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Research article

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Non-stationary Navier–Stokes equations in $2D$ power cusp domain. II. Existence of the solution

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Abstract: The initial boundary value problem for the non-stationary Navier-Stokes equations is studied in 2D bounded domain with a power cusp singular point O on the boundary. We consider the case where the boundary value has a nonzero flux over the boundary. In this case there is a source/sink in O and the solution necessary has infinite energy integral. In the first part of the paper the formal asymptotic expansion of the solution near the singular point was constructed. In this, second part, the constructed asymptotic decomposition is justified, i.e., existence of the solution which is represented as the sum of the constructed asymptotic expansion and a term with finite energy norm is proved. Moreover, it is proved that the solution represented in this form is unique.

Keywords: Nonstationary Navier-Stokes problem, power cusp domain, singular solutions, asymptotic expansion

MSc: 35Q30, 35A20, 76M45, 76D03, 76D10

1 Introduction

In this paper we continue to study the boundary value problem for the non-stationary Navier–Stokes system

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{u} = 0, \\ \mathbf{u}|_{\partial\Omega\setminus O} = \mathbf{a}, \\ \mathbf{u}(x,0) = \mathbf{b}(x) \end{cases} \tag{1.1}$$

in a two-dimensional bounded domain Ω with the cusp point $O = (0,0)$ at the boundary: $\Omega = G_H \cup \Omega_0$, where $G_H = \left\{ x \in \mathbb{R}^2 : |x_1| < \varphi(x_2),\ x_2 \in (0,H] \right\}$, $\varphi(x_2) = \gamma_0 x_2^{\lambda}$, $\gamma_0 = \text{const}$, $\lambda > 1$ (see Figure 1). For simplicity we assume that the boundary $\partial\Omega \cap \partial\Omega_0$ is infinitely smooth. Here $\mathbf{u} = (u_1, u_2)$ stands for the velocity field, p stands for the pressure, $\nu > 0$ is the constant kinematic viscosity.

It is supposed that $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega$, i.e., the support of the boundary value $\mathbf{a} \in L^2(0,T; W^{3/2,2}(\partial\Omega))$ is separated from the cusp point O. We also assume that the flux $F(t)$ of $\mathbf{a}$ is nonzero:

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = -F(t), \quad F(t) \not\equiv 0, \quad F(0) = 0. \tag{1.2}$$

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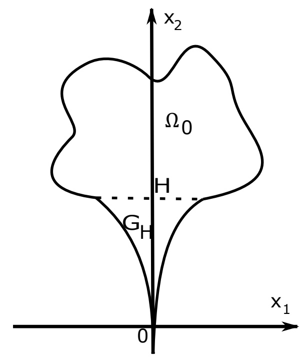

Fig. 1: Domain Ω

The initial velocity $\mathbf{b} \in W^{1,2}(\Omega)$ and the boundary value $\mathbf{a}$ have to satisfy the necessary compatibility conditions

$$\operatorname{div} \mathbf{b}(x) = 0, \quad \mathbf{b}(x)|_{\partial\Omega} = \mathbf{a}(x, 0). \tag{1.3}$$

The solution $\mathbf{u}$ of (1.1) has to satisfy the condition

$$\int\limits_{\sigma(h)} \mathbf{u} \cdot \mathbf{n}\, dS + \int\limits_{\partial\Omega \cap \partial\Omega_0} \mathbf{a} \cdot \mathbf{n}\, dS = 0$$

where $\sigma(h) = \{x \in G_H : x_2 = h = \text{const}\}$, which means that the total flux of the fluid is equal to zero. Thus,

$$\int\limits_{\sigma(h)} \mathbf{u} \cdot \mathbf{n}\, dx_1 = F(t) \not\equiv 0, \tag{1.4}$$

and we can regard the cusp point O as a source (or a sink) of intensity $F(t)$.

