Research article

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Non-stationary Navier–Stokes equations in $2D$ power cusp domain. I. Construction of the formal asymptotic decomposition

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Abstract: The initial boundary value problem for the non-stationary Navier-Stokes equations is studied in 2D bounded domain with a power cusp singular point O on the boundary. The case of the boundary value with a nonzero flow rate is considered. In this case there is a source/sink in O and the solution necessary has infinite energy integral. In the first part of the paper the formal asymptotic expansion of the solution near the singular point is constructed. The justification of the asymptotic expansion and the existence of a solution are proved in the second part of the paper.

Keywords: Nonstationary Navier-Stokes problem, power cusp domain, singular solutions, asymptotic expansion

MSC: 35Q30, 35A20, 76M45, 76D03, 76D10

1 Introduction

The point source/sink approach is widely used in physics and astronomy. For example, stars are routinely treated as point sources. Pulsars are treated as point sources when observed using radio telescopes. Generally, a source of light can be considered as a point source, for example, light passing through a pinhole or other small aperture, viewed from a distance much greater than the size of the hole. In nuclear physics, a "hot spot" is a point source of radiation. Sources of various types of pollution are often considered as point sources in large-scale studies of pollution. Sound is an oscillating pressure wave. As the pressure oscillates up and down, an audio point source acts in turn as a fluid point source and then a fluid point sink. (Such an object does not exist physically, but is often a good simplified model for calculations.)

Fluid point sources and sinks are commonly used also in fluid dynamics and aerodynamics. Point source-sink pairs are often used as simple models for driving flow through a gap in a wall. The use of localized suction to control vortices around aerofoil sections is one of such problems. In oceanography, it is common to use point sources to model the influx of fluid from channels and holes. There are also applications of pulsed source-sink systems in the study of chaotic advection and many others.

The asymptotic behaviour of the solutions to the Stokes and Navier–Stokes equations in singularly perturbed domains become of growing interest during the last fifty years. There is an extensive literature con-

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cerning these issues for various elliptic problems, see, e.g., [1–12]. In particular, the steady Navier–Stokes equations are studied in a punctured domain $\Omega = \Omega_0 \setminus \{O\}$ with $O \in \Omega_0$ assuming that the point O is a sink or source of the fluid [13–15] (see also [16] for the review of these results). Although the steady Navier–Stokes equations in singularly perturbed domains are well studied, there are few papers studying the initial boundary value problem for the non-stationary Navier-Stokes equations in such domains (e.g., [17–19]). We can also mention the recent paper [20] where the Dirichlet problem for the non-stationary Stokes system is studied in a three-dimensional cone and the paper [21] where the solvability of the steady state Navier–Stokes problem with a sink or source in the cusp point O was proved for arbitrary data.

In recent papers [22, 23] the authors have studied existence of singular solutions to the stationary, time-periodic and initial boundary value problems for the linear Stokes equations in domains having a power-cusp (peak type) singular point on the boundary. The case where the flux of the boundary value is nonzero was considered. Therefore, there is a sink or source in the cusp point O and the solution is necessary singular. In [22, 23] by constructing the formal asymptotic decomposition of a solution, we reduced the linear problem with singular data to one with regular right-hand side and then applied the well known solvability results for the Stokes system. Constructing the asymptotic representation we followed the ideas proposed in the paper [24] where the asymptotic behaviour of solutions to the stationary Stokes and Navier–Stokes problems was studied in unbounded domains with paraboloidal outlets to infinity. In turn, the method used in [24] was a variant of the algorithm of constructing the asymptotic representation of solutions to elliptic equations in slender domains (see, [25–28] for arbitrary elliptic problems; [29, 30] for the stationary Stokes and Navier–Stokes equations).

In this paper we study the non-stationary Navier–Stokes equations in a two dimensional power cusp domain. To be precise, we consider the initial boundary value problem

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{a}(x,t), \\ \mathbf{u}(x,0) = \mathbf{b}(x) \end{cases} \tag{1.1}$$

in the 2D bounded domain $\Omega = G_H \cup \Omega_0$, where $G_H = \{ x \in \mathbb{R}^2 : |x_1| < \varphi(x_2),\ x_2 \in (0,H] \}$, $\varphi(x_2) = \gamma_0 x_2^\lambda$, $\gamma_0 = \text{const}$, $\lambda > 1$, and $\partial\Omega \cap \partial\Omega_0$ is C^2 (see Figure 1). Here $\mathbf{u} = (u_1, u_2)$ stands for the velocity field, p stands for the pressure, $\nu > 0$ is the constant kinematic viscosity. We assume that the initial velocity $\mathbf{b} \in W^{1,2}(\Omega)$ and the support of the boundary value $\mathbf{a} \in L^2(0,T; W^{1/2,2}(\partial\Omega))$ is separated from the cusp point O, $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$. We also suppose that the flux of $\mathbf{a}$ is nonzero, i.e.,

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = -F(t), \quad F(0) = 0, \tag{1.2}$$

where $\mathbf{n}$ is the unit outward (with respect to Ω) normal to $\partial\Omega$. Moreover, the initial velocity $\mathbf{b}$ and the boundary value $\mathbf{a}$ have to satisfy the necessary compatibility conditions

$$\operatorname{div}\mathbf{b}(x) = 0, \quad \mathbf{b}(x)|_{\partial\Omega} = \mathbf{a}(x,0). \tag{1.3}$$

From (1.2) it also follows that

$$\int_{\partial\Omega} \mathbf{b}\cdot\mathbf{n}\,dS = 0.$$

The solution $\mathbf{u}$ of (1.1) has to satisfy the condition[1]

$$\int_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS + \int_{\partial\Omega\cap\partial\Omega_0} \mathbf{a}\cdot\mathbf{n}\,dS = 0,$$

where $\sigma(h)$ is a cross-section of G_H, i.e., $\sigma(h) = \{x \in G_H : x_n = h = \text{const}\}$. Thus,

$$\int_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dx_1 = F(t) \not\equiv 0, \tag{1.4}$$

1 This condition means that the total flux of the fluid is equal to zero.

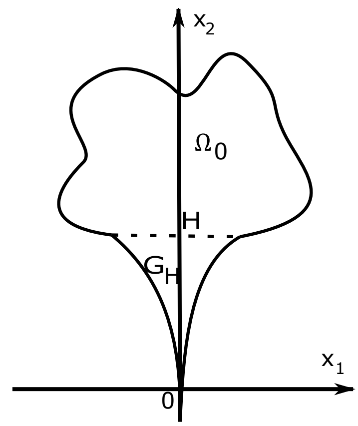

Fig. 1: Domain Ω

and we can regard the cusp point O as a source (or a sink) of intensity $F(t)$.

Notice that problem (1.1) cannot have a solution with the finite Dirichlet integral. Indeed, by (1.4) and the definition of G_H, we have

$$|F(t)|^2 = \Big| \int_{\sigma(h)} u_2(x,t)\,dx_1 \Big|^2 \le 2\varphi(x_2) \int_{\sigma(h)} |u_2(x,t)|^2\,dx_1$$
$$\le c\varphi^3(x_2) \int_{\sigma(h)} \Big| \frac{\partial u_2(x,t)}{\partial x_1} \Big|^2\,dx_1.$$

Dividing this inequality by $\varphi^3(x_2)$ and integrating over x_2 from 0 to H, we get

$$|F(t)|^2 \int_0^H \frac{dx_2}{\varphi^3(x_2)} \le c \int_0^H \int_{\sigma(h)} |\nabla u_2(x,t)|^2\,dx_1 dx_2 \le c \int_{G_H} |\nabla \mathbf{u}(x,t)|^2\,dx.$$

Let $F(t) \neq 0$. Then the Dirichlet integral of $\mathbf{u}$ can be finite only if $\int_0^H \frac{dx_2}{\varphi^3(x_2)} < \infty$, but this is not the case for $\varphi(x_2) = \gamma_0 x_2^\lambda$ with $\lambda > 1$. Thus, the solution $\mathbf{u}$ of (1.1) satisfying condition (1.4) is necessary singular in the cusp point O and its singularity depends on cusp's power λ. Rather, roughly speaking, the divergence in the cusp of the normal component of the velocity field of the fluid holds from the flux, that is, making to zero the "surface portion" near the cusp.

In order to prove the solvability of such solution, we first construct the formal asymptotic expansion of it near the singular point. It contains both outer and inner (boundary layer-in-time) asymptotic expansions and has the following form

$$\mathbf{U}^{[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t, \tau\right) = \mathbf{U}^{O,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t\right) + \mathbf{U}^{B,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, \tau\right),$$
$$P^{[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t, \tau\right) = P^{O,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t\right) + P^{B,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, \tau\right).$$

The pair $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is an outer asymptotics of the solution, the "slow" time variable t plays the role of a parameter and the initial condition is not satisfied in general case. The pair $(\mathbf{U}^{B,[J]}, P^{B,[J]})$ is the boundary layer corrector (the inner part of the asymptotic expansion) which compensate the discrepancy in the initial condition and exponentially vanishes as $\tau \to \infty$. Note that the fast time variable $\tau = \frac{t}{x_2^{2\lambda}}$ in our case depends on x_2, i.e., the fast time variable τ is changing dependently of the distance to the cusp point O. The construction of the boundary layer-in-time is based on the ideas proposed in [17]–[19], where an asymptotic expansion of solutions to the non-stationary Navier–Stokes equations is constructed in thin structures.

Both outer and inner parts of the asymptotic expansions are of the form of finite sums in powers of x_2. We construct these sums up to the terms which leave in equations (1.1) the discrepancy belonging to $L^2(\Omega)$ and then the solution of problem (1.1) is constructed as the sum of the asymptotic expansion and the term with finite energy.

The paper is divided into two parts: the construction of the formal asymptotics and the proof of the existence of a remainder (the existence of a part with the finite energy norm). This is done because otherwise the article becomes too long and bearing in mind that the construction of asymptotics and the proof of existence use different techniques and these parts can be read separately.

Let G be a bounded domain in $\mathbb{R}^n$. In this article, we use usual notations of functional spaces (e.g., [31]). By $L^p(G)$ and $W^{m,p}(G)$, $1 \leq p < \infty$, we denote the usual Lebesgue and Sobolev spaces, respectively. The norms in $L^p(G)$ and $W^{m,p}$ are indicated by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$. We denote by $C^\infty(G)$ the set of all infinitely differentiable functions defined on G and by $C_0^\infty(G)$ the subset of all functions from $C^\infty(G)$ with compact supports in G. By $\mathring{W}^{k,q}(G)$ we denote the completion of the $C_0^\infty(G)$ in the $\|\cdot\|_{W^{m,p}}$ norm. The space $L^p(0,T;X)$ consists of all measurable functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} = \left(\int_0^T \|\mathbf{u}(t)\|^p\,dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty.$$

2 The leading-order term

In the paper we construct a formal asymptotic decomposition of the solution $(\mathbf{u}, p)$ near the cuspidal point $0 \in G_H$. It has the following form

$$\begin{aligned} \mathbf{u}(\tfrac{x_1}{x_2^\lambda}, x_2, t, \tau) &= \mathbf{u}^o(\tfrac{x_1}{x_2^\lambda}, x_2, t) + \mathbf{u}^b(\tfrac{x_1}{x_2^\lambda}, x_2, \tau), \\ p(\tfrac{x_1}{x_2^\lambda}, x_2, t, \tau) &= p^o(\tfrac{x_1}{x_2^\lambda}, x_2, t) + p^b(\tfrac{x_1}{x_2^\lambda}, x_2, \tau), \end{aligned} \tag{2.1}$$

where $\tau = t/x_2^{2\lambda}$; the pair $(\mathbf{u}^o, p^o)$ is the outer part of asymptotic expansion and $(\mathbf{u}^b, p^b)$ is the boundary-layer-in-time corrector (the inner part of the asymptotic expansion) which compensate the discrepancy in the initial condition.

Consider homogeneous problem (1.1) in the domain G_H (remind that $\mathbf{u}|_{\partial G_H \cap \partial\Omega} = 0$). We formally put (2.1) into (1.1) and then separate the result into two problems

$$\begin{cases} \mathbf{u}_t^o - \nu\Delta\mathbf{u}^o + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \nabla p^o = 0, \\ \operatorname{div}\mathbf{u}^o = 0, \\ \mathbf{u}^o|_{\partial G_H \cap \partial\Omega} = 0, \\ \int\limits_{\sigma(h)} \mathbf{u}\cdot\mathbf{n}\,dS = F(t), \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mathbf{u}_t^b - \nu\Delta\mathbf{u}^b + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^b + (\mathbf{u}^b \cdot \nabla)\mathbf{u}^o + (\mathbf{u}^b \cdot \nabla)\mathbf{u}^b + \nabla p^b = 0, \\ \operatorname{div}\mathbf{u}^b = 0, \\ \mathbf{u}|_{\partial G_H \cap \partial\Omega} = 0, \quad \mathbf{u}^b(x, 0) = \mathbf{b}(x) - \mathbf{u}^o(x, 0). \end{cases} \tag{2.3}$$

The terms $(\mathbf{u}^o \cdot \nabla)\mathbf{u}^b$, $(\mathbf{u}^b \cdot \nabla)\mathbf{u}^o$ in (2.3) depend not only on the fast time variable τ but also on the slow time t.

2.1 The leading-order term of the outer asymptotic decomposition

Consider problem (2.2) in the domain G_H. Rewriting (2.2) in coordinates $y_1 = x_1 x_2^{-\lambda}$, $y_2 = x_2$, $t = t$, we obtain the initial boundary value problem in the domain $\Pi = \{y \in \mathbb{R}^2 : |y_1| < \gamma_0,\ y_2 \in (0, H)\}$:

$$\begin{cases} \partial_t u_1^o - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_1^o + (\mathbf{u}^o \cdot \mathfrak{N})u_1^o + y_2^{-\lambda}\partial_1 p^o = 0, & y \in \Pi, \\ \partial_t u_2^o - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_2^o + (\mathbf{u}^o \cdot \mathfrak{N})u_2^o + \mathfrak{D}p^o = 0, & y \in \Pi, \\ y_2^{-\lambda}\partial_1 u_1^o + \mathfrak{D}u_2^o = 0, \\ \mathbf{u}^o|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.4}$$

where $\partial_k = \dfrac{\partial}{\partial y_k}$, $k = 1, 2$, $\partial_t = \dfrac{\partial}{\partial t}$, $\mathfrak{D} = \partial_2 - \lambda y_2^{-1} y_1 \partial_1$, $\mathfrak{N} = \begin{pmatrix} y_2^{-\lambda}\partial_1 \\ \mathfrak{D} \end{pmatrix}$.

The leading-order term for the outer asymptotic decomposition is the same as for the time-periodic Stokes problem (see [22]) or nonstationary Stokes problem (see [23]). In particular, it was shown in [22] that the leading-order asymptotic term $(\mathbf{U}_{\mu_0}, P_{\mu_0})$ has the form

$$\begin{aligned} U_{1,\mu_0}(y_1, y_2, t) &= y_2^{\mu_0+3\lambda-2}\mathscr{U}_{1,\mu_0}(y_1, t), \\ U_{2,\mu_0}(y_1, y_2, t) &= \frac{F(t)}{\kappa_0} y_2^{\mu_0+2\lambda-1}\Phi(y_1), \\ P_{\mu_0}(y_1, y_2, t) &= \frac{F(t)}{\kappa_0\mu_0} y_2^{\mu_0} + y_2^{\mu_0+2\lambda-2}\mathscr{Q}_{\mu_0}(y_1, t), \end{aligned} \tag{2.5}$$

where

$$\mu_0 = 1 - 3\lambda, \tag{2.6}$$

the function Φ is the solution to

$$\begin{cases} \nu\partial_1^2\Phi(y_1) = 1, & |y_1| < \gamma_0, \\ \Phi(y_1) = 0, & |y_1| = \gamma_0, \end{cases} \quad \text{i.e., } \Phi(y_1) = \tfrac{1}{2\nu}\left(|y_1|^2 - \gamma_0^2\right), \tag{2.7}$$

$$\kappa_0 := \int_{-\gamma_0}^{\gamma_0} \Phi(y_1)\, dy_1 = -\tfrac{2}{3\nu}\gamma_0^3 < 0 \tag{2.8}$$

and $(\mathscr{U}_{1,\mu_0}, \mathscr{Q}_{\mu_0})$ is the solution of the Stokes type problem

$$\begin{cases} -\nu\partial_1^2\mathscr{U}_{1,\mu_0}(y_1, t) + \partial_1\mathscr{Q}_{\mu_0}(y_1, t) = 0, & |y_1| < \gamma_0, \\ \partial_1\mathscr{U}_{1,\mu_0}(y_1, t) = \mathscr{G}_0(y_1, t), \\ \mathscr{U}_{1,\mu_0}(y_1, t)|_{|y_1|=\gamma_0} = 0, \end{cases} \tag{2.9}$$

with $\mathscr{G}_0(y_1, t) = \lambda\kappa_0^{-1}F(t)(1 + y_1 \cdot \partial_1)\Phi(y_1)$. Moreover, by construction, the following compatibility condition for problem (2.9)

$$\int_{-\gamma_0}^{\gamma_0} \mathscr{G}_0(y_1, t)\, dy_1 = 0$$

holds[2].

Since in (2.9) the time variable t is included only as a parameter, in general, the vector function $(\mathbf{U}_{1,\mu_0}, U_{2,\mu_0})$ does not satisfy the initial condition. In order to compensate the discrepancy $\mathbf{u}(x, 0) = -\mathbf{U}_{\mu_0}(y_1, y_2, 0)$, we have to construct a boundary layer near the point $t = 0$[3].

2 Hereafter we assume that all arising initial boundary value problems admit smooth solutions. The solvability results, regularity and estimates of these solutions are discussed in Section 4.2.

3 Notice that on this step we do not satisfy the "regular" part **b** of the initial condition, we just compensate the discrepancy appearing in the initial condition because of the inner asymptotic decomposition.

