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Joint Universality of the Zeta-Functions of Cusp Forms

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Abstract: Let F be the normalized Hecke-eigen cusp form for the full modular group and $\zeta(s,F)$ be the corresponding zeta-function. In the paper, the joint universality theorem on the approximation of a collection of analytic functions by shifts $(\zeta(s+ih_1\tau,F),\dots,\zeta(s+ih_r\tau,F))$ is proved. Here, $h_1,\dots,h_r$ are algebraic numbers linearly independent over the field of rational numbers.

Keywords: Hecke-eigen cusp form; joint universality; universality; zeta-function

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1. Introduction

The series of the types

$$\sum_{m=1}^{\infty}\frac{a_m}{m^s}\quad\text{and}\quad\sum_{m=1}^{\infty}a_m \mathrm{e}^{-\lambda_m s},\quad s=\sigma+it,$$

where $\{\lambda_m\}$ is a nondecreasing sequence of real numbers and $\lim_{m\to\infty}\lambda_m=+\infty$ are called Dirichlet series. The majority of zeta-functions are meromorphic functions in some half-plane defined by Dirichlet series having a certain arithmetic sense. The most important of zeta-functions is the Riemann zeta-function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},\quad \sigma>1.$$

In [1], Voronin discovered a very interesting and important property of $\zeta(s)$ to approximate a wide class of analytic functions by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$, and called it universality. Later, it turned out that some other zeta-functions also are universal in the Voronin sense. This paper is devoted to the universality of zeta-functions of certain cusp forms.

Let

$$SL(2,\mathbb{Z})=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}:\ a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}$$

be the full modular group. If the function $F(z)$ is holomorphic in the upper half-plane $\mathrm{Im}z>0$, and for all elements of $SL(2,\mathbb{Z})$ with some $\kappa\in 2\mathbb{N}$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right)=(cz+d)^{\kappa}F(z), \tag{1}$$

where $F(z)$ is called a modular form of weight κ for the full modular group. Then, $F(z)$ has Fourier series expansion

$$F(z)=\sum_{m=-\infty}^{\infty}c(m)\mathrm{e}^{2\pi imz}.$$

If $c(m) = 0$ for all $m \leqslant 0$, then $F(z)$ is a cusp form of weight κ. The corresponding zeta-function (or L-function) $\zeta(s, F)$ is defined for $\sigma > \frac{\kappa+1}{2}$ by the Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and has the analytic continuation to an entire function. Additionally, we suppose that $F(z)$ is a simultaneous eigenfunction of all Hecke operators T_m

$$T_m F(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^\kappa} \sum_{b(\bmod d)} F\left(\frac{az+b}{d}\right), \quad m \in \mathbb{N}.$$

In this case, $c(1) \neq 0$; therefore, the form $F(z)$ can be normalized, and thus, we may suppose that $c(1) = 1$.

Now, we suppose that $F(z)$ is a normalized Hecke-eigen cusp form of weight κ for the full modular group. Then, the zeta-function $\zeta(s, F)$ can be written, for $\sigma > \frac{\kappa+1}{2}$, as a product over primes

$$\zeta(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying the equality $\alpha(p) + \beta(p) = c(p)$.

In the paper [2], the universality of the function $\zeta(s, F)$ was proved. Let $D_\kappa = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$, $\mathcal{K}_F$ be the class of compact subsets of the strip D_κ with connected complements, and $H_{0,F}(K)$, $K \in \mathcal{K}_F$ the class of continuous nonvanishing functions on K that are analytic in the interior of K. Moreover, let $\text{meas}A$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, in [2], the following theorem was obtained.

Theorem 1. *Suppose that $K \in \mathcal{K}_F$ and $f(s) \in H_{0,F}(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 1 shows that there are infinitely many shifts $\zeta(s + i\tau, F)$ approximating a given function $f(s) \in H_{0,F}$. In the shifts $\zeta(s + i\tau, F)$ of Theorem 1, τ takes arbitrary real values; therefore, the theorem is of continuous type. Further, discrete universality theorems for the function $\zeta(s, F)$ are known. In [3,4], the discrete universality theorems with shifts $\zeta(s + ikh, F)$, $k \in \mathbb{N}$, $h > 0$ being a fixed number, were proved. Denote by $H(D_\kappa)$ the space of analytic on D_κ functions endowed with the topology of uniform convergence on compacta. The paper [5] is devoted to the universality for compositions $\Phi(\zeta(s, F))$ with certain operators $\Phi : H(D_\kappa) \to H(D_\kappa)$. The results of the latter paper were applied in [6] for the functional independence of the compositions $\Phi(\zeta(s, F))$.

