (7) *must satisfy*

$$\xi(d) \ge \frac{1}{2d+2}\left(\left(\frac{3+\sqrt{5}}{2}\right)^d + \left(\frac{3-\sqrt{5}}{2}\right)^d\right)$$

and

$$\eta(d) \ge \frac{1}{2d+2}\left(\left(\frac{1+\sqrt{5}}{2}\right)^d + \left(\frac{1-\sqrt{5}}{2}\right)^d\right).$$

In the next theorem, we compare the lengths of two polynomials P, Q, which have not necessarily the same degree but the deviation of P from zero in an interval is smaller than that of Q.

Theorem 2. *Let $P, Q \in \mathbb{R}[x]$ be polynomials satisfying*

$$\max_{0\le x\le 1}|P(x)| \le \max_{0\le x\le 1}|Q(x)|. \tag{11}$$

Then,

$$L(P) \le \frac{1}{2}((3+2\sqrt{2})^d + (3-2\sqrt{2})^d)L(Q), \tag{12}$$

where $d = \deg P$. Furthermore, if $P, Q \in \mathbb{R}[x]$ are polynomials satisfying

$$\max_{-1\le x\le 1}|P(x)| \le \max_{-1\le x\le 1}|Q(|x|)| \tag{13}$$

and $d = \deg P$, then

$$L(P) \le \frac{1}{2}((1+\sqrt{2})^d + (1-\sqrt{2})^d)L(Q). \tag{14}$$

From Theorem 2 it is easy to find some explicit $\xi(d)$ and $\eta(d)$ for which (4) and (6) hold under the corresponding assumptions (5) and (7). Indeed, by (1), we have $H(P) \le L(P)$ and $L(Q) \le (\deg Q + 1)H(Q)$. Therefore, (12) implies that

$$H(P) \le \frac{\deg Q + 1}{2}((3+2\sqrt{2})^d + (3-2\sqrt{2})^d)H(Q).$$

In particular, for $\deg Q = d$, since (5) implies (11), we see that, under assumption (5), inequality (4) holds with the constant

$$\xi(d) = \frac{d+1}{2}((3+2\sqrt{2})^d + (3-2\sqrt{2})^d).$$

Likewise, in view of (14), as (7) implies (13), under assumption (7), we derive that inequality (6) holds with the constant

$$\eta(d) = \frac{d+1}{2}((1+\sqrt{2})^d + (1-\sqrt{2})^d).$$

In the next theorem, we will improve these bounds as follows:

Theorem 3. *Under assumption* (5), *inequality* (4) *holds with the constant*

$$\xi(d) = 3^{3d/2}\sqrt{d},$$

while under assumption (7), *inequality* (6) *holds with the constant*

$$\eta(d) = 3^{3d/4}\sqrt{7d}.$$

Note that

$$3^{3/2} = 5.196152\ldots < 3 + 2\sqrt{2} = 5.828427\ldots$$

and

$$3^{3/4} = 2.279507\ldots < 1 + \sqrt{2} = 2.414213\ldots.$$

By Theorem 1, the constants $3^{3/2}$ and $3^{3/4}$ cannot be replaced by the constants smaller than $\frac{3+\sqrt{5}}{2} = 2.618033\ldots$ and $\frac{1+\sqrt{5}}{2} = 1.618033\ldots$, respectively, so there is still a gap for improvement of those constants in one or the other direction.

In Section 2, we give some auxiliary lemmas and then prove Theorems 1–3 in Section 3. The proofs are self-contained except for some basic properties of Chebyshev polynomials and Lemma 4.

2 Auxiliary results related to Chebyshev polynomials

Let $T_d(x) = \cos(d \arccos x)$, where $-1 \le x \le 1$ be the Chebyshev polynomial of the first kind. Below, we shall use the following standard formulas:

$$T_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \binom{d}{2k} (x^2-1)^k x^{d-2k}, \tag{15}$$

$$T_d(x) = \frac{d}{2} \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \frac{(d-k-1)!}{k!\,(d-2k)!} (2x)^{d-2k} \tag{16}$$

for all $x \in \mathbb{R}$ and

$$T_d(x) = \frac{1}{2}((x+\sqrt{x^2-1})^d + (x-\sqrt{x^2-1})^d) \tag{17}$$

for $x \ge 1$ that can be found, for instance, in [13, Chapter 22]. See also [14] and [15].