More information and references concerning the Navier–Stokes equations in domains with singular boundaries are given in the introduction to the first part of the paper, see [1]. In [1] the formal asymptotic decomposition of the solution $(\mathbf{u}, p)$ near the cusp point was constructed. This asymptotic expansion has the form

$$\begin{aligned} \mathbf{U}^{[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, t, \tau\right) &= \mathbf{U}^{O,[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, t\right) + \mathbf{U}^{B,[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, \tau\right), \\ P^{[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, t, \tau\right) &= P^{O,[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, t\right) + P^{B,[J]}\left(\tfrac{x_1}{x_2^\lambda}, x_2, \tau\right), \end{aligned} \tag{1.5}$$

where the pair $(\mathbf{U}^{O,[J]}, P^{O,[J]})$ is an approximate solution (outer asymptotic expansion) of the Navier–Stokes problem in variables $y_1 = x_1 x_2^{-\lambda}$, $y_2 = x_2$, $t = t$; the "slow" time variable t plays the role of a parameter and, in general, the initial condition is not satisfied. The pair $(\mathbf{U}^{B,[J]}, P^{B,[J]})$ is the boundary layer corrector (the inner part of the asymptotic expansion) which compensate the discrepancy in the initial condition. Notice that $(\mathbf{U}^{B,[J]}, P^{B,[J]})$ exponentially vanishes as the fast time $\tau = \frac{t}{x_2^{2\lambda}}$ tends to infinity. The number J is taken so large that the discrepancy $\mathbf{H}^{[J]}$ of $(\mathbf{U}^{[J]}, P^{[J]})$ in the Navier–Stokes equations belongs to the space $L^2(0, T; L^2(\Omega))$, while the discrepancy in the initial condition is zero (see [1] for details). Moreover, in order to ensure the existence of all terms of asymptotic decomposition up to the order J, we have to assume that

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0, T; W^{1/2,2}(\partial\Omega)), \quad l = 1, 2, \ldots, J + 1.$$

Here we justify this asymptotics. We prove that there exists a solution of problem (1.1) which is represented as a sum of the singular part (the constructed asymptotic decomposition) and the function having finite energy. To be more precise, we construct a solenoidal extension $\mathbf{V}$ of the boundary value $\mathbf{a}$ which coincides with $\mathbf{U}^{[J]}$ near the cusp point and we look for the solution $(\mathbf{u}, p)$ of (1.1) in the form $\mathbf{u} = \mathbf{V} + \mathbf{v}$, $p = \zeta P^{[J]} + q$, where ζ is a smooth cut-off function localising the asymptotical part of the pressure near the cusp point O. Then for

$(\mathbf{v}, q)$ we obtain the problem

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{V}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{V} + \nabla q := \widehat{\mathbf{f}}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{\partial\Omega} = 0, \\ \mathbf{u}(x,0) = \mathbf{b}(x) - \mathbf{V}(x,0) := \widehat{\mathbf{u}}_0(x) \end{cases} \tag{1.6}$$

with $\widehat{\mathbf{f}} \in L^2(0,T;L^2(\Omega))$, $\widehat{\mathbf{u}}_0 \in \mathring{W}^{1,2}(\Omega)$. In the paper we prove the existence of a unique regular solution $\mathbf{v}$ to (1.6).

The existence of singular solutions to the time-periodic and the non-stationary Stokes problem in the domain with a cusp point was studied in [2, 3]. We can also mention the recent paper [4] where the Dirichlet problem for the non-stationary Stokes system is studied in a three-dimensional cone. The non-stationary Navier–Stokes equations in tube-structures were studied in [5, 6]. The solvability of the stationary Navier–Stokes system in the cusp domain with source or sink in the cusp point was proved in [7]. The steady Navier-Stokes equations are also studied in a punctured domain $\Omega = \Omega_0 \setminus \{O\}$ with $O \in \Omega_0$ assuming that the point O is a sink or source of the fluid, see [8–10] and [11] for the review of these results. We also mention the papers [12–15] where the stationary Navier–Stokes equations were studied in domains with paraboloidal outlets to infinity. Such geometry has similarities with the cusp domains, the difference is that in the case of a domain with outlet to infinity $x_2 \to \infty$, while in the cusp domain $x_2 \to 0$.

The paper is organised as follows. In Section 2 we introduce the main notation, function spaces and prove certain inequalities needed in subsequent sections. In Section 3 we study the Stokes problem and the Stokes operator in the cusp domain. Finally, the main result of the paper, the unique solvability of problem (1.6), is proved in Section 4.

