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Zeros of Dirichlet Polynomials belonging to the Extended Selberg Class

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Introduction

We write $s = \sigma + it$. In the first half of the 18th century Euler analyzed the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

for real values of s. He managed to evaluate the series when s is a positive even integer, thus solving the Basel problem in 1734 (the problem was posed in 1650). That is he showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Additionally, in 1737, Euler proved the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

when $s > 1$. Expressions like the one in the right hand side of the above equation are called Eurler's products.

Definition 1. Suppose $f(s)$ is an analytic function defined on a non-empty open subset $U \in \mathbb{C}$. If $F(s)$ is an analytic function defined on an open connected subset $V \in \mathbb{C}$, $U \subset V$ and $F(s) = f(s)$, for all $s \in U$ then $F(s)$ is called an analytic continuation of $f(s)$. Analytic continuations are unique.

In 1859, Riemann [9] analyzed the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

in the complex plane. He managed to find an analytic continuation of the above equation to the whole of $\mathbb{C}$ except at $s = 1$ where it has a simple pole. This analytic continuation is called the Riemann zeta function and is denoted by $\zeta(s)$. Additionally, in his paper Riemann showed that the following functional equation holds for $\zeta(s)$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

where $\Gamma(s)$ is the gamma function (for definition and properties see [13, Chapter 1.86]), for all $s \in \mathbb{C}$.

From the functional equation it is easy to see that $\zeta(s)$ is equal to zero when s is a negative even number, these zeros are called trivial. Riemann showed that all non-trivial zeros of $\zeta(s)$ lie in the strip $0 \leq \sigma \leq 1$, this strip is called the critical strip. He also conjectured that all non-trivial zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$, which is called the critical line. This conjecture is called the Riemann's hypothesis (RH) and it is yet to be proven or disproven. RH is an important statement to mathematics, especially number theory, since it has implications on the distribution of primes.

In [11], Selberg introduced the class of functions S that have similar properties to $\zeta(s)$:

Definition 2. $F(s)$ belongs to the Selberg class S if:

(I) For $\sigma > 1$, F(s) is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

(II) There exists $k \in \mathbb{N}$ such that $(s-1)^k F(s)$ is an entire function of finite order.

(III) $F(s)$ satisfies the functional equation:

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})},$$

where $\Phi(s) = F(s) Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, with $Q, \lambda_j, \Re(\mu_j) \geq 0$ and $|\omega| = 1$.

(IV) For every $\varepsilon > 0$, $|a_n| < n^{\varepsilon}$.

(V) For σ sufficiently large $F(s)$ has an Euler product, that is

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

where $b_n = 0$ unless n is a positive power of a prime and $|b_n| < n^{\theta}$ for some $\theta < 1/2$.

Let $F(s) \in S$, we say that a zero of $F(s)$ is trivial if it arises from the poles of the gamma function factors in the functional equation. If all non-trivial zeros of $F(s)$ are on the critical line we say that $F(s)$ satisfies RH.

In [11], Selberg conjectured that all functions from S satisfy RH. By relaxing the conditions $(IV)-(V)$ in Definition 2 we get a class of zeta functions that definitely contains functions that do not satisfy RH:

Definition 3. A non-identically vanishing function $F(s)$ belongs to the extended Selberg class $S^{\#}$ if it satisfies $(I)-(III)$ in Definition 2.

In this work we study functions from $S^{\#}$ that satisfy a simpler functional equation:

$$Q^s F(s) = \omega \overline{Q^{1-\bar{s}} F(1-\bar{s})}.$$

A function from $S^{\#}$ satisfies this functional equation when the quantity $d_F = 2\sum_{j=1}^{r} \lambda_j$ is equal to zero. d_F is an invariant and is called the degree of $F(s)$. Denote $S_d^{\#}$ the subclass of $S^{\#}$ of functions with degree d. It turns out that $S_0 = \{1\}$ (see Conrey and Ghosh [2]), however $S_0^{\#} \neq \{1\}$.
The full characterization of functions from $S_0^{\#}$ is given by Kaczorowski and Perelli in [5]. For functions from $S_0^{\#}$ the quantity Q is an invariant. Additionally, $Q^2 = q$ is a positive integer and is called the conductor of $F(s)$, we denote it by q_F. Also these functions take the form of Dirichlet polynomials

$$F(s) = \sum_{n|q_F} \frac{a_n}{n^s}$$

and the functional equation can be equivalently rewritten as a condition for coefficients of $F(s)$:

$$a_n = \omega \frac{n\overline{a_{q_F/n}}}{\sqrt{q_F}},\ n \mid q_F.$$

The main goal of this work is to give sufficient conditions under which a function from $S_0^{\#}$ satisfies RH. A well know equivalent condition for the Riemann zeta function to satisfy RH is given by Speiser in [12]. He showed that RH is equivalent to the derivative of the Riemann zeta function having no zeros left of the critical line.