Let, for a fixed $l \in \mathbb{N}$,

$$\Gamma_0(l) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0(\bmod l)\right\}$$

denote the Hecke subgroup of the group $SL(2, \mathbb{Z})$. If $F(z)$ satisfies (1) for all elements of $\Gamma_0(l)$, then $F(z)$ is called a cusp form of weight κ and level l. The form $F(z)$ is called a new form if it is not a cusp form of level $l_1 \mid l$. In [7], a universality theorem was obtained for zeta-functions of new forms.

The universality theorem of [2] was generalized in [8] for shifts $\zeta(s + i\varphi(\tau), F)$ with differentiable function $\varphi(\tau)$ satisfying the estimates $(\varphi'(\tau))^{-1} = o(\tau)$ and $\varphi(2\tau) \max\limits_{\tau\leqslant t\leqslant 2\tau} (\varphi'(t))^{-1}$

$\ll \tau$ as $\tau \to \infty$. The discrete version of results of [8] is given in [9]. In [10], the shifts $\zeta(s + i\gamma_k, F)$, where $\{\gamma_k : k \in \mathbb{N}\}$ is the sequence of nontrivial zeros of $\zeta(s)$, are used.

The joint universality of zeta- and L-functions is a more complicated problem of analytic number theory. In this case, a collection of analytic functions are simultaneously approximated by a collection of shifts of zeta-functions. The first result in this direction also belongs to Voronin. He considered [11] the functional independence of Dirichlet L-functions $L(s, \chi)$ with pairwise nonequivalent Dirichlet characters χ and, for this, he obtained their joint universality. The paper [12] is devoted to the joint universality for zeta-functions of new forms twisted by Dirichlet characters, i.e., for the functions

$$\sum_{m=1}^{\infty} \frac{c(m)\chi(m)}{m^s}, \quad \sigma > \frac{\kappa + 1}{2},$$

with pairwise nonequivalent Dirichlet characters $\chi_1, \dots, \chi_r$.

Joint universality theorems with generalized shifts $\zeta(s + i\varphi_j(k), F)$, $j = 1, \dots, r$, with some differentiable functions $\varphi_j(\tau)$ can be found in [13]. Continuous and discrete joint universality theorems for more general zeta-functions are given in [14–16].

Our aim is to obtain a joint universality theorem for zeta-functions of normalized Hecke-eigen cusp forms by using different shifts. The first of the denseness results for shifts of a universal function were discussed in [17].

The main result of the paper is the following statement.

Theorem 2. *Suppose that $h_1, \dots, h_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}_F$ and $f_j(s) \in H_{0,F}(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + ih_j\tau, F) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

For the proof of Theorem 2, we will apply the probabilistic approach based on a limit theorem in the space of analytic functions.

2. Mean Square Estimates

Recall the metric in the space $H(D_\kappa)$. Let $\{K_l : l \in \mathbb{N}\} \subset D_\kappa$ be a sequence of compact subsets such that

$$D_\kappa = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for $l \in \mathbb{N}$, and if $K \subset D_\kappa$ is a compact, then $K \subset K_l$ for some l. For example, we can take K_l closed rectangles. Then

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D_\kappa),$$

is a metric in $H(D_\kappa)$ inducing the topology of uniform convergence on compacta.

Let

$$H^r(D_\kappa) = (\underbrace{H(D_\kappa) \times \cdots \times H(D_\kappa)}_{r}).$$

For $\underline{g}_j = (g_{j1}, \dots, g_{jr}) \in H^r(D_\kappa)$, $j = 1, 2$, define

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j}).$$

Then, $\underline{\rho}$ is a metric in $H^r(D_\kappa)$ inducing the product topology.

Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n} \right)^\theta \right\}, \quad m, n \in \mathbb{N}.$$

Then, the series

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s},$$

in view of the estimate

$$c(m) \ll m^{\frac{\kappa-1}{2}+\varepsilon},$$

is absolutely convergent in every fixed half plane $\sigma > \hat{\sigma}$. However, for our aim, this convergence is sufficient only for $\sigma > \frac{\kappa}{2}$.

