VILNIAUS UNIVERSITETAS

MATEMATIKOS IR INFORMATIKOS FAKULTETAS

MATEMATIKOS MAGISTRANTŪROS STUDIJŲ PROGRAMA

Magistro baigiamasis darbas

Epšteino dzeta funkcijų suma virš Rymano dzeta funkcijos nulių
Sum of the Epstein Zeta over the Riemann Zeta Zeros

Tomas Kondratavičius

Darbo vadovas Prof. dr. Ramūnas Garunkštis
(darbo vadovo pedagoginis vardas, vadovo vardas ir pavardė)

Vilnius, 2021

Contents

SUM OF THE EPSTEIN ZETA OVER THE RIEMANN ZETA ZEROS

Abstract. We investigate the sum of values of the Epstein zeta function $Z(s,\alpha)$ at non-trivial zeros of the Riemann zeta function $\zeta(s)$, for $\alpha^2 \in \mathbb{N}$

1. Introduction

Let $s = \sigma + it$ denote a complex variable. Let T be a sufficiently large positive number throughout this paper. First define the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 0.$$

The Riemann zeta function can be analytically continued throughout the whole complex plane except the point $s = 1$, which is a simple pole with residue 1. Also denote the Riemann zeta zeros in the critical strip $0 < \sigma < 1$ by $\varrho = \beta + i\gamma$, the non-trivial zeros of the Riemann zeta function. In addition, by (a, b) we mean the greatest common divisor of integers a and b.

The Epstein zeta function is given by

$$Z_Q(s) = \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{1}{Q(m,n)^s}, \quad \sigma > 1,$$

where $Q(m,n) = am^2 + bmn + cn^2$ is a positive definite binary quadratic form with discriminant $\Delta := b^2 - 4ac < 0$. The dash on the sign of summation indicates that the part where $m = n = 0$ is omitted. Additionally, the Epstein zeta function has an analytic continuation to all complex plane except the point $s = 1$, which is a simple pole ([12]) with the residue $\frac{2\pi}{\sqrt{-\Delta}}$ ([24]). Moreover, the Epstein zeta function is closely related to another zeta function, namely, Dedekind's zeta function.

Let $\mathbb{K}$ be an algebraic number field, then the Dedekind zeta function of $\mathbb{K}$ is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where the sum is over all ideals $\mathfrak{a} \subset \mathcal{O}_{\mathbb{K}}$, with $\mathcal{O}_{\mathbb{K}}$ being the ring of integers of $\mathbb{K}$. $\mathfrak{N}$ is the norm of an ideal defined as a number of elements of the quotient ring $O_{\mathbb{K}}/\mathfrak{a}$. Note, when $\mathbb{K}$ is a quadratic field, in other words $\mathbb{K} = \mathbb{Q}(\sqrt{m})$, with m square-free, Dedekind's zeta function satisfies the identity ([4])

$$\zeta_{\mathbb{Q}(\sqrt{m})}(s) = \zeta(s)L(s,\chi),$$

where $L(s,\chi)$ is a Dirichlet L function associated with Dirichlet character χ

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Let G be a group of reduced residue classes mod q, for some integer q. For each character f of G we define a function χ as:

$$\chi(n) = f(\hat{n}) \quad \text{if } (n,k) = 1,$$
$$\chi(n) = 0 \qquad \text{if } (n,k) > 1.$$

The function χ is called a Dirichlet character mod q ([2]). Also, we will call χ_o a principle Dirichlet character if:

$$\chi_0 = \begin{cases} 1 & \text{if} \quad (n,k) = 1, \\ 0 & \text{if} \quad (n,k) > 1. \end{cases}$$

In addition, $\chi(n)$ is a multiplicative function,

$$\chi(nm) = \chi(n)\chi(m) \quad , \text{for } m, n \in \mathbb{N}.$$

On the other hand, Dedekind's zeta function associated with a quadratic imaginary field $Q(\sqrt{m})$, $m < 0$, is a sum of the Epstein zeta functions ([20],[25])

$$\zeta_{\mathbb{Q}(\sqrt{m})}(s) = \frac{1}{l} \sum_{r} \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{Q_r(m,n)^s},$$

where l is the number of units in $Q(\sqrt{m})$ and $Q_r(m,n)$ runs through all classes of forms of discriminant Δ, where Δ is a discriminant of $Q(\sqrt{m})$. The number of classes of forms of discriminant Δ is called a class number - $h(\Delta)$. Therefore for the class number $h(d) = 1$ the Epstein zeta function becomes a multiple of Dedekind's zeta function and therefore a multiple of the Riemann zeta function. It is known, for $d < 0$, that

$$d = -3,\ -4,\ -7,\ -8,\ -11,\ -19,\ -43,\ -67,\ -163 \tag{1}$$

are the only values for which $h(d) = 1$ ([3]). Hence, if Δ corresponds to one of the 9 values above, then $h(d) = 1$ and every non-trivial zero of the Riemann zeta function is also a zero of the Epstein zeta function. With this in mind we consider a sum

$$\sum_{1<\gamma<T} Z(\varrho, \alpha),$$

taken over non-trivial zeros of the Riemann zeta function. Where $Z(s,\alpha)$ is the special Epstein zeta function $Z(s,Q)$ with $Q = m^2 + \alpha^2 n^2$. Similar sums were considered by many authors ([5],[8],[9],[10]). In 1984 S.M. Gonek ([10]), by assuming Riemann hypothesis, proved, for α any real number satisfying $|\alpha| \leq \frac{1}{2}L$,

$$\sum_{1\leq\gamma\leq T} |\zeta(\frac{1}{2} + i(\gamma + \alpha L^{-1}))|^2 = \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right) \frac{T}{2\pi} \log^2 T + O(T \log T). \tag{2}$$

In 2000 R. Garunkštis and J. Steuding ([8]), under the assumption of the Riemann hypothesis, for fixed $\alpha \neq 1$, proved

$$\sum_{1\leq\gamma\leq T} \zeta(\varrho,\alpha) = -\left(\Lambda\left(\frac{1}{\alpha}\right) + L(1,-\alpha)\right)\frac{T}{2\pi} + O(T^{\frac{1}{2}+\frac{16}{237}+\varepsilon}), \tag{3}$$

where

$$\Lambda(x) = \begin{cases} \log p & \text{if } x = p^k, p \text{ is prime}, k \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

is the von Mangoldt Λ-function, and

$$L(s,\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^s}\exp(2\pi i\alpha n).$$