Definition 4. Let $f, g : \mathbb{R} \to \mathbb{R}$. We say that $f(x) = \mathcal{O}(g(x))$, if there exists constants $C > 0$ and $x_0 \in \mathbb{R}$, such that $|f(x)| < C|g(x)|$, when $x > x_0$.

Let $F(s) \in S_0^{\#}$ and $q_F \geq 2$. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $0 \leq t < T$, $\sigma < 1/2$. In [3], Garunkštis and Šimėnas prove that $N(T) = N_1(T) + \mathcal{O}(1)$. In this work we show that there exists arbitrarily large values of $T > 0$, such that $N(T) = N_1(T)$. This implies that a function from $S_0^{\#}$ satisfies RH if and only if its derivative does not have zeros left of the critical line. Thus we see that Speiser's result for the Riemann zeta function holds for functions from $S_0^{\#}$ as well.
Additionally, we show that there exists a connection between functions from $S_0^{\#}$ and self-inversive polynomials.

Definition 5. A polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

is called self-inversive if

$$p(z) = \omega p^*(z),$$

where ω is fixed, $|\omega| = 1$ and

$$p^*(z) = \sum_{j=0}^{n} \overline{a_{n-j}} z^j,$$

for all $z \in \mathbb{C}$.
Equivalently, $p(z)$ is self-inversive if $p_j = \omega \overline{p_{n-j}}$ for $j = 0, ..., n$.

Self-inversive polynomials have interesting properties. For example, roots of a self-inversive polynomial are symmetric with respect to the unit circle, that is if r is a root of a self-inversive polynomial then $1/\overline{r}$ is also a root of the same polynomial. Additionally, if all zeros of a polynomial are on the unit circle then it is necessarily self-inversive (see [8]). An important result about self-inversive polynomials is Cohn's theorem (proved by Cohn in [1]):

Theorem 1. *Let $p(z) = \sum_{j=1}^{n} a_j z^j$ be a self-inversive polynomial. Then $p(z)$ and $q(z) = \sum_{j=0}^{n-1} (n-j)\overline{a_{n-j}} z^j$ have the same number of roots in $|z| < 1$.*

Schinzel in [10] and Lakatos and Losonczi in [8] proved sufficient conditions for a self-inversive polynomial to have all roots on the unit circle:

Theorem 2. *Let $p(z) = \sum_{j=1}^{n} A_j z^j$ be a self-inversive polynomial. If either of*

$$|A_n| \geq \inf_{\substack{c,d \in \mathbb{C} \\ |d|=1}} \sum_{j=0}^{n} |cA_j - d^{m-j} A_n|,$$

$$|A_n| \geq \frac{1}{2} \sum_{j=1}^{n-1} |A_j|$$

are satisfied then all zeros of $p(z)$ are on the unit circle.

The main result of this work is that if equivalent inequalities are satisfied for coefficients of a function from $S_0^{\#}$ then the function satisfies RH.

Chapter 1

Statement of results

Definition 6. We denote two subsets of $S_0^\#$:

$$S_0^\#[a,q] := \{F(s) \in S_0^\# : \max_{n|q_F} |a_n| \leq a, q_F = q, |a_1| = 1\},$$

and

$$S_0^\#[a,q,T_0,\eta] := \{F(s) \in S_0^\#[a,q] : \min\{\min_{\sigma\in(-\infty,1/2)} |F(\sigma{+}iT_0)|, \min_{\sigma\in(-\infty,1/2)} |F'(\sigma{+}iT_0)|\} \geq \eta\}.$$

Firstly, we improved the result of Garunkštis and Šimėnas [3, Theorem 3] by giving a bound uniform in a and q.

Theorem 3. *Let $F(s) \in S_0^\#[a,q]$, $q \geq 2$ and $a \geq \sqrt{q}$. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $0 \leq t < T$, $\sigma < 1/2$. Then*

$$|N(T) - N_1(T)| = \mathcal{O}(\log a + \log q + \log q \log 2aq).$$

Next, we showed that a function from $S_0^\#$ satisfies RH if and only if its derivative does not have zeros left of the critical line.

Theorem 4. *Let $N(T,T_0)$ and $N_1(T,T_0)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $T_0 < t < T$, $\sigma < 1/2$. For any $T_0 \in \mathbb{R}$, $a, \eta > 0$, $q \geq 2$ there exists a monotonic sequence $\{T_j\}$, $T_j \to \infty$, such that*

$$N(T_j,T_0) = N_1(T_j,T_0),\ \forall\, j \in \mathbb{N},$$

if $F(s) \in S_0^\#[a,q,T_0,\eta]$.