For brevity, let $\underline{h} = (h_1, \dots, h_r)$,

$$\underline{\zeta}(s + i\underline{h}\tau, F) = (\zeta(s + ih_1\tau, F), \dots, \zeta(s + ih_r\tau, F))$$

and

$$\underline{\zeta}_n(s + i\underline{h}\tau, F) = (\zeta_n(s + ih_1\tau, F), \dots, \zeta_n(s + ih_r\tau, F)).$$

Lemma 1. *For all $\underline{h}$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\Big(\underline{\zeta}(s + i\underline{h}\tau, F), \underline{\zeta}_n(s + i\underline{h}\tau, F)\Big) \mathrm{d}\tau = 0.$$

Proof. By the definitions of the metrics ρ and $\underline{\rho}$, it suffices to show that, for every $h \in \mathbb{R}$ and compact set $K \subset D_\kappa$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s + ih\tau, F), \zeta_n(s + ih\tau, F))\mathrm{d}\tau = 0.$$

It is well known that for fixed $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$,

$$\int_{-T}^{T} |\zeta(\sigma + it, F)|^2 \mathrm{d}t \ll_\sigma T,$$

where $\ll_\sigma$ means that the implied constant depends on σ. Therefore,

$$\int_{-T}^{T} |\zeta(\sigma + iht, F)|^2 \mathrm{d}t \ll_{\sigma,h} T,$$

and, for $v \in \mathbb{R}$,

$$\frac{1}{T} \int_0^T |\zeta(\sigma + ih\tau + iv, F)|^2 \mathrm{d}v \ll_{\sigma,h} 1 + |v|. \tag{2}$$

Let

$$l_n(s) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) n^z,$$

where $\Gamma(z)$ denotes the Euler gamma-function and θ is a number from the definition of $v_n(m)$. Using the Mellin formula

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(s)\alpha^s \mathrm{d}s = \mathrm{e}^{-\alpha}, \quad \alpha, \beta > 0,$$

we find that

$$\exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\} = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{1}{\theta}\Gamma\left(\frac{1}{\theta}\right)\left(\frac{m}{n}\right)^{-s}\mathrm{d}s.$$

Therefore, in virtue of the definition of the function $v_n(m)$, we obtain that, for $\sigma > \frac{\kappa}{2}$,

$$\begin{aligned}
\zeta_n(s,F) &= \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{c(m)}{m^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m}{n}\right)^{-z}\frac{\mathrm{d}z}{z} \\
&= \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\left(\frac{l_n(z)}{z}\sum_{m=1}^{\infty}\frac{c(m)}{m^{s+z}}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,F)l_n(z)\frac{\mathrm{d}z}{z}.
\end{aligned} \tag{3}$$

Let $K \in D_\kappa$ be a fixed compact set. Then, there exists $\varepsilon > 0$ such that, for all $s = \sigma + it \in K$, the inequalities $\frac{\kappa}{2} + 2\varepsilon < \sigma < \frac{\kappa+1}{2} - \varepsilon$ are satisfied. We take, for such σ,

$$\theta_1 = \frac{\kappa}{2} + \varepsilon - \sigma.$$

Then, $\theta_1 < 0$. Therefore, by the residue theorem and (3),

$$\zeta_n(s,F) - \zeta(s,F) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty}\zeta(s+z,F)l_n(z)\frac{\mathrm{d}z}{z}.$$

Hence, for all $s \in K$,

$$\begin{aligned}
\zeta(s+ih\tau,F) - \zeta_n(s+ih\tau,F) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{\kappa}{2}+\varepsilon+it+ih\tau+iv,F\right)\frac{l_n\left(\frac{\kappa}{2}+\varepsilon-\sigma+iv\right)}{\frac{\kappa}{2}+\varepsilon-\sigma+iv}\mathrm{d}v \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{\kappa}{2}+\varepsilon+ih\tau+iv,F\right)\frac{l_n\left(\frac{\kappa}{2}+\varepsilon-s+iv\right)}{\frac{\kappa}{2}+\varepsilon-s+iv}\mathrm{d}v \\
&\ll \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{\kappa}{2}+\varepsilon+ih\tau+iv,F\right)\right|\sup_{s\in K}\left|\frac{l_n\left(\frac{\kappa}{2}+\varepsilon-s+iv\right)}{\frac{\kappa}{2}+\varepsilon-s+iv}\right|\mathrm{d}v.
\end{aligned}$$

Thus, in view of (2),

$$\begin{aligned}
&\frac{1}{T}\int_{0}^{\infty}\sup_{s\in K}|\zeta(s+ih\tau,F)-\zeta_n(s+ih\tau,F)|\mathrm{d}\tau \\
&\ll \int_{-\infty}^{\infty}\left(\left(\frac{1}{T}\int_{0}^{\infty}\left|\zeta\left(\frac{\kappa}{2}+\varepsilon+ih\tau+iv\right)\right|^2\mathrm{d}\tau\right)^{1/2}\sup_{s\in K}\left|\frac{l_n\left(\frac{\kappa}{2}+\varepsilon-s+iv\right)}{\frac{\kappa}{2}+\varepsilon-s+iv}\right|\right)\mathrm{d}v
\end{aligned}$$

$$\ll_{\varepsilon,h,K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|v|)\exp\{-c_1|v|\}\mathrm{d}v \ll_{\varepsilon,h,K} n^{-\varepsilon} \tag{4}$$