For $\sigma > 1$, $\zeta(\varrho,\alpha)$ is the Hurwitz zeta function

$$\zeta(\varrho,\alpha) = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}.$$

In 2010 R. Garunkštis, J. Kalpokas and J. Steuding showed ([9]), for ψ, χ Dirichlet characters mod$Q$ and q accordingly, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, with A and B being positive constants,

$$\begin{aligned}\sum_{1<\gamma_\chi\leq T} L(\varrho_\chi,\psi) = \frac{T}{2\pi}\log\frac{Tq}{2\pi\,\mathrm{e}} - \delta(q,Q)L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\varphi(Q)}\frac{T}{2\pi} \\ + \frac{L'}{L}(1,\psi\overline{\chi})\frac{T}{2\pi} + O(T\exp(-c\log^{\frac{1}{4}-\varepsilon}T)),\end{aligned} \tag{4}$$

where $\delta(q,Q) = 1$ if $q|Q$ and $\delta(q,Q) = 0$ otherwise. Symbol ψ_0 is the principal Dirichlet character $\bmod Q$ and c is a positive absolute constant. The main idea is to consider the sum as a contour integral. Which can then be estimated by using variations of lemmas introduced by Gonek in ([10]). It is, therefore, natural to expect similar results in the case of the Epstein zeta function.

Let Gauss sum associated with a Dirichlet character χ mod$q$ be defined by

$$G(\chi,k) = \sum_{a=1}^{q}\chi(a)\exp\left(2\pi i\frac{ak}{q}\right).$$

2. Results

Theorem 1. *Let $\varrho = \beta + i\gamma$ be non-trivial zeros of the Riemann zeta function, then for α fixed*

$$\sum_{1<\gamma<T} Z(\varrho,\alpha) \ll T\log^3 T.$$

Remark. It is important to note that according to the results (2), (3), (4) and Gonek's method we would expect to get something of the form

$$\sum_{1<\gamma<T} Z(\varrho,\alpha) = M(T) + E(T),$$

where $M(T)$ is an explicit main term and $E(T)$ an error term. However, in the proof below we obtained

$$M(T) = \frac{2\alpha}{\varphi(4\alpha^2)} G(\chi_0,1)\, \mathrm{e}^{\frac{\pi i}{2}} T,$$

with $G(\chi_0,1)=0$ in our case. Furthermore, compared to (2), (3), (4), the bound $E(T) \ll T\log^3 T$ of the error term is larger. This is because the functional equation (18) of $Z(s,\alpha)$ already yields a quicker grow. Indeed, $Z(s,\alpha) \sim t^{1-2\sigma}\log t$, for $\sigma<0$, compared to $\zeta(s) \sim t^{\frac{1}{2}-\sigma}\log t$, for $\sigma<0$ (see (22)).

In view of (1), $h(\Delta)=1$, only when

$$\alpha = \frac{\sqrt{3}}{2},\ 1,\ \frac{\sqrt{7}}{2},\ \sqrt{2},\ \frac{\sqrt{11}}{2},\ \frac{\sqrt{19}}{2},\ \frac{\sqrt{43}}{2},\ \frac{\sqrt{67}}{2},\ \frac{\sqrt{163}}{2}.$$

Hence, for the above values of α, we have

$$\sum_{1<\gamma<T} Z(\varrho,\alpha) = 0.$$

However, the result in Theorem 1 does not include such a connection with the class number - $h(\Delta)$. Therefore, in the future, we are planning to investigate a more difficult case. In particular, the sum

$$\sum_{1<\gamma<T} |Z(s,\alpha)|^2$$

instead. Which, we hope, will distinguish cases, when $h(\Delta)=1$ from cases, when $h(\Delta)>1$.

3. Lemmas

Lemma 2. *For sufficiently large A, uniformly in b,*

$$\frac{1}{2\pi}\int_A^B \left(\frac{t}{2\pi}\right)^{b-\frac{1}{2}} \exp\left(it\log\frac{t}{er}\right) dt \tag{5}$$
$$= \begin{cases} \left(\frac{r}{2\pi}\right)^b \exp\left(\frac{\pi i}{4} - ir\right) + E(r,A) & \text{if} \quad A \le r \le B \le 2A, \\ E(r,A) & \text{if} \quad r < A \text{ or } r > B, \end{cases}$$

where

$$E(r,A) = O(A^{b-\frac{1}{2}}) + O\left(\frac{A^{b+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right).$$

Proof. The proof is given by Gonek in [10].

Lemma 3. *Let* $1 < a \le 1 + \frac{1}{\log T}$*, then*

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2 n^2)^a} E(2\pi(m^2+\alpha^2n^2), T) \ll T^{a-\frac{1}{2}}(a-1)^{-1},$$

where

$$E(r, A) = O(A^{a-\frac{1}{2}}) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right).$$

Proof. By the Laurent expansion at $s = 1$, see [24],

$$Z(s,\alpha) = \frac{\frac{\pi}{\alpha}}{s-1} + A_0 + O(s-1), \quad (s \mapsto 1).$$

Hence, we have

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2 n^2)^a} \ll (a-1)^{-1}.$$

Therefore

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2 n^2)^a} O(T^{a-\frac{1}{2}}) \ll T^{a-\frac{1}{2}}(a-1)^{-1}.$$