Corollary 5. *RH is true for $F(s) \in S_0^\#$ if and only if $F'(s)$ has no zeros in $\sigma < 1/2$.*

In the general theory of zeta functions it is expected that an Euler product and RH are closely related to each other. However, this is not true for a function from $S_0^\#$, as Corollary 5 lets us produce an example of a function from $S_0^\#$ that does not have an Euler product but satisfies RH:

$$F(s) = 1 + \frac{1}{2^s} + \frac{6/\sqrt{12}}{6^s} + \frac{\sqrt{12}}{12^s}.$$

$F(s)$ satisfies RH due to the fact that following inequality holds:

$$\left|\frac{\log(12)\sqrt{12}}{12^s}\right| > \left|\frac{\log(2)}{2^s}\right| + \left|\frac{6\log(6)/\sqrt{12}}{6^s}\right|$$

when $\Re(s) \leq 1/2$.
Using a similar idea as in the proof of Theorem 4 we were able to show that the formula for the number of zeros of a function from $S_0^\#$, proved by Kaczorowski and Perelli in [6], is exact for certain values of $T > 0$.

Proposition 6. *Let $F(s) \in S_0^\#$, $q_F \geq 2$. Let $N(T)$ be the number of zeros of $F(s)$ in the region $0 \leq t < T$ (counting with multiplicities). There exists a monotonic sequence $\{T_j\}$ (dependent on $F(s)$), $T_j \to \infty$, such that*

$$N(T_j) = \frac{\log q_F}{2\pi} T_j,\ \forall\, j \in \mathbb{N}.$$

Next, we proved conditions for the coefficients of a function from $S_0^\#$ that are sufficient for the function to satisfy RH.

Theorem 7. *Let $F(s) \in S_0^\#$, $q_F \geq 2$. Let $A_n = a_n/\sqrt{n}$ if $n \leq \sqrt{q_F}$ and $A_n = \omega\overline{A_{q_F/n}}$ for $n \mid q_F$. If either of*

$$|A_{q_F}| \geq \inf_{\substack{c,d\in\mathbb{C} \\ |d|=1}} \sum_{n|q_F} |cA_n - d^{q_F/n}A_{q_F}| \tag{1.1}$$

$$|A_{q_F}| \geq \frac{1}{2} \sum_{\substack{n|q_F \\ n\neq 1,q_F}} |A_n| \tag{1.2}$$

are satisfied then $F(s)$ satisfies RH.

We illustrate Theorem 7 with the following examples. Firstly, take the example function used previously

$$F_1(s) = 1 + \frac{1}{2^s} + \frac{6/\sqrt{12}}{6^s} + \frac{\sqrt{12}}{12^s}.$$

For this function $A_1 = A_{12} = 1$ and $A_2 = A_6 = \sqrt{2}/2$. Thus, since the coefficients of $F_1(s)$ satisfy both (1.1) (take c, $d = 1$) and (1.2) it satisfies RH as shown previously. As a next example take

$$F_2(s) = 1 + \frac{2}{2^s} + \frac{\sqrt{12}}{6^s} + \frac{\sqrt{12}}{12^s}.$$

For this function $A_1 = A_{12} = 1$ and $A_2 = A_6 = \sqrt{2}$. We see that the coefficients do not satisfy (1.2), however they satisfy (1.1) (take c, $d = 1$). Thus, $F_2(s)$ satisfies RH. Next take

$$F_3(s) = 1 + \frac{3}{2^s} + \frac{18/\sqrt{12}}{6^s} + \frac{\sqrt{12}}{12^s}.$$

For this function $A_1 = A_{12} = 1$ and $A_2 = A_6 = 3\sqrt{2}/2$. The coefficients do not satisfy (1.2). They also do not satisfy (1.1) because $F_3(s)$ has a zero at $1.7376 + 4.7700i$.

Definition 7. Let $F(s) \in S_0^\#$, $q_F \geq 2$. Define

$$\mu_F = \inf_{\rho_1 \neq \rho_2} |\rho_1 - \rho_2|,$$

where ρ_1 and ρ_2 run through all the zeros of $F(s)$ (if $F(s)$ has non-simple zeros μ_F can still be strictly bigger than 0).

Lastly, we showed that given $\mu_F > 0$ it is enough to check a finite region of $\mathbb{C}$ to determine if a function from $S_0^\#$ satisfies RH. The condition $\mu_F > 0$ is non-trivial as we were not able to show that there does not exist a function from $S_0^\#$ and a sequence of disks with radii shrinking to 0 such that all the disks in the sequence contain atleast two distinct zeros of the function.