2.2 The leading-order term for the boundary layer

Rewriting (2.3) in fast coordinates $y_1 = x_1 x_2^{-\lambda}$, $y_2 = x_2$, $\tau = t x_2^{-2\lambda}$, we get

$$
\begin{cases}
y_2^{-2\lambda}\partial_\tau u_1^b - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}_b^2)u_1^b + (\mathbf{u}^o\cdot\mathfrak{N}_b)u_1^b + (\mathbf{u}^b\cdot\mathfrak{N}_b)u_1^o \\
+(\mathbf{u}^b\cdot\mathfrak{N}_b)u_1^b + y_2^{-\lambda}\partial_1 p^b = 0, \quad y\in\Pi, \\
y_2^{-2\lambda}\partial_\tau u_2^b - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}_b^2)u_2^b + (\mathbf{u}^o\cdot\mathfrak{N}_b)u_2^b + (\mathbf{u}^b\cdot\mathfrak{N}_b)u_2^o \\
+(\mathbf{u}^b\cdot\mathfrak{N}_b)u_2^b + \mathfrak{D}_b p^b = 0, \quad y\in\Pi, \\
y_2^{-\lambda}\partial_1 u_1^b + \mathfrak{D}_b u_2^b = 0, \\
\mathbf{u}^b|_{|y_1|=\gamma_0} = 0, \qquad \mathbf{u}^b(y_1,y_2,0) = -\mathbf{U}_{\mu_0}(y_1,y_2,0),
\end{cases}
\tag{2.10}
$$

where $\mathfrak{D}_b = \partial_2 - \lambda y_2^{-1}y_1\partial_1 - 2\lambda y_2^{-1}\tau\partial_\tau$, $\partial_\tau = \dfrac{\partial}{\partial\tau}$, $\mathfrak{N}_b = \begin{pmatrix} y_2^{-\lambda}\partial_1 \\ \mathfrak{D}_b \end{pmatrix}$.

We look for a solution $(\mathbf{U}_{\mu_0}^b, P_{\mu_0}^b)$ of (2.10) in the form

$$
\begin{aligned}
&P_{\mu_0}^b(y_1,y_2,\tau) = y_2^{\mu_0} g_{\mu_0}^b(\tau) + \mathcal{Q}_{\mu_0}^b(y_1,y_2,\tau), \\
&\mathbf{U}_{\mu_0}^b(y_1,y_2,\tau) = \left(U_{1,\mu_0}^b(y_1,y_2,\tau), U_{2,\mu_0}^b(y_1,y_2,\tau)\right),
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{aligned}
U_{1,\mu_0}^b(y_1,y_2,\tau) &= y_2^{\mu_0+3\lambda-2}\mathcal{U}_{1,\mu_0}^b(y_1,\tau), \\
U_{2,\mu_0}^b(y_1,y_2,\tau) &= y_2^{\mu_0+2\lambda-1}\mathcal{U}_{2,\mu_0}^b(y_1,\tau), \\
\mathcal{Q}_{\mu_0}^b(y_1,y_2,\tau) &= y_2^{\mu_0+2\lambda-2}\mathcal{Q}_{\mu_0}^b(y_1,\tau)
\end{aligned}
$$

with

$$
\mathcal{U}_{2,\mu_0}^b(y_1,\tau) = \Phi_{\mu_0}^b(y_1,\tau)
$$

and μ_0 is described in (2.6). Substituting solution (2.11) into (2.10), collecting the terms with the same powers of y_n, and having in mind that $F(0) = 0$ (see (1.2)), we get the following problems

$$
\begin{cases}
\partial_\tau\Phi_{\mu_0}^b(y_1,\tau) - \nu\partial_1^2\Phi_{\mu_0}^b(y_1,\tau) = s_{\mu_0}^b(\tau), \quad |y_1|<\gamma_0, \\
\Phi_{\mu_0}^b(y_1,\tau)|_{|y_1|=\gamma_0} = 0, \quad \Phi_{\mu_0}^b(y_1,0) = 0, \\
\int\limits_{-\gamma_0}^{\gamma_0}\Phi_{\mu_0}^b(y_1,\tau)dy_1 = 0,
\end{cases}
\tag{2.12}
$$

and

$$
\begin{cases}
\partial_\tau\mathcal{U}_{1,\mu_0}^b(y_1,\tau) - \nu\partial_1^2\mathcal{U}_{1,\mu_0}^b(y_1,\tau) + \partial_1\mathcal{Q}_{\mu_0}^b(y_1,\tau) = 0, \quad |y_1|<\gamma_0, \\
\partial_1\mathcal{U}_{1,\mu_0}^b(y_1,\tau) = \lambda A_b(y_1,\tau,\partial_1,\partial_\tau)\mathcal{U}_{2,\mu_0}^b(y_1,\tau), \\
U_{1,\mu_0}^b|_{|y_1|=\gamma_0} = 0, \quad U_{1,\mu_0}^b(y_1,0) = -U_{1,\mu_0}(y_1,0) := u_{1,\mu_0}^b(y_1),
\end{cases}
\tag{2.13}
$$

where

$$
A_b(y_1,\tau,\partial_1,\partial_\tau) = 1 + y_1\partial_1 + 2\tau\partial_\tau. \tag{2.14}
$$

The homogeneous inverse problem (2.12) has a unique trivial solution $(\Phi_{\mu_0}^b, s_{\mu_0}^b) = (0,0)$ (see, e.g., [32]) and therefore $U_{2,\mu_0}^b = 0$ and so the right-hand side of equation $(2.13)_2$ is zero.

For the function g we get the following ODE

$$
2\lambda\tau\frac{dg_{\mu_0}^b(\tau)}{d\tau} - \mu_0 g_{\mu_0}^b(\tau) = 0. \tag{2.15}
$$

(because $s_{\mu_0}^b(\tau) = 0$). The pair $(U_{1,\mu_0}^b, Q_{\mu_0}^b)$ solves the 1-dimensional non-stationary Stokes type problem (2.13). The solvability conditions of problem (2.13)

$$
\begin{gathered}
\int\limits_{-\gamma_0}^{\gamma_0} A_b(y_1,\tau,\partial_1,\partial_\tau)U_{2,\mu_0}^b(y_1,\tau)\,dy_1 = 0, \\
0 = \lambda A_b(y_1,\tau,\partial_1,\partial_\tau)U_{2,\mu_0}^b(y_1,0) = -\partial_1 U_{1,\mu_0}(y_1,0) = 0,
\end{gathered}
$$

are satisfied automatically.

The function $g^b_{\mu_0}$ is the solution to ODE (2.15) and has the form

$$g^b_{\mu_0}(\tau) = C\tau^{\frac{1}{2\lambda}-\frac{3}{2}},$$

and we set the constant $C = 0$, in order to have finite boundary layer pressure $P^b_{\mu_0}$ at point $\tau = 0$.

Consider "mixed" terms, i.e., the terms $(\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b)\,U^b_{1,\mu_0}$, $\left(\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b\right)U_{1,\mu_0}$, $(\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b)\,U^b_{2,\mu_0}$ and $\left(\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b\right)U_{2,\mu_0}$. As it is said before, these terms depend not only on the fast time variable τ but also on slow time t. We expand these terms in Taylor's series with respect to the variable t and then replace t in obtained expression by the product $\tau y_2^{2\lambda}$. As a result, we get (recall that $\mathbf{U}^b_{2,\mu_0} = 0$)

$$\begin{gathered}
(\mathbf{U}_{\mu_0}(y,t)\cdot\mathfrak{N}_b)\,U^b_{1,\mu_0}(y,\tau) = \\
\left(\left[\mathbf{U}_{\mu_0}(y,0) + y_2^{2\lambda}\frac{\tau}{1!}\frac{\partial\mathbf{U}_{\mu_0}}{\partial t}(y,0) + \ldots + \frac{\tau^k y_2^{2\lambda k}}{k!}\frac{\partial^k\mathbf{U}_{\mu_0}}{\partial t^k}(y,0) + \ldots\right]\cdot\mathfrak{N}_b\right)U^b_{1,\mu_0}(y,\tau) \\
= y_2^{\mu_0+2\lambda-3}T^{(1)}_{1,\mu_0}(y_1,0,\tau) + y_2^{\mu_0+4\lambda-3}\tau\frac{\partial T^{(1)}_{1,\mu_0}}{\partial t}(y_1,0,\tau) + \ldots + \\
y_2^{\mu_0+2(k+1)\lambda-3}\frac{\tau^k}{k!}\frac{\partial^k T^{(1)}_{1,\mu_0}}{\partial t^k}(y_1,0,\tau) + \ldots = y_2^{\mu_0+2\lambda-3}T^{(1)}_{1,\mu_0}(y_1,0,\tau) + \bar{T}^{(1)}_{1,\mu_0}(y,0,\tau),
\end{gathered}$$

where $y = (y_1, y_2)$, $\mathfrak{N}_b$ is defined in the beginning of the present section,

$$T^{(1)}_{1,\mu_0}(y_1,0,\tau) = \left(\mathscr{U}_{\mu_0}(y_1,0)\cdot\mathfrak{N}_{bb}\right)\mathscr{U}^b_{1,\mu_0}(y_1,\tau),$$

$$\mathfrak{N}_{bb} = \begin{pmatrix}\partial_1\\ \mu_0 + 3\lambda - 2 - \lambda y_1\partial_1 - 2\lambda\tau\partial_\tau\end{pmatrix},$$

and by $\bar{T}^{(1)}_{1,\mu_0}$ we denote the collection of the remaining terms that belong to L^2-space[4] and, therefore, we are not interested in their detailed expression. Similarly,

$$\left(\mathbf{U}^b_{\mu_0}(y,\tau)\cdot\mathfrak{N}_b\right)U_{1,\mu_0}(y,t) = y_2^{\mu_0+2\lambda-3}T^{(2)}_{1,\mu_0}(y_1,0,\tau) + \bar{T}^{(2)}_{1,\mu_0}(y_1,y_2,0,\tau),$$

where

$$T^{(2)}_{1,\mu_0}(y_1,0,\tau) = \left(\mathscr{U}^b_{\mu_0}(y_1,\tau)\cdot\mathfrak{N}_{bb}\right)\mathscr{U}_{1,\mu_0}(y_1,0)$$

and $\bar{T}^{(2)}_{1,\mu_0}$ is in L^2-space. The same argument gives[5]

$$(\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b)\,U^b_{2,\mu_0} + \left(\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b\right)U_{2,\mu_0} =$$

$$y_2^{\mu_0+\lambda-2}\left(T^{(1)}_{2,\mu_0}(y_1,0,\tau) + T^{(2)}_{2,\mu_0}(y_1,0,\tau)\right) + \bar{T}^{(1)}_{2,\mu_0}(y,0,\tau) + \bar{T}^{(2)}_{2,\mu_0}(y,0,\tau),$$

where

$$\begin{aligned}
T^{(1)}_{2,\mu_0}(y_1,0,\tau) &= \left(\mathscr{U}_{\mu_0}(y_1,0)\cdot\mathfrak{N}_{bb}\right)\mathscr{U}^b_{2,\mu_0}(y_1,\tau) = 0,\\
T^{(2)}_{2,\mu_0}(y_1,0,\tau) &= \left(\mathscr{U}^b_{\mu_0}(y_1,\tau)\cdot\mathfrak{N}_{bb}\right)\mathscr{U}_{2,\mu_0}(y_1,0),
\end{aligned}$$

with $\bar{T}^{(1)}_{2,\mu_0} = 0$ and $\bar{T}^{(2)}_{2,\mu_0}$ is in L^2-space (since $\mathbf{U}^b_{2,\mu_0} = 0$).

4 Hereafter, by L^2-space we mean the space $L^2(0,T;L^2(G_H))$.

5 The term $(\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b)\,U^b_{2,\mu_0}$ of course is equal to zero, and we write the calculations containing it only in order to explain what kind of terms could appear and to have the same notations for all approximations.

Functions $\mathbf{U}_{\mu_0}, Q_{\mu_0}, \mathbf{U}^b_{\mu_0}, Q^b_{\mu_0}$ leave in equations $(2.4)_1$, $(2.4)_2$, $(2.10)_1$, $(2.10)_2$ the discrepancies $H_{1,\mu_0} = H_{1,\mu_0}(y_1, y_2, t, \tau)$, $H_{2,\mu_0} = H_{2,\mu_0}(y_1, y_2, t, \tau)$:

$$
\begin{aligned}
H_{1,\mu_0} = {} & \left(\nu\mathfrak{D}^2 - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N}) - (\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b) - \partial_t\right) U_{1,\mu_0} \\
& + \left(\nu\mathfrak{D}_b^2 - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b) - (\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b)\right) U^b_{1,\mu_0} \\
= {} & y_2^{\mu_0+3\lambda-4}\widehat{\mathscr{F}}_{1,\mu_0}(y_1,t) + y_2^{\mu_0+2\lambda-3}\mathscr{N}_{1,\mu_0}(y_1,t) \\
& + y_2^{\mu_0+3\lambda-2}\widetilde{\mathscr{F}}_{1,\mu_0}(y_1,t) + y_2^{\mu_0+3\lambda-4}\mathscr{F}^b_{1,\mu_0}(y_1,\tau) \\
& + y_2^{\mu_0+2\lambda-3}\mathscr{N}^b_{1,\mu_0}(y_1,\tau) + F_{1,\mu_0}(y_1,y_2,0,\tau) \\
:= {} & F^o_{1,\mu_0}(y_1,y_2,t) + F^b_{1,\mu_0}(y_1,y_2,\tau) + F_{1,\mu_0}(y_1,y_2,0,\tau), \\
H_{2,\mu_0} = {} & \left(\nu\mathfrak{D}^2 - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N}) - (\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b) - \partial_t\right) U_{2,\mu_0} - \mathfrak{D}\mathscr{Q}_{\mu_0} \\
& + \left(\nu\mathfrak{D}_b^2 - (\mathbf{U}_{\mu_0}\cdot\mathfrak{N}_b) - (\mathbf{U}^b_{\mu_0}\cdot\mathfrak{N}_b)\right) U^b_{2,\mu_0} - \mathfrak{D}_b\mathscr{Q}^b_{\mu_0} \\
= {} & y_2^{\mu_0+2\lambda-3}\widehat{\mathscr{F}}_{2,\mu_0}(y_1,t) + y_2^{\mu_0+\lambda-2}\mathscr{N}_{2,\mu_0}(y_1,t) \\
& + y_2^{\mu_0+2\lambda-1}\widetilde{\mathscr{F}}_{2,\mu_0}(y_1,t) + y_2^{\mu_0+2\lambda-3}\mathscr{F}^b_{2,\mu_0}(y_1,\tau) \\
& + y_2^{\mu_0+\lambda-2}\mathscr{N}^b_{2,\mu_0}(y_1,\tau) + F_{2,\mu_0}(y_1,y_2,0,\tau) \\
:= {} & F^o_{2,\mu_0}(y_1,y_2,t) + F^b_{2,\mu_0}(y_1,y_2,\tau) + F_{2,\mu_0}(y_1,y_2,0,\tau).
\end{aligned}
\tag{2.16}
$$

In order to explain formula (2.16), we represent it schematically:

$\mathbf{F}^b_{\mu_0}$ = *terms including* $\{\mathscr{F}^b + \mathscr{N}^b\}$ *is a collection of discrepancies arising from the leading-order term of the boundary layer construction.* $\Rightarrow$ Where $\mathscr{N}^b$ denotes the discrepancies arising from the nonlinear terms in equations $(2.10)_{1,2}$ and $\mathscr{F}^b$ are the discrepancies arising from the linear part of equations $(2.10)_{1,2}$.

$\Uparrow$

$\mathbf{H}_{\mu_0} = \mathbf{F}^o_{\mu_0} + \mathbf{F}^b_{\mu_0} + \mathbf{F}_{\mu_0}$ $\Rightarrow$ $\mathbf{F}_{\mu_0}$ *is a collection of all terms that belong to* L^2*-space .*

$\Downarrow$

$\mathbf{F}^o_{\mu_0}$ *is a collection of discrepancies arising from the leading-order term of the outer asymptotic expansion:*
$\mathbf{F}^o_{\mu_0}$ = *terms including* $\{\widehat{\mathscr{F}} + \mathscr{N} + \widetilde{\mathscr{F}}\}$ $\Rightarrow$ Where $\widetilde{\mathscr{F}}$ denotes the discrepancies arising from $\partial_t\mathbf{U}_{\mu_0}$, $\mathscr{N}$ denotes the discrepancies arising from the nonlinear term $(\mathbf{U}_{\mu_0}\cdot\mathfrak{N})\mathbf{U}_{\mu_0}$ in equations $(2.4)_{1,2}$ and $\widehat{\mathscr{F}}$ -terms arising from the linear part of equations $(2.4)_{1,2}$.

Our goal is to construct such an asymptotic decomposition of the solution that discrepancies would belong to the L^2-space. However, since $\lambda > 1$, neither $\mathbf{F}^o_{\mu_0}$ nor $\mathbf{F}^b_{\mu_0}$ satisfies this condition and we need to construct higher-order terms of the asymptotic decomposition.

3 Higher-order terms of the asymptotic decomposition

3.1 Outer asymptotics

In order to construct the solution of problem (1.1), we have to ensure that discrepancies in equation (1.1) belong to L^2-space. However, this is not the case having only the leading order asymptotic term. Therefore, we have to compensate the singular terms in the expressions of discrepancies (2.16). To do this, we construct the higher order asymptotic terms. They leave some new discrepancies which also may be singular. So, we compensate them in the same way and continue this process until the discrepancies are from L^2-space.