To evaluate the second term we split the range of summation in the following three sets

$$\begin{aligned} A:&\quad |T - 2\pi(m^2+\alpha^2n^2)| > \frac{T}{2},\\ B:&\quad T^{\frac{1}{2}} \le |T - 2\pi(m^2+\alpha^2n^2)| \le \frac{T}{2},\\ C:&\quad |T - 2\pi(m^2+\alpha^2n^2)| < T^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{A} \frac{1}{(m^2+\alpha^2n^2)^a} \frac{T^{a+\frac{1}{2}}}{|T-2\pi(m^2+\alpha^2n^2)| + T^{\frac{1}{2}}} &\ll \frac{T^{a+\frac{1}{2}}}{\frac{T}{2}+T^{\frac{1}{2}}} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2n^2)^a}\\ &\ll T^{a-\frac{1}{2}}(a-1)^{-1}; \end{aligned}$$

$$\begin{aligned} &\sum_{B} \frac{1}{(m^2+\alpha^2n^2)^a} \frac{T^{a+\frac{1}{2}}}{|T-2\pi(m^2+\alpha^2n^2)| + T^{\frac{1}{2}}}\\ &\ll T^{a+\frac{1}{2}} \sum_{T^{\frac{1}{2}} \le |T-2\pi(m^2+\alpha^2n^2)| \le \frac{T}{2}} \frac{1}{(m^2+\alpha^2n^2)^a |T-2\pi(m^2+\alpha^2n^2)|}. \end{aligned}$$

To continue, assume that

$$T^{\frac{1}{2}} \le 2\pi(m^2+\alpha^2n^2) - T \le \frac{T}{2}.$$

Next, we split the sum over B further into $\ll \log T$ sums of the type

$$T + P \le 2\pi(m^2+\alpha^2n^2) \le T + 2P,$$

where $T^{\frac{1}{2}} \ll P \ll T$. With this, we have

$$\frac{1}{2\pi(m^2+\alpha^2 n^2)-T} \ll P^{-1}.$$

Therefore

$$\sum_{T^{\frac{1}{2}} \leq 2\pi(m^2+\alpha^2n^2)-T\leq \frac{T}{2}} \frac{1}{(m^2+\alpha^2n^2)^a|T-2\pi(m^2+\alpha^2n^2)|}$$
$$\ll P^{-1}\log T \sum_{T+P\leq 2\pi(m^2+\alpha^2n^2)\leq T+2P} \frac{1}{(m^2+\alpha^2n^2)^a} \ll T^{-a}\log T.$$

Similarly, we get the bound for the range $T^{\frac{1}{2}} \leq T-2\pi(m^2+\alpha^2n^2) \leq \frac{T}{2}$. Hence

$$\sum_B \ll T^{a-\frac{1}{2}}(a-1)^{-1};$$

$$\sum_C \frac{T^{a+\frac{1}{2}}}{(m^2+\alpha^2n^2)^a(|T-2\pi(m^2+\alpha^2n^2)|+T^{\frac{1}{2}})}$$
$$\ll T^a \sum_{|T-2\pi(m^2+\alpha^2n^2)|<T^{\frac{1}{2}}} \frac{1}{(m^2+\alpha^2n^2)^a}$$
$$\ll T^{\frac{1}{2}}.$$

This and bounds for the other ranges prove the lemma.

Lemma 4. *Let* $1 < a \leq 1+\frac{1}{\log T}$*, then*

$$\frac{1}{2\pi}\int_1^T \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{t}{2\pi e}\right) \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2n^2)^s}dt$$
$$= \sum_{1\leq m^2+\alpha^2n^2\leq \frac{T}{2\pi}} 1 + O(T^{a-\frac{1}{2}}\log T).$$

Proof. We follow the proof of the Lemma 5 of Gonek [10]. First, we note that

$$\frac{1}{2\pi}\int_{\frac{T}{2}}^T \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{t}{2\pi e}\right) \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2n^2)^s}dt \tag{6}$$
$$= \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2n^2)^a}\left(\frac{1}{2\pi}\int_{\frac{T}{2}}^T \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{t}{2\pi e(m^2+\alpha^2n^2)}\right)dt\right),$$

where the inversion of summation and integration is justified by absolute convergence of the series. Applying Lemma 2 with $A=\frac{\tau}{2}$, $B=\tau$ and $r=2\pi(m^2+\alpha^2n^2)$, the left hand side of (6) is

$$\mathrm{e}^{\frac{\pi i}{4}} \sum_{\frac{\tau}{4\pi}\leq m^2+\alpha^2n^2\leq\frac{\tau}{2\pi}} 1 + \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2n^2)^a}E(2\pi(m^2+\alpha^2n^2),\tau).$$

By Lemma 3 the second term, in the last expression, is

$$\ll \tau^{a-\frac{1}{2}} \log \tau. \tag{7}$$

Therefore (6) is equal to

$$\mathrm{e}^{\frac{\pi i}{4}} \sum_{\frac{\tau}{4\pi} \le m^2+\alpha^2 n^2 \le \frac{\tau}{2\pi}} 1 + O(\tau^{a-\frac{1}{2}} \log \tau).$$

Let l be an integer such that $T_0 \le \frac{T}{2^l} < 2T_0$, where T_0 is some fixed positive number. In view of (7), with $\tau = \frac{T}{2^{j-1}}$, for $j = 1, \dots, l$; we find

$$\begin{aligned}&\frac{1}{2\pi} \int_{\frac{T}{2^l}}^{T} \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it \log \frac{t}{2\pi e}\right) \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2 n^2)^s} dt \\ &= \mathrm{e}^{\frac{\pi i}{4}} \sum_{\frac{T}{2^{l+1}\pi} \le m^2+\alpha^2 n^2 \le \frac{T}{2\pi}} 1 + O(T^{a-\frac{1}{2}} \log T).\end{aligned}$$

Finally, note that

$$\frac{1}{2\pi} \int_{1}^{\frac{T}{2^l}} \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it \log \frac{t}{2\pi e}\right) \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m^2+\alpha^2 n^2)^s} dt \ll 1$$

and

$$\sum_{1 \le m^2+\alpha^2 n^2 \le \frac{T}{2^l \pi}} 1 \ll 1.$$

This completes the proof.

Lemma 5. *(Perron's formula) Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $a_n \ll \psi(n)$, $\psi(n)$ non-decreasing and

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \ll (\sigma - 1)^{-\alpha},$$

as $\sigma \mapsto 1$. Then if $c > 0$, $\sigma + c > 1$, x is not an integer and N is the integer nearest to x,

$$\begin{aligned}\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) \\ + O\left(\frac{\psi(2x) x^{1-\sigma} \log x}{T}\right) + O\left(\frac{\psi(N) x^{1-\sigma}}{T|x-N|}\right).\end{aligned}$$

Proof. See Titchmarsh [23].