Proposition 8. *Let $F(s) \in S_0^\#$, $q_F \geq 2$. Assume $\mu_F > 0$. Let $0 < \delta < \mu_F/2$. Let $N_\delta(s)$ be the number of zeros of $F(s)$ in the set $\{z : |s - z| < \delta\}$ (counting with multiplicities). Then for any zero ρ of $F(s)$ there exists a monotonic sequence $\{T_j\}$ (independent of ρ), $T_j \to \infty$, such that $N_\delta(\rho + iT_j) = N_\delta(\rho)$ $\forall\, j \in \mathbb{N}$.*

Corollary 9. *Let $F(s) \in S_0^\#$, $q_F \geq 2$. Assume $\mu_F > 0$. Let T_1 be the first member of the sequence defined in Propositon 8 (taking $\delta = \mu_F/4$). Then $F(s)$ satisfies RH if its zeros lie on the line $\sigma = 1/2$ for $0 \leq t < T_1 + \mu_F/4$.*

Chapter 2

Proofs

Lemma 10. *Let $F(s) \in S_0^{\#}[a, q]$, $q \geq 2$. Then, for $\sigma > \log(2aq)/\log 2$,*

$$\left|\frac{F'}{F}(s)\right| < \log q.$$

Proof. We see that

$$\left|\sum_{\substack{n|q \\ n\neq 1}} \frac{a_n}{n^s}\right| < \frac{1}{2},$$

when $\sigma > \log(2aq)/\log 2$. Suppose $\sigma > \log(2aq)/\log 2$, then $|F(s)| > 1/2$ and by solving the inequality

$$\left|\frac{F'(s)}{F(s)}\right| < \log q$$

we obtain that $\sigma > \log(2aq)/\log 2$, which proves the lemma. □

Proof of Theorem 3. By taking the logarithmic derivative in the functional equation we derive that

$$\frac{F'}{F}(s) = -\log q - \overline{\frac{F'}{F}(1-\overline{s})}. \tag{2.1}$$

Let R be a rectangle with vertices $1/2-\delta$, $1/2-\delta+iT$, $-\sigma_0+iT$, $-\sigma_0$, where $\delta > 0$ is sufficiently small and it will be chosen later, also $\sigma_0 = \log(2aq)/\log 2 + 1$. Suppose $F(s) = 0$ or $F'(s) = 0$ when $t = 0$, then it is obvious that we can find a rectangle R' with a bottom side slightly lower than 0 such, that R and R' would contain exactly the same zeros of $F(s)$ and $F'(s)$. We argue analogously for the top side of R and thus we can assume that $F(s) \neq 0$ and $F'(s) \neq 0$ on the top and bottom sides of the rectangle R. We can also pick δ such that $F(s) \neq 0$ and $F'(s) \neq 0$ on the right side of R.

We see that $|F(s)| > 1/2$ when $\sigma \geq \sigma_0$ and because zeros of $F(s)$ are symmetric across the critical strip due to the functional equation, we have that $F(s) \neq 0$ when

$\sigma \leq -\sigma_0$. By formula (2.1) and Lemma 10, it is easy to see that

$$\Re \frac{F'}{F}(\sigma + it) < 0, \tag{2.2}$$

when $\sigma \leq -\sigma_0$, thus $F'(s) \neq 0$ when $\sigma \leq -\sigma_0$. Hence to prove the theorem it is enough to show that the change of $\arg F'/F(s)$ along the rectangle R is $\mathcal{O}(\log a + \log q + \log q \log 2aq)$ as $T \to \infty$.

By formula (2.2) the argument change of $F'/F(s)$ along the left side of R is less than π.

We consider the right-hand side $1/2 - \delta + it$, $0 \leq t \leq T$ of R. By equality (2.1), we see that

$$\Re \frac{F'}{F}\left(\frac{1}{2} + it\right) = -\frac{\log q}{2}$$

if $1/2 + it$ is not a zero of $F(s)$. We claim that there is a sufficiently small $\delta = \delta(T)$ such that, for $0 \leq t \leq T$,

$$\Re \frac{F'}{F}\left(\frac{1}{2} - \delta + it\right) \leq -\frac{\log q}{4}. \tag{2.3}$$

To prove this inequality, it is enough to consider the case when $1/2 - \delta + it$ is in the neighborhood of a zero $\rho = 1/2 + i\gamma$. We have

$$\frac{F'}{F}(s) = \frac{m}{s - \rho} + m' + \mathcal{O}(|s - \rho|),$$

where m is the multiplicity of ρ. Hence taking $s = 1/2 - \delta + it$, we see that

$$\Re \frac{F'}{F}(s) = -\frac{m\delta}{|s - \rho|^2} + \Re(m') + \mathcal{O}(|s - \rho|).$$

Thus $\Re m' = -\log(q)/2$. This proves the inequality (2.3). Therefore, the argument change along the right side of the contour is less than π.