Here, we used the estimate

$$\Gamma\left(\frac{1}{\theta}\left(\frac{\kappa}{2}+\varepsilon-s+iv\right)\right) \ll \exp\left\{-\frac{c}{\theta}|v-t|\right\} \ll_{\kappa} \exp\{-c_1|v|\}, \quad c_1>0.$$

Estimate (4) proves the lemma. □

Let $\mathbb{P}$ be the set of all prime numbers, and $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. Define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p.$$

Then, the torus Ω with product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\mathbb{X})$ is the Borel σ-field of the space $\mathbb{X}$), the probability Haar measure m_H can be defined. Moreover, let

$$\underline{\Omega} = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1,\ldots,r$. Once again, $\underline{\Omega}$ is a compact topological Abelian group. Therefore, on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$ the probability Haar measure $\underline{m}_H$ exists. This gives the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$. Denote by m_{jH} the Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1,\ldots,r$. Then, $\underline{m}_H$ is the product of the measures $m_{1H},\ldots,m_{rH}$. Now, denote by $\underline{\omega} = (\omega_1,\ldots,\omega_r)$ the elements of $\underline{\Omega}$, where $\omega_j \in \Omega_j$, $j = 1,\ldots,r$. Let $\omega_j(p)$ be the pth component of an element $\omega_j \in \Omega_j$, $j = 1,\ldots,r$, $p \in \mathbb{P}$. Extend elements $\omega_j(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N},$$

and define $H(D_\kappa)$-valued random element

$$\zeta(s,\omega_j,F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_j(m)}{m^s}, \quad j = 1,\ldots,r.$$

The later series is uniformly convergent on compact subsets of D_κ for almost all ω_j. Moreover, for fixed $\sigma \in \left(\frac{\kappa}{2}, \frac{\kappa+1}{2}\right)$

$$\int_{-T}^{T} \left|\zeta(s+it,\omega_j,F)\right|^2 \mathrm{d}t \ll_\sigma T \tag{5}$$

for almost all ω_j, $j = 1,\ldots,r$ [18]. Define one more series

$$\zeta_n(s,\omega_j,F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_j(m)v_n(m)}{m^s}, \quad j = 1,\ldots,r,$$

which also, as $\zeta_n(s,F)$, are absolutely convergent for $\sigma > \frac{\kappa}{2}$. Let

$$\underline{\zeta}(s+i\underline{h}\tau,\underline{\omega},F) = (\zeta(s+ih_1\tau,\omega_1,F),\ldots,\zeta(s+ih_r\tau,\omega_1,F))$$

and

$$\underline{\zeta}_n(s+i\underline{h}\tau,\underline{\omega},F) = (\zeta_n(s+ih_1\tau,\omega_1,F),\ldots,\zeta_n(s+ih_r\tau,\omega_r,F)).$$

Then, repeating the proof of Lemma 1 and using estimate (5), we arrive to the following statement.

Lemma 2. *For all $\underline{h}$ and almost all $\underline{\omega}$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}\Big(\underline{\zeta}(s+i\underline{h}\tau,\underline{\omega},F),\underline{\zeta}_n(s+i\underline{h}\tau,\underline{\omega},F)\Big)\mathrm{d}\tau = 0.$$

3. Limit Theorems

On the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),\underline{m}_H)$, define $H(D_\kappa)$-valued random element

$$\underline{\zeta}(s,\underline{\omega},F) = (\zeta(s,\omega_1,F),\dots,\zeta(s,\omega_1,F))$$

and denote by $P_{\underline{\zeta},F}$ its distribution, i.e.,

$$P_{\underline{\zeta},F}(A) = \underline{m}_H\Big\{\underline{\omega}\in\underline{\Omega} : \underline{\zeta}(s,\underline{\omega},F)\in A\Big\},\quad A\in\mathcal{B}(H^r(D_\kappa)).$$

Theorem 3. *Suppose that $h_1,\dots,h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and*

$$P_{T,F}(A) \overset{def}{=} \frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T] : \underline{\zeta}(s+i\underline{h}\tau,F)\in A\Big\},\quad A\in\mathcal{B}(H^r(D_\kappa)).$$

Then, $P_{T,F}$ converges weakly to $P_{\underline{\zeta},F}$ as $T\to\infty$.