In this subsection we compensate the terms arising from construction of the outer asymptotic decomposition. At each step of this process we obtain the same equations with the right-hand sides having similar structure. Therefore, we first consider the equations

$$\begin{cases} \partial_t u_1 - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_1 + (\mathbf{u}\cdot\mathfrak{N})u_1 + y_2^{-\lambda}\partial_1 p = Z_1(\varphi_1,\varphi_2), \\ \partial_t u_2 - \nu(y_2^{-2\lambda}\partial_1^2 + \mathfrak{D}^2)u_2 + (\mathbf{u}\cdot\mathfrak{N})u_2 + \mathfrak{D}p = Z_2(\varphi_1,\varphi_2), \\ y_2^{-\lambda}\partial_1 u_1 + \mathfrak{D}u_2 = 0, \\ \mathbf{u}|_{|y_1|=\gamma_0} = 0, \qquad \mathbf{u}(y_1,y_2,0) = 0, \end{cases} \tag{3.1}$$

with "abstract" right-hand sides $\big(Z_1(\varphi_1,\varphi_2), Z_2(\varphi_1,\varphi_2)\big)$ having the form of one of the following expressions

$$\big(Z_1(\varphi_1,\varphi_2), Z_2(\varphi_1,\varphi_2)\big) = \begin{cases} \big(\nu\mathfrak{D}^2\varphi_1, \nu\mathfrak{D}^2\varphi_2 - \mathfrak{D}p_\varphi\big)\,, \\ \text{or} \\ -\big((\boldsymbol{\varphi}\cdot\mathfrak{N})\varphi_1, (\boldsymbol{\varphi}\cdot\mathfrak{N})\varphi_2\big)\,, \\ \text{or} \\ -\left(\partial_t\varphi_1, \partial_t\varphi_2\right), \end{cases} \tag{3.2}$$

where the functions $\boldsymbol{\varphi} = (\varphi_1,\varphi_2)$ and p_φ are specified below.

Let

$$\begin{aligned} \varphi_{1,\mu}(y_1,y_2,t) &= y_2^{\mu+3\lambda-2}\mathscr{U}_{1,\mu}(y_1,t), \\ \varphi_{2,\mu}(y_1,y_2,t) &= y_2^{\mu+2\lambda-1}\mathscr{U}_{2,\mu}(y_1,t), \\ p_{\varphi,\mu}(y_1,y_2,t) &= y_2^{\mu}g_\mu(t) + y_2^{\mu+2\lambda-2}\mathscr{Q}_\mu(y_1,t), \end{aligned} \tag{3.3}$$

g_μ be arbitrary functions and μ belong to a certain set of indices M. Substituting expressions (3.3) into (3.2), we derive

$$Z_1(\varphi_{1,\mu},\varphi_{2,\mu}) = $$
$$= \begin{cases} \nu\mathfrak{D}^2\varphi_{1,\mu} \sim y_2^{M_L+\lambda-2}\widehat{\mathscr{F}}_{1,\mu}(y_1,t), \\ -(\boldsymbol{\varphi}_\mu\cdot\mathfrak{N})\varphi_{1,\mu} = -(y_2^{-\lambda}\varphi_{1,\mu}\cdot\partial_1 + \varphi_{2,\mu}\mathfrak{D})\varphi_{1,\mu} \\ \sim y_2^{M_N+\lambda-2}\mathscr{N}_{1,\mu}(y_1,t), \\ -\partial_t\varphi_{1,\mu} \sim y_2^{M_T+\lambda-2}\widetilde{\mathscr{F}}_{1,\mu}(y_1,t), \end{cases}$$

$$Z_2(\varphi_{1,\mu},\varphi_{2,\mu}) = $$
$$= \begin{cases} \nu\mathfrak{D}^2\varphi_{2,\mu} - \mathfrak{D}p_{\varphi,\mu} = \nu\mathfrak{D}^2\varphi_{2,\mu} - \mathfrak{D}\left(y_2^{\mu+2\lambda-2}\mathscr{Q}_\mu\right) \sim y_2^{M_L-1}\widehat{\mathscr{F}}_{2,\mu}(y_1), \\ -(\boldsymbol{\varphi}_\mu\cdot\mathfrak{N})\varphi_{2,\mu} = -(y_2^{-\lambda}\boldsymbol{\varphi}_{1,\mu}\cdot\partial_1 + \varphi_{2,\mu}\mathfrak{D})\varphi_{2,\mu} \sim y_2^{M_N-1}\mathscr{N}_{2,\mu}(y_1), \\ -\partial_t\varphi_{2,\mu} \sim y_2^{M_T-1}\widetilde{\mathscr{F}}_{2,\mu}(y_1), \end{cases}$$

where

$$M_L = \mu + 2\lambda - 2, \quad M_N = 2\mu + 4\lambda - 2, \quad M_T = \mu + 2\lambda. \tag{3.4}$$

From (3.4) we obtain the following rules for elements of the set M

$$\begin{aligned} &\mu \in M \Rightarrow \mu + 2\lambda - 2 \in M, \\ &\mu_1,\mu_2 \in M \Rightarrow \mu_1 + \mu_2 + 4\lambda - 2 \in M, \\ &\mu \in M \Rightarrow \mu + 2\lambda \in M. \end{aligned} \tag{3.5}$$

In the lemma below we describe the set M which is the most narrow set of indices satisfying (3.5).

Lemma 3.1. *1. If parameter $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, $N = 1,2,\dots$, then*

$$M = \big\{1 - 3\lambda + k(\lambda - 1) : \ k = 0,1,\dots\big\};$$

2. If parameter $\lambda = \frac{N+4}{N}$, $N = 1, 2, \dots$, *then*

$$M = \big\{1 - 3\lambda + k(\lambda - 1) : \ k = 0, 1, \dots\big\} \cup$$
$$\big\{1 - 3\lambda + k(\lambda - 1) + 2 : \ k = 0, 1, \dots\big\} := M_1 \cup M_2;$$

3. In other cases

$$M = \big\{1 - 3\lambda + 2i + 2j\lambda + k(\lambda - 1) : \ i, j, k = 0, 1, \dots\big\}; \tag{3.6}$$

For the reader convenience the proof of Lemma 3.1 is given in Appendix B. The proof itself is irrelevant for the construction of the asymptotic expansion, however it explains why the three cases described in Lemma 3.1 appear.

Assume that $(\mathbf{U}^{O,[M]}, P^{O,[M]})$ is represented in the form

$$\begin{aligned} U_1^{O,[M]}(y_1, y_2, t) &= \sum_{\mu \in M} y_2^{\mu+3\lambda-2} \mathscr{U}_{1,\mu}(y_1, t), \\ U_2^{O,[M]}(y_1, y_2, t) &= \sum_{\mu \in M} y_2^{\mu+2\lambda-1} \mathscr{U}_{2,\mu}(y_1, t), \\ P^{O,[M]}(y_1, y_2, t) &= \sum_{\mu \in M} y_2^{\mu} g_\mu(t) + y_2^{\mu+2\lambda-2} \mathscr{Q}_\mu(y_1, t), \end{aligned} \tag{3.7}$$

where M is the set of indices described in Lemma 3.1; the pair of functions $(\mathscr{U}_{1,\mu}, \mathscr{Q}_\mu)$ is the solution of

$$\begin{cases} -\nu \partial_1^2 \mathscr{U}_{1,\mu}(y_1, t) + \partial_1 \mathscr{Q}_\mu(y_1, t) = Z_1(\mathscr{U}_{1,\bar\mu}, \mathscr{U}_{2,\bar\mu}), & |y_1| < \gamma_0, \\ \partial_1 \mathscr{U}_{1,\mu}(y_1, t) = -A(\mu) \mathscr{U}_{2,\mu}(y_1, t), \\ \mathscr{U}_{1,\mu}\big|_{|y_1| = \gamma_0}(y_1, t) = 0, \end{cases} \tag{3.8}$$

where $\mu, \ \bar\mu \in M$[6],

$$A(\mu) = \mu + 2\lambda - 1 - \lambda y_1 \partial_1, \tag{3.9}$$
$$\mathscr{U}_{2,\mu}(y_1, t) = g_\mu(t) \mu \Phi(y_1) + \mathscr{U}^\star_{2,\mu}(y_1, t);$$

Φ is the solution to problem (2.7), the function $\mathscr{U}^\star_{2,\mu}$ satisfy the equations

$$\begin{cases} -\nu \partial_1^2 \mathscr{U}^\star_{2,\mu}(y_1, t) = Z_2(\mathscr{U}_{1,\bar\mu}, \mathscr{U}_{2,\bar\mu}), & |y_1| < \gamma_0, \\ \mathscr{U}^\star_{2,\mu}\big|_{|y_1| = \gamma_0}(y_1, t) = 0. \end{cases}$$

Functions g_μ are uniquely determined from the following solvability condition for problem (3.8)

$$\int_{-\gamma_0}^{\gamma_0} A(\mu) \mathscr{U}_{2,\mu}(y_1, t)\, dy_1 = 0. \tag{3.10}$$

Indeed, using (2.8) and the equality

$$\int_{-\gamma_0}^{\gamma_0} y_1 \cdot \partial_1 \Phi(y_1)\, dy_1 = -\int_{-\gamma_0}^{\gamma_0} \Phi(y_1)\, dy_1 = -\kappa_0.$$

we rewrite (3.10) in the form

$$g_\mu(t) \mu \kappa_0 (\mu + 3\lambda - 1) = -\int_{-\gamma_0}^{\gamma_0} A(\mu) \mathscr{U}^\star_{2,\mu}(y_1, t)\, dy_1,$$

Thus, if $\mu \neq 0$ and $\mu \neq \mu_0$, then

$$g_\mu(t) = -\frac{1}{\mu \kappa_0} \int_{-\gamma_0}^{\gamma_0} \mathscr{U}^\star_{2,\mu}(y_1, t)\, dy_1.$$

6 Numbers $\mu, \ \bar\mu \in M$ are different.

In the case $\mu = \mu_0$ we have

$$g_{\mu_0}(t) = \frac{F(t)}{\mu_0 \kappa_0}.$$

Hereafter we assume that $\mu \neq 0$.

First of all we will study the first case of Lemma 3.1, i.e., $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+1}{N}$, and then, without going into details, the other two cases.

3.1.1 Outer asymptotics. Case $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$.

If $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, then from (3.7) and Lemma 3.1 we get

$$\begin{aligned}
U_1^{O,[J]}(y_1, y_2, t) = &\quad y_2^{-1} \mathscr{U}_{1,0}(y_1, t) + \sum_{k=1}^{J} y_2^{-1+k(\lambda-1)} \mathscr{U}_{1,k}(y_1, t),\\
U_2^{O,[J]}(y_1, y_2, t) = &\quad \frac{F(t)}{\kappa_0} y_2^{-\lambda} \Phi(y_1) + \sum_{k=1}^{J} y_2^{-\lambda+k(\lambda-1)} \mathscr{U}_{2,k}(y_1, t),\\
P^{O,[J]}(y_1, y_2, t) = &\quad \frac{F(t)}{\kappa_0(1-3\lambda)} y_2^{1-3\lambda} + y_2^{-1-\lambda} \mathscr{Q}_0(y_1, t) +\\
&\quad \sum_{k=1}^{J} y_2^{1-3\lambda+k(\lambda-1)} g_k(t) + y_2^{-1-\lambda+k(\lambda-1)} \mathscr{Q}_k(y_1, t),
\end{aligned} \tag{3.11}$$

where $\mathbf{U}_0 = \mathbf{U}_{\mu_0}$, the pair $(\mathscr{U}_{1,0}, \mathscr{Q}_0)$ solves problem (2.9), Φ is defined by (2.7) and the functions $(\mathscr{U}_{1,k}, \mathscr{Q}_k)$ solve the following problems

$$\begin{cases}
-\nu \partial_1^2 \mathscr{U}_{1,k}(y_1, t) + \partial_1 \mathscr{Q}_k(y_1, t) = \mathscr{Z}_{1,k}(y_1, t), \quad |y_1| < \gamma_0,\\
\partial_1 \mathscr{U}_{1,k}(y_1, t) = -A(1 - 3\lambda + k(\lambda-1)) \mathscr{U}_{2,k}(y_1, t),\\
\mathscr{U}_{1,k}|_{|y_1|=\gamma_0}(y_1, t) = 0,
\end{cases} \tag{3.12}$$

with A described in (3.9),

$$\mathscr{U}_{2,k}(y_1, t) = g_k(t)(1 - 3\lambda + k(\lambda-1))\Phi(y_1) + \mathscr{U}^{\star}_{2,k}(y_1, t)$$

and $\mathscr{U}^{\star}_{2,k}$ satisfying the equation

$$\begin{cases}
-\nu \partial_1^2 \mathscr{U}^{\star}_{2,k}(y_1, t) = \mathscr{Z}_{2,k}(y_1, t), \quad |y_1| < \gamma_0,\\
\mathscr{U}^{\star}_{2,k}|_{|y_1|=\gamma_0}(y_1, t) = 0.
\end{cases} \tag{3.13}$$

The functions g_k are uniquely determined from the solvability condition for problem (3.12):

$$\int_{-\gamma_0}^{\gamma_0} A(1 - 3\lambda + k(\lambda-1)) \mathscr{U}_{2,k}(y_1, t)\, dy_1 = 0, \tag{3.14}$$

and arguing as above, we find

$$g_k(t) = -\frac{1}{\kappa_0(1-3\lambda+k(\lambda-1))} \int_{-\gamma_0}^{\gamma_0} \mathscr{U}^{\star}_{2,k}(y_1, t)\, dy_1, \tag{3.15}$$

$k = 1, 2, \ldots$, and

$$g_0(t) = \frac{F(t)}{\kappa_0(1-3\lambda)} \tag{3.16}$$

(see Section 2.1). Note that $1 - 3\lambda + k(\lambda-1) \neq 0$ due to the assumption $\mu \neq 0$.

The right-hand sides $\mathscr{Z}_k(y_1, t) = (\mathscr{Z}_{1,k}(y_1, t), \mathscr{Z}_{2,k}(y_1, t))$ contain the most singular terms which we compensate at the step $k = 1, 2, \ldots$. Notice that writing down problems (3.12), (3.13), we multiplied both sides of (3.1) by $y_2^{2\lambda}$. Therefore, $\mathbf{Z}_k(y_1, y_2, t) = y_2^{2\lambda} \mathbf{H}_k^{\nabla}(y_1, y_2, t)$, where $\mathbf{H}_k^{\nabla}$ is equal to the most singular term in the discrepancies $\mathbf{H}_k$, i.e., the functions $\mathscr{Z}_k = (\mathscr{Z}_{1,k}, \mathscr{Z}_{2,k})$ are equal to $\widehat{\mathscr{F}}_j = (\widehat{\mathscr{F}}_{1,j}, \widehat{\mathscr{F}}_{2,j})$, $\mathscr{N}_j = (\mathscr{N}_{1,j}, \mathscr{N}_{2,j})$, $\widetilde{\mathscr{F}}_j = (\widetilde{\mathscr{F}}_{1,j}, \widetilde{\mathscr{F}}_{2,j})$ or a sum of them, $j = 0, 1, \dots$[7]. We compensate them by the following rule:

$$\begin{aligned} &\mathscr{N}_0 \to \widehat{\mathscr{F}}_0 + \mathscr{N}_1 \to \widehat{\mathscr{F}}_1 + \mathscr{N}_2 \to \dots \to \widehat{\mathscr{F}}_{\frac{2}{\lambda-1}} + \mathscr{N}_{\frac{2}{\lambda-1}+1} + \widetilde{\mathscr{F}}_0 \to \dots \\ &\to \widehat{\mathscr{F}}_{\frac{2}{\lambda-1}+j} + \mathscr{N}_{\frac{2}{\lambda-1}+j+1} + \widetilde{\mathscr{F}}_j \to \dots \end{aligned} \tag{3.17}$$

$j = 1, 2, \dots$, $\mathscr{N}_k = (\mathscr{N}_{1,k}, \mathscr{N}_{2,k})$, $\widehat{\mathscr{F}}_k = (\widehat{\mathscr{F}}_{1,k}, \widehat{\mathscr{F}}_{2,k})$, $\widetilde{\mathscr{F}}_k = (\widetilde{\mathscr{F}}_{1,k}, \widetilde{\mathscr{F}}_{2,k})$ and

$$\begin{aligned} &\widehat{\mathscr{F}}_{1,k}(y_1, t) = y_2^{-(-3+k(\lambda-1))} \nu \mathfrak{D}^2 U_{1,k}(y_1, y_2, t), \\ &\widehat{\mathscr{F}}_{2,k}(y_1, t) = y_2^{-(-\lambda-2+k(\lambda-1))} \left[\nu \mathfrak{D}^2 U_{2,k}(y_1, y_2, t) - \mathfrak{D}\left(y_2^{-\lambda-1+k(\lambda-1)} \mathscr{Q}_k(y_1, t) \right) \right], \\ &\mathscr{N}_{1,k}(y_1, t) = -y_2^{-(-\lambda-2+k(\lambda-1))} \sum_{i+j=k} (\mathbf{U}_i \cdot \mathfrak{N}) U_{1,j}(y_1, y_2, t), \\ &\mathscr{N}_{2,k}(y_1, t) = -y_2^{-(-2\lambda-1+k(\lambda-1))} \sum_{i+j=k} (\mathbf{U}_i \cdot \mathfrak{N}) U_{2,j}(y_1, y_2, t), \\ &\widetilde{\mathscr{F}}_{1,k}(y_1, t) = -y_2^{-(-1+k(\lambda-1))} \partial_t U_{1,k}(y_1, y_2, t), \\ &\widetilde{\mathscr{F}}_{2,k}(y_1, t) = -y_2^{-(-\lambda+k(\lambda-1))} \partial_t U_{2,k}(y_1, y_2, t), \end{aligned}$$

where $i + j = k$, k, i, $j = 0, 1, 2, \dots$. Scheme (3.17) means that the functions $(\mathscr{U}_{1,1}, \mathscr{Q}_1)$, $\mathscr{U}^{\star}_{2,1}$ solve problems (3.12), (3.13) with the right-hand side $\mathscr{Z}_1 = \mathscr{N}_0$; the functions $(\mathscr{U}_{1,2}, \mathscr{Q}_2)$, $\mathscr{U}^{\star}_{2,2}$ solve problems (3.12), (3.13) with the right-hand side $\mathscr{Z}_2 = \widehat{\mathscr{F}}_0 + \mathscr{N}_1$ and so on.