Define $G(s) = (F(s + iT) + \overline{F(\overline{s} + iT)})/2$, then $G(\sigma) = \Re F(\sigma + iT)$. Let $b = -\sigma_0/2 + 1/4$ and $K = \sigma_0/2 + 1/4$ then, by the argument principle, the argument change of F(s) along the top side of R is less or equal to $2\pi n(K)$, where $n(K)$ is the number of zeros of $G(s)$ in $|s - b| < K$. We have

$$\int_0^{2K} \frac{n(k)}{k} dk \geq n(K) \int_K^{2K} \frac{1}{k} dk = n(K) \log 2.$$

Thus, by Jensen's formula

$$n(K) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |G(a + 2Ke^{i\theta})| d\theta - \frac{\log |G(a)|}{\log 2} \leq \frac{2}{\log 2} \max_{|s-b| \leq 2K} \log |G(s)|. \tag{2.4}$$

Also

$$\max_{|s-b|\leq 2K} |G(s)| \leq aq^{1-b+2K}. \tag{2.5}$$

By combining (2.4), (2.5) we get that the argument change of $F(s)$ along the top side of R is $\mathcal{O}(\log a + \log q + \log q \log 2aq)$.

For the argument change of $F'(s)$ along the top side of R we can apply the same reasoning as we did for $F(s)$. The only thing that changes is that the inequality (2.5) changes to

$$\max_{|s-b|\leq 2K} |G(s)| \leq aq^{1-b+2K} \max\{1, \log q\}.$$

We see that the bound stays the same as for $F(s)$. Thus the combined argument change of $F'/F(s)$ is $\mathcal{O}(\log a + \log q + \log q \log 2aq)$.

We evaluate the argument change along the bottom side of R the same way we evaluated the top side and get an analogous bound. □

Lemma 11. *Let $\sigma_0 \in \mathbb{R}$. Let $F(s) \in S_0^{\#}[a, q]$, where $a > 0$, $q \geq 2$, and let $T_0 \in \mathbb{R}$. Then $\forall\ \varepsilon > 0$ there exists a monotonic sequence $\{T_j\}$ (dependent only on ε, a, q and σ_0), $T_j \to \infty$, such that*

$$\max_{j\in\mathbb{N}}\{|F(s+iT_0)-F(s+iT_j)|, |F'(s+iT_0)-F'(s+iT_j)|, |F''(s+iT_0)-F''(s+iT_j)|\} < \varepsilon,$$

when $\sigma \geq \sigma_0$.

Proof. First we note that we can assume without loss of generality that $T_0 = 0$ due to the following inequality:

$$\max_{j\in\mathbb{N}}\{|F(s+iT_0) - F(s+iT_j)|, |F'(s+iT_0) - F'(s+iT_j)|, |F''(s+iT_0) - F''(s+iT_j)|\}$$
$$\leq a \max\{1, \log^2 q\} \max_{j\in\mathbb{N}} \sum_{n|q} \left| \frac{1}{n^{s+iT_0}} - \frac{1}{n^{s+iT_j}} \right|.$$

Dirichlet's approximation theorem states that, for any $\delta > 0$ there exists $k \in \mathbb{N}$ such, that

$$\max_{n|q} \left| \left\lfloor k\frac{\log(n)}{\log(q)} \right\rceil - k\frac{\log(n)}{\log(q)} \right| \leq \delta, \tag{2.6}$$

where $\lfloor x \rceil$ is the closest integer to $x \in \mathbb{R}$ rounding up in case of half values (see [7] for a discussion of bounds for k). Then

$$\sum_{n|q} \left| \frac{1}{n^s} - \frac{1}{n^{s+i2k\pi/\log(q)}} \right| \leq q \max\{1, e^{-\log(q)\sigma_0}\}|1 - e^{i\delta}|. \tag{2.7}$$

Fix any $\delta > 0$, for each $j \in \mathbb{N}$ take one $k \in \mathbb{N}$ that satisfies (2.6) with δ/j instead of δ. Denote this k by k_j. We can assume without loss of generality that $\{k_j\}$ is a non-decreasing sequence. If the sequence $\{k_j\}$ is bounded then $F(s)$ is periodic in the direction of the imaginary axis and the existence of $\{T_j\}$ is obvious. Otherwise we can take $T_j = 2k_j\pi/\log(q)$ and the sequence satisfies the lemma's condition for some $\varepsilon > 0$ due to inequality (2.7). By taking $\delta \to 0$ we can find a sequence for all $\varepsilon > 0$. □