We divide the proof of Theorem 3 into several lemmas.

Lemma 3. *Suppose that $\lambda_1,\dots,\lambda_r$ are algebraic numbers such that the system $\log\lambda_1,\dots,\log\lambda_r$ is linearly independent over $\mathbb{Q}$. Then, for arbitrary algebraic numbers $\beta_0,\beta_1,\dots,\beta_r$ that are not all zeros, the inequality*

$$|\beta_0+\beta_1\log\lambda_1+\dots+\beta_r\log\lambda_r| > h^{-c}$$

holds. Here, h denotes the height of the numbers $\beta_0,\beta_1,\dots,\beta_r$, and c is an effective constant depending on r, $\lambda_1,\dots,\lambda_r$ and maximum of degrees of the numbers $\beta_0,\beta_1,\dots,\beta_r$.

The lemma is a Baker result on linear forms of logarithm; see, for example, ref. [19]. For $A\in\mathcal{B}(\underline{\Omega})$, define

$$Q_T(A) = \frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T] : \Big(\Big(p^{-ih_1\tau} : p\in\mathbb{P}\Big),\dots,\Big(p^{-ih_r\tau} : p\in\mathbb{P}\Big)\Big)\in A\Big\}.$$

Lemma 4. *Let $\lambda_1,\dots,\lambda_r$ be the same as in Theorem 3. Then, Q_T converges weakly to the Haar measure $\underline{m}_H$ as $T\to\infty$.*

Proof. We apply the Fourier transform method. Denote by $g_T(\underline{k}_1,\dots,\underline{k}_r)$, $\underline{k}_j = \{k_{pj} : k_{pj}\in\mathbb{Z},\ p\in\mathbb{P}\}$, $j=1,\dots,r$ the Fourier transform of Q_T. By the definition of Q_T, we have

$$\begin{aligned} g_T(\underline{k}_1,\dots,\underline{k}_r) &= \int_{\underline{\Omega}} \prod_{j=1}^r \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{pj}}(p)\,\mathrm{d}Q_T \\ & \frac{1}{T}\int_0^T \exp\Bigg\{-i\tau\sum_{j=1}^k \sum_{p\in\mathbb{P}}{}^{*} h_j k_{pj}\log p\Bigg\}\mathrm{d}\tau, \end{aligned} \tag{6}$$

where the star shows that only a finite number of integers k_{pj} are not zero. Obviously,

$$g_T(\underline{0},\dots,\underline{0}) = 1. \tag{7}$$

Now, suppose that $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Then, there exists a prime number p such that $k_{pj} \neq 0$ for some j. Therefore,

$$\beta_p \stackrel{def}{=} \sum_{j=1}^{r} h_j k_{pj} \neq 0$$

because the numbers $h_1, \dots, h_r$ are linearly independent over Q. Thus, in view of Lemma 3,

$$B_{\underline{k}_1,\dots,\underline{k}_r} \stackrel{def}{=} \sum_{j=1}^{k} \sum_{p\in\mathbb{P}}^{*} h_j k_{pj} \log p = \sum_{p\in\mathbb{P}}^{*} \beta_p \log p \neq 0.$$

This and (6) imply

$$g_T(\underline{k}_1, \dots, \underline{k}_r) = \frac{1 - \exp\{-iTB_{\underline{k}_1,\dots,\underline{k}_r}\}}{iTB_{\underline{k}_1,\dots,\underline{k}_r}}.$$

Therefore, by (7),

$$\lim_{T\to\infty} g_T(\underline{k}_1, \dots, \underline{k}_r) \stackrel{def}{=} \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}), \end{cases}$$

and this proves the lemma. □

For $A \in \mathcal{B}(H^r(D_\kappa))$, define

$$P_{T,n,F}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \underline{\zeta}_n(s + i\underline{h}\tau, F) \in A\Big\}$$

and

$$P_{T,n,\underline{\Omega},F}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \underline{\zeta}_n(s + i\underline{h}\tau, \underline{\omega}, F) \in A\Big\}.$$

Moreover, let the mapping $u_n : \underline{\Omega} \to H^r(D_\kappa)$ be given by

$$u_{n,F}(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\omega}, F),$$

and $V_{n,F} = \underline{m}_H u_{n,F}^{-1}$, where

$$V_{n,F}(A) = \underline{m}_H\Big(u_{n,F}^{-1}A\Big), \quad A \in \mathcal{B}(H^r(D_\kappa)).$$

Since the series for $\zeta_n(s, \omega_j, F)$ are absolutely convergent for $\sigma > \frac{\kappa}{2}$, the mapping $u_{n,F}$ is continuous. Moreover, by the definitions of Q_T and $P_{T,n,F}$, we have $P_{T,n,F} = Q_T u_{n,F}^{-1}$. This equality, continuity of $u_{n,F}$, Lemma 4, the well-known properties of weak convergence, and the invariance of the Haar measure $\underline{m}_H$ lead to the following lemma.