4. Proof of the theorem

Let $a = 1 + \frac{1}{\log T}$ and define the contour $\mathfrak{C}$ to be the rectangle with vertices $a+i$, $a+iT$, $1-a+iT$, $1-a+i$. Then Cauchy Theorem gives

$$\sum_{1<\gamma<\tau} Z(\varrho,\alpha) = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s) Z(s,\alpha)ds. \tag{8}$$

Thus

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s) Z(s,\alpha)ds\\ &= \frac{1}{2\pi i}\left(\int_{a+i}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a-i}^{a+i}\right)\frac{\zeta'}{\zeta}(s)Z(s,\alpha)ds\\ &=: \sum_{j=1}^{4} I_j.\end{aligned}$$

In the half-plane $\sigma > 1$ of absolute convergence we may rewrite the integrand in the above formula as a Dirichlet series and interchange summation and integration. Then using

$$\frac{\zeta'}{\zeta}(s) = -\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^s},$$

we get

$$\mathfrak{I}_1 = -\frac{1}{2\pi}\sum_{k=2}^{\infty}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{\Lambda(k)}{(k(m^2+\alpha^2n^2))^a}\int_1^T\frac{dt}{(k(m^2+\alpha^2n^2))^{it}}. \tag{9}$$

Since $k(m^2+\alpha^2n^2) \neq 1$, we have

$$\int_1^T \frac{dt}{(k(m^2+\alpha^2n^2))^{it}} \ll 1.$$

Next, by the Laurent expansions at $s=1$ (see [24]),

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \gamma + O(s-1), \tag{10}$$

$$Z(s,\alpha) = \frac{\frac{\pi}{\alpha}}{s-1} + A_0 + O(s-1), \tag{11}$$

valid for $s \to 1$, we get

$$\sum_{k=2}^{\infty}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{\Lambda(k)}{(k(m^2+\alpha^2n^2))^a} \ll \frac{\zeta'}{\zeta}(a)Z(a,1) \ll \log^2 T.$$

Thus

$$\mathfrak{I}_1 \ll \log^2 T. \tag{12}$$

We continue by estimating integrals on the horizontal paths. For the logarithmic derivative we have the partial fraction decomposition (see [14])

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma|\le 1}\frac{1}{s-\varrho} + O(\log|t+2|) \quad \text{for} \quad -1\le\sigma\le 2, |t|\ge 1. \tag{13}$$

In addition, by the Riemann-von Mangoldt formula ([23]) for the number of nontrivial zeros of $\zeta(s)$,

$$N(T) := \#\{\varrho = \beta + i\gamma \,:\, 0 < \gamma \leq T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{14}$$

Hence, we have

$$N(T+1) - N(T) \ll (T+1)\log(T+1) - T\log T \ll \log T. \tag{15}$$

From the above it follows that the zeros ϱ cannot lie too dense: for any given $T_0 > 1$ there exists a $T \in (T_0, T_0+1]$ such that

$$\min_{\gamma} |T - \gamma| \gg \frac{1}{\log T}. \tag{16}$$

With regard to (15) and (16), and the partial fraction decomposition (13) it follows that

$$\begin{aligned}
\frac{\zeta'}{\zeta}(\sigma + it) &= \sum_{|t-\gamma|\leq 1} \frac{1}{\sigma + it - \beta - i\gamma} + O(\log|t+2|) \\
&\ll \sum_{|t-\gamma|\leq 1} \frac{1}{\sqrt{(\sigma-\beta)^2 + (t-\gamma)^2}} + O(\log|t+2|) \\
&\ll \log|t+2| \sum_{|t-\gamma|\leq 1} 1 + O(\log|t+2|) \\
&\ll \log^2|t+2|.
\end{aligned}$$

Thus

$$\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2 \qquad \text{for} \quad -1 \leq \sigma \leq 2, T \geq 1. \tag{17}$$

It is known that the Epstein zeta function associated with a positive definite binary quadratic form Q satisfies the functional equation (see [18])

$$\left(\frac{\sqrt{-D}}{2\pi}\right)^{1-s} \Gamma(1-s) Z_Q(1-s) = \left(\frac{\sqrt{-D}}{2\pi}\right)^{s} \Gamma(s) Z_Q(s),$$

where D is a discriminant of Q. In our case $D = -4\alpha^2$, therefore

$$\left(\frac{\alpha}{\pi}\right)^{1-s} \Gamma(1-s) Z(1-s,\alpha) = \left(\frac{\alpha}{\pi}\right)^{s} \Gamma(s) Z(s,\alpha). \tag{18}$$

Furthermore, recall Stirling's formula

$$\log\Gamma(s) = (s - \frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \tag{19}$$

valid for $|s| \geq \frac{1}{2}$ and $|\arg s| < \pi - \delta$. Using Stirling's formula for $|t| \geq 1$

$$\begin{aligned}\frac{\Gamma(s)}{\Gamma(1-s)} &= \frac{(\sigma+it)^{\sigma-\frac{1}{2}+it}\,\mathrm{e}^{-\sigma-it}\sqrt{2\pi}}{(1-\sigma-it)^{\frac{1}{2}-\sigma-it}\,\mathrm{e}^{\sigma-1+it}\sqrt{2\pi}}\left(1+O\left(\frac{1}{t}\right)\right)\\ &= (-itit)^{\sigma-\frac{1}{2}+it}(\frac{\sigma}{it}+1)^{it}(\frac{\sigma-1}{it}+1)^{it}\,\mathrm{e}^{1-2\sigma-2it}\left(1+O\left(\frac{1}{t}\right)\right)\\ &= t^{2\sigma-1+2it}\,\mathrm{e}^{-2it}\left(1+O\left(\frac{1}{t}\right)\right).\end{aligned}$$