Proof of Theorem 4. Take any $F(s) \in S_0^{\#}[a, q, T_0, \eta]$ and let $\delta > 0$ be defined as in proof of Theorem 3. Let $\sigma_0 < 1/2 - \delta$ be such that the argument change of $F'/F(s)$ along any vertical strip $\sigma = \sigma_0$ be less than $\pi/2$, existence of σ_0 can be easily deduced from (2.1). For $\varepsilon > 0$ sufficiently small let $\{T_j\}$ be the sequence defined in Lemma 11.

For each $j \in \mathbb{N}$ define a rectangle R_j with vertices $1/2-\delta+iT_0$, $1/2-\delta+iT_0+iT_j$, $\sigma_0 + iT_0 + iT_j$, $\sigma_0 + iT_0$.

Let

$$H = \sup_{F(s)\in S_0^{\#}[a,q],\sigma\geq\sigma_0} |F(s)| + \sup_{F(s)\in S_0^{\#}[a,q],\sigma\geq\sigma_0} |F'(s)|.$$

For $\varepsilon < \eta/2$, $F(s)$ and $F'(s) \neq 0$ on the top side of R_j. Rewrite $F(s+iT_0+iT_j) = F(s+iT_0)+\theta(s+iT_0;T_j)$, then $|\theta(s+iT_0;T_j)| < \varepsilon$. The combined argument change of $F(s)$ along the top and bottom sides of R_j can be made sufficiently small, as shown by the inequality:

$$\left| \int_{\sigma_0}^{1/2-\delta} \frac{F'(z+iT_0)+\theta'(z+iT_0;T_j)}{F(z+iT_0)+\theta(z+iT_0;T_j)} - \frac{F'(z+iT_0)}{F(z+iT_0)} dz \right| =$$

$$\int_{\sigma_0}^{1/2-\delta} \left| \frac{F(F'+\theta') - F'(F+\theta)}{F(F+\theta)} \right| dz \leq \int_{\sigma_0}^{1/2-\delta} \left| 2\frac{F\theta' - F'\theta}{\eta^2} \right| dz \leq$$

$$(1/2-\delta-\sigma_0)\frac{4H\epsilon}{\eta^2}.$$

In the same way we can prove that combined argument change of $F'(s)$ along the top and bottom sides of R_j can be made sufficiently small.

The argument change of $F'/F(s)$ along the right side of R_j does not exceed π as shown in the proof of Theorem 3. Thus, the argument change of $F'/F(s)$ along the right and left sides of R_j does not exceed $3\pi/2$ which implies that the argument change of $F'/F(s)$ along R_j is equal to zero.

□

Proof of Proposition 6. We can assume without loss of generality that $F(s) \neq 0$, when $t = 0$. For $\varepsilon > 0$ sufficiently small let $\{T_j\}$ be the sequence defined in Lemma 11 (taking $\sigma_0 = 1/2$). There exists a constant $K > 0$ such that $F(s) \neq 0$, when

$\sigma \geq K$ or $\sigma \leq 1-K$. Then

$$N(T) = \frac{1}{2\pi i}\int\limits_{1-K}^{K} + \int\limits_{K}^{K+iT} + \int\limits_{K+iT}^{1-K+iT} + \int\limits_{1-K+iT}^{1-K} \frac{F'(z)}{F(z)}dz. \tag{2.8}$$

Using formula (2.1) we get

$$\int\limits_{1-K+iT}^{1-K} \frac{F'(z)}{F(z)}dz = iT\log q_F - \int\limits_{K}^{K+iT} \overline{\frac{F'(z)}{F(z)}}dz,$$

then

$$\left(\int\limits_{K}^{K+iT} + \int\limits_{1-K+iT}^{1-K}\right)\frac{F'(z)}{F(z)}dz = iT\log q_F + \int\limits_{K}^{K+iT} \frac{F'(z)}{F(z)} - \overline{\frac{F'(z)}{F(z)}}dz = iT\log q_F + 2i\int\limits_{K}^{K+iT} \Im\frac{F'(z)}{F(z)}dz. \tag{2.9}$$

Note that

$$\lim_{K\to\infty}\left|2i\int\limits_{K}^{K+iT}\Im\frac{F'(z)}{F(z)}dz\right| = 0$$

for every fixed $T \in \mathbb{R}$. Also applying the functional equation (2.1) we get

$$\left(\int\limits_{1-K}^{K} + \int\limits_{K+iT}^{1-K+iT}\right)\frac{F'(z)}{F(z)}dz = 2i\left(\int\limits_{1/2}^{K} + \int\limits_{K+iT}^{1/2+iT}\right)\Im\frac{F'(z)}{F(z)}dz. \tag{2.10}$$