Lemma 5. *Let $h_1, \dots, h_r$ be the same as Theorem 3. Then, $P_{T,n,F}$ and $P_{T,n,\underline{\Omega},F}$ both converge weakly to the measure $V_{n,F}$ as $T \to \infty$.*

Additionally to $P_{T,F}$, define

$$P_{T,\underline{\Omega},F}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \underline{\zeta}(s + i\tau, \underline{\omega}, F) \in A\Big\}, \quad A \in \mathcal{B}(H^r(D_\kappa)).$$

Lemma 6. *Let $h_1, \dots, h_r$ be the same as Theorem 3. Then, on $(H^r(D_\kappa), \mathcal{B}(H^r(D_\kappa)))$, there exists a probability measure P_F such that $P_{T,F}$ and $P_{T,\underline{\Omega},F}$ both converge weakly to P_F as $T \to \infty$.*

Proof. Since the series for $\zeta_n(s,F)$ is absolutely convergent, by a standard way it follows—see, for example [14,18]—that the sequence $\{V_{n,F} : m \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K \subset H^r(D_\kappa))$ such that

$$V_{n,F}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, by the Prokhorov theorem, see [20], the sequence $\{V_{n,F}\}$ is relatively compact, i.e., each of its subsequences contains a subsequence $\{V_{n_k,F}\}$ such that $V_{n_k,F}$ converges weakly to a certain probability measure P_F on $(H^r(D_\kappa), \mathcal{B}(H^r(D_\kappa)))$ as $k \to \infty$.

Let ξ_T be a random variable defined on a certain probability space with measure ν and uniformly distributed on $[0,T]$. Define the $H^r(D_\kappa)$-valued random element

$$\underline{X}_{T,n,F} = \underline{X}_{T,n,F}(s) = \underline{\zeta}_n(s + i\underline{h}\xi_T, F)$$

and denote by $\underline{X}_{n,F} = \underline{X}_{n,F}(s)$ the $H^r(D_\kappa)$-valued random element having the distribution $V_{n,F}$. Then, by Lemma 5, we have

$$\underline{X}_{T,n,F} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_{n,F}, \tag{8}$$

where $\xrightarrow[T\to\infty]{\mathcal{D}}$ means the convergence in distribution. Moreover, since $V_{n_k,F}$ converges weakly to P_F, the relation

$$\underline{X}_{n_k,F} \xrightarrow[k\to\infty]{\mathcal{D}} P_F \tag{9}$$

is true. Let

$$\underline{X}_{T,F} = \underline{X}_{T,F}(s) = \underline{\zeta}(s + i\underline{h}\xi_T, F).$$

Then, using Lemma 1, we find that for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \nu\Big\{\underline{\varrho}(\underline{X}_{T,F}, \underline{X}_{T,n,F}) \geqslant \varepsilon\Big\} \\
&\qquad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_0^T \underline{\varrho}\Big(\underline{\zeta}(s + i\underline{h}\tau, F), \underline{\zeta}_n(s + i\underline{h}\tau, F)\Big) \mathrm{d}\tau = 0.
\end{aligned}$$

The later equality together with (8) and (9), and Theorem 4.2 of [20] lead to the relation

$$X_{T,F} \xrightarrow[T\to\infty]{\mathcal{D}} P_F. \tag{10}$$

This proves that $P_{T,F}$ converges weakly to P_F as $T \to \infty$.

The relation (10) shows that the limit measure P_F is independent of the subsequence $\{n_k\}$. Therefore, we have

$$\underline{X}_{n,F} \xrightarrow[n\to\infty]{\mathcal{D}} P_F. \tag{11}$$

Define the $H^r(D_\kappa)$-valued random elements

$$\underline{X}_{T,n,\underline{\Omega},F} = \underline{X}_{T,n,\underline{\Omega},F}(s) = \underline{\zeta}_n(s + i\underline{h}\xi_T, \underline{\omega}, F)$$

an

$$\underline{X}_{T,\underline{\Omega},F} = \underline{X}_{T,\underline{\Omega},F}(s) = \underline{\zeta}(s + i\underline{h}\xi_T, \underline{\omega}, F).$$