The most useful property of Chebyshev polynomials that we will use is the inequality

$$|T_d(x)| \le 1 \quad \text{for } -1 \le x \le 1. \tag{18}$$

Some basic properties of Chebyshev polynomials were established by Chebyshev himself, Markov and Bernstein (see [16]). We first state the following lemma, which is a combination of Theorem 2.1 (with $\xi = 0$), Corollary 2.1 and Theorem 2.2 in the paper of Osipov and Sazhin [17]. (Its partial case has been proved in [18]. See also [19].)

Lemma 4. *Let $P \in \mathbb{R}[x]$ be a degree d polynomial satisfying $|P(x)| \le 1$ for $x \in [0, a]$. Then, for each k, $0 \le k \le d$, the absolute value of the kth coefficient of $P(x)$ does not exceed that of the kth coefficient of the polynomial $T_d(2x/a - 1)$. In particular,*

$$L(P) \le L(T_d(2x/a-1)). \tag{19}$$

Moreover, if $P \in \mathbb{R}[x]$ is a degree d polynomial satisfying $|P(x)| \le 1$ for $x \in [-1, 1]$, then the length of the polynomial $P(x)$ does not exceed that of $T_d(x)$.

Next, we calculate the lengths of some shifted Chebyshev polynomials.

Lemma 5. *For each $a > 0$, we have*

$$L(T_d(2x/a-1)) = \frac{1}{2}\left(\left(1+\frac{2}{a}+\frac{2}{a}\sqrt{a+1}\right)^d + \left(1+\frac{2}{a}-\frac{2}{a}\sqrt{a+1}\right)^d\right) \tag{20}$$

and

$$L(T_d(x/a)) = \frac{1}{2}\left(\left(\frac{1+\sqrt{a^2+1}}{a}\right)^d + \left(\frac{1-\sqrt{a^2+1}}{a}\right)^d\right). \tag{21}$$

Proof. In view of $\arccos(2y-1) = 2\arccos(\sqrt{y})$ for $y \in [0,1]$, we obtain $T_d(2y-1) = T_{2d}(\sqrt{y})$. Hence, $T_d(2x/a-1) = T_{2d}(\sqrt{x/a})$ and from (16) it follows that

$$T_d(2x/a-1) = d \sum_{k=0}^{d} (-1)^k \left(\frac{4}{a}\right)^{d-k} \frac{(2d-k-1)!}{k!\,(2d-2k)!} x^{d-k} \tag{22}$$

for $x \in [0, a]$. By the uniqueness theorem for holomorphic functions (see [20, p. 122]), this formula must hold for each $x \in \mathbb{C}$.

From (22) we see that the coefficients of the polynomial $T_d(2x/a-1)$ have alternating signs, so its length equals the modulus of its value at $x=-1$, namely,

$$L(T_d(2x/a-1)) = |T_d(-2/a-1)| = |T_d(2/a+1)|.$$

Inserting $x = 2/a+1$ into (17) we find that $T_d(2/a+1)$ is equal to the right hand side of (20), whence the result.

Similarly, by (16), the length of $T_d(x/a)$ is equal to $|T_d(i/a)|$. Since

$$((i/a)^2-1)^k = (-1)^k(1+1/a^2)^k \quad \text{and} \quad (i/a)^{d-2k} = i^d(-1)^k a^{2k-d},$$

inserting $x = i/a$ into (15), we find that

$$L(T_d(x/a)) = |T_d(i/a)| = a^{-d}\sum_{k=0}^{\lfloor d/2\rfloor}\binom{d}{2k}(a^2+1)^k = \frac{1}{2a^d}((1+\sqrt{a^2+1})^d + (1-\sqrt{a^2+1})^d),$$

which completes the proof of (21). □

Combining (19) with (20) we obtain the following:

Lemma 6. *Let $a>0$ and let $P \in \mathbb{R}[x]$ be a degree d polynomial satisfying $|P(x)| \le 1$ for $0 \le x \le a$. Then,*

$$L(P) \le \frac{1}{2}\left(\left(1+\frac{2}{a}+\frac{2}{a}\sqrt{a+1}\right)^d + \left(1+\frac{2}{a}-\frac{2}{a}\sqrt{a+1}\right)^d\right).$$

Likewise, combining the last statement of Lemma 4 with (21), where $a=1$, we obtain the following:

Lemma 7. *Let $P \in \mathbb{R}[x]$ be a degree d polynomial satisfying $|P(x)| \le 1$ for $-1 \le x \le 1$. Then,*

$$L(P) \le \frac{1}{2}((1+\sqrt{2})^d + (1-\sqrt{2})^d).$$

In the next lemma, we give the bounds for the modulus of the polynomial $P \in \mathbb{R}[x]$ of degree d and the modulus of its reciprocal polynomial $P^*(x) = x^d P(1/x)$ in the interval $[0,1]$ under assumptions of (5) and (7).