Hence

$$\frac{\Gamma(s)}{\Gamma(1-s)} = t^{2\sigma-1}\exp\left(2it\log\frac{t}{e}\right)\left(1+O\left(\frac{1}{t}\right)\right), \tag{20}$$

for σ fixed and $|t| \geq 1$. Therefore from the Laurent expansion of the Epstein zeta function

$$Z(1+\frac{1}{\log T}+it,\alpha) \ll \log T$$

and using functional equation (18), and asymptotic (20)

$$\begin{aligned}Z(-\frac{1}{\log T}+it,\alpha) &= \left(\frac{\alpha}{\pi}\right)^{\frac{2}{\log T}+2it-1}\frac{\Gamma(s)}{\Gamma(1-s)}Z(1+\frac{1}{\log T}+it,\alpha)\\ &\ll T\log T.\end{aligned}$$

Thus, an application of the Phragmen-Lindelöf principle yields the estimate

$$Z(s,\alpha) \ll |t|\log|t+2| \qquad \text{for} \quad -\frac{1}{\log T} \leq \sigma \leq 1+\frac{1}{\log T}, |t| \geq 1,$$

for $|t| \ll T$. Hence, together with estimate (17), we see that

$$\mathfrak{I}_2, \mathfrak{I}_4 \ll T(\log T)^3. \tag{21}$$

It remains to estimate $\mathfrak{I}_3$. We substitute $s \mapsto 1-\overline{s}$ and get

$$\mathfrak{I}_3 = -\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(1-\overline{s})Z(1-\overline{s},\alpha)ds.$$

Conjugating above gives

$$\overline{\mathfrak{I}}_3 = -\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(1-s)Z(1-s,\alpha)ds.$$

Next, we shall use the functional equations. To begin with, the functional equation for $\zeta(s)$ can be written as ([23])

$$\zeta(s) = \Delta(s)\zeta(1-s), \tag{22}$$

where

$$\Delta(s) := \frac{(2\pi)^s}{2\Gamma(s)\cos\frac{\pi s}{2}}.$$

By this and the functional equation (18) we obtain

$$\begin{aligned}\overline{\mathfrak{I}}_3 =& -\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\Delta'}{\Delta}(s)\left(\frac{\alpha}{\pi}\right)^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}Z(s,\alpha)ds\\ &+\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)\left(\frac{\alpha}{\pi}\right)^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}Z(s,\alpha)ds \\ =&:\mathfrak{F}_1+\mathfrak{F}_2,\ \text{say.}\end{aligned} \tag{23}$$

We will also require

$$\tan\frac{\pi s}{2} = -i\frac{\mathrm{e}^{-\frac{\pi t}{2}+\frac{\sigma\pi i}{2}} - \mathrm{e}^{\frac{\pi t}{2}-\frac{\sigma\pi i}{2}}}{\mathrm{e}^{-\frac{\pi t}{2}+\frac{\sigma\pi i}{2}} + \mathrm{e}^{\frac{\pi t}{2}-\frac{\sigma\pi i}{2}}} = i+O\left(\mathrm{e}^{-\pi t}\right).$$

In addition, using Stirling's formula (19),

$$\begin{aligned}\frac{\Gamma'}{\Gamma}(s) &= \log(\sigma+it)+O\left(\frac{1}{t}\right)\\ &= \log it+\log\left(\frac{\sigma}{it}+1\right)+O\left(\frac{1}{t}\right)\\ &= \log t+\frac{\pi i}{2}+O\left(\frac{1}{t}\right).\end{aligned}$$

By the above estimates, we obtain that

$$\begin{aligned}\frac{\Delta'}{\Delta}(s) &= \log(2\pi)-\frac{\Gamma'}{\Gamma}(s)+\frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right)\\ &= -\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right),\end{aligned}$$

for $t\geq 1$. Therefore

$$\mathfrak{F}_1 = \int_1^T\left(-\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)d\mathfrak{J}, \tag{24}$$

where

$$\mathfrak{J} = \frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{\alpha}{\pi}\right)^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}Z(s,\alpha)ds.$$

By the estimate (20) we get

$$\begin{aligned}\mathfrak{J} =& \frac{1}{2\pi}\int_1^\tau\left(\frac{\alpha}{\pi}\right)^{2a+2it-1}t^{2a-1}e^{2it\log\frac{t}{e}}Z(a+it,\alpha)dt\\ &+O\left(\frac{1}{2\pi}\int_1^\tau\left|\left(\frac{\alpha}{\pi}\right)^{2a+2it-1}t^{2a-1}e^{2it\log\frac{t}{e}}Z(a+it,\alpha)\right|\frac{1}{t}dt\right).\end{aligned} \tag{25}$$

Moreover, the estimate $Z(a+it,\alpha)\ll\log T$ implies that the second term in (25) is $O\left(\tau^{2a-1}\log\tau\right)$. Furthermore, we can rewrite (25) as

$$\mathfrak{J} = \alpha^{2a-1}2^{2a-1}\int_1^\tau\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)d\mathfrak{G}+O\left(\tau^{2a-1}\log\tau\right), \tag{26}$$

where

$$\mathfrak{G} = \frac{1}{2\pi}\int_1^t\left(\frac{u}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{u}{2\pi e}\right)Z(a+iu,\alpha)du.$$

By Lemma 4,

$$\mathfrak{G} = e^{\frac{\pi i}{4}} \sum_{1\le m^2+\alpha^2 n^2 \le \frac{t}{2\pi}} 1 + O(t^{a-\frac{1}{2}} \log T). \tag{27}$$

To continue with, note that

$$\sum_{m,n\in\mathbb{Z}}' \frac{1}{(m^2+\alpha^2 n^2)^s} = \sum_{k=1}^{\infty} \frac{r(k)}{k^s}, \tag{28}$$

where

$$r(k) = \sum_{k=m^2+\alpha^2 n^2} 1 \le \sum_{k=m^2+n^2} 1 \ll d(n) \ll n^{\varepsilon},$$

and $d(n) = \sum_{d|n} 1$. The last two estimates are given in [11]. By Perron's formula (Lemma 5) with $\psi(n) = n^{\varepsilon}$ and $s = 0$,

$$\sum_{1\le m^2+\alpha^2 n^2 \le \frac{t}{2\pi}} 1 = \frac{1}{2\pi i} \int_{a-iU}^{a+iU} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{t^{1+\varepsilon} \log T}{U}\right).$$