3.2 Boundary layer.

Consider equations (2.10) with right-hand sides having special form

$$\begin{cases} y_2^{-2\lambda} \partial_\tau u_1^b - \nu(y_2^{-2\lambda} \partial_1^2 + \mathfrak{D}^2) u_1^b + (\mathbf{u}^o \cdot \mathfrak{N}_b) u_1^b + (\mathbf{u}^b \cdot \mathfrak{N}_b) u_1^o \\ +(\mathbf{u}^b \cdot \mathfrak{N}_b) u_1^b + y_2^{-\lambda} \partial_1 p^b = Z_{1,k}^b(y, \tau), \quad y \in \Pi, \\ y_2^{-2\lambda} \partial_\tau u_2^b - \nu(y_2^{-2\lambda} \partial_1^2 + \mathfrak{D}^2) u_2^b + (\mathbf{u}^o \cdot \mathfrak{N}_b) u_2^b + (\mathbf{u}^b \cdot \mathfrak{N}_b) \\ u_2^o + (\mathbf{u}^b \cdot \mathfrak{N}_b) u_2^b + \mathfrak{D}_b p^b = Z_{2,k}^b(y, \tau), \quad y \in \Pi, \\ y_2^{-\lambda} \partial_1 u_1^b + \mathfrak{D}_b u_2^b = 0, \\ \mathbf{u}^b|_{|y_1| = \gamma_0} = 0, \qquad \mathbf{u}^b(y_1, y_2, 0) = -\mathbf{U}_{\mu_0}(y_1, y_2, 0), \end{cases} \tag{3.18}$$

where functions $\mathbf{Z}_k^b(y, \tau)$, $k = 1, 2, \dots$, depend on the case whether $\lambda = \frac{N+1}{N}$ (or $\lambda = \frac{N+2}{N}$) or λ has another value.

3.2.1 Boundary layer. Case: $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$.

If $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, then the functions $\mathbf{Z}_k^b(y, \tau)$, $k = 1, 2, \dots$, in (3.18) have the following representation

$$\mathbf{Z}_k^b(y, \tau) = y_2^{2\lambda} \left(y_2^{1-2\lambda+k(\lambda-1)} \mathscr{Z}_{1,k}^b(y_1, \tau), y_2^{2-3\lambda+k(\lambda-1)} \mathscr{Z}_{2,k}^b(y_1, \tau) \right),$$

and are described by the following rule

$$\begin{aligned} &\mathscr{N}_0^b \to \mathscr{F}_0^b + \mathscr{N}_1^b \to \mathscr{F}_1^b + \mathscr{N}_2^b \to \dots \to \mathscr{F}^b_{\frac{2}{\lambda-1}} + \mathscr{N}^b_{\frac{2}{\lambda-1}+1} \to \dots \\ &\to \mathscr{F}^b_{\frac{2}{\lambda-1}+j} + \mathscr{N}^b_{\frac{2}{\lambda-1}+j+1} \to \dots \end{aligned}$$

7 Recall that functions denoted by italic letters do not depend on y_2.

$j = 1, 2, \dots, \mathscr{N}_k^b = (\mathscr{N}_{1,k}^b, \mathscr{N}_{2,k}^b)$, $\mathscr{F}_k^b = (\mathscr{F}_{1,k}^b, \mathscr{F}_{2,k}^b)$ and

$$
\begin{aligned}
\mathscr{F}_{1,k}^b(y_1,\tau) &= y_2^{-(-3+k(\lambda-1))}\nu\mathfrak{D}^2 U_{1,k}^b(y,\tau),\\
\mathscr{F}_{2,k}^b(y_1,\tau) &= y_2^{-(-\lambda-2+k(\lambda-1))}\left[\nu\mathfrak{D}^2 U_{2,k}^b(y,\tau) - \mathfrak{D}\left(y_2^{-\lambda-1+k(\lambda-1)}\mathscr{Q}_k^b(y_1,\tau)\right)\right],\\
\mathscr{N}_{1,k}^b(y_1,\tau) &= -\Bigg(y_2^{-(-\lambda-2+k(\lambda-1))}\sum_{i+j=k}(\mathbf{U}_i^b\cdot\mathfrak{N})U_{1,j}^b(y,\tau) + T_{1,k}(y_1,0,\tau)\\
&\qquad + T_{1,k}^b(y_1,0,\tau)\Bigg),\\
\mathscr{N}_{2,k}^b(y_1,\tau) &= -\Bigg(y_2^{-(-2\lambda-1+k(\lambda-1))}\sum_{i+j=k}(\mathbf{U}_i^b\cdot\mathfrak{N})U_{2,j}^b(y,\tau) + T_{2,k}(y_1,0,\tau)\\
&\qquad + T_{2,k}^b(y_1,0,\tau)\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{T}_{1,k}^b(y_1,0,\tau) &= \left(\mathscr{U}_i(y_1,0)\cdot\begin{pmatrix}\partial_1\\ -1+j(\lambda-1)-\lambda y_1\partial_1-2\lambda\tau\partial_\tau\end{pmatrix}\right)\mathscr{U}_{1,j}^b(y_1,\tau),\\
\mathbf{T}_{1,k}(y_1,0,\tau) &= \left(\mathscr{U}_i^b(y_1,\tau)\cdot\begin{pmatrix}\partial_1\\ -1+j(\lambda-1)-\lambda y_1\partial_1-2\lambda\tau\partial_\tau\end{pmatrix}\right)\mathscr{U}_{1,j}(y_1,0),\\
\mathbf{T}_{2,k}^b(y_1,0,\tau) &= \left(\mathscr{U}_i(y_1,0)\cdot\begin{pmatrix}\partial_1\\ -1-2\lambda+j(\lambda-1)-\lambda y_1\partial_1-2\lambda\tau\partial_\tau\end{pmatrix}\right)\mathscr{U}_{2,j}^b(y_1,\tau),\\
\mathbf{T}_{2,k}(y_1,0,\tau) &= \left(\mathscr{U}_i^b(y_1,\tau)\cdot\begin{pmatrix}\partial_1\\ -1-2\lambda+j(\lambda-1)-\lambda y_1\partial_1-2\lambda\tau\partial_\tau\end{pmatrix}\right)\mathscr{U}_{2,j}(y_1,0),
\end{aligned}
$$

$i + j = k,\ k,\ i,\ j = 0, 1, 2, \dots$.

We look for the boundary layer asymptotic expansion in the form:

$$
\begin{aligned}
U_1^{B,[J]}(y,\tau) &= \sum_{k=0}^{J} y_2^{-1+k(\lambda-1)}\mathscr{U}_{1,k}^b(y_1,\tau),\\
U_2^{B,[J]}(y,\tau) &= \sum_{k=0}^{J} y_2^{-\lambda+k(\lambda-1)}\mathscr{U}_{2,k}^b(y_1,\tau),\\
P^{B,[J]}(y,\tau) &= \sum_{k=0}^{J} y_2^{1-3\lambda+k(\lambda-1)}g_k^b(\tau) + y_2^{-1-\lambda+k(\lambda-1)}\mathscr{Q}_k^b(y_1,\tau),
\end{aligned}
\tag{3.19}
$$

where $(\mathscr{U}_{1,k}^b, \mathscr{Q}_k^b)$, $k = 1, 2, \dots$, are solutions to the problems

$$
\begin{cases}
\partial_\tau\mathscr{U}_{1,k}^b - \nu\partial_1^2\mathscr{U}_{1,k}^b + \partial_1\mathscr{Q}_k^b = \mathscr{Z}_{1,k}^b, \quad |y_1| < \gamma_0,\\
\partial_1\mathscr{U}_{1,k}^b = \left[\lambda A_b(y_1,\tau,\partial_1,\partial_\tau) - k(\lambda-1)\right]\mathscr{U}_{2,k}^b,\\
\mathscr{U}_{1,k}^b|_{|y_1|=\gamma_0} = 0, \quad \mathscr{U}_{1,k}^b(y_1,0) = -\mathscr{U}_{1,k}(y_1,0),
\end{cases}
\tag{3.20}
$$

$$
\mathscr{U}_{2,k}^b(y_1,\tau) = \Phi_k^b(y_1,\tau) + \mathscr{U}_{2,k}^\diamond(y_1,\tau),
\tag{3.21}
$$

the operator A_b is described by formula (2.14); the functions $\mathscr{U}_{2,k}^\diamond$ solve the problems

$$
\begin{cases}
\partial_\tau\mathscr{U}_{2,k}^\diamond - \nu\partial_1^2\mathscr{U}_{2,k}^\diamond = \mathscr{Z}_{2,k}^b, \quad |y_1| < \gamma_0,\\
\mathscr{U}_{2,k}^\diamond|_{|y_1|=\gamma_0} = 0, \quad \mathscr{U}_{2,k}^\diamond(y_1,0) = 0,
\end{cases}
\tag{3.22}
$$

while the Φ_k^b, s_k^b, $k = 1, 2, \dots$, are solutions to the problems

$$
\begin{cases}
\partial_\tau\Phi_k^b(y_1,\tau) - \nu\partial_1^2\Phi_k^b(y_1,\tau) := s_k^b(\tau), \quad |y_1| < \gamma_0,\\
\Phi_k^b(y_1,\tau)|_{|y_1|=\gamma_0} = 0, \quad \Phi_k^b(y_1,0) = -\mathscr{U}_{2,k}(y_1,0),\\
\int_{-\gamma_0}^{\gamma_0}\Phi_k^b(y_1,\tau)\,dy_1 = -\int_{-\gamma_0}^{\gamma_0}\mathscr{U}_{2,k}^\diamond(y_1,\tau)\,dy_1.
\end{cases}
\tag{3.23}
$$

Notice, that (3.23) is the inverse problem, the function $s_k^b(\tau)$ is not known and we have to find it in order to satisfy the flux condition $(3.23)_3$, i.e., the solution to problem (3.23) is the pair (Φ_k^b, s_k^b).

Finally, $g_k^b(\tau)$ are found from ODE's,

$$s_k^b(\tau) = -c_k g_k^b(\tau) + 2\lambda\tau \frac{dg_k^b(\tau)}{d\tau},$$

where $c_k = 1 - 3\lambda + k(\lambda - 1)$. Note, that by construction, $\int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}_{2,k}(y_1, t)\, dy_1 = 0$. Therefore, the solvability condition

$$\int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}_{2,k}^{\diamond}(y_1, 0)\, dy_1 = \int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}_{2,k}(y_1, 0)\, dy_1$$

holds automatically. Remind, that by assumption $\mu \neq 0$. Therefore, $1 - 3\lambda + k(\lambda - 1) \neq 0$ and we find

$$g_k^b(\tau) = \left(\frac{1}{2\lambda} \int\limits_0^\tau s_k^b(t) t^{-M_k - 1}\, dt \right) \tau^{M_k}, \quad \text{if } M_k > 0, \tag{3.24}$$

and

$$g_k^b(\tau) = \left(\frac{1}{2\lambda} \int\limits_\tau^\infty s_k^b(t) t^{-M_k - 1}\, dt \right) \tau^{M_k}, \quad \text{if } M_k < 0, \tag{3.25}$$

where $M_k = \frac{c_k}{2\lambda} \neq 0$.

Finally a compatibility condition for problem (3.20)

$$\left[\lambda A_b(y_1, \tau, \partial_1, \partial_\tau) - k(\lambda - 1)\right] \mathscr{U}_{2,k}^b(y_1, 0) = -\partial_1 \mathscr{U}_{1,k}(y_1, 0)$$

is satisfied automatically due to the construction.

3.2.2 Discrepancies. Case: $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$.

The pair of functions $(\mathbf{U}_k, Q_k)$ leaves in equations (3.1) discrepancies $\mathbf{H}_k = (H_{1,k}, H_{2,k})$:

$$\begin{aligned} \mathbf{H}_k(y, t, \tau) &= \widehat{\mathbf{F}}_{k-1}(y, t) + \widehat{\mathbf{F}}_k(y, t) + \mathbf{N}_k(y, t) \\ &+ \sum_{j=\max\{0, k - \frac{2}{\lambda - 1} - 1\}}^{k} \widetilde{\mathbf{F}}_j(y, t) + \sum_{j=1}^{k} \mathbf{F}_j(y, t) + \mathbf{F}_{k-1}^b(y, \tau) \\ &+ \mathbf{F}_k^b(y, \tau) + \mathbf{N}_k^b(y, \tau), \end{aligned} \tag{3.26}$$

where $k = 0, 1, \ldots$, $\widehat{\mathbf{F}}_{-1}(y, t) = 0$, $\mathbf{F}_{-1}^b(y, \tau) = 0$,

$$\begin{aligned} \widehat{\mathbf{F}}_k(y, t) &= \left(y_2^{-3 + k(\lambda - 1)} \widehat{\mathscr{F}}_{1,k}(y_1, t), y_2^{-\lambda - 2 + k(\lambda - 1)} \widehat{\mathscr{F}}_{2,k}(y_1, t) \right), \\ \mathbf{N}_k(y, t) &= \left(y_2^{-\lambda - 2 + k(\lambda - 1)} \mathscr{N}_{1,k}(y_1, t), y_2^{-2\lambda - 1 + k(\lambda - 1)} \mathscr{N}_{2,k}(y_1, t) \right), \\ \widetilde{\mathbf{F}}_k(y, t) &= \left(y_2^{-1 + k(\lambda - 1)} \widetilde{\mathscr{F}}_{1,k}(y_1, t), y_2^{-\lambda + k(\lambda - 1)} \widetilde{\mathscr{F}}_{2,k}(y_1, t) \right), \\ \mathbf{F}_k^b(y, \tau) &= \left(y_2^{-3 + k(\lambda - 1)} \mathscr{F}_{1,k}^b(y_1, \tau), y_2^{-\lambda - 2 + k(\lambda - 1)} \mathscr{F}_{2,k}^b(y_1, \tau) \right), \\ \mathbf{N}_k^b(y, \tau) &= \left(y_2^{-\lambda - 2 + k(\lambda - 1)} \mathscr{N}_{1,k}^b(y_1, \tau), y_2^{-2\lambda - 1 + k(\lambda - 1)} \mathscr{N}_{2,k}^b(y_1, \tau) \right), \end{aligned}$$

and $\sum\limits_{j=1}^{k} \mathbf{F}_j$ is the collection of terms belonging to L^2-space.

3.3 Case $\lambda = \frac{N+4}{N}$.

3.3.1 Outer asymptotics. Case $\lambda = \frac{N+4}{N}$.

As in the previous section, $\mathbf{Z}_k(y_1, y_2, t) = y_2^{2\lambda} \mathbf{H}_k^{\nabla}(y_1, y_2, t)$, where $\mathbf{H}_k^{\nabla}$ is equal to the most singular term in the discrepancies $\mathbf{H}_k$, $k = 1, 2, \ldots$, i.e., the function $\mathscr{Z}_k = (\mathscr{Z}_{1,k}, \mathscr{Z}_{2,k})$ is equal to the most singular term which we compensate at the step k, and is described by the following rule[8]

$$
\begin{aligned}
\mathbf{N}_0 \to \begin{cases} \widetilde{\mathbf{N}}_0 \to \widehat{\mathbf{F}}_0 + \mathbf{N}_1, \text{ if } \lambda > 3, \\ \widehat{\mathbf{F}}_0 + \mathbf{N}_1 \to \widetilde{\mathbf{N}}_0, \text{ if } \lambda < 3. \end{cases} \to \widehat{\mathbf{F}}_1 + \mathbf{N}_2 \to \cdots \to \\
\widehat{\mathbf{F}}_{\lfloor \frac{2}{\lambda-1} \rfloor} + \mathbf{N}_{\lfloor \frac{2}{\lambda-1} \rfloor+1} + \widetilde{\widetilde{N}}_{\lfloor \frac{2}{\lambda-1} \rfloor+1-\frac{4}{\lambda-1}} \to \widetilde{\mathbf{F}}_0 + \widetilde{\mathbf{N}}_1 \to \\
\widehat{\mathbf{F}}_{\lfloor \frac{2}{\lambda-1} \rfloor+1} + \mathbf{N}_{\lfloor \frac{2}{\lambda-1} \rfloor+2} + \widetilde{\widetilde{N}}_{\lfloor \frac{2}{\lambda-1} \rfloor+2-\frac{4}{\lambda-1}} \to \cdots \to \\
\widehat{\mathbf{F}}_{\lfloor \frac{2}{\lambda-1} \rfloor+j} + \mathbf{N}_{\lfloor \frac{2}{\lambda-1} \rfloor+1+j} + \widetilde{\widetilde{N}}_{\lfloor \frac{2}{\lambda-1} \rfloor+1-\frac{4}{\lambda-1}+j} \to \widetilde{\mathbf{F}}_j + \widetilde{\mathbf{N}}_{j+1} \to \\
\widehat{\mathbf{F}}_{\lfloor \frac{2}{\lambda-1} \rfloor+j+1} + \mathbf{N}_{\lfloor \frac{2}{\lambda-1} \rfloor+j+2} + \widetilde{\widetilde{N}}_{\lfloor \frac{2}{\lambda-1} \rfloor+1-\frac{4}{\lambda-1}+j+1} \to \cdots
\end{aligned}
\tag{3.27}
$$

where $\lfloor x \rfloor$ is the integer part of the number x, $j \in \mathbb{N}$, $\mathbf{N} = (\mathbf{U}\cdot\mathfrak{N})\mathbf{U}$, $\widetilde{\mathbf{N}} = (\widetilde{\mathbf{U}}\cdot\mathfrak{N})\mathbf{U} + (\mathbf{U}\cdot\mathfrak{N})\widetilde{\mathbf{U}}$, $\widetilde{\widetilde{\mathbf{N}}} = (\widetilde{\mathbf{U}}\cdot\mathfrak{N})\widetilde{\mathbf{U}}$ and

$$
\begin{aligned}
\widehat{\mathbf{F}}_k(y,t) &= \left(y_2^{-3+k(\lambda-1)}\widehat{\mathscr{F}}_{1,k}(y_1,t), y_2^{-\lambda-2+k(\lambda-1)}\widehat{\mathscr{F}}_{2,k}(y_1,t)\right), \\
\mathbf{N}_k(y,t) &= \left(y_2^{-\lambda-2+k(\lambda-1)}\mathscr{N}_{1,k}(y_1,t), y_2^{-2\lambda-1+k(\lambda-1)}\mathscr{N}_{2,k}(y_1,t)\right), \\
\widetilde{\mathbf{N}}_k(y,t) &= \left(y_2^{-\lambda+k(\lambda-1)}\widetilde{\mathscr{N}}_{1,k}(y_1,t), y_2^{1-2\lambda+k(\lambda-1)}\widetilde{\mathscr{N}}_{2,k}(y_1,t)\right), \\
\widetilde{\widetilde{\mathbf{N}}}_k(y,t) &= \left(y_2^{2-\lambda+k(\lambda-1)}\widetilde{\widetilde{\mathscr{N}}}_{1,k}(y_1,t), y_2^{3-2\lambda+k(\lambda-1)}\widetilde{\widetilde{\mathscr{N}}}_{2,k}(y_1,t)\right), \\
\widetilde{\mathbf{F}}_k(y,t) &= \left(y_2^{-1+k(\lambda-1)}\widetilde{\mathscr{F}}_{1,k}(y_1,t), y_2^{-\lambda+k(\lambda-1)}\widetilde{\mathscr{F}}_{2,k}(y_1,t)\right),
\end{aligned}
\tag{3.28}
$$