Lemma 8. *Let $P, Q \in \mathbb{R}[x]$ be two degree d polynomials satisfying* (5)*. Then, $|P(x)| \le (d+1)H(Q)$ and $|P^*(x)| \le (d+1)H(Q)$ for $x \in [0,1]$.*

Furthermore, if $P, Q \in \mathbb{R}[x]$ are two degree d polynomials satisfying (7)*, then*

$$\frac{1}{2}|P(x)+P(-x)|, \frac{1}{2}|P(x)-P(-x)| \le (d+1)H(Q) \tag{23}$$

and

$$\frac{1}{2}|P^*(x)+P^*(-x)|, \frac{1}{2}|P^*(x)-P^*(-x)| \le (d+1)H(Q) \tag{24}$$

for $x \in [-1,1]$.

Proof. By (1) and (2), we clearly have $|Q(x)| \le L(Q) \le (d+1)H(Q)$ for $x \in [0,1]$, which implies the first inequality in view of (5). To see that the second inequality holds, observe that (5) and $\deg P = \deg Q$ implies $|P^*(x)| \le |Q^*(x)|$ for each $x \ge 0$. In particular, this inequality holds for $x \in [0,1]$. Now, the second inequality of the lemma follows by $|Q^*(x)| \le (d+1)H(Q^*)$ for $x \in [0,1]$ and $H(Q^*) = H(Q)$.

For the second part, assume that P, Q are two degree d polynomials satisfying (7). For $x \in [-1,1]$, we have $|Q(|x|)| \le L(Q) \le (d+1)H(Q)$, which implies $|P(x)| \le (d+1)H(Q)$. Also, from (7) it follows that $|P^*(x)| \le |Q^*(|x|)|$ for $|x| \le 1$, and hence

$$|P^*(x)| \le |Q^*(|x|)| \le (d+1)H(Q^*) = (d+1)H(Q).$$

Furthermore, since the pair of polynomials $P(-x)$, $Q(x)$ also satisfies (7), we obtain $|P(-x)|, |P^*(-x)| \leq (d+1)H(Q)$ for $x \in [-1, 1]$. This implies the bounds (23) and (24). □

We conclude this section with the following inequality:

Lemma 9. *For $d \in \mathbb{N}$ and k, $0 \leq k \leq d$, set*

$$f(k, d) := \frac{d}{d+k}\binom{d+k}{2k}4^k. \tag{25}$$

Then,

$$\max_{0 \leq k \leq \lfloor d/2 \rfloor} f(k, d) = f(\lfloor d/2 \rfloor, d) < 0.47 \cdot \frac{3^{3d/2}}{\sqrt{d}}. \tag{26}$$

Furthermore, for $d \geq 3$ odd we have

$$f((d+1)/2, d) = \frac{9d-3}{d+1} f((d-1)/2, d-1). \tag{27}$$

Proof. Inequality (26) can be easily verified directly for d in the range $1 \leq d \leq 7$. Then, the maximum of the left hand side of (26) is indeed attained at $k = \lfloor d/2 \rfloor$ and is equal to

$$1, 8, 18, 160, 400, 3584, 9408$$

for $d = 1, 2, 3, 4, 5, 6, 7$, respectively. In each case, it is smaller than $0.47 \cdot 3^{3d/2}/\sqrt{d}$ for the corresponding value of d.

Fix any $d \geq 8$. By (25), for k in the range $0 \leq k \leq \lfloor d/2 \rfloor - 1$, we have

$$\frac{f(k+1, d)}{f(k, d)} = 4\frac{(d+k)(d-k)}{(2k+1)(2k+2)} > \frac{4(d^2-k^2)}{(2k+2)^2} = \frac{d^2-k^2}{(k+1)^2} > 1,$$

because $k^2 + (k+1)^2 < 2(k+1)^2 \leq 2\lfloor d/2 \rfloor^2 < d^2$ for $d \geq 2$. Thus, the maximum of $f(k, d)$ in (26) is attained at $k = \lfloor d/2 \rfloor$.