Next ,the calculus of residues yields

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{ds}{s} =$$
$$\frac{1}{2\pi i} \left\{ \int_{a-iU}^{b-iU} + \int_{b-iU}^{b+iU} + \int_{b+iU}^{a+iU} \right\} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{ds}{s}$$
$$+ \mathrm{Res}_{\ s=1} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{1}{s},$$

where $b = \frac{1}{2} + \frac{1}{\log T}$. Therefore from the Laurent expansion of the Epstein zeta function (11)

$$\mathrm{Res}_{\ s=1} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{1}{s} = \frac{t}{2\alpha}.$$

Additionally, it was shown in [21] that, for $t \ge 10$, we have

$$Z(\sigma+it,\alpha) \ll t^{1-\sigma} \log t,$$

uniformly for $0 \le \sigma \le 1$. By this

$$\int_{a\pm iU}^{b\pm iU} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{ds}{s} \ll tU^{-\frac{1}{2}} \log U$$

and

$$\int_{b-iU}^{b+iU} Z(s,\alpha) \left(\frac{t}{2\pi}\right)^s \frac{ds}{s} \ll t^{\frac{1}{2}} U^{\frac{1}{2}} \log U.$$

By choosing $U = t$, we get

$$\sum_{1\le m^2+\alpha^2 n^2 \le \frac{t}{2\pi}} 1 = \frac{1}{2\alpha} t + O(t^{\frac{1}{2}} \log t).$$

From this we obtain

$$\mathfrak{G} = e^{\frac{\pi i}{4}} \frac{1}{2\alpha} t + O(t^{\frac{1}{2}} \log t).$$

After inserting $\mathfrak{G}$, the main term of (26) is

$$\begin{aligned} &\alpha^{2a-1}2^{2a-1}\int_1^\tau \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{2t\alpha^2}{\pi e}\right) d\left(\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}t+O(t^{\frac{1}{2}}\log T)\right) \\ &=\alpha^{2a-1}2^{2a-1}\,\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}\int_1^\tau \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{2t\alpha^2}{\pi e}\right) dt \\ &+O\left(\int_1^\tau \left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} \exp\left(it\log\frac{2t\alpha^2}{\pi e}\right) t^{-\frac{1}{2}}\log T dt\right). \end{aligned} \tag{29}$$

Again by applying Lemma 2, the main term in (29) is

$$\frac{\pi}{\alpha}\,\mathrm{e}^{\frac{\pi i}{2}\left(1-\frac{1}{\alpha^2}\right)}+O(\tau^{\frac{1}{2}}).$$

And the error term in (29) is

$$O(\tau^a\log\tau).$$

Therefore

$$\mathfrak{J}=O(\tau^a\log\tau).$$

Hence

$$\begin{aligned} \mathfrak{F}_1&=\int_1^T\left(-\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)d\mathfrak{J}(\tau) \\ &=-\log\frac{T}{2\pi}\mathfrak{J}(T)+\frac{1}{2\pi}\int_1^T\frac{1}{\tau}\mathfrak{J}(\tau)d\tau+O\left(\frac{1}{T}\mathfrak{J}(T)+\int_1^T\frac{1}{\tau^2}\mathfrak{J}(\tau)d\tau\right) \\ &=O\left(T^a\log^2 T\right)+O\left(\int_1^T\tau^{a-1}\log\tau d\tau\right)+O(T^{a-1}\log T)+O\left(\int_1^T\tau^{a-2}\log\tau d\tau\right) \\ &\ll T^a\log^2 T \\ &\ll T\log^2 T. \end{aligned} \tag{30}$$

Similarly,

$$\mathfrak{F}_2=\alpha^{2a-1}2^{2a-1}\int_1^T\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)\frac{\zeta'}{\zeta}(a+iu)d\mathfrak{G}+O(T^{2a-1}\log^2 T),$$

where

$$\mathfrak{G}=\frac{1}{2\pi}\int_1^t\left(\frac{u}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{u}{2\pi e}\right)Z(a+iu,\alpha)du.$$

For this we already obtained an asymptotic bound

$$\mathfrak{G}=\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}t+O(t^{\frac{1}{2}}\log t).$$

Therefore the main term of $\mathfrak{F}_2$ is

$$\begin{aligned}
&\alpha^{2a-1}2^{2a-1}\int_1^T\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)\frac{\zeta'}{\zeta}(a+iu)d\left(\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}t+O(t^{\frac{1}{2}}\log T)\right)\\
&=\alpha^{2a-1}2^{2a-1}\,\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}\int_1^T\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)\frac{\zeta'}{\zeta}(a+iu)dt \qquad (31)\\
&+O\left(\int_1^T\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)\frac{\zeta'}{\zeta}(a+iu)t^{-\frac{1}{2}}\log T dt\right).
\end{aligned}$$

Obviously, the error term in (31) is $\ll T^{\alpha}\log^2 T$. Moreover, by slightly modified Lemma 4 (for the proof see [10]), the main term in (31) is

$$\begin{aligned}
&\alpha^{2a-1}2^{2a-1}\,\mathrm{e}^{\frac{\pi i}{4}}\frac{1}{2\alpha}\int_1^T\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{2t\alpha^2}{\pi e}\right)\frac{\zeta'}{\zeta}(a+iu)dt\\
&=\frac{\pi}{\alpha}\,\mathrm{e}^{\frac{\pi i}{2}}\sum_{k\le\frac{T2\alpha^2}{\pi}}\Lambda(k)\exp\left(-2\pi i\frac{k}{4\alpha^2}\right)+O(T^{a-\frac{1}{2}}\log T).
\end{aligned}$$

By collecting all the terms, we have

$$\mathfrak{F}_2=\frac{\pi}{\alpha}\,\mathrm{e}^{\frac{\pi i}{2}}\sum_{k\le\frac{T2\alpha^2}{\pi}}\Lambda(k)\exp\left(-2\pi i\frac{k}{4\alpha^2}\right)+O(T^{2a-1}\log^2 T). \qquad (32)$$