2 Notation, function spaces and auxiliary results

Let G be a domain in $\mathbb{R}^n$. We use usual notation of functional spaces (e.g., [16]). By $L^p(G)$ and $W^{m,p}(G)$, $1 \le p < \infty$, we denote the Lebesgue and Sobolev spaces, respectively. The norm in a Banach space X is denoted by $\|\cdot\|_X$. $C^\infty(G)$ is the set of all infinitely differentiable functions defined on G and $C_0^\infty(G)$ - the subset of all functions from $C^\infty(G)$ having compact supports in G. By $\mathring{W}^{k,q}(G)$ we denote the completion of the $C_0^\infty(G)$ in the $\|\cdot\|_{W^{m,p}(G)}$-norm and by $W^{m-1/p,p}(\partial G)$ the space of traces on ∂G of functions from $W^{m,p}(G)$. The space $L^p(0,T;X)$ consists of all measurable functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} = \left(\int_0^T \|\mathbf{u}(t)\|_X^p dt\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

We do not distinguish in notation the spaces of vector and scalar functions; from the context it will be clear which space we have in mind.

Denote $J_0^\infty(G) = \{\mathbf{v} \in C_0^\infty(G) : \operatorname{div}\mathbf{v} = 0\}$ the set of all divergence free vector fields from $C_0^\infty(G)$ and by $J_0(G)$ be the closure of $J_0^\infty(G)$ in $L^2(G)$-norm. Let $H(G) = \{\mathbf{v} \in \mathring{W}^{1,2}(G) : \operatorname{div}\mathbf{v} = 0\}$. If G is a bounded domain with Lipschitz boundary, then $H(G)$ coincides with the closure of $J_0^\infty(G)$ in the norm $\|\cdot\|_{W^{1,2}(G)}$ (see [17]).

Let us consider the cusp domain Ω. Let $h_0 = H$, $h_k = h_{k-1} - \frac{\varphi(h_{k-1})}{2L}$, where L is a Lipschitz constant for the function φ, $k = 1, 2, \ldots$. The sequence $\{h_k\}$ is decreasing and bounded from below. Assume that the limit of this sequence is $a_0 \neq 0$. From the definition of the sequence it follows that $a_0 = a_0 - \frac{\varphi(a_0)}{2L}$. Then $\varphi(a_0) = 0$. However, $\varphi(a_0) \neq 0$ for $a_0 \neq 0$ and, hence the limit $a_0 = 0$. Since the sequence is decreasing and the limit is equal to 0, all its elements are positive.

Denote $\omega_l = \{x \in \mathbb{R}^2 : |x_1| < \varphi(x_2), x_2 \in (h_l, h_{l-1})\}$, $l = 1, \ldots$. Note that

$$\tfrac{1}{2}\varphi(h_l) \le \varphi(t) \le \tfrac{3}{2}\varphi(h_l), \quad t \in [h_{l+1}, h_l]. \tag{2.1}$$

Define the transformation $y = \mathcal{P}_l x$ by the formulas

$$y_1 = \frac{2Lx_1}{\varphi(x_2)}, \quad y_2 = \frac{2L(x_2 - h_{l-1})}{\varphi(h_{l-1})} \tag{2.2}$$

and introduce the domains

$$G_0 = \{y \in \mathbb{R}^2 : |y_1| < 2L, -1 < y_2 < 0\},$$
$$G_1 = \{y \in \mathbb{R}^2 : |y_1| < 2L,\ -1 - g(y_1) < y_2 < -1\},$$
$$G_2 = \{y \in \mathbb{R}^2 : |y_1| < 2L, -2 < y_2 < 1\}.$$

In the definition of G_1 the function $g \in C^\infty$ satisfies the conditions $g(\pm 2L) = 0$, $0 < g(y_1) < 1$ for $|y_1| < 2L$ and it is such that the curve $\{y : y_1 = -2L\} \cup \{y : y_1 = 2L\} \cup \{y : |y_1| < 2L,\ y_2 = -1 - g(y_1)\}$ is infinitely smooth.