Assume first that $d \geq 8$ is even. Set

$$B_d := \frac{2^{d+1}(3d/2)!}{3(d/2)!d!}. \tag{28}$$

Then, $\lfloor d/2 \rfloor = d/2$, and hence

$$\max_{0 \leq k \leq \lfloor d/2 \rfloor} f(k, d) = f(d/2, d) = \frac{d}{d+d/2}\binom{3d/2}{d/2}4^{d/2} = B_d. \tag{29}$$

Applying (10) to $n = 3d/2$, $d/2$ and d, by (28), we derive that for each even $d \geq 8$

$$\begin{aligned} B_d &= \frac{2^{d+1}\sqrt{3\pi d}\,(3d/2e)^{3d/2}e^{\theta_{3d/2}}}{3\sqrt{\pi d}\,(d/2e)^{d/2}e^{\theta_{d/2}}\sqrt{2\pi d}\,(d/e)^d e^{\theta_d}} = \sqrt{\frac{2}{3\pi d}} \cdot 3^{3d/2} \cdot e^{\theta_{3d/2}-\theta_{d/2}-\theta_d} \\ &< e^{1/144} \cdot \sqrt{\frac{2}{3\pi}} \cdot \frac{3^{3d/2}}{\sqrt{d}} < 0.47 \cdot \frac{3^{3d/2}}{\sqrt{d}}. \end{aligned}$$

So, by (29), this implies (26) for each even $d \geq 8$.

If $d \geq 9$ is odd, then $D := d - 1 \geq 8$ is even and $\lfloor d/2 \rfloor = (d-1)/2$. Accordingly, we have

$$\max_{0 \leq k \leq \lfloor d/2 \rfloor} f(k, d) = f((d-1)/2, d).$$

Thus, in view of

$$f((d-1)/2, d) = \frac{d2^d}{3d-1} \cdot \frac{((3d-1)/2)!}{(d-1)!((d+1)/2)!} = \frac{(D+1)2^{D+1}}{3D+2} \cdot \frac{(3D/2+1)!}{D!(D/2+1)!} = \frac{D+1}{D+2} \cdot 3B_D,$$

where the last equality follows from (28), we derive that the left hand side of (26) does not exceed $3B_D$. Using the upper bound for B_D for even $D = d - 1 \geq 8$ we derive that

$$3B_D < 3 \cdot 0.47 \cdot \frac{3^{3D/2}}{\sqrt{D}} = 1.41 \cdot 3^{-3/2} \cdot \frac{3^{3d/2}}{\sqrt{d-1}} < 0.29 \cdot \frac{3^{3d/2}}{\sqrt{d}}$$

for each odd $d \geq 9$. This completes the proof of (26).

The verification of (27) is straightforward. The left hand side of (27) divided by the binomial coefficient $\binom{3(d-1)/2}{(d-1)/2}$ equals

$$\frac{f((d+1)/2, d)}{\binom{3(d-1)/2}{(d-1)/2}} = \frac{2d(3d/2 + 1/2)(3d/2 - 1/2)2^{d+1}}{(3d+1)(d+1)d} = \frac{(3d-1)2^d}{d+1},$$

while, by (29), the right hand side divided by the same binomial coefficient equals

$$\frac{9d-3}{d+1} \cdot \frac{f((d-1)/2, d-1)}{\binom{3(d-1)/2}{d-1}} = \frac{9d-3}{d+1} \cdot \frac{2 \cdot 4^{(d-1)/2}}{3} = \frac{(3d-1)2^d}{d+1},$$

which is the same. □

3 Proofs of the main results

Proof of Theorem 1. Consider the pair of degree d polynomials

$$P(x) = T_d(x/2 - 1) \quad \text{and} \quad Q(x) = 1 + x^d.$$

We first show that they satisfy condition (5). Indeed, for $0 \leq x \leq 4$, we have $|T_d(x/2 - 1)| \leq 1$ by (18), so $|P(x)| \leq 1 \leq 1 + x^d = Q(x)$. Assume that $x \geq 4$. We will show that then

$$|T_d(x/2 - 1)| = T_d(x/2 - 1) \leq x^d.$$

Indeed, setting $y := x/2 - 1 \geq 1$ and using (17) we deduce

$$\frac{1}{2}((y + \sqrt{y^2 - 1})^d + (y - \sqrt{y^2 - 1})^d) < \frac{1}{2}((2y)^d + 1) < (2y + 2)^d.$$

Hence, these two polynomials satisfy (5) for each $x \geq 0$. This implies the first part of the theorem, because $H(Q) = 1$ and $H(P)$ is greater than or equal to

$$\frac{L(P)}{d+1} = \frac{L(T_d(x/2 - 1))}{d+1} = \frac{1}{2d+2}\left(\left(\frac{3+\sqrt{5}}{2}\right)^d + \left(\frac{3-\sqrt{5}}{2}\right)^d\right)$$

by (1) and (20) with $a = 4$.