We require the identity

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\chi(n)=\begin{cases}1 & \text{if}\quad n\equiv a\pmod q,\\ 0 & \text{otherwise,}\end{cases} \qquad (33)$$

where χ is the Dirichlet character modulo some fixed q, $(a,q)=1$ and $\varphi(q)$ is Euler's totient function. In view of (33), we can split the sum in (32) into two sums with $(k,4\alpha^2)>1$ and $(k,4\alpha^2)=1$. If $(k,4\alpha^2)>1$, then

$$\sum_{\substack{k\le\frac{T2\alpha^2}{\pi}\\(k,4\alpha^2)>1}}\Lambda(k)\exp\left(-2\pi i\frac{k}{4\alpha^2}\right)=O\left(\sum_{k\le\frac{T2\alpha^2}{\pi}}\Lambda(k)\right)=O(\log T).$$

If $(k, 4\alpha^2) = 1$, then we can apply Dirichlet character identity (33) to get

$$\begin{aligned}
&\sum_{\substack{k\leq \frac{T2\alpha^2}{\pi}\\(k,4\alpha^2)=1}} \Lambda(k)\exp\left(-2\pi i\frac{k}{4\alpha^2}\right)\\
&=\sum_{a=1}^{4\alpha^2}\exp\left(-2\pi i\frac{a}{4\alpha^2}\right)\sum_{\substack{k\leq \frac{T2\alpha^2}{\pi}\\(k,4\alpha^2)=1\\k\equiv a \bmod 4\alpha^2}}\Lambda(k)\\
&=\frac{1}{\varphi(4\alpha^2)}\sum_{\chi \bmod 4\alpha^2}\sum_{a=1}^{4\alpha^2}\exp\left(-2\pi i\frac{a}{4\alpha^2}\right)\overline{\chi}(a)\sum_{k\leq \frac{T2\alpha^2}{\pi}}\Lambda(k)\chi(k)\\
&=\frac{1}{\varphi(4\alpha^2)}G(\overline{\chi}_0,-1)\sum_{k\leq \frac{T2\alpha^2}{\pi}}\Lambda(k)\chi_0(k)\\
&+\frac{1}{\varphi(4\alpha^2)}\sum_{\substack{\chi \bmod 4\alpha^2\\\chi\neq\chi_0}}G(\overline{\chi},-1)\sum_{k\leq \frac{T2\alpha^2}{\pi}}\Lambda(k)\chi(k),
\end{aligned}$$

where χ_0 is the principle Dirichlet character $\bmod 4\alpha^2$ and G is a Gauss sum defined as

$$G(\overline{\chi},-1)=\sum_{a=1}^{4\alpha^2}\exp\left(-2\pi i\frac{a}{4\alpha^2}\right)\overline{\chi}(a).$$

Next, by Perron's formula (Lemma 5)

$$\begin{aligned}
&\sum_{k\leq \frac{T2\alpha^2}{\pi}}\Lambda(k)\chi(k)\\
&=-\frac{1}{2\pi i}\int_{c-iU}^{c+iU}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{ds}{s}+\left(\frac{T\log^2 T}{U}\right),
\end{aligned}$$

where $L(s, \chi)$ is the Dirichlet L function associated with Dirichlet character χ

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

It is important to note, that for the principal character χ the function $L(s, \chi)$ has a simple pole at $s = 1$ otherwise it is an entire function.

Next we follow [9]. The zero-free region of the function $L(s, \chi)$ is given by (for proof see [19], chapter 8, Theorem 6.2)

$$L(s,\chi)\neq 0 \quad \text{for} \quad \sigma > 1-\frac{c}{\log^{\frac{3}{4}+\varepsilon}T},$$

where c is an absolute positive constant. By the Cauchy residue theorem

$$-\frac{1}{2\pi i}\int_{c-iU}^{c+iU}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{ds}{s}=$$
$$-\frac{1}{2\pi i}\left\{\int_{a-iU}^{b-iU}+\int_{b-iU}^{b+iU}+\int_{b+iU}^{a+iU}\right\}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{ds}{s}$$
$$+\operatorname{Res}_{s=1}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{1}{s},$$

where we choose $b=1-c/\log^{\frac{3}{4}+\varepsilon}T$. Recall, that for the principal Dirichlet character χ_0, we have

$$\frac{L'}{L}(s,\chi_0)=-\frac{1}{s-1}+\gamma+\sum_{p|4\alpha^2}\frac{\log p}{p^s-1}+O(s-1).$$

Hence

$$\operatorname{Res}_{s=1}\frac{L'}{L}(s,\chi_0)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{1}{s}=-\frac{T2\alpha^2}{\pi}$$

and

$$\operatorname{Res}_{s=1}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{1}{s}=0,$$

if χ is not the principle Dirichlet character. As in the case of the Epstein zeta function, for $\chi \bmod 4\alpha^2$ and $t\geq 0$ (see Prachar [19])

$$N_\chi(t+1)-N_\chi(t):=\{\varrho_\chi=\beta_\chi+i\gamma_\chi:\quad t<\gamma_\chi\leq t+1\}\leq\log(t+2).$$

Thus, for any given $t_0\geq 1$, there exists t, $t\in(t_0,t_0+1]$, such that

$$\min_{\gamma_\chi}|t-\gamma_\chi|\gg\frac{1}{\log t}. \tag{34}$$

Further, we have the partial fraction decomposition

$$\frac{L'}{L}(s,\chi)=\sum_{|t-\gamma_\chi|\leq 1}\frac{1}{s-\varrho_\chi}+O(\log T)\quad\text{for}-1\leq\sigma\leq 2,\quad t\geq 1. \tag{35}$$

Similarly, as in the case (17), (34) and (35) gives

$$\frac{L'}{L}(s,\chi)\ll\log^2(T)\quad\text{for}-1\leq\sigma\leq 2,\quad t\geq 1. \tag{36}$$

This yields

$$\int_{a\pm iU}^{b\pm iU}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{ds}{s}\ll\frac{T^a}{U}\log^2 U.$$