Since $\lambda \neq \frac{N+1}{N}$ and $\lambda \neq \frac{N+2}{N}$, $N = 1, 2, \ldots$, from (3.7) and Lemma 3.1 it follows that

$$
\begin{aligned}
U_1^{O,[J]}(y,t) = {}& y_2^{-1}\mathscr{U}_{1,0}(y_1,t) + \sum_{k=1}^{K} y_2^{-1+k(\lambda-1)}\mathscr{U}_{1,k}(y_1,t) \\
& + \sum_{k=1}^{L} y_2^{1+k(\lambda-1)}\widetilde{\mathscr{U}}_{1,k}(y_1,t), \\
U_2^{O,[J]}(y,t) = {}& \frac{F(t)}{\kappa_0}y_2^{-\lambda}\Phi(y_1) + \sum_{k=1}^{K} y_2^{-\lambda+k(\lambda-1)}\mathscr{U}_{2,k}(y_1,t) \\
& + \sum_{k=1}^{L} y_2^{-\lambda+2+k(\lambda-1)}\widetilde{\mathscr{U}}_{2,k}(y_1,t), \\
P^{O,[J]}(y,t) = {}& \frac{F(t)}{\kappa_0(1-3\lambda)}y_2^{1-3\lambda} + y_2^{-1-\lambda}\mathscr{Q}_0(y_1,t) \\
& + \sum_{k=1}^{K}\left[y_2^{1-3\lambda+k(\lambda-1)}g_k(t) + y_2^{-1-\lambda+k(\lambda-1)}\mathscr{Q}_k(y_1,t)\right] \\
& + \sum_{k=1}^{L}\left[y_2^{3-3\lambda+k(\lambda-1)}\tilde{g}_k(t) + y_2^{1-\lambda+k(\lambda-1)}\widetilde{\mathscr{Q}}_k(y_1,t)\right]
\end{aligned}
\tag{3.29}
$$

where $J \in \mathbb{N}$, $K = \min\left\{J, \left\lfloor \frac{J+\frac{2}{\lambda-1}}{2} \right\rfloor\right\}$, $L = \left\lfloor \frac{J-\frac{2}{\lambda-1}+1}{2} \right\rfloor$, the pair $(\mathscr{U}_{1,0}, \mathscr{Q}_0)$ solves problem (2.9), Φ is defined in (2.7), the functions $(\mathscr{U}_{1,k}, \mathscr{Q}_k)$ solve problems (3.12) with the right-hand sides $\widehat{\mathscr{F}}_{1,k-1} + \mathscr{N}_{1,k} + \widetilde{\widetilde{\mathscr{N}}}_{1,k-\frac{4}{\lambda-1}}$, the functions $\mathscr{U}_{2,k}^{\star}$ satisfy equations (3.13) with the right-hand sides $\widehat{\mathscr{F}}_{2,k-1} + \mathscr{N}_{2,k} + \widetilde{\widetilde{\mathscr{N}}}_{2,k-\frac{4}{\lambda-1}}$; and g_k are uniquely determined from the compatibility condition (3.14) and are given by either by (3.15) or (3.16).

The functions $(\widetilde{\mathscr{U}}_{1,k}, \widetilde{\mathscr{Q}}_k)$, $k = 1, 2, \ldots$, are solutions to the problems

$$
\begin{cases}
-\nu\partial_1^2\widetilde{\mathscr{U}}_{1,k} + \partial_1\widetilde{\mathscr{Q}}_k = \widetilde{\mathscr{F}}_{1,k-1} + \widetilde{\mathscr{N}}_{1,k}, \\
\partial_1\widetilde{\mathscr{U}}_{1,k} = A(3-3\lambda+k(\lambda-1))\widetilde{\mathscr{U}}_{2,k}, \\
\widetilde{\mathscr{U}}_{1,k}\big|_{|y_1|=\gamma_0} = 0,
\end{cases}
\tag{3.30}
$$

8 In this section we assume that $\lambda = \frac{N+4}{N}$, so $\frac{4}{\lambda-1} \in \mathbb{N}$.

$$\widetilde{\mathscr{U}}_{2,k}(y_1,t) = \widetilde{g}_k(t)(3-3\lambda+k(\lambda-1))\Phi(y_1) + \widetilde{\mathscr{U}}^{\star}_{2,k}(y_1,t),$$

$\widetilde{\mathscr{U}}^{\star}_{2,k}$ satisfy equations (3.13) with the right-hand sides $\widetilde{\mathscr{F}}_{2,k-1} + \widetilde{\mathscr{N}}_{2,k}$, functions $\widetilde{g}_k$ are uniquely determined from the solvability condition for problem (3.30) which is equivalent to the equation

$$\int_{-\gamma_0}^{\gamma_0} A(3-3\lambda+k(\lambda-1))\widetilde{\mathscr{U}}^{\star}_{2,k}(y_1,t)\,dy_1 = 0.$$

Remark 3.1. The functions $\widetilde{\mathscr{U}}_k$ in fact also produce some discrepancies. However, one part of it is already in L^2-space and the other part has the same powers of y_2 as discrepancies produced by $\mathscr{U}_k$. Therefore, in order to keep the notations as simple as possible, we do not write these terms explicitly in the scheme (3.27).

3.3.2 Boundary layer. Case $\lambda = \frac{N+4}{N}$.

In this case the outer asymptotic expansion (see (3.29)) includes both functions $\mathbf{U}_k(y,t)$ and $\widetilde{\mathbf{U}}_k(y,t)$. Therefore now we have to compensate both initial values $\mathbf{U}_k(y,0)$ and $\widetilde{\mathbf{U}}_k(y,0)$. Thus, the right-hand sides for the boundary layer problems are alternating in a similar way as the outer asymptotic ones: the functions $\mathbf{Z}^b_k(y,\tau)$, $k=1,2,\dots$, in (3.18) obey the following rule

$$\begin{gathered}
\mathbf{N}^b_0 \to \begin{cases} \widetilde{\mathbf{N}}^b_0 \to \widehat{\mathbf{F}}^b_0 + \mathbf{N}^b_1, & \text{if } \lambda > 3, \\ \widehat{\mathbf{F}}^b_0 + \mathbf{N}^b_1 \to \widetilde{\mathbf{N}}^b_0, & \text{if } \lambda < 3. \end{cases} \to \widehat{\mathbf{F}}^b_1 + \mathbf{N}^b_2 \to \dots \to \\
\widehat{\mathbf{F}}^b_{\lfloor\frac{2}{\lambda-1}\rfloor} + \mathbf{N}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1} + \widetilde{\widetilde{\mathbf{N}}}{}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1-\frac{4}{\lambda-1}} \to \widetilde{\mathbf{F}}^b_0 + \widetilde{\mathbf{N}}^b_1 \to \\
\widehat{\mathbf{F}}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1} + \mathbf{N}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+2} + \widetilde{\widetilde{\mathbf{N}}}{}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+2-\frac{4}{\lambda-1}} \to \dots \to \\
\widehat{\mathbf{F}}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+j} + \mathbf{N}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1+j} + \widetilde{\widetilde{\mathbf{N}}}{}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1-\frac{4}{\lambda-1}+j} \to \widetilde{\mathbf{F}}^b_j + \widetilde{\mathbf{N}}^b_{j+1} \to \\
\widehat{\mathbf{F}}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+j+1} + \mathbf{N}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+j+2} + \widetilde{\widetilde{\mathbf{N}}}{}^b_{\lfloor\frac{2}{\lambda-1}\rfloor+1-\frac{4}{\lambda-1}+j+1} \to \dots
\end{gathered} \tag{3.31}$$

where $j \in \mathbb{N}$, the functions $\widehat{\mathbf{F}}_k$, $\mathbf{N}_k$, $\widetilde{\mathbf{F}}_k$, $\widetilde{\mathbf{N}}_k$, $\widetilde{\widetilde{\mathbf{N}}}_k$ are described in (3.28) and

$$\begin{aligned}
\widehat{\mathbf{F}}^b_k(y,\tau) &= \left(y_2^{-3+k(\lambda-1)}\widehat{\mathscr{F}}^b_{1,k}(y_1,\tau),\, y_2^{-\lambda-2+k(\lambda-1)}\widehat{\mathscr{F}}^b_{2,k}(y_1,\tau)\right), \\
\mathbf{N}^b_k(y,\tau) &= \left(y_2^{-\lambda-2+k(\lambda-1)}\mathscr{N}^b_{1,k}(y_1,\tau),\, y_2^{-2\lambda-1+k(\lambda-1)}\mathscr{N}^b_{2,k}(y_1,\tau)\right), \\
\widetilde{\mathbf{N}}^b_k(y,\tau) &= \left(y_2^{-\lambda+k(\lambda-1)}\widetilde{\mathscr{N}}^b_{1,k}(y_1,\tau),\, y_2^{1-2\lambda+k(\lambda-1)}\widetilde{\mathscr{N}}^b_{2,k}(y_1,\tau)\right), \\
\widetilde{\widetilde{\mathbf{N}}}{}^b_k(y,\tau) &= \left(y_2^{2-\lambda+k(\lambda-1)}\widetilde{\widetilde{\mathscr{N}}}{}^b_{1,k}(y_1,\tau),\, y_2^{3-2\lambda+k(\lambda-1)}\widetilde{\widetilde{\mathscr{N}}}{}^b_{2,k}(y_1,\tau)\right), \\
\widetilde{\mathbf{F}}^b_k(y,\tau) &= \left(y_2^{-1+k(\lambda-1)}\widetilde{\mathscr{F}}^b_{1,k}(y_1,\tau),\, y_2^{-\lambda+k(\lambda-1)}\widetilde{\mathscr{F}}^b_{2,k}(y_1,\tau)\right).
\end{aligned} \tag{3.32}$$

Remark 3.2. Here, as in the previous section, we expand terms $(\mathbf{U}\cdot\mathfrak{N}_b)\mathbf{U}^b$, $(\mathbf{U}^b\cdot\mathfrak{N}_b)\mathbf{U}$, $(\widetilde{\mathbf{U}}\cdot\mathfrak{N}_b)\mathbf{U}^b$, $(\mathbf{U}^b\cdot\mathfrak{N}_b)\widetilde{\mathbf{U}}$, $(\mathbf{U}\cdot\mathfrak{N}_b)\widetilde{\mathbf{U}}^b$, $(\widetilde{\mathbf{U}}^b\cdot\mathfrak{N}_b)\mathbf{U}$, $(\widetilde{\mathbf{U}}\cdot\mathfrak{N}_b)\widetilde{\mathbf{U}}^b$, $(\widetilde{\mathbf{U}}^b\cdot\mathfrak{N}_b)\widetilde{\mathbf{U}}$ in Taylor's series with respect to the time variable t and then replace t in this expansion by the product $\tau y_2^{2\lambda}$. In order to keep notations as simple as possible, we do not write out these terms explicitly. We collect the terms having the same power of y_2 and denote the corresponding sums by $\mathbf{N}$, $\widetilde{\mathbf{N}}$, $\widetilde{\widetilde{\mathbf{N}}}$.

We look for the boundary layer asymptotic expansion in the form:

$$
\begin{aligned}
U_1^{B,[J]}(y,\tau) &= \sum_{k=0}^{K} y_2^{-1+k(\lambda-1)}\mathscr{U}_{1,k}^b(y_1,\tau) + \sum_{k=1}^{L} y_2^{1+k(\lambda-1)}\widetilde{\mathscr{U}}_{1,k}^b(y_1,\tau),\\
U_2^{B,[J]}(y,\tau) &= \sum_{k=0}^{K} y_2^{-\lambda+k(\lambda-1)}\mathscr{U}_{2,k}^b(y_1,\tau) + \sum_{k=1}^{L} y_2^{2-\lambda+k(\lambda-1)}\widetilde{\mathscr{U}}_{2,k}^b(y_1,\tau),\\
P^{B,[J]}(y,\tau) &= \sum_{k=0}^{K} \Big[y_2^{1-3\lambda+k(\lambda-1)}g_k^b(\tau) + y_2^{-1-\lambda+k(\lambda-1)}\mathscr{Q}_k^b(y_1,\tau)\Big]\\
&+\sum_{k=1}^{L} \Big[y_2^{3-3\lambda+k(\lambda-1)}\tilde{g}_k^b(\tau) + y_2^{1-\lambda+k(\lambda-1)}\widetilde{\mathscr{Q}}_k^b(y_1,\tau)\Big]
\end{aligned}
\tag{3.33}
$$

where $(\mathscr{U}_{1,k}^b, \mathscr{Q}_k^b)$, $k = 1, 2, \dots$, are solutions to problems (3.20) with the corresponding right-hand sides described in (3.31), the functions $\mathscr{U}_{2,k}^b$ are described by (3.21) with $\mathscr{U}_{2,k}^\diamond$ being the solutions to problems (3.22) with right-hand sides described in the scheme (3.31) and the functions (Φ_k^b, s_k^b) solve inverse problems (3.23). Since by assumption $1 - 3\lambda + k(\lambda - 1) \neq 0$, the functions g_k^b are gyven by (3.24) or (3.25).

The functions $\widetilde{\mathscr{U}}_{1,k}^b$, $\widetilde{\mathscr{Q}}_k^b$, $k = 1, 2, \dots$, are solutions to

$$
\begin{cases}
\partial_\tau\widetilde{\mathscr{U}}_{1,k}^b - \nu\partial_1^2\widetilde{\mathscr{U}}_{1,k}^b + \partial_1\widetilde{\mathscr{Q}}_k^b = \mathscr{L}_{1,k}^b, \quad |y_1| < \gamma_0,\\
\partial_1\widetilde{\mathscr{U}}_{1,k}^b = \big[\lambda A_b(y_1,\tau,\partial_1,\partial_\tau) - k(\lambda-1)\big]\,\widetilde{\mathscr{U}}_{2,k}^b,\\
\widetilde{\mathscr{U}}_{1,k}^b|_{|y_1|=\gamma_0} = 0, \quad \widetilde{\mathscr{U}}_{1,k}^b(y_1,0) = -\widetilde{\mathscr{U}}_{1,k}(y_1,0).
\end{cases}
\tag{3.34}
$$

The right-hand sides in (3.34) are functions from the scheme (3.31) corresponding to terms with "$\sim$",

$$
\widetilde{\mathscr{U}}_{2,k}^b(y_1,\tau) = \widetilde{\Phi}_k^b(y_1,\tau) + \widetilde{\mathscr{U}}_{2,k}^\diamond(y_1,\tau),
$$

the operator A_b is described by formula (2.14); the functions $\widetilde{\mathscr{U}}_{2,k}^\diamond$ satisfy the equations

$$
\begin{cases}
\partial_\tau\widetilde{\mathscr{U}}_{2,k}^\diamond - \nu\partial_1^2\widetilde{\mathscr{U}}_{2,k}^\diamond = \mathscr{L}_{2,k}^b, \quad |y_1| < \gamma_0,\\
\widetilde{\mathscr{U}}_{2,k}^\diamond|_{|y_1|=\gamma_0} = 0, \quad \widetilde{\mathscr{U}}_{2,k}^\diamond(y_1,0) = 0,
\end{cases}
\tag{3.35}
$$

while the functions $(\widetilde{\Phi}_k^b, \tilde{s}_k^b)$, $k = 1, 2, \dots$, are solutions to the inverse problems

$$
\begin{cases}
\partial_\tau\widetilde{\Phi}_k^b(y_1,\tau) - \nu\partial_1^2\widetilde{\Phi}_k^b(y_1,\tau) := \tilde{s}_k^b(\tau), \quad |y_1| < \gamma_0,\\
\widetilde{\Phi}_k^b(y_1,\tau)|_{|y_1|=\gamma_0} = 0, \quad \widetilde{\Phi}_k^b(y_1,0) = -\widetilde{\mathscr{U}}_{2,k}(y_1,0),\\
\int\limits_{-\gamma_0}^{\gamma_0} \widetilde{\Phi}_k^b(y_1,\tau)\,dy_1 = -\int\limits_{-\gamma_0}^{\gamma_0} \widetilde{\mathscr{U}}_{2,k}^\diamond(y_1,\tau)\,dy_1.
\end{cases}
\tag{3.36}
$$

Finally, the functions $\tilde{g}_k^b(\tau)$ are solutions to the following ODE's

$$
\tilde{s}_k^b(\tau) = -c_k\tilde{g}_k^b(\tau) + 2\lambda\tau\frac{d\tilde{g}_k^b(\tau)}{d\tau}
$$

with $c_k = 1 - 3\lambda + k(\lambda - 1)$. Notice, that by construction, $\int\limits_{-\gamma_0}^{\gamma_0} \widetilde{\mathscr{U}}_{n,k}(y_1,t)\,dy_1 = 0$. Therefore, the solvability condition

$$
\int\limits_{-\gamma_0}^{\gamma_0} \widetilde{\mathscr{U}}_{n,k}^\diamond(y_1,0)\,dy_1 = \int\limits_{-\gamma_0}^{\gamma_0} \widetilde{\mathscr{U}}_{n,k}(y_1,0)\,dy_1
$$

holds automatically. Remind, that by assumption, $\mu \neq 0$. Therefore, $1 - 3\lambda + k(\lambda - 1) \neq 0$ and

$$
\tilde{g}_k^b(\tau) = \Big(\tfrac{1}{2\lambda}\int\limits_0^\tau \tilde{s}_k^b(t)t^{-M_k-1}\,dt\Big)\,\tau^{M_k}, \quad \text{if } M_k > 0,
$$

and

$$
\tilde{g}_k^b(\tau) = \Big(\tfrac{1}{2\lambda}\int\limits_\tau^\infty \tilde{s}_k^b(t)t^{-M_k-1}\,dt\Big)\,\tau^{M_k}, \quad \text{if } M_k < 0,
$$

where $M_k = \frac{c_k}{2\lambda} \neq 0$.