For the second part, take $P(x) = T_d(x/2)$ and $Q(x) = 1 + x^d$. Then, for $-2 \leq x \leq 2$, we have $|T_d(x/2)| \leq 1$ by (18), and hence $|P(x)| \leq 1 \leq 1 + |x|^d = Q(|x|)$. Since $|T_d(x)| = |T_d(-x)|$, in order to verify (7) it suffices to check that $|T_d(x/2)| = T_d(x/2) \leq x^d$ for $x \geq 2$. This is indeed the case, because

$$T_d(x/2) = \frac{1}{2}((x/2 + \sqrt{(x/2)^2 - 1})^d + (x/2 - \sqrt{(x/2)^2 - 1})^d) < \frac{1}{2}(x^d + 1) < x^d$$

by (17). Consequently, these two polynomials satisfy (7) for each $x \in \mathbb{R}$. As above, this implies the second part of the theorem, because $H(Q) = 1$ and $H(P)$ is greater than or equal to

$$\frac{L(P)}{d+1} = \frac{L(T_d(x/2))}{d+1} = \frac{1}{2d+2}\left(\left(\frac{1+\sqrt{5}}{2}\right)^d + \left(\frac{1-\sqrt{5}}{2}\right)^d\right)$$

by (21) with $a = 2$. □

Proof of Theorem 2. Suppose that $P, Q \in \mathbb{R}[x]$ are two polynomials satisfying (11). Then, by (2), we have $|P(x)| \le L(Q)$ for each $x \in [0,1]$. Applying Lemma 6 to $a = 1$ and the polynomial $P(x)/L(Q)$ of length $L(P)/L(Q)$ we find that

$$\frac{L(P)}{L(Q)} \le \frac{1}{2}((3+2\sqrt{2})^d + (3-2\sqrt{2})^d),$$

which implies (12).

Similarly, by (2) and (13), we obtain $|P(x)| \le L(Q)$ for each $x \in [-1,1]$. So, applying Lemma 7 to the polynomial $P(x)/L(Q)$ of length $L(P)/L(Q)$ we derive that

$$\frac{L(P)}{L(Q)} \le \frac{1}{2}((1+\sqrt{2})^d + (1-\sqrt{2})^d),$$

which gives (14). □

Proof of Theorem 3. Let P and Q be two degree d polynomials satisfying (5). Then, by Lemma 8, we have

$$|P(x)| \le (d+1)H(Q) \quad \text{and} \quad |P^*(x)| \le (d+1)H(Q) \tag{30}$$

for $x \in [0,1]$. From (22) with $a = 1$, it follows that

$$T_d(2x-1) = \sum_{k=0}^{d} (-1)^{d-k} \frac{d}{d+k} \binom{d+k}{2k} 4^k x^k.$$

Write

$$P(x) = a_0 + a_1x + \cdots + a_dx^d \quad \text{and} \quad P^*(x) = a_d + a_{d-1}x + \cdots + a_0x^d.$$

Then, using (30), by the first part of Lemma 4 with $a = 1$, we deduce that for each $0 \le k \le \lfloor d/2 \rfloor$

$$\frac{\max\{|a_k|, |a_{d-k}|\}}{(d+1)H(Q)} \le \frac{d}{d+k}\binom{d+k}{2k}4^k,$$

which is less than $0.47 \cdot 3^{3d/2}/\sqrt{d}$ by Lemma 9. Hence, from

$$H(P) = \max_{0\le k\le d} |a_k| = \max_{0 \le k \le \lfloor d/2 \rfloor} \max\{|a_k|, |a_{d-k}|\}$$

we conclude that

$$H(P) < (d+1)H(Q)\frac{3^{3d/2}}{2\sqrt{d}} \le 2 \cdot 3^{3d/2-1}\sqrt{d}\,H(Q), \tag{31}$$

for each $d \ge 3$. Note that (31) also holds for $d = 1$ and $d = 2$ by (8) and (9). This implies the first part of the theorem with $\xi(d) = 2 \cdot 3^{3d/2-1}\sqrt{d}$, which is better than claimed.