Similarly, in view of the estimate (36), we get

$$\int_{b-iU}^{b+iU}\frac{L'}{L}(s,\chi)\left(\frac{T2\alpha^2}{\pi}\right)^s\frac{ds}{s}\ll T^b\log^2 U.$$

Next, by choosing $U=T^{1-b}$, we obtain

$$\mathfrak{F}_2=-\frac{2\alpha}{\varphi(4\alpha^2)}G(\overline{\chi}_0,-1)\,\mathrm{e}^{\frac{\pi i}{2}}T+O(T^{a-1}\log^2 T). \tag{37}$$

Finally, combining the estimate (30) of $\mathfrak{F}_1$ and the asymptotic (37) of $\mathfrak{F}_2$

$$\begin{aligned}\overline{\mathfrak{I}}_3 &= \mathfrak{F}_1 + \mathfrak{F}_2 \\ &= -\frac{2\alpha}{\varphi(4\alpha^2)} G(\overline{\chi}_0, -1)\, \mathrm{e}^{\frac{\pi i}{2}} T + O(T \log^2 T).\end{aligned}$$

After conjugating

$$\mathfrak{I}_3 = \frac{2\alpha}{\varphi(4\alpha^2)} G(\chi_0, 1)\, \mathrm{e}^{\frac{\pi i}{2}} T + O(T \log^2 T).$$

Hence from this, (12) and (21)

$$\sum_{1<\gamma<\tau} Z(\varrho, \alpha) = \frac{2\alpha}{\varphi(4\alpha^2)} G(\chi_0, 1)\, \mathrm{e}^{\frac{\pi i}{2}} T + O(T \log^3 T).$$

This proves Theorem 1.

5. Summary

The sum of the special Epstein zeta functions was investigated in this paper. The special Epstein zeta function is given by $Z(s,\alpha) = \sum'_{m,n\in\mathbb{Z}} (m^2 + \alpha^2 n^2)^{-s}$. We prove that, for non-trivial zeros $\varrho = \beta + i\gamma$ of the Riemann zeta function, $\sum_{1<\gamma<T} Z(s,\alpha) \ll T \log^3 T$. The main idea was to use Gonek's method. Specifically, interpret the sum as a contour integral.

References

[1] R.F. Arenstorf and L.L. Brewer, A study of the motion of zeros of the Epstein zeta function associated to $m^2 + y^2 n^2$ as y varies from 1 to $\sqrt{6}$, *Computers & Mathematics with Applications*, **26**(5) (1993), 57-69.

[2] T.M. Apostol, *Introduction to analytic number theory.* Springer-Verlag, New York, 1976.

[3] H. Cohen, *A Course in Computational Algebraic Number Theory.* Springer-Verlag, Berlin, Springer-Verlag.

[4] H. Cohen, *Number theory, Volume II: Analytic and Modern Tools.* Springer-Verlag, New York, 2007.

[5] J.B. Conrey, A. Ghosh, S.M. Gonek, Simple zeros of the Riemann zeta-function, *Proceedings of the London Mathematical Society*, **76**(3) (1998), 497-522.

[6] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Springer-Verlag, New York, 1980.

[7] A. Fujii, On a mean value theorem in the theory of the Riemann zeta function, *Commentarii Mathematici Universitatis Sancti Pauli*, **44**(1) (1995), 59-67.

[8] R. Garunkštis, J. Steuding, On the Distribution of Zeros of the Hurwitz Zeta-Function, *Mathematics of Computation*, **76**(257) (2007), 323-337.

[9] R. Garunkštis, J. Kalpokas, J. Steuding, Sum of the Dirichlet L-function over nontrivial zeros of another Dirichlet L-function, *Acta Mathematica Hungarica*, **128**(3) (2010), 287-298.

[10] S. M. Gonek, Mean values of the Riemann zeta function and its derivatives, *Inventiones Mathematicae*, **75**(1) (1984), 123-141.

[11] G. H. Hardy, E. M. Wright , *An Introduction to the Theory of Numbers*, 4th ed., Clarendon press, Oxford, 1979.

[12] D.A. Hejhal, Zeros of Epstein zeta functions and supercomputers, *Proceedings of the International Congress of Mathematicians*, **2** (1987), 1362-1384.

[13] D.A. Kaptan, Y. Karabulut, C.Y. Yildirim, Some mean value theorems for the Riemann zeta-function and Dirichlet L-functions, *Commentarii Mathematici Universitatis Sancti Pauli*, **60**(1-2) (2011), 83-87.

[14] A.A. Karatsuba, S.M. Voronin, *The Riemann zeta-function*, de Gruyter, New York, 1992.

[15] N. Laaksonen, Y.N. Petridis, On the value distribution of two Dirichlet L-functions, *Functiones et Approximatio. Commentarii Mathematici*, **58**(1) (2018), 43-68.

[16] E. Landau, Über Imaginär-quadratischer Zahlkörper, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, **1918** (1918), 285-295.

[17] W. G. Nowak, Primitive lattice points inside an ellipse, *Czechoslovak Mathematical Journal*, **55**(2) (2005), 519-530.

[18] H. S. A. Potter, Approximate equations for the Epstein zeta function, *Proceedings of the London Mathematical Society*, **36**(1) (1934), 501-515.

[19] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Wiena, 1957.

[20] H. S. A. Potter, E. C. Titchmarsh, The Zeros of Epstein's Zeta-Functions, *Proceedings of the London Mathematical Society*, **39**(1) (1935), 372-384.

[21] A. Sankaranarayanan , Zeros of quadratic zeta-functions on the critical line, *Acta Arithmetica*, **69** (1995), 21-38.

[22] J. Steuding, On the Value Distribution of Hurwitz Zeta-Functions at the Nontrivial Zeros of the Riemann Zeta-Function, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, **71**(1) (2001), 113-121.

[23] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., revised by D.R. Heath-Brown, Oxford University Press, Oxford, 1986.

[24] Z. Nan-Yue & K. S. Williams, On the epstein zeta function, *Tamkang Journal of Mathematics*, **26**(2) (1995), 165-176.

[25] D. Zagier, A Kronecker limit formula for real quadratic fields, *Mathematische Annalen*, **213**(2) (1975), 153-184.