Obviously the transformation $\mathcal{P}_l^{-1}$ maps G_0 onto ω_l. Consider the domain $\omega_l^* = \mathcal{P}_l^{-1} G_2$. Then $\omega_l \subset \omega_l^* = \{x \in \mathbb{R}^2 :\ |x_1| < \varphi(x_2),\ h_{l-1} - \frac{\varphi(h_{l-1})}{L} < x_2 < h_{l-1} + \frac{\varphi(h_{l-1})}{2L}\} \subset \omega_{l+1} \cup \omega_l \cup \omega_{l-1}$, $l = 2, 3, \dots$ (see Figure 2). It is easy to see that

$$\Omega = \Omega_0 \cup \Big(\bigcup_{l=1}^{\infty} \omega_l\Big) = \Omega_0 \cup \omega_1 \cup \Big(\bigcup_{l=2}^{\infty} \omega_l^*\Big).$$

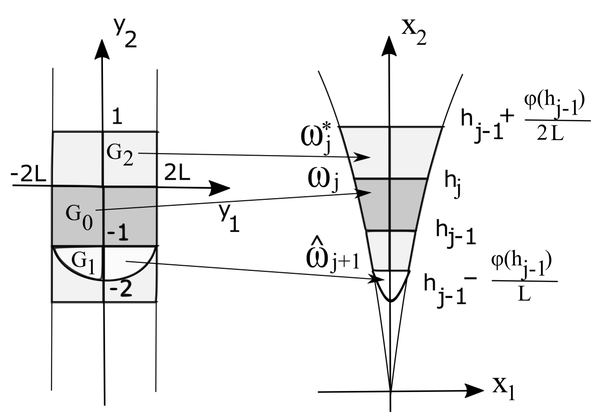

Fig. 2: Domains ω_j, ω_j^* and $\hat{\omega}_{j+1}$ in different coordinate systems

Let us fix $K \geq 2$ (sufficiently large) and define $\widehat{\omega}_K = \mathcal{P}_{K-1}^{-1} G_1$, $\widehat{\omega}_K^* = \mathcal{P}_{K-1}^{-1}(G_0 \cup G_1)$. Obviously, $\widehat{\omega}_K \subset \omega_K$ and $\widehat{\omega}_K \subset \widehat{\omega}_K^*$. Let

$$\Omega_K = \Omega_0 \cup \Big(\bigcup_{l=1}^{K-1} \omega_l\Big) \cup \widehat{\omega}_K.$$

The boundary of Ω_K consist of $\partial\Omega \cap \bar{\Omega}_K$ and the curve Γ_K which is defined as $\Gamma_K = \mathcal{P}_{K-1}^{-1} E_0$, where $E_0 = \{y : |y_1| < 2L, y_2 = -1 - g(y_1)\}$. By construction, the boundary $\partial\Omega_K$ is smooth (see Figure 3).

We can take also the other covering of the domain Ω_K. Namely,

$$\Omega_K = \Omega_0 \cup \omega_1 \cup \Big(\bigcup_{l=1}^{K-2} \omega_l\Big) \cup \widehat{\omega}_{K-1}^* \cup \widehat{\omega}_K^*,$$

where $\widehat{\omega}_{K-1}^* = \mathcal{P}_{K-1}^{-1}\big(\{y \in \mathbb{R}^2 : |y_1| < 2L, -1 - g(y_1) < y_2 < -1\}\big) = \mathcal{P}_{K-1}^{-1}(G_1)$.

We also introduce domains $\Omega_l^\sharp = \Omega_0 \bigcup \Big(\bigcup_{j=1}^{l} \omega_j\Big)$, $l = 1, 2, \dots$ (see Figure 4).

In Ω_K we define the function space $L_1^2(\Omega_K)$ with the weighted norm

$$\|\mathbf{f}\|^2_{L_1^2(\Omega_K)} = \int_{\Omega_0} |\mathbf{f}(x)|^2 dx + \int_{\Omega_K \setminus \Omega_0} \varphi^2(x_2)|\mathbf{f}(x)|^2 dx.$$

Let us prove some auxiliary inequalities for functions defined in Ω.

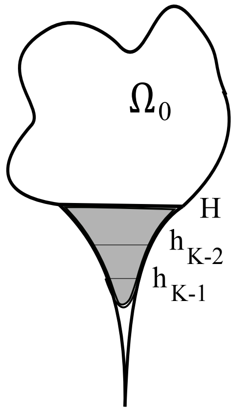

Fig. 3: Domains Ω_K.

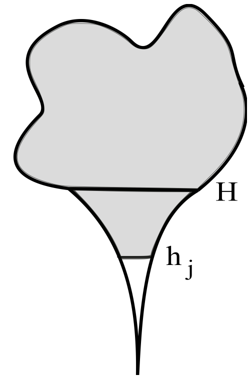

Fig. 4: Domains $\Omega_j^\sharp$.