Then, repeating the above arguments using Lemmas 2 and 5, and relation (11), we obtain that

$$X_{T,n,F} \xrightarrow[T\to\infty]{\mathcal{D}} P_F,$$

and this is equivalent to weak convergence of $P_{T,\underline{\Omega},F}$ to P_F as $T \to \infty$. The lemma is proved. □

To prove Theorem 3, it remains to show that $P_F = P_{\underline{\zeta},F}$. For this, we will apply some elements of the ergodic theory. For brevity, let

$$\underline{h}_\tau = \left(\left(p^{-ih_1\tau} : p \in \mathbb{P}\right), \dots, \left(p^{-ih_r\tau} : p \in \mathbb{P}\right)\right), \quad \tau \in \mathbb{R}.$$

Define the transformation of $\underline{\Omega}$

$$\varphi_\tau(\underline{\omega}) = \underline{h}_\tau \underline{\omega}, \quad \underline{\omega} \in \underline{\Omega}.$$

Since the Haar measure $\underline{m}_H$ is invariant, the transformation φ_τ is measure-preserving and $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group. A set $A \in \mathcal{B}(\underline{\Omega})$ is called invariant with respect to the group $\{\varphi_\tau\}$ if the sets A and $\varphi_\tau(A)$, $\tau \in \mathbb{R}$, differ one from another at most by a set of $\underline{m}_H$-measure zero.

Lemma 7. *Let $h_1, \dots, h_r$ be the same as Theorem 3. Then, the group $\{\varphi_\tau\}$ is ergodic, i.e., the σ-field of invariant sets consists of sets having $\underline{m}_H$-measure 1 or 0.*

Proof. The characters χ of the group Ω are of the form

$$\chi(\underline{\omega}) = \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}^{*} \omega_j^{k_{pj}}(p).$$

This fact already was used in the proof of Lemma 4. Let A be an arbitrary invariant set, I_A its indicator function, and χ be a nontrivial character. Preserving the notation of the proof of Lemma 4, we have $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$ and $B_{\underline{k}_1,\dots,\underline{k}_r} \neq 0$. Therefore, there exists $\tau_0 \in \mathbb{R}$ such that

$$\chi(\underline{h}_\tau) = \exp\{-i\tau_0 B_{\underline{k}_1,\dots,\underline{k}_r}\} \neq 1. \tag{12}$$

Moreover, in view of the invariance of A, we have

$$I_A(\underline{h}_{\tau_0}\underline{\omega}) = I_A(\underline{\omega}) \tag{13}$$

for almost all $\underline{\omega} \in \underline{\Omega}$. Denote by $\hat{I}_A$ the Fourier transform of I_A. Then, by (13),

$$\hat{I}_A(\chi) = \chi(\underline{h}_{\tau_0}) \int\limits_{\underline{\Omega}} I_A(\underline{h}_{\tau_0}\underline{\omega})\chi(\underline{\omega})\mathrm{d}\underline{m}_H = \chi(\underline{h}_{\tau_0})\hat{I}_A(\chi).$$

This and (12) show that

$$\hat{I}_A(\chi) = 0. \tag{14}$$

Now, let χ_0 denote the trivial character of $\underline{\Omega}$, and suppose that $\hat{I}_A(\chi_0) = \alpha$. Then, in view of (14), we find that

$$\hat{I}_A(\chi) = \alpha \int\limits_{\underline{\Omega}} \chi(\underline{\omega})\mathrm{d}\underline{m}_H = \hat{\alpha}(\chi).$$

Hence, $I_A(\underline{\omega}) = \alpha$ for almost all $\underline{\omega} \in \underline{\Omega}$. Since I_A is the indicator function, $I_A(\underline{\omega}) = 1$ or $I_A(\underline{\omega}) = 0$ for almost all $\underline{\omega}$. Thus, $\underline{m}_H(A) = 1$ or $\underline{m}_H(A) = 0$, and the lemma is proved. □

Proof of Theorem 3. We have mentioned that it suffices to show that $P_F = P_{\underline{\zeta},F}$. By Lemma 6 and the equivalent of weak convergence in terms of continuity sets, we have

$$\lim_{T \to \infty} P_{T,\underline{\Omega},F}(A) = P_F(A) \tag{15}$$

for a continuity set A of the measure P_F, i.e., $P_F(\partial A) = 0$, where ∂A is the boundary of A. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the random variable

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if} \quad \underline{\zeta}(s, \underline{\omega}, F) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7 implies the ergodicity of the random process $\xi(\varphi_\tau(\underline{\omega}))$. Therefore, by the classical Birkhoff–Khintchine ergodic theorem, see, for example [21],

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\varphi_\tau(\underline{\omega}))\mathrm{d}\tau = \mathbb{E}\xi = P_{\underline{\zeta},F}(A), \tag{16}$$

where $\mathbb{E}\xi$ is the expectation of ξ.