Compatibility condition for problem (3.34)

$$\left[\lambda A_b(y_1, \tau, \partial_1, \partial_\tau) - k(\lambda - 1)\right] \widetilde{\mathscr{U}}^b_{2,k}(y_1, 0) = -\partial_1 \widetilde{\mathscr{U}}_{1,k}(y_1, 0)$$

is satisfied automatically.

3.3.3 Discrepancies. Case $\lambda = \frac{N+4}{N}$.

Let us denote the elements of sequence (3.27) by $\mathbf{S}_k$, $k = 0, 1, 2, \dots$, i.e., $\mathbf{S}_0 = \mathbf{N}_0$ and so on, and by $\mathbf{S}^b_k$ the elements of sequence (3.31). Then the discrepancies $\mathbf{H}_k = (H_{1,k}, H_{2,k})$ left by the functions $(\mathbf{U}_k, Q_k)$ in equations (3.1) can be written in the form

$$\begin{gathered}\mathbf{H}_k(y, t, \tau) = \sum_{i=0}^{k} \Big(\widehat{\mathbf{F}}_i(y, t) + \mathbf{N}_i(y, t) + \widetilde{\mathbf{F}}_i(y, t) + \widetilde{\mathbf{N}}_i(y, t) + \\ \widetilde{\widetilde{\mathbf{N}}}_i(y, t) + \widehat{\mathbf{F}}^b_i(y, \tau) + \mathbf{N}^b_i(y, \tau) + \widetilde{\mathbf{F}}^b_i(y, \tau) + \widetilde{\mathbf{N}}^b_i(y, \tau) + \widetilde{\widetilde{\mathbf{N}}}{}^b_i(y, \tau) \Big) \\ - \sum_{i=0}^{k-1} \left(\mathbf{S}_i + \mathbf{S}^b_i \right) + \mathbf{F}_k,\end{gathered} \tag{3.37}$$

where $\widehat{\mathbf{F}}_k$, $\mathbf{N}_k$, $\widetilde{\mathbf{F}}_k$, $\widetilde{\mathbf{N}}_k$, $\widetilde{\widetilde{\mathbf{N}}}_k$, $\widehat{\mathbf{F}}^b_k$, $\mathbf{N}^b_k$, $\widetilde{\mathbf{F}}^b_k$, $\widetilde{\mathbf{N}}^b_k$, $\widetilde{\widetilde{\mathbf{N}}}{}^b_k$ are described in (3.28) and (3.32); the functions $\mathbf{F}_k$ belong to the L^2-space. Formula (3.37) means that we sum up all the discrepancies and then subtract the discrepancies which are already compensated.

Note that the most singular term in (3.37) is equivalent to

$$\left(y_2^{-3+k(\lambda-1)} \mathscr{H}_1(y_1, t, \tau),\ y_2^{-\lambda-2+k(\lambda-1)} \mathscr{H}_2(y_1, t, \tau) \right).$$

3.4 Other value of the parameter λ

3.4.1 Outer asymptotics

If $\lambda \neq \frac{N+1}{N}$, $\lambda \neq \frac{N+2}{N}$ and $\lambda \neq \frac{N+4}{N}$, $N = 1, 2, \dots$, then from (3.7) and Lemma 3.1 we get

$$\begin{aligned} U_1^{O,[I,J,K]}(y, t) = &\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} y_2^{-1+2i+2j\lambda+k(\lambda-1)} \mathscr{U}_{1,\{i,j,k\}}(y_1, t), \\ U_2^{O,[I,J,K]}(y, t) = &\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} y_2^{-\lambda+2i+2j\lambda+k(\lambda-1)} \mathscr{U}_{2,\{i,j,k\}}(y_1, t), \\ P^{O,[I,J,K]}(y, t) = &\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \Big[y_2^{1-3\lambda+2i+2j\lambda+k(\lambda-1)} g_{\{i,j,k\}}(t) \\ &+ y_2^{-1-\lambda+2i+2j\lambda+k(\lambda-1)} \mathscr{Q}_{\{i,j,k\}}(y_1, t) \Big], \end{aligned} \tag{3.38}$$

where the functions $(\mathscr{U}_{1,\{i,j,k\}}, \mathscr{Q}_{\{i,j,k\}})$ solve the problems

$$\begin{cases} -\nu \partial_1^2 \mathscr{U}_{1,\{i,j,k\}}(y_1, t) + \partial_1 \mathscr{Q}_{\{i,j,k\}}(y_1, t) = \mathscr{L}_{1,\{i,j,k\}}(y_1, t), \\ \partial_1 \mathscr{U}_{1,\{i,j,k\}}(y_1, t) = -A(\Theta_{\{i,j,k\}}) \mathscr{U}_{2,\{i,j,k\}}(y_1, t), \\ \mathscr{U}_{1,\{i,j,k\}}(y_1, t)\big|_{|y_1| = \gamma_0} = 0. \end{cases} \tag{3.39}$$

$A(\Theta_{\{i,j,k\}})$ is defined in (3.9), $\Theta_{\{i,j,k\}} = 1 - 3\lambda + 2i + 2j\lambda + k(\lambda - 1)$,

$$\mathscr{U}_{2,\{i,j,k\}}(y_1, t) = g_{\{i,j,k\}}(t) \Theta_{\{i,j,k\}} \Phi(y_1) + \mathscr{U}^{\star}_{2,\{i,j,k\}}(y_1, t),$$

the functions $\mathscr{U}^{\star}_{2,\{i,j,k\}}$ satisfy equations

$$\begin{cases} -\nu\partial_1^2\mathscr{U}^{\star}_{2,\{i,j,k\}}(y_1,t) = \mathscr{Z}_{2,\{i,j,k\}}(y_1,t), & |y_1| < \gamma_0, \\ \mathscr{U}^{\star}_{2,\{i,j,k\}}(y_1,t)|_{|y_1|=\gamma_0} = 0. \end{cases} \tag{3.40}$$

The functions $g_{\{i,j,k\}}$ are uniquely determined from the following solvability condition for problem (3.39)

$$\int_{-\gamma_0}^{\gamma_0} A(\Theta_{\{i,j,k\}})\mathscr{U}_{2,\{i,j,k\}}(y_1,t)\,dy_1 = 0.$$

Similarly as above,

$$g_{\{i,j,k\}}(t) = -\frac{1}{\kappa_0\Theta_{\{i,j,k\}}}\int_{-\gamma_0}^{\gamma_0} \mathscr{U}^{\star}_{2,\{i,j,k\}}(y_1,t)\,dy_1,$$

$k = 1, 2, \dots$. Note that the condition $\mu \neq 0$ is equivalent to $1 - 3\lambda + 2i + 2j\lambda + k(\lambda - 1) \neq 0$. The right-hand sides $\mathbf{Z}_{i,j,k}(y_1, y_2, t) = y_2^{2\lambda}\mathbf{H}^{\triangledown}_{i,j,k}(y_1, y_2, t)$, where $\mathbf{H}^{\triangledown}_{i,j,k}$ are equal to the most singular terms which we compensate at the given step $\{i, j, k\}$, $i, j, k = 1, 2, \dots$.

3.4.2 Boundary layer

We look for the boundary layer asymptotic expansion in the form:

$$\begin{aligned} U_1^{B,[I,J,K]}(y,\tau) = {} & \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} y_2^{-1+2i+2j\lambda+k(\lambda-1)}\mathscr{U}^{b}_{1,\{i,j,k\}}(y_1,\tau), \\ U_2^{B,[I,J,K]}(y,\tau) = {} & \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} y_2^{-\lambda+2i+2j\lambda+k(\lambda-1)}\mathscr{U}^{b}_{2,\{i,j,k\}}(y_1,\tau), \\ P^{B,[I,J,K]}(y,\tau) = {} & \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K} \Big[y_2^{1-3\lambda+2i+2j\lambda+k(\lambda-1)}g^{b}_{\{i,j,k\}}(\tau) \\ & + y_2^{-1-\lambda+2i+2j\lambda+k(\lambda-1)}\mathscr{Q}^{b}_{\{i,j,k\}}(y_1,\tau)\Big], \end{aligned} \tag{3.41}$$

where $(\mathscr{U}^{b}_{1,\{i,j,k\}}, \mathscr{Q}^{b}_{\{i,j,k\}})$, $i,\ j,\ k = 0, 1, \dots$, are solutions to

$$\begin{cases} \partial_\tau\mathscr{U}^{b}_{1,\{i,j,k\}} - \nu\partial_1^2\mathscr{U}^{b}_{1,\{i,j,k\}} + \partial_1\mathscr{Q}^{b}_{\{i,j,k\}} = \mathscr{Z}^{b}_{1,\{i,j,k\}}, & |y_1| < \gamma_0, \\ \partial_1\mathscr{U}^{b}_{1,\{i,j,k\}} = \left[\lambda A_b(y_1,\tau,\partial_1,\partial_\tau) - 2i - 2j\lambda - k(\lambda-1)\right]\mathscr{U}^{b}_{2,\{i,j,k\}}, \\ \mathscr{U}^{b}_{1,\{i,j,k\}}|_{|y_1|=\gamma_0} = 0, \quad \mathscr{U}^{b}_{1,\{i,j,k\}}(y_1,0) = -\mathscr{U}_{1,\{i,j,k\}}(y_1,0), \end{cases} \tag{3.42}$$

$$\mathscr{U}^{b}_{2,\{i,j,k\}}(y_1,\tau) = \Phi^{b}_{\{i,j,k\}}(y_1,\tau) + \mathscr{U}^{\diamond}_{2,\{i,j,k\}}(y_1,\tau),$$

operator A_b is described by formula (2.14); the functions $\mathscr{U}^{\diamond}_{2,\{i,j,k\}}$ satisfy the equations

$$\begin{cases} \partial_\tau\mathscr{U}^{\diamond}_{2,\{i,j,k\}} - \nu\partial_1^2\mathscr{U}^{\diamond}_{2,\{i,j,k\}} = \mathscr{Z}^{b}_{2,\{i,j,k\}}, \\ \mathscr{U}^{\diamond}_{2,\{i,j,k\}}|_{|y_1|=\gamma_0} = 0, \quad \mathscr{U}^{\diamond}_{2,\{i,j,k\}}(y_1,0) = 0. \end{cases} \tag{3.43}$$

The right-hand sides $\mathbf{Z}^{b}_{\{i,j,k\}}(y_1, y_2, \tau) = y_2^{2\lambda}\mathbf{H}^{\triangledown}_{\{i,j,k\}}(y_1, y_2, \tau)$, where $\mathbf{H}^{\triangledown}_{\{i,j,k\}}$ is equal to the most singular terms which we compensate at the step $\{i, j, k\}$, $i, j, k = 1, 2, \dots$.

Further, the functions $\left(\Phi^{b}_{\{i,j,k\}}, s^{b}_{\{i,j,k\}}\right)$ are solutions of the inverse problems

$$\begin{cases} \partial_\tau\Phi^{b}_{\{i,j,k\}}(y_1,\tau) - \nu\partial_1^2\Phi^{b}_{\{i,j,k\}}(y_1,\tau) = s^{b}_{\{i,j,k\}}(\tau), & |y_1| < \gamma_0, \\ \Phi^{b}_{\{i,j,k\}}(y_1,\tau)|_{|y_1|=\gamma_0} = 0, \\ \Phi^{b}_{\{i,j,k\}}(y_1,0) = -\mathscr{U}_{2,\{i,j,k\}}(y_1,0), \\ \int_{-\gamma_0}^{\gamma_0} \Phi^{b}_{\{i,j,k\}}(y_1,\tau)\,dy_1 = -\int_{-\gamma_0}^{\gamma_0} \mathscr{U}^{\diamond}_{2,\{i,j,k\}}(y_1,\tau)\,dy_1. \end{cases} \tag{3.44}$$

Finally, $g^b_{\{i,j,k\}}(\tau)$ are found as solutions to ODEs

$$s^b_{\{i,j,k\}}(\tau) = -c_{\{i,j,k\}} g^b_{\{i,j,k\}}(\tau) + 2\lambda\tau \frac{dg^b_{\{i,j,k\}}(\tau)}{d\tau},$$

where $c_{\{i,j,k\}} = 1 - 3\lambda + 2i + 2j\lambda + k(\lambda - 1)$. Since $\int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}_{2,\{i,j,k\}}(y_1, \tau)\, dy_1 = 0$, the solvability condition

$$\int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}^{\diamond}_{2,\{i,j,k\}}(y_1, 0)\, dy_1 = \int\limits_{-\gamma_0}^{\gamma_0} \mathscr{U}_{2,\{i,j,k\}}(y_1, 0)\, dy_1$$

is satisfied. Moreover,

$$g^b_{\{i,j,k\}}(\tau) = \left(\frac{1}{2\lambda} \int\limits_0^{\tau} s^b_{\{i,j,k\}}(t) t^{-M_{\{i,j,k\}}-1}\, dt \right) \tau^{M_{\{i,j,k\}}}, \quad \text{if } M_{\{i,j,k\}} > 0,$$

and

$$g^b_{\{i,j,k\}}(\tau) = \left(\frac{1}{2\lambda} \int\limits_{\tau}^{\infty} s^b_{\{i,j,k\}}(t) t^{-M_{\{i,j,k\}}-1}\, dt \right) \tau^{M_{\{i,j,k\}}}, \quad \text{if } M_{\{i,j,k\}} \le 0,$$

where $M_{\{i,j,k\}} = \frac{c_{\{i,j,k\}}}{2\lambda}$.

Compatibility condition for problem (3.42)

$$\left[\lambda A_b(y_1, \tau, \partial_1, \partial_\tau) - 2i - 2j\lambda - k(\lambda - 1)\right] \mathscr{U}^b_{2,\{i,j,k\}} = -\partial_1 \mathscr{U}_{1,\{i,j,k\}}(y_1, 0)$$

is satisfied automatically.

3.4.3 Discrepancies

Functions $\mathbf{U}_{\{I,J,K\}}$, $Q_{\{I,J,K\}}$ leave in equations (3.1) the following discrepancies $\mathbf{H}_{\{I,J,K\}} = (H_{1,\{I,J,K\}}, H_{2,\{I,J,K\}})$:

$$\begin{aligned} \mathbf{H}_{\{I,J,K\}}(y, t, \tau) = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \Big[& \nu \mathfrak{D}^2 \mathbf{U}_{\{i,j,k\}}(y, t) - \partial_t \mathbf{U}_{\{i,j,k\}}(y, t) \\ & -(\mathbf{U}_{\{i,j,k\}} \cdot \mathfrak{N}) \mathbf{U}_{\{i,j,k\}}(y, t) + \nu \mathfrak{D}^2 \mathbf{U}^b_{\{i,j,k\}}(y, \tau) \\ & -(\mathbf{U}^b_{\{i,j,k\}} \cdot \mathfrak{N}_b) \mathbf{U}^b_{\{i,j,k\}}(y, \tau) - \left[\mathbf{T}_{\{i,j,k\}} + \mathbf{T}^b_{\{i,j,k\}}\right](y, 0, \tau) \\ & -\mathbf{S}_{\{I,J,K\}}(y, t) - \mathbf{S}^b_{\{I,J,K\}}(y, \tau) \Big], \end{aligned} \tag{3.45}$$

where $\mathbf{S}_{\{I,J,K\}}$, $\mathbf{S}^b_{\{I,J,K\}}$ are the sums of all already compensated terms for outer and boundary layers parts, respectively.

As before, we expand the terms $(\mathbf{U}_{\{i,j,k\}} \cdot \mathfrak{N}_b)\mathbf{U}^b_{\{i,j,k\}}$, $(\mathbf{U}^b_{\{i,j,k\}} \cdot \mathfrak{N}_b)\mathbf{U}_{\{i,j,k\}}$ in Taylor's series with respect to the time variable t and then replace t by the product $\tau y_2^{2\lambda}$. These expansions are denoted by $\mathbf{T}_{\{i,j,k\}}$ and $\mathbf{T}^b_{\{i,j,k\}}$.

The most singular term in formula (3.45) can be written in the form

$$\left(y_2^{-3+2i+2j\lambda+k(\lambda-1)} \mathscr{H}_{1,}(y_1, t, \tau),\ y_2^{-\lambda+2(i-1)+2j\lambda+k(\lambda-1)} \mathscr{H}_2(y_1, t, \tau) \right).$$

4 Regularity, existence and estimates

4.1 Regularity conditions

Consider the asymptotic expansion

$$\begin{aligned} \mathbf{U}^{[J]}(x,t) &= \mathbf{U}^{O,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t\right) + \mathbf{U}^{B,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, \frac{t}{x_2^{2\lambda}}\right), \\ P^{[J]}(x,t) &= P^{O,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t\right) + P^{B,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, \frac{t}{x_2^{2\lambda}}\right), \end{aligned} \tag{4.1}$$

where $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is the outer asymptotic expansion given by formula (3.11) if $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$ (by formula (3.29) if $\lambda = \frac{N+4}{N}$ and by formula (3.38) if the parameter λ has other value); $(\mathbf{U}^{B,[J]}, P^{B,[J]})$ is the boundary layer-in-time expansion given by (3.19) if $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$ (by (3.33) if $\lambda = \frac{N+4}{N}$ and by (3.41) if parameter λ has other value). $(\mathbf{U}^{[J]}, P^{[J]})$ is an approximate solution of problem (1.1) and the corresponding discrepancies $\mathbf{H}_J(y', y_n, t, \tau)$ are given by formulas (3.26), (3.37), (3.45). Constructing the above asymptotic representations we were solving problems (3.12), (3.13), (3.20), (3.22), (3.30), (3.34), (3.35), (3.36), (3.39), (3.40), (3.42), (3.43), (3.44). Therefore, it is necessary to have at each step sufficient regularity of the data which is needed for the solvability of the corresponding problems. Examining the right-hand sides of these problems we see the loss of one time derivative on each step of the outer asymptotic construction. Therefore, in order to ensure the existence of all terms of asymptotic expansion up to the order J, we have to assume that the flux

$$F(t) \in W^{J+1,2}(0,T).$$

Since the flux $F(t)$ is the integral of the normal component of the boundary value $\mathbf{a}(x,t)$ over $\partial\Omega$, the last requirement imply, the following regularity conditions for $\mathbf{a}$:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0,T; W^{1/2,2}(\partial\Omega)), \quad l = 0, 1, 2, \dots, J+1.$$

The boundary layer construction does not cause any loss of regularity, and it is enough to suppose that $\mathbf{b} \in W^{1,2}(\Omega)$.