Lemma 2.1. *(Poincaré inequalities). Let $u \in W^{1,2}_{loc}(\bar\Omega)$, $u|_{\partial\Omega} = 0$. Then the following inequalities*

$$\int_{\omega_l} |u(x)|^2 dx \le \frac{9}{\pi^2}\varphi^2(h_{l-1}) \int_{\omega_l} |\nabla u(x)|^2 dx, \tag{2.3}$$

$$\int_h^H \int_{-\varphi(x_2)}^{\varphi(x_2)} |\varphi(x_2)|^{\kappa-2}|u(x)|^2 dx \le \frac{4}{\pi^2} \int_h^H \int_{-\varphi(x_2)}^{\varphi(x_2)} |\varphi(x_2)|^{\kappa}|\nabla u(x)|^2 dx \tag{2.4}$$

hold for any $\kappa \in \mathbb{R}$ and any $h \in (0, H)$.

Proof. By the classical Poincaré inequality on the interval $\big(-\varphi(x_2), \varphi(x_2)\big)$, we have

$$\int_{-\varphi(x_2)}^{\varphi(x_2)} |u(x_1, x_2)|^2 dx_1 \le \frac{4}{\pi^2}\varphi^2(x_2) \int_{-\varphi(x_2)}^{\varphi(x_2)} \left|\frac{\partial u(x_1,x_2)}{\partial x_1}\right|^2 dx_1.$$

Integrating this inequality over (h_l, h_{l-1}) and applying (2.1) we derive (2.3). To prove (2.4) it is enough to multiply the above inequality by $|\varphi(x_2)|^{\kappa-2}$ and integrate over the interval (h, H). ■

Lemma 2.2. *Let $u \in W^{1,2}_{loc}(\bar\Omega)$. Then*

$$\|u\|^4_{L^4(\omega_l)} \le c\varphi^{-2}(h_{l-1})\|u\|^2_{L^2(\omega_l)}\Big(\|u\|^2_{L^2(\omega_l)} + \varphi^2(h_{l-1})\|\nabla u\|^2_{L^2(\omega_l)}\Big) \tag{2.5}$$

holds with a constant c independent of l. In particular, if $u \in W^{2,2}_{loc}(\bar\Omega)$, then the following estimate

$$\|\nabla u\|^4_{L^4(\omega_l)} \le c\varphi^{-2}(h_{l-1})\|\nabla u\|^2_{L^2(\omega_l)}\Big(\|\nabla u\|^2_{L^2(\omega_l)} + \varphi^2(h_{l-1})\|\nabla^2 u\|^2_{L^2(\omega_l)}\Big)$$

holds.

Proof. After the transformation $\mathcal{P}_l$, the domain ω_l is transformed into the domain $G_0 = \{y : \ |y_1| < 1, -1 < y_2 < 0\}$ which is independent of l. In G_0 holds the inequality (see [18])

$$\|u\|^4_{L^4(G_0)} \le c\|u\|^2_{L^2(G_0)}\Big(\|u\|^2_{L^2(G_0)} + \|\nabla u\|^2_{L^2(G_0)}\Big).$$

Passing in the last inequality to variables x we obtain

$$\begin{aligned}
&\int_{\omega_l} |u|^4 \frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})} dx \\
&\le c\Big(\int_{\omega_l} |u|^2 \frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})} dx\Big)\Big(\int_{\omega_l} |u|^2 \frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})} dx + \int_{\omega_l} |\nabla_y u|^2 \frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})} dx\Big)
\end{aligned}$$

$$\le c\int\limits_{\omega_l}|u|^2\frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})}dx\Big(\int\limits_{\omega_l}|u|^2\frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})}dx+$$
$$\int\limits_{\omega_l}\Big[|\frac{\partial u}{\partial x_1}|^2\Big(\varphi^2(x_2)+|x_1|^2\frac{\varphi^2(h_{l-1})}{\varphi^2(x_2)}|\varphi'(x_2)|^2\Big)+|\frac{\partial u}{\partial x_2}|^2\varphi^2(h_{l-1})\Big]\frac{4L^2}{\varphi(x_2)\varphi(h_{l-1})}dx\Big).$$

Since $|x_1|<\varphi(x_2)$ and $|\varphi'(x_2)|\le$ const, from the last inequality using (2.1) we derive (2.5). ■
From (2.3) and (2.5) we obtain

Lemma 2.3. *Let $u\in W^{1,2}_{loc}(\bar\Omega)$, $u|_{\partial\Omega}=0$. Then*

$$\|u\|_{L^4(\omega_l)}\le c\|u\|^{1/2}_{L^2(\omega_l)}\|\nabla u\|^{1/2}_{L^2(\omega_l)}\le c\varphi^{1/2}(h_{l-1})\|\nabla u\|_{L^2(\omega_l)} \tag{2.6}$$

with a constant c is independent of l.