However, by the definitions of φ_τ and ξ,

$$\frac{1}{T} \int_0^T \xi(\varphi_\tau(\underline{\omega}))\mathrm{d}\tau = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \underline{\zeta}(s + i\underline{h}\tau, \underline{\omega}, F) \in A\Big\} = P_{T,\underline{\Omega},F}(A).$$

This and (16) show that

$$\lim_{T\to\infty} P_{T,\underline{\Omega},F}(A) = P_{\underline{\zeta},F}(A).$$

Therefore, by (15), we obtain that $P_F(A) = P_{\underline{\zeta},F}(A)$ for all continuity sets A of $P_F(A)$. Hence, $P_F = P_{\underline{\zeta},F}$, and the theorem is proved. □

4. Proof of Theorem 2

Recall that the support of the measure $P_{\underline{\zeta},F}$ is a minimal closed set $S_F \subset H^r(D_\kappa)$ such that $P_{\underline{\zeta},F}(S_F) = 1$.

Lemma 8. *The support of the measure $P_{\underline{\zeta},F}$ is the set* $(\{g \in H(D_\kappa) : g(s) \neq 0 \text{ or } g(s) \equiv 0\})^r$.

Proof. Since the space $H^r(D_\kappa)$ is separable, we have [20],

$$\mathcal{B}(H^r(D_\kappa)) = \underbrace{(\mathcal{B}(H(D_\kappa)) \times \cdots \times \mathcal{B}(H(D_\kappa)))}_{r}.$$

Therefore, it suffices to consider the measure $P_{\underline{\zeta},F}$ on the rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \ldots, A_r \in H(D_\kappa).$$

Let $\zeta(s, \omega, F)$ be the $H(D_\kappa)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H is the Haar measure. Then, it is known [10] that the support of the distribution of $\zeta(s, \omega, F)$ is the set $\{g \in H(D_\kappa) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Thus, the same set is the support of the distributions of $\zeta(s, \omega_j, F)$, $j = 1, \ldots, r$. Since the measure $\underline{m}_H$ is the product of the measures m_{jH}, $j = 1, \ldots, r$, we have

$$\underline{m}_H\Big\{\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\omega}, F) \in A\Big\} = \prod_{j=1}^r m_{jH}\{\omega_j \in \Omega_j : \zeta(s, \omega_j, F) \in A_j\}.$$

This equality, the minimality of the support, and the support of the distributions of $\zeta(s, \omega_j, F)$ prove the lemma. □

Proof of Theorem 2. By the Mergelyan theorem on the approximation of analytic functions by polynomials [22], there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\le j\le r}\sup_{s\in K_j}\left|f_j(s)-e^{p_j(s)}\right|<\frac{\varepsilon}{2}. \tag{17}$$

Define the set

$$G_\varepsilon=\left\{(g_1,\dots,g_r)\in H^r(D_\kappa):\ \sup_{1\le j\le r}\sup_{s\in K_j}\left|g_j(s)-e^{p_j(s)}\right|<\frac{\varepsilon}{2}\right\}.$$

In view of Lemma 8, the set G_ε is an open neighborhood of an element $(e^{p_1(s)},\dots,e^{p_r(s)})$ in support of the measure $P_{\underline{\zeta},F}$. Hence,

$$P_{\underline{\zeta},F}(G_\varepsilon)>0. \tag{18}$$

This, Theorem 3 and the equivalent of weak convergence in terms of open sets, and the definitions of $P_{T,F}$ and G_ε prove the theorem with "lim inf". Define one more set

$$\hat{G}_\varepsilon=\left\{(g_1,\dots,g_r)\in H^r(D_\kappa):\ \sup_{1\le j\le r}\sup_{s\in K_j}\left|g_j(s)-f_j(s)\right|<\varepsilon\right\},$$

There $\partial\hat{G}_{\varepsilon_1}\bigcap\partial\hat{G}_{\varepsilon_2}=\emptyset$ for $\varepsilon_1\ne\varepsilon_2$. This shows that $P_{\underline{\zeta},F}\left(\partial\hat{G}_\varepsilon\right)=0$ for all but, for those countable, many $\varepsilon>0$. Moreover, (17) and (18) imply that $P_{\underline{\zeta},F}\left(\hat{G}_\varepsilon\right)>0$. This, Theorem 3 and the equivalent of weak convergence of probability measures in terms of continuity sets, and the definitions of $P_{T,F}$ and $\hat{G}_\varepsilon$ prove the theorem with "lim". □

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