Fix $\varepsilon > 0$ sufficiently small. Let $k > 0$, $T \in \mathbb{R}$ and $A = \max_{n|q_F}|a_n|$. For k sufficiently big we have $|F(s)| > 1/2$ and

$$\begin{aligned}\lim_{K\to\infty}\left|2i\int\limits_{k}^{K} + \int\limits_{K+iT}^{k+iT}\Im\frac{F'(z)}{F(z)}dz\right| &\leq \lim_{K\to\infty} 2\int\limits_{k}^{K} + \int\limits_{k+iT}^{K+iT} 2\,|F'(z)|\,dz \\ &\leq \lim_{K\to\infty} 8A\int\limits_{k}^{K}\frac{1}{2^\sigma}d\sigma < \varepsilon.\end{aligned} \tag{2.11}$$

Let $\{T_j\}$ be the sequence defined in Lemma 11 (taking ε/k instead of ε and $\sigma_0 = 1/2$). Similarly as in proof of Theorem 4 there exists a constant C (not dependent on $j \in \mathbb{N}$), such that

$$\left|2i\int\limits_{1/2}^{k} + \int\limits_{k+iT_j}^{1/2+iT_j}\Im\frac{F'(z)}{F(z)}dz\right| < C\varepsilon(k-1/2)/k < C\varepsilon \tag{2.12}$$

Combining formulas (2.9), (2.10) and inequalities (2.11), (2.12) we get

$$\begin{aligned}N(T_j) = \lim_{K\to\infty}\frac{1}{2\pi}\left|T_j\log q_F + 2\int\limits_{K}^{K+iT_j} + \int\limits_{1/2}^{k} + \int\limits_{k+iT_j}^{1/2+iT_j} + \int\limits_{k}^{K} + \int\limits_{K+iT}^{k+iT}\Im\frac{F'(z)}{F(z)}dz\right| &\leq \\ \frac{\log q_F}{2\pi}T_j + \frac{(C+1)}{2\pi}\varepsilon.&\end{aligned}$$

By construction of $\{T_j\}$ we see that $(\log(q_F)/2\pi)T_j \in \mathbb{N}$ for all $j \in \mathbb{N}$. □

Proof of Theorem 7. Suppose $F(s)$ does not satisfy RH, then there exists a zero ρ of $F(s)$, such that $\Re\rho > 1/2$. Let D be a disc around ρ, such that D would not contain any other zeros of $F(s)$ and the line $\sigma = 1/2$ would not pass through D. Let $\varepsilon > 0$ be sufficiently small and let $\{T_j\}$ be a sequence defined in Lemma 11. By construction of $\{T_j\}$ there exists a $k \in \mathbb{N}$, such that $T_j = 2k\pi/\log(q_F)$. Note that by construction of $\{T_j\}$

$$\max_{n|q_F}\left|\left\lfloor k\frac{\log(n)}{\log(q_F)}\right\rceil - k\frac{\log(n)}{\log(q_F)}\right| \leq \varepsilon,$$

where $\lfloor x\rceil$ is the closest integer to $x \in \mathbb{R}$ rounding up in case of half values. Define $G(s) = \sum_{n|q_F} A_n q_F^{((1/2-s)/k)\lfloor k\log(n)/\log(q_F)\rceil}$. Then $G(s)$ is a self-inversive polynomial in terms of $q_F^{(1/2-s)/k}$. Let s_0 be any zero of $G(s)$. Then by Theorem 2 $\left|q_F^{(1/2-s_0)/k}\right| = 1$ because (1.1) or (1.2) is satisfied. Thus, $\Re s_0 = 1/2$. We can similarly rewrite $F(s) = \sum_{n|q_F} A_n q_F^{((1/2-s)/k)(k\log(n)/\log(q_F))}$.
Let $\eta = \min_{s\in D}|F(s)|$ and $A = \max_{n|q_F}|A_n|$. Then

$$\max_{s\in D}|F(s) - G(s)| = \max_{s\in D}\left|\sum_{n|q_F} A_n\left(q_F^{((1/2-s)/k)(k\log(n)/\log(q_F))} - q_F^{((1/2-s)/k)\lfloor k\log(n)/\log(q_F)\rceil}\right)\right|$$
$$\leq \max_{s\in D} A\left|1 - q_F^{((1/2-s)/k)\varepsilon}\right|\sum_{n|q_F}\left|q_F^{((1/2-s)/k)(k\log(n)/\log(q_F))}\right| \leq \max_{s\in D} Aq_F\left|1 - q_F^{((1/2-s)/k)\varepsilon}\right| < \eta$$

when ε sufficiently small. Thus by Rouche's theorem (see [13, Section 3.42.])