Next, suppose that $P(x) = a_0 + a_1x + \cdots + a_dx^d$ and $Q(x)$ satisfy (7). Assume first that d is even. Then, we can rewrite (23) in the form

$$|a_0 + a_2x^2 + \cdots + a_dx^d|, |a_1x + a_3x^3 + \cdots + a_{d-1}x^{d-1}| \le (d+1)H(Q).$$

Replacing x^2 by $y \in [0,1]$ and using

$$|a_1x^2 + a_3x^4 + \cdots + a_{d-1}x^d| \le |a_1x + a_3x^3 + \cdots + a_{d-1}x^{d-1}|,$$

we deduce

$$|a_0 + a_2y + \cdots + a_dy^{d/2}|, |a_1y + a_3y^2 + \cdots + a_{d-1}y^{d/2}| \le (d+1)H(Q)$$

for $y \in [0,1]$. In the same fashion, from (24) it follows that

$$|a_d + a_{d-2}y + \cdots + a_0y^{d/2}|, |a_{d-1}y + a_{d-3}y^2 + \cdots + a_1y^{d/2}| \le (d+1)H(Q)$$

for $y \in [0,1]$.

Now, by the first part of Lemma 4 with $a = 1$ and d replaced by $d/2$, we derive that $\max\{|a_0|, |a_d|\} \le (d+1)H(Q)$ and for each $1 \le k \le \lfloor d/4 \rfloor$

$$\frac{\max\{|a_{2k-1}|, |a_{2k}|, |a_{d-2k}|, |a_{d-2k+1}|\}}{(d+1)H(Q)} \le \frac{d/2}{d/2+k}\binom{d/2+k}{2k}4^k,$$

which is less than $0.47 \cdot 3^{3d/4}/(\sqrt{d/2})$ by Lemma 9. Consequently, the moduli of all coefficients of P except possibly for $a_{d/2}$ (which happens for d even but not divisible by 4) are bounded above by

$$(d+1)H(Q)\frac{3^{3d/4}}{2\sqrt{d/2}} < 3^{3d/4}\sqrt{2d}\,H(Q). \tag{32}$$

In particular, this implies the bound better than claimed for d divisible by 4.

If d is of the form $4l-2$, $l \in \mathbb{N}$, then the coefficient $a_{d/2} = a_{2l-1}$ occurs for y^l. So, by the first part of Lemma 4 with $a = 1$ and d replaced by $d/2$, we get

$$\frac{|a_{d/2}|}{(d+1)H(Q)} \le \frac{d/2}{d/2+l}\binom{d/2+l}{2l}4^l = f((d+2)/4, d/2).$$

For $d = 2$, this equals 2 and gives the bound $|a_1| \le 6H(Q)$ implying $H(P) \le 6H(Q)$, which is better than required. For $d \ge 6$, by the identity (27) with d replaced by $d/2$, we see that the right hand side is equal to

$$\frac{9d-6}{d+2}f((d-2)/4, d/2-1).$$

In view of (26) this is less than

$$0.47 \cdot \frac{9d-6}{d+2} \cdot \frac{3^{3d/4-3/2}}{\sqrt{d/2-1}} = 0.47 \cdot \sqrt{6} \cdot \frac{d-2/3}{d+2} \cdot \frac{3^{3d/4}}{\sqrt{d-2}},$$

and hence

$$|a_{d/2}| < 0.47 \cdot \sqrt{6} \cdot \frac{(d-2/3)(d+1)}{d+2} \cdot \frac{3^{3d/4}}{\sqrt{d-2}}H(Q).$$

Now, by

$$(d+1)(d-2/3) < (d+2)\sqrt{d(d-2)}$$

and $0.47 \cdot \sqrt{6} < 1.16$, we obtain $|a_{d/2}| < 1.16 \cdot 3^{3d/4}\sqrt{d}$ for $d \ge 6$. Combining this with (32), we complete the proof of the inequality $H(P) < 3^{3d/4}\sqrt{2d}\,H(Q)$ for each $d \ge 2$ of the form $4l-2$, $l \in \mathbb{N}$.