Note that these regularity conditions are the same as for the construction of the asymptotic expansion for the non-stationary Stokes problem in a power cusp domains (see [23]).

4.2 Estimates of asymptotic decomposition

Let us first formally summarize the types of problems we were dealing with while constructing asymptotics. As for the outer asymptotic part, we faced with the following problems:

$$\begin{cases} -\nu\partial_1^2 U_2(y_1) = Z_2(y_1), \\ U_2(-\gamma_0) = U_2(\gamma_0) = 0, \end{cases} \tag{T1}$$

and

$$\begin{cases} -\nu\partial_1^2 U_1(y_1) - \partial_1 Q(y_1) = Z_1(y_1), \\ \partial_1 U_1(y_1) = G(y_1), \\ U_1(-\gamma_0) = U_1(\gamma_0) = 0, \end{cases} \tag{T2}$$

where the solvability condition $\int\limits_{-\gamma_0}^{\gamma_0} G(y_1)\,dy_1 = 0$ is satisfied.

Problems of type (T1) have explicit solutions

$$U_2(y_1) = ay_1 + b - \frac{1}{\nu}\int\limits_{-\gamma_0}^{y_1} Z_2(s)\,ds, \tag{4.2}$$

where a and b are found to satisfy boundary conditions.

From $(\mathrm{T2})_2$ we find that

$$U_1(y_1) = \int_{-\gamma_0}^{y_1} G(s)\,ds, \tag{4.3}$$

and, therefore,

$$Q(y_1) = -\nu G(y_1) - \int_{-\gamma_0}^{y_1} Z_1(s)\,ds. \tag{4.4}$$

As for the boundary layer construction, we meet three types of problems:

$$\begin{cases} \partial_\tau \Phi(y_1,\tau) - \nu\partial_1^2\Phi(y_1,\tau) = s(\tau), \\ \Phi|_{|y_1|=\gamma_0} = 0, \quad \Phi(y_1,0) = u_2^b(y_1), \\ \int_{-\gamma_0}^{\gamma_0} \Phi(y_1\tau)\,dy_1 = \mathcal{F}(\tau), \end{cases} \tag{T3}$$

where the flux $\mathcal{F}(t)$ and the initial data $u_2^b(y_1)$ satisfy the conditions $\mathcal{F}(0) = 0$ and $\int_{-\gamma_0}^{\gamma_0} u_2^b\,dy_1 = 0$. Notice that the necessary compatibility condition for problem (T3) is $\mathcal{F}(0) = \int_{-\gamma_0}^{\gamma_0} u_2^b\,dy_1$;

$$\begin{cases} \partial_\tau U_1(y_1,\tau) - \nu\partial_1^2 U_1(y_1,\tau) + \partial_1 Q(y_1,\tau) = Z_1(y_1,\tau), \\ \partial_1 U_1(y_1,\tau) = G(y_1,\tau), \\ U_1(-\gamma_0,\tau) = U_1(\gamma_0,\tau) = 0, \\ U_1(y_1,0) = u_1^b, \end{cases} \tag{T4}$$

with two compatibility conditions $\int_{-\gamma_0}^{\gamma_0} G(y_1,\tau)\,dy_1 = 0$ and $G(y_1,0) = \partial_1 u_1^b(y_1)$ that are satisfied due to the construction; and

$$\begin{cases} \partial_\tau U_2(y_1,\tau) - \nu\partial_1^2 U_2(y_1,\tau) = Z_2(y_1,\tau), \\ U_2(-\gamma_0,\tau) = U_2(\gamma_0,\tau) = 0, \\ U_2(y_1,0) = 0. \end{cases} \tag{T5}$$

Results concerning the regularity and estimates of solutions of boundary layer problems (T3), (T4), (T5) follow either from classical results concerning heat and Stokes equations, or from results about inverse problems.

Problem (T5) is the initial boundary value problem (with zero initial value) for the classical heat equation and its solution satisfies the estimates (e.g., [33])

$$\begin{aligned} \sup_{\tau\in[0,\infty)} \|U_2(\cdot,\tau)\|_{W^{1,2}(\Upsilon)} + \|U_2\|_{L^2(0,\infty;W^{2,2}(\Upsilon))} + \\ + \|\partial_\tau U_2\|_{L^2(0,\infty;L^2(\Upsilon))} \le C\|Z_2\|_{L^2(0,\infty;L^2(\Upsilon))}, \end{aligned} \tag{4.5}$$

where $\Upsilon = (-\gamma_0,\gamma_0)$. If in addition $\partial_\tau Z_2 \in L^2(0,\infty;L^2(\Upsilon))$, then $U_2 \in L^\infty(0,\infty;W^{2,2}(\Upsilon))$, $\partial_\tau U_2 \in L^\infty(0,\infty;L^2(\Upsilon))$ and

$$\begin{aligned} \sup_{\tau\in[0,\infty)} \|U_2(\cdot,\tau)\|_{W^{2,2}(\Upsilon)} + \sup_{\tau\in[0,\infty)} \|\partial_\tau U_2\|_{L^2(\Upsilon)} + \\ \le C\Big(\|Z_2\|_{L^2(0,\infty;L^2(\Upsilon))} + \|\partial_\tau Z_2\|_{L^2(0,\infty;L^2(\Upsilon))}\Big). \end{aligned} \tag{4.6}$$

If the right-hand side Z_2 of (T5) exponentially vanishes as $\tau \to \infty$, then the solution U_2 also exponentially vanishes and

$$\begin{aligned} \sup_{\tau\in[0,\infty)} \Big(e^{2\mu t}\|U_2(\cdot,\tau)\|^2_{W^{1,2}(\Upsilon)}\Big) + \|e^{\mu t}U_2\|^2_{L^2(0,\infty;W^{1,2}(\Upsilon))} \\ + \|e^{\mu t}\partial_\tau U_2\|^2_{L^2(0,\infty;L^2(\Upsilon)} \le C\|e^{\mu t}Z\|^2_{L^2(0,\infty;L^2(\Upsilon))}, \end{aligned} \tag{4.7}$$

where $\mu \in (0,\mu_*)$ with sufficiently small μ_*.

Problem (T4) is a nonstationary 1-dimensional problem of the Stokes type; the corresponding existence theory is well known (e.g., [34]), problem (T4) admits a unique weak solution $U_1 \in L^2(0,\infty;\mathring{W}^{1,2}(\Upsilon)\cap W^{2,2}(\Upsilon))$ with $\partial_\tau U_1 \in L^2(0,\infty;L^2(\Upsilon))$, $\partial_1 Q \in L^2(0,\infty;L^2(\Upsilon))$ such that

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\|U_1(\cdot,\tau)\|_{W^{1,2}(\Upsilon)}+\|U_1\|_{L^2(0,\infty;W^{2,2}(\Upsilon))}+\\&\quad+\|\partial_\tau U_1\|_{L^2(0,\infty;L^2(\Upsilon))}+\|\partial_1 Q\|_{L^2(0,\infty;L^2(\Upsilon))}\\&\le C\Big(\|Z_1\|_{L^2(0,\infty;L^2(\Upsilon))}+\|G\|_{L^2(0,\infty;W^{1,2}(\Upsilon))}+\|u_1^b\|_{W^{1,2}(\Upsilon)}\Big).\end{aligned}\tag{4.8}$$

If, in addition $u_1^b \in W^{2,2}(\Upsilon)$, $\partial_\tau G \in L^2(0,\infty;L^2(\Upsilon))$, $\partial_\tau Z_1 \in L^2(0,\infty;L^2(\Upsilon))$, then

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\|U_1(\cdot,\tau)\|_{W^{2,2}(\Upsilon)}+\sup_{\tau\in[0,\infty)}\|\partial_\tau U_1\|_{L^2(\Upsilon)}\\&\le C\Big(\|Z_1\|_{W^{1,2}(0,\infty;L^2(\Upsilon))}+\|G\|_{W^{1,2}(0,\infty;W^{1,2}(\Upsilon))}+\|u_1^b\|_{W^{2,2}(\Upsilon)}\Big).\end{aligned}\tag{4.9}$$

The solution of (T4) exponentially vanishes in the integral sense as $\tau\to\infty$, provided that data exponentially vanishes. For sufficiently small $\mu>0$ there holds the estimate

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\Big(e^{2\mu t}\|U_1(\cdot,\tau)\|^2_{W^{1,2}(\Upsilon)}\Big)+\|e^{\mu t}U_1\|^2_{L^2(0,\infty;W^{1,2}(\Upsilon))}\\&\quad+\|e^{\mu t}\partial_\tau U_1\|^2_{L^2(0,\infty;L^2(\Upsilon)}\\&\le C\Big(\|e^{\mu t}Z_1\|_{L^2(0,\infty;L^2(\Upsilon))}+\|e^{\mu t}G\|_{L^2(0,\infty;W^{1,2}(\Upsilon))}+\|u_1^b\|_{W^{1,2}(\Upsilon)}\Big).\end{aligned}\tag{4.10}$$

Problem (T3) is the inverse problem for the heat equation. It admits a unique weak solution $(\Phi,s)\in L^2(0,\infty;\mathring{W}^{1,2}(\Upsilon)\cap W^{2,2}(\Upsilon))\times L^2(0,\infty)$ with $\partial_\tau\Phi\in L^2(0,\infty;L^2(\Upsilon))$ provided $u_0^b\in\mathring{W}^{1,2}(\Upsilon)$, $\mathcal{F}\in W^{1,2}(0,\infty)$ and the compatibility condition $\mathcal{F}(0)=\int\limits_{-\gamma_0}^{\gamma_0}u_2^b\,dy_1$ holds. Moreover, the following estimate

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\|\Phi(\cdot,\tau)\|^2_{W^{1,2}(\Upsilon)}+\|\Phi\|^2_{L^2(0,\infty;W^{2,2}(\Upsilon))}+\|\partial_\tau\Phi\|^2_{L^2(0,\infty;L^2(\Upsilon)}\\&\quad+\|s\|^2_{L^2(0,\infty)}\le C\Big(\|u_2^b\|^2_{W^{1,2}(\Upsilon)}+\|\mathcal{F}\|^2_{W^{1,2}(0,\infty)}\Big)\end{aligned}\tag{4.11}$$

is valid. If the flux $\mathcal{F}$ exponentially vanishes, then for sufficiently small $\mu>0$ we additionally have the estimate

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\Big(e^{2\mu t}\|\Phi(\cdot,\tau)\|^2_{W^{1,2}(\Upsilon)}\Big)+\|e^{\mu t}\Phi\|^2_{L^2(0,\infty;W^{1,2}(\Upsilon))}\\&\quad+\|e^{\mu t}\partial_\tau\Phi\|^2_{L^2(0,\infty;L^2(\Upsilon)}+\|e^{\mu t}s\|^2_{L^2(0,\infty)}\\&\quad\le C\Big(\|u_2^b\|^2_{W^{1,2}(\Upsilon)}+\|e^{\mu t}\mathcal{F}\|^2_{W^{1,2}(0,\infty)}\Big),\end{aligned}\tag{4.12}$$

If the data are more regular $u_2^b\in\mathring{W}^{1,2}(\Upsilon)\cap W^{2,2}(\Upsilon)$, $\mathcal{F}\in W^{2,2}(0,\infty)$ and $\mathcal{F}(0)=\int\limits_{-\gamma_0}^{\gamma_0}u_2^b(y_1)dy_1=0$, then the solution also has the improved regularity, $\Phi\in L^\infty(0,\infty;W^{2,2}(\Upsilon))$, $\Phi_\tau\in L^\infty(0,\infty;L^2(\Upsilon))$, $s\in L^\infty(0,\infty)$ and

$$\begin{aligned}&\sup_{\tau\in[0,\infty)}\|\Phi(\cdot,\tau)\|^2_{W^{2,2}(\Upsilon)}+\sup_{\tau\in[0,\infty)}\|\partial_\tau\Phi\|^2_{L^2(\Upsilon)}\\&+\sup_{\tau\in[0,\infty)}|s(\tau)|^2\le C\Big(\|u_2^b\|^2_{W^{2,2}(\Upsilon)}+\|F\|^2_{W^{2,2}(0,\infty)}\Big).\end{aligned}\tag{4.13}$$

The unique solvability of problem (T3) and estimates (4.11), (4.12) are proved in [32, 35]. The proof of estimate (4.13) is given in Appendix A.

Define

$$\mathbf{U}^{[J]}(x,t)=\mathbf{U}^{O,[J]}(x_1/x_2^\lambda,x_2,t)+\mathbf{U}^{B,[J]}(x_1/x_2^\lambda,x_2,t/x_2^{2\lambda}),$$
$$P^{[J]}(x,t)=P^{O,[J]}(x_1/x_2^\lambda,x_2,t)+P^{B,[J]}(x_1/x_2^\lambda,x_2,t/x_2^{2\lambda}),$$

where $\mathbf{U}^{O,[J]}$, $\mathbf{U}^{B,[J]}$, $P^{O,[J]}$, $P^{B,[J]}$ are given either by (3.11), (3.19) or by (3.29), (3.33), or by (3.38), (3.41) depending on the value of λ. By construction,

$$\operatorname{div}\mathbf{U}^{[J]}(x,t)=0\ \text{ in }\ G_H,\quad \mathbf{U}^{[J]}(x,t)=0\ \text{ on }\ \partial G_H\cap\partial\Omega,$$

$$\mathbf{U}^{[J]}(x,0)=0 \text{ in } G_H, \qquad \int\limits_{\sigma(h)} \mathbf{U}^{[J]}\cdot\mathbf{n}\,dx_1 = F(t).$$

Let us start with estimates of leading-order term of the asymptotic decomposition. Problem (2.9) is of type (T2) with the solution depending on t as a parameter. Moreover, the right-hand side $G(y_1,\tau)$ in (2.9) is equal to $\lambda\kappa_0^{-1}F(t)(1+y_1\cdot\partial_1)\Phi(y_1)$, where $\Phi=\frac{1}{2\nu}(|y_1|^2-\gamma_0^2)$ is a solution of problem (2.7) (which is of type (T1)) with $Z_2(y_1)=1$. Clearly, the leading asymptotic term (see (2.5)) satisfies the following estimates

$$\begin{gathered}
\left|\frac{\partial^k U_{1,0}}{\partial y_1^k}\right| \le C y_2^{-1}|F(t)|, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^k U_{1,0}}{\partial y_1^k}\right| \le C y_2^{-1}|F'(t)|,\\
\left|\frac{\partial^k U_{2,0}}{\partial y_1^k}\right| \le C y_2^{-\lambda}|F(t)|, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^k U_{2,0}}{\partial y_1^k}\right| \le C y_2^{-\lambda}|F'(t)|,\\
\left|\frac{\partial^k U_{1,0}}{\partial y_2^k}\right| \le C y_2^{-1-k}|F(t)|, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^k U_{1,0}}{\partial y_2^k}\right| \le C y_2^{-1-k}|F'(t)|,\\
\left|\frac{\partial^k U_{2,0}}{\partial y_2^k}\right| \le C y_2^{-\lambda-k}|F(t)|, \qquad \left|\frac{\partial}{\partial t}\frac{\partial^k U_{2,0}}{\partial y_2^k}\right| \le C y_2^{-\lambda-k}|F'(t)|,
\end{gathered} \tag{4.14}$$

$k=0,1,2,\dots$.

Using estimates (4.14) of the leading asymptotic term, estimates (4.5)-(4.13) of solutions to problems (T3)–(T5) and following the scheme of construction of the asymptotic decomposition we obtain, by induction, the following estimates

$$\begin{gathered}
\sup_{t\in[0,T]}\|\mathbf{U}^{[J]}(\cdot,y_2,t)\|^2_{W^{1,2}(\Upsilon)} + \|\mathbf{U}^{[J]}\|^2_{L^2(0,T;W^{2,2}(\Upsilon)}\\
+\|\mathbf{U}_t^{[J]}\|^2_{L^2(0,T;L^2(\Upsilon))} \le \frac{c}{\varphi^2(y_2)}\int_0^T |||F|||^2_{J+1}\,dt,\\
\sup_{t\in[0,T]}\|\mathbf{U}^{[J]}(\cdot,y_2,t)\|^2_{W^{2,2}(\Upsilon)} + \sup_{t\in[0,T]}\|\mathbf{U}_t^{[J]}\|^2_{L^2(\Upsilon)}\\
+\|\nabla\mathbf{U}_t^{[J]}\|^2_{L^2(0,\infty;L^2(\Upsilon))} \le \frac{c}{\varphi^2(y_2)}\int_0^T |||F|||^2_{J+2}\,dt,\\
\sup_{t\in[0,T]}\left\|\frac{\partial\mathbf{U}^{[J]}(\cdot,y_2,t)}{\partial y_2}\right\|^2_{W^{1,2}(\Upsilon)} \le \frac{c}{\varphi^4(y_2)}\int_0^T |||F|||^2_{J+1}\,dt,
\end{gathered} \tag{4.15}$$

where $|||F|||_J^2=\sum\limits_{k=0}^{J}\left|\frac{\partial^k F(t)}{\partial t^k}\right|^2$, $\varphi(y_2)=\gamma_0 y_2^\lambda$.