Theorem 12. *Let $K \subset G \subset \mathbb{C}$ be a bounded region with continuous boundary ∂K. Let $f(s)$, $g(s)$ be two holomorphic functions in G such that $|g(s)| < |f(s)|$ holds when $s \in \partial K$. Then $f(s)$ and $f(s) + g(s)$ have the same number of zeros (counting with multiplicities) in K.*

$F(s)$ and $G(s)$ have the same number of zeros in D (taking $f(s) = F(s)$ and $g(s) = G(s) - F(s)$) which is a contradiction. □

Lemma 13. *Let $F(s) \in S_0^{\#}$, $q_F \geq 2$. Let $\sigma_0 \in \mathbb{R}$ and $\Re s \geq \sigma_0$. Suppose $F(s) \neq 0$ and d is the distance from s to the nearest zero of $F(s)$. Then*

$$\frac{1}{|F(s)|} < e^{-C_1\log d + C_2},$$

where C_1, C_2 are positive constants that depend on σ_0 and $F(s)$.

Proof. We use the following lemma proved in [4]:

Lemma 14. *If $f(s)$ is regular and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \tag{2.13}$$

in $\{s : |s-s_0| \le r\}$ with $M > 1$, then

$$\left|\frac{f(s_0)}{f(s)}\prod_\rho \frac{s-\rho}{s_0-\rho}\right| < e^{CM}$$

for $|s-s_0| \le (3/8)r$, where C is some constant and ρ runs through the zeros of $f(s)$ such that $|\rho - s_0| \le (1/2)r$.

By Lemma 10, there exist $\sigma_1 > \sigma_0$ such that $|F(s)| > 1/2$, when $\Re s = \sigma_1$.
Let $r = (8/3)|\sigma_0 - \sigma_1|$ and $s_0 = \sigma_1 + it$. Then (2.13) is satisfied, because $|F(s)| \le H$ in the region $\Re(s) \ge \sigma_1 - r$ for some $H > 0$.
Let D be a circle of radius r in the complex plane and $N(D)$ the number of zeros of $|F(s)|$ in D (counting with multiplicities). From Proposition 6 we see that $N(D)$ is bounded, that is there exists $N \in \mathbb{N}$ such that $N(D) < N$ for any circle in $\mathbb{C}$ with a fixed radius. Then

$$\frac{1}{2}\left|\frac{1}{F(s)}\right|\left(\frac{d}{r}\right)^N < \left|\frac{F(s_0)}{F(s)}\right|\left(\frac{d}{r}\right)^N < \left|\frac{F(s_0)}{F(s)}\prod_\rho \frac{s-\rho}{s_0-\rho}\right|$$

which implies

$$\left|\frac{1}{F(s)}\right| < 2\left(\frac{r}{d}\right)^N e^c < e^{-C_1 \log d + C_2}.$$

□

Proof of Proposition 8. Let $\varepsilon > 0$ be sufficiently small and let $\{T_j\}$ be a sequence defined in Lemma 11. Denote $D = \{s : |\rho - s| = \delta\}$. By Lemma 13 there exists $\eta > 0$ such that $|F(s)| > \eta$, when $s \in D$ for any ρ. When $\varepsilon < \eta/2$ we have

$$\max_{s\in D} |F(s) - F(s+iT_j)| < \eta < \min_{s \in D} |F(s)|,$$

for any $j \in \mathbb{N}$. Thus by Rouche's theorem $N_\delta(\rho) = N_\delta(\rho + iT_j)$ for all $j \in \mathbb{N}$. □

Proof of Corollary 9. Suppose $F(s)$ does not satisfy RH and let ρ be the zero of $F(s)$ that is not on the critical line and that has the smallest positive imaginary part. Suppose $\Im\rho > T_1 + \mu_F/4$. By Proposition 8 $F(s)$ has a zero with imaginary part between $\Im\rho - T_1 - \mu_F/4$ and $\Im\rho - T_1 + \mu_F/4$, which is a contradiction to the minimality of $\Im\rho$. We note that this second zero cannot be on the critical line since $|\Re\rho - 1/2| > \mu_F/2$ due to zeros of $F(s)$ being symmetric across the critical line.
Thus, we have shown that if $F(s)$ does not satisfy RH it must have a zero in the region $0 \le t < T_1 + \mu_F/4$ that is not on the critical line, which proves the corollary.

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