It remains to consider the case when d is odd. For $d = 1$, we have $H(P) \le H(Q)$ by (8), so we can assume that $d \ge 3$. Note that, by an argument as above, (23) implies

$$|a_0 + a_2x^2 + \dots + a_{d-1}x^{d-1}|, |a_1x + a_3x^3 + \dots + a_dx^d| \le (d+1)H(Q),$$

while (24) yields

$$|a_{d-1} + a_{d-3}x^2 + \dots + a_1x^{d-1}|, |a_dx + a_{d-2}x^3 + \dots + a_0x^d| \le (d+1)H(Q).$$

Following the above argument, replacing x^2 by $y \in [0, 1]$, we find that

$$|a_0 + a_2y + \dots + a_{d-1}y^{\frac{d-1}{2}}|, |a_1y + a_3y^2 + \dots + a_dy^{\frac{d+1}{2}}| \le (d+1)H(Q)$$

and

$$|a_{d-1} + a_{d-3}y + \dots + a_1y^{\frac{d-1}{2}}|, |a_dy + a_{d-2}y^2 + \dots + a_0y^{\frac{d+1}{2}}| \le (d+1)H(Q)$$

for $y \in [0, 1]$.

Applying the first part of Lemma 4 with $a = 1$ to the above 4 polynomials of degree at most $(d+1)/2$, we derive that $\max\{|a_0|, |a_{d-1}|\} \le (d+1)H(Q)$, whereas for each $1 \le k \le \lfloor (d+1)/4 \rfloor$

$$\frac{\max\{|a_{2k-1}|, |a_{2k}|, |a_{d-2k-1}|, |a_{d-2k+2}|\}}{(d+1)H(Q)}$$

does not exceed

$$\frac{(d+1)/2}{(d+1)/2+k}\binom{(d+1)/2+k}{2k}4^k.$$

By Lemma 9, this is less than $3^{3(d+1)/4}/\sqrt{2d+2}$. It follows that the moduli of all the coefficients of P (except possibly for $a_{(d+1)/2}$ when $d = 4l+1$, $l \in \mathbb{N}$) are less than

$$(d+1)\frac{3^{3(d+1)/4}}{\sqrt{2d+2}}H(Q) = \frac{3^{3/4}\sqrt{d+1}}{\sqrt{2}}3^{3d/4}H(Q) < 2\sqrt{d}\cdot 3^{3d/4}H(Q)$$

for $d \geq 3$. This completes the proof of the second part of the theorem for all odd d except for those of the form $d = 4l+1$, $l \in \mathbb{N}$.

We now turn to the case when $d = 4l+1$. Then, the only coefficient $a_{(d+1)/2} = a_{2l+1}$ that remains to be dealt with occurs for y^{l+1}. In that case, by the first part of Lemma 4 with $a = 1$ and d replaced by $(d+1)/2$, and by identity (27) with d replaced by $(d+1)/2$, we find that

$$\frac{|a_{(d+1)/2}|}{(d+1)H(Q)} \leq \frac{(d+1)/2}{(d+1)/2+l+1}\binom{(d+1)/2+l+1}{2l+2}4^{l+1} = f((d+3)/4,(d+1)/2) = \frac{9d+3}{d+3}f((d-1)/4,(d-1)/2).$$

Using the upper bound (26) on $f((d-1)/4,(d-1)/2)$ we deduce

$$\frac{|a_{(d+1)/2}|}{H(Q)} < (d+1)\frac{0.47(9d+3)}{d+3}\cdot\frac{3^{3d/4-3/4}}{\sqrt{(d-1)/2}}.$$

Here, the right hand side is less than $3^{3d/4}\sqrt{7d}$, for each $d \geq 5$, because

$$(d+1)(d+1/3) < (d+3)\sqrt{d^2-d}$$

and $0.47\cdot 9\cdot 3^{-3/4}\cdot\sqrt{2} < 2.63 < \sqrt{7}$. Hence, $|a_{(d+1)/2}| < 3^{3d/4}\sqrt{7d}\,H(Q)$. Since we already proved that the moduli of all other coefficients a_k, $k \neq (d+1)/2$, are less than $2\sqrt{d}\cdot 3^{3d/4}H(Q)$, this completes the proof of the inequality $H(P) < 3^{3d/4}\sqrt{7d}\,H(Q)$ for each d of the form $4l+1$. This finishes the proof of the theorem. □

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