Since $W^{1,2}(\Upsilon)\subset C(\Upsilon)$, we also have

$$\begin{aligned}
&\sup_{\substack{t\in(0,\infty)\\ y_1\in\Upsilon}}\left(|\mathbf{U}^{[J]}(y_1,y_2,t)|^2+\left|\frac{\partial\mathbf{U}^{[J]}(y_1,y_2,t)}{\partial y_1}\right|^2\right)\\
&\le c\sup_{t\in(0,\infty)}\int_0^T\|\mathbf{U}^{[J]}(\cdot,y_2,t)\|^2_{W^{2,2}(\Upsilon)} \le \frac{c}{\varphi^2(y_2)}\int_0^T |||F|||^2_{J+2}\,dt,
\end{aligned}$$

$$\begin{aligned}
\sup_{\substack{t\in(0,\infty)\\ y_1\in\Upsilon}}\left|\frac{\partial\mathbf{U}^{[J]}(y_1,y_2,t)}{\partial y_2}\right|^2 dt &\le c\sup_{t\in(0,\infty)}\int_0^T\left\|\frac{\partial\mathbf{U}^{[J]}(\cdot,y_2,t)}{\partial y_2}\right\|^2_{W^{1,2}(\Upsilon)}\\
&\le \frac{c}{\varphi^4(y_2)}\int_0^T |||F|||^2_{J+1}\,dt.
\end{aligned}$$

Passing to the coordinates x yields

$$\begin{gathered}
\sup_{\substack{t\in(0,\infty)\\ x_1\in(-\varphi(x_2),\varphi(x_2))}}|\mathbf{U}^{[J]}(x_1,x_2,t)|^2 dt \le \frac{c}{\varphi^2(x_2)}\int_0^T |||F|||^2_{J+1}\,dt,\\
\sup_{\substack{t\in(0,\infty)\\ x_1\in(-\varphi(x_2),\varphi(x_2))}}|\nabla_x\mathbf{U}^{[J]}(x_1,x_2,t)|^2 \le \frac{c}{\varphi^4(x_2)}\int_0^T |||F|||^2_{J+1}\,dt.
\end{gathered} \tag{4.16}$$

4.3 Estimates of discrepancies

Functions $\mathbf{U}^{[J]}$, $P^{[J]}$ satisfy the Navier–Stokes equations

$$\begin{cases} \mathbf{U}_t^{[J]} - \nu\Delta\mathbf{U}^{[J]} + (\mathbf{U}^{[J]}\cdot\nabla)\mathbf{U}^{[J]} + \nabla P^{[J]} &= \mathbf{H}_J, \\ \operatorname{div}\mathbf{U}^{[J]} &= 0, \\ \mathbf{U}^{[J]}|_{\partial G_H\cap\partial\Omega} &= 0, \\ \mathbf{U}^{[J]}(x,0) &= 0. \end{cases} \tag{4.17}$$

The estimates of discrepancies depend on the value of λ. If $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, then, by construction,

$$\begin{aligned} \|\mathbf{H}_J\|^2_{L^2(\Upsilon)} \le c y_2^{-a_1}\Big(&\|\widehat{\mathscr{F}}_{J-1}\|^2_{L^2(\Upsilon)} + \|\widehat{\mathscr{F}}_J\|^2_{L^2(\Upsilon)} \\ &+\|\mathscr{N}_J\|^2_{L^2(\Upsilon)} + \sum_{k=\max\{0,J-1-\frac{2}{\lambda-1}\}}^{J}\|\widetilde{\mathscr{F}}_k\|^2_{L^2(\Upsilon)} + \sum_{k=1}^{J}\|\mathscr{F}_k\|^2_{L^2(\Upsilon)} \\ &+\|\mathscr{F}^b_{J-1}\|^2_{L^2(\Upsilon)} + \|\mathscr{F}^b_J\|^2_{L^2(\Upsilon)} + \|\mathscr{N}^b_J\|^2_{L^2(\Upsilon)}\Big). \end{aligned}$$

In the case $\lambda = \frac{N+4}{N}$,

$$\begin{aligned} \|\mathbf{H}_J\|^2_{L^2(\Upsilon)} \le c y_2^{-a_2}\Big\|\sum_{k=0}^{J}\Big(&\widehat{\mathbf{F}}_k + \mathbf{N}_k + \widetilde{\mathbf{F}}_k + \widetilde{\mathbf{N}}_k \\ &+\widetilde{\widetilde{\mathbf{N}}}_k + \widehat{\mathbf{F}}^b_k + \mathbf{N}^b_k + \widetilde{\mathbf{F}}^b_k + \widetilde{\mathbf{N}}^b_k + \widetilde{\widetilde{\mathbf{N}}}{}^b_k\Big) - \sum_{k=0}^{J-1}\Big(\mathbf{S}_k + \mathbf{S}^b_k\Big) + \mathbf{F}_k\Big\|^2_{L^2(\Upsilon)}. \end{aligned}$$

If $\lambda \ne \frac{N+1}{N}$, $\lambda \ne \frac{N+2}{N}$, $\lambda \ne \frac{N+4}{N}$, then

$$\begin{aligned} \|\mathbf{H}_{\{I,J,K\}}\|^2_{L^2(\Upsilon)} \le c y_2^{-a_3}\Big\|\sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K}\Big(&\nu\mathfrak{D}^2\mathbf{U}_{\{i,j,k\}}(y,t) - \partial_t\mathbf{U}_{\{i,j,k\}}(y,t) \\ &-(\mathbf{U}_{\{i,j,k\}}\cdot\mathfrak{N})\mathbf{U}_{\{i,j,k\}}(y,t) + \nu\mathfrak{D}^2\mathbf{U}^b_{\{i,j,k\}}(y,\tau) - (\mathbf{U}^b_{\{i,j,k\}}\cdot\mathfrak{N}_b)\mathbf{U}^b_{\{i,j,k\}}(y,\tau) \\ &-\Big[\mathbf{T}_{\{i,j,k\}} + \mathbf{T}^b_{\{i,j,k\}}\Big](y,0,\tau) - \mathbf{S}_{\{I,J,K\}}(y,t) - \mathbf{S}^b_{\{I,J,K\}}(y,\tau)\Big)\Big\|^2_{L^2(\Upsilon)}, \end{aligned}$$

where $0 < a_i < 2$[9]. Passing to the variables x we obtain for all three cases the following estimate

$$\int_0^T \|\mathbf{H}_J\|^2_{L^2(G_H)}dt \le c\int_0^T |||F|||^2_{J+2}\,dt. \tag{4.18}$$

In the last case we chose all three numbers I, J, K so big that the discrepancy $\mathbf{H}_{\{I,J,K\}}$ belongs to L^2, but, for simplicity, we denote $\mathbf{H}_{\{I,J,K\}}$ just by $\mathbf{H}_J$.

Appendix A

Proof of estimate (4.13)**.** Differentiating equation (T3) with respect to τ we get

$$\partial_\tau\Phi_\tau(y_1,\tau) - \nu\partial_1^2\Phi_\tau(y_1,\tau) = s'(\tau). \tag{A.1}$$

Multiplying (A.1) by Φ_τ and integrating over the interval $(-\gamma_0,\gamma_0)$ yields

$$\frac{1}{2}\frac{d}{d\tau}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau|^2dy_1 + \int_{-\gamma_0}^{\gamma_0}|\nabla\Phi_\tau|^2dy_1 = s'(\tau)\int_{-\gamma_0}^{\gamma_0}\Phi_\tau dy_1 = s'(\tau)\mathcal{F}'(\tau).$$

9 We chose J or I, J, K big enough to satisfy this condition.

So, integrating from 0 to τ, we get

$$\begin{aligned}\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1+\int_0^\tau\int_{-\gamma_0}^{\gamma_0}|\nabla\Phi_r(y_1,r)|^2dy_1dr\\=\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,0)|^2dy_1+\int_0^\tau s'(r)\mathcal{F}'(r)dr\\=\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,0)|^2dy_1-\int_0^\tau s(r)\mathcal{F}''(r)dr+s(\tau)\mathcal{F}'(\tau)-s(0)\mathcal{F}'(0).\end{aligned}$$

Since $\mathcal{F}(0)=0$, we have $s(0)=0$ (see [32]) and hence $\Phi_\tau(y_1,0)=\nu\partial_1^2\Phi(y_1,0)=\nu\partial_1^2u_2^b(y_1)$. Therefore, using (4.12) we obtain

$$\begin{aligned}\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1+\int_0^\tau\int_{-\gamma_0}^{\gamma_0}|\nabla\Phi_r(y_1,r)|^2dy_1dr\\\le\frac{\nu}{2}\int_{-\gamma_0}^{\gamma_0}|\partial_1^2u_2^b(y_1))|^2dy_1+\frac{1}{2}\int_0^\tau|s(r)|^2dr+\frac{1}{2}\int_0^\tau|\mathcal{F}''(r)|^2dr+\varepsilon|s(\tau)|^2\\+c_\varepsilon|\mathcal{F}'(\tau)|^2\le c\Big(\|\partial_1^2u_2^b\|^2_{W^{2,2}((-\gamma_0,\gamma_0))}+\|\mathcal{F}\|^2_{W^{2,2}(0,\infty)}\Big)+\varepsilon|s(\tau)|^2.\end{aligned}\tag{A.2}$$

From equation (T3) we have

$$s'(\tau)=\Phi_{\tau\tau}(y_1,\tau)-\nu\partial_1^2\Phi_\tau(y_1,\tau),$$

and because of (4.12) we can write

$$s(\tau)=-\int_\tau^\infty s'(r)dr=-\int_\tau^\infty\big(\Phi_{rr}(y_1,r)-\nu\partial_1^2\Phi_r(y_1,r)\big)dr.$$

Multiplying this relation by $v_0(y_1)$, where v_0 is solution of the problem

$$\begin{cases}\nu\partial_1^2v_0=1,\\ v_0|_{|y_1|=\gamma_0}=0,\end{cases}$$

integrating this relation over the interval $(-\gamma_0,\gamma_0)$ and integrating by parts, we obtain

$$\begin{aligned}s(\tau)\kappa_0=-\int_\tau^\infty\int_{-\gamma_0}^{\gamma_0}\big(\Phi_{rr}(y_1,r)v_0(y_1)-\nu\partial_1^2\Phi_r(y_1,r)v_0(y_1)\big)dy_1dr\\=-\int_\tau^\infty\frac{1}{2}\frac{d}{dr}\int_{-\gamma_0}^{\gamma_0}\Phi_r(y_1,r)v_0(y_1)dy_1dr+\nu\int_\tau^\infty\int_{-\gamma_0}^{\gamma_0}\Phi_r(y_1,r)\partial_1^2v_0(y_1)dy_1dr\\=\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}\Phi_\tau(y_1,\tau)v_0(y_1)dy_1+\int_\tau^\infty\frac{1}{2}\frac{d}{dr}\int_{-\gamma_0}^{\gamma_0}\Phi(y_1,r)dy_1dr\\=\frac{1}{2}\int_{-\gamma_0}^{\gamma_0}\Phi_\tau(y_1,\tau)v_0(y_1)dy_1-\frac{1}{2}\mathcal{F}(\tau),\end{aligned}$$

where $\kappa_0=\int_{\gamma_0}^{\gamma_0}v_0(y_1)dy_1<0$. Therefore,

$$\begin{aligned}|s(\tau)|^2\le c\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1+c|\mathcal{F}(\tau)|^2\\\le c\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1+c\|\mathcal{F}\|^2_{W^{1,2}(0,\infty)}.\end{aligned}\tag{A.3}$$

From (A.2), (A.3) follows the inequality

$$\begin{aligned}\sup_{\tau\in[0,\infty)}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1+\sup_{\tau\in[0,\infty)}|s(\tau)|^2\\\le c\Big(\|\partial_1^2u_2^b\|^2_{W^{2,2}(-\gamma_0,\gamma_0)}+\|\mathcal{F}\|^2_{W^{2,2}(0,\infty)}\Big).\end{aligned}\tag{A.4}$$

Finally, from equation (T3) we get

$$\begin{aligned}\sup_{\tau\in[0,\infty)}\int_{-\gamma_0}^{\gamma_0}|\partial_1^2\Phi(y_1,\tau)|^2dy_1\le\sup_{\tau\in[0,\infty)}\int_{-\gamma_0}^{\gamma_0}|\Phi_\tau(y_1,\tau)|^2dy_1\\+\sup_{\tau\in[0,\infty)}|s(\tau)|^2\le c\Big(\|\partial_1^2u_2^b\|^2_{W^{2,2}((-\gamma_0,\gamma_0))}+\|\mathcal{F}\|^2_{W^{2,2}(0,\infty)}\Big).\end{aligned}\tag{A.5}$$

Estimates (A.4), (A.5) together with (4.11) imply (4.13). ∎

Remark 4.1. The above proof of a priori estimate (4.13) for the solution of problem (T3) contains inaccuracy. The solution which we have in hands does not possess enough regularity to perform all computations in the proof. However, these reasonings can be justified in a usual way by using Galerkin approximations.

Appendix B

Proof of Lemma 3.1. Remind that we start from $\mu_0 = 1 - 3\lambda \in M$ (see (2.6)).

1. If $\mu_1 = \mu_2 = \mu_0$, then from $(3.5)_2$ we get

$$\mu_0 + \mu_0 + 4\lambda - 2 = \mu_0 + \lambda - 1 = (1 - 3\lambda) + \lambda - 1.$$

If

$$\mu_1 = 1 - 3\lambda + i(\lambda - 1), \qquad \mu_2 = 1 - 3\lambda + j(\lambda - 1),$$

it follows from $(3.5)_{1,2}$ that

$$\mu_1 + 2\lambda - 2 = 1 - 3\lambda + (i + 2)(\lambda - 1),$$
$$\mu_1 + \mu_2 + 4\lambda - 2 = 1 - 3\lambda + (i + j + 1)(\lambda - 1).$$

Obviously, elements constructed following the rule $(3.5)_3$ belong to the set $T = \{1 - 3\lambda + 2j\lambda : j = 0, 1, \dots\}$. Elements from the set T belong to M if $k = j\left(2 + \frac{2}{\lambda-1}\right)$ is a natural number, i.e., if either $\frac{2}{\lambda-1} \in \mathbb{N}$ or $\frac{1}{\lambda-1} \in \mathbb{N}$ ($\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, $N = 1, 2, \dots$). Indeed, from $(3.5)_3$ we have

$$M \ni 1 - 3\lambda + j\left(2 + \frac{2}{\lambda - 1}\right)(\lambda - 1) = 1 - 3\lambda + 2j\lambda \in T.$$

Thus, if $\lambda = \frac{N+1}{N}$ or $\lambda = \frac{N+2}{N}$, $N = 1, 2, \dots$, then M is the most narrow set of indices, satisfying (3.5).

2. If $\lambda = \frac{N+4}{N}$, but $\lambda \neq \frac{N+1}{N}$ and $\lambda \neq \frac{N+2}{N}$, $N = 1, 2, \dots$, then $\frac{4}{\lambda-1} \in \mathbb{N}$, however, $\frac{2}{\lambda-1}$ and $\frac{1}{\lambda-1}$ are not natural numbers. In this case an element μ_3 which obeys the rule $(3.5)_3$ can be expressed as $\mu_1 + 2$, where μ_1 obeys $(3.5)_1$. Now, analogically to the first case, we show that $\mu_1 \in M_1$ and μ_2, $\mu_3 \in M_2$,

$$\mu_1 = 1 - 3\lambda + i(\lambda - 1), \quad \mu_2 = 1 - 3\lambda + j(\lambda - 1) + 2, \quad \mu_3 = 1 - 3\lambda + k(\lambda - 1) + 2,$$

obey the rules (3.5), i.e. we show that $M = M_1 \cup M_2$. It is is already proved in first part that $\mu_1 \in M$. From $(3.5)_1$ we get

$$\mu_2 + 2\lambda - 2 = 1 - 3\lambda + (j + 2)(\lambda - 1) + 2 \in M_2;$$

and from $(3.5)_2$ it follows that

$$\mu_1 + \mu_2 + 4\lambda - 2 = 1 - 3\lambda + (i + j + 1)(\lambda - 1) + 2 \in M_2;$$
$$\mu_2 + \mu_3 + 4\lambda - 2 = 1 - 3\lambda + (j + k + 4)(\lambda - 1) + 6.$$

However, since $\frac{4}{\lambda-1} \in \mathbb{N}$, one can easily check that the last element belongs to M_2, i.e.,

$$1 - 3\lambda + (j + k + 4)(\lambda - 1) + 6 = 1 - 3\lambda + l(\lambda - 1) + 2 \in M_2 \iff$$
$$(j + k + 4)(\lambda - 1) + 4 = l(\lambda - 1) \iff l = j + k + 4 + \frac{4}{\lambda - 1};$$

Finally, from $(3.5)_3$ we obtain

$$\mu_2 + 2\lambda = 1 - 3\lambda + (j + 2)(\lambda - 1) + 4.$$

Since $\frac{4}{\lambda-1} \in \mathbb{N}$, we easily check, similarly as before, that the last element belongs to M_1, i.e.,

$$1 - 3\lambda + (j + 2)(\lambda - 1) + 4 = 1 - 3\lambda + i(\lambda - 1) \in M_1 \iff i = j + 2 + \frac{4}{\lambda - 1}.$$

3. If

$$\mu_1 = 1 - 3\lambda + 2i_1 + 2j_1\lambda + k_1(\lambda - 1), \quad \mu_2 = 1 - 3\lambda + 2i_2 + 2j_2\lambda + k_2(\lambda - 1),$$

then from $(3.5)_1$ we get that

$$\mu_1 + 2\lambda - 2 = 1 - 3\lambda + 2i_1 + 2j_1\lambda + (k_1 + 2)(\lambda - 1);$$

from $(3.5)_2$ it follows that

$$\mu_1 + \mu_2 + 4\lambda - 2 = 1 - 3\lambda + 2(i_1 + i_2) + 2(j_1 + j_2)\lambda + (k_1 + k_2 + 1)(\lambda - 1);$$

and finally, from $(3.5)_3$ we obtain

$$\mu_1 + 2\lambda = 1 - 3\lambda + 2i_1 + 2(j_1 + 1)\lambda + k_1(\lambda - 1). \qquad \blacksquare$$

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