Let us consider in ω_l the divergence problem

$$\begin{cases}\operatorname{div}\mathbf{v}=g & \text{in } \omega_l,\\ \mathbf{v}=0 & \text{on } \partial\omega_l.\end{cases} \tag{2.7}$$

Lemma 2.4. *Let $g\in L^2(\omega_l)$ and*

$$\int\limits_{\omega_l}g(x)\,dx=0. \tag{2.8}$$

Then there exists a solution $\mathbf{v}\in\mathring{W}^{1,2}(\omega_l)$ of (2.7) satisfying the estimate

$$\|\mathbf{v}\|_{L^2(\omega_l)}\le c\|g\|_{L^2(\omega_l)} \tag{2.9}$$

with a constant c independent of l.

Proof. The transformation $\mathcal{P}_l$ (see (2.2)) maps the domain ω_l onto $G_0=\{y:\ |y_1|<1,-1<y_2<0\}$. Because of (2.8),

$$\int\limits_{G_0}g(\mathcal{P}_l^{-1}(y))J_l^{-1}(\mathcal{P}_l^{-1}(y))dy=\int\limits_{\omega_l}g(x)\,dx=0,$$

where $J_l(x)=\frac{4L^2}{\varphi(h_{l-1})\varphi(x_2)}$ is the Jacobian. Therefore (see [17]), there exists a function $\hat{\mathbf{v}}\in\mathring{W}^{1,2}(G_0)$ such that

$$\operatorname{div}_y\hat{\mathbf{v}}(y)=J_l^{-1}(\mathcal{P}_l^{-1}(y))g(\mathcal{P}_l^{-1}(y))$$

and

$$\|\nabla_y\hat{\mathbf{v}}\|_{L^2(G_0)}\le c\|\hat{J}_l^{-1}\hat{g}\|_{L^2(G_0)}, \tag{2.10}$$

where $\hat g(y)=g(\mathcal{P}_l^{-1}(y))$, etc. Let us define the vector field $\mathbf{v}(x)$ with the components

$$v_1(x)=\hat v_1(y)|_{y=\mathcal{P}_l(x)}\frac{2L}{\varphi(h_{l-1})}+\hat v_2(y)|_{y=\mathcal{P}_l(x)}\frac{2Lx_1\varphi'(x_2)}{\varphi^2(x_2)},$$
$$v_2(x)=\hat v_2(y)|_{y=\mathcal{P}_l(x)}\frac{2L}{\varphi(x_2)}.$$

Then it is straightforward to verify that

$$\operatorname{div}_x\mathbf{v}=\frac{4L^2}{\varphi(h_{l-1})\varphi(x_2)}\operatorname{div}_y\hat{\mathbf{v}}(y)|_{y=\mathcal{P}_l(x)}=g(x).$$

Thus, we have only to show estimate (2.9). Let us estimate the norm $\|\nabla_x\mathbf{v}\|_{L^2(\omega_l)}$. Using inequality (2.1) for φ, Poincaré inequality (2.3) and the relations $|\varphi'(x_2)|\le$ const, $|\varphi''(x_2)\varphi^2(x_2)|+|\varphi'^2(x_2)\varphi(x_2)|\le$ const (recall that $\varphi(x_2)=\gamma_0x_2^\lambda$, $\lambda>1$) we obtain

$$\int\limits_{\omega_l}|\nabla_xv_1(x)|^2dx\le\frac{c}{\varphi^2(h_{l-1})}\int\limits_{\omega_l}\Big(|\nabla_x\hat v_1(\mathcal{P}_l(x))|^2+|\nabla_x\hat v_2(\mathcal{P}_l(x))|^2+\frac{|\hat v_2(\mathcal{P}_l(x))|^2}{\varphi^2(x_2)}\Big)dx$$
$$\le\frac{c}{\varphi^2(h_{l-1})}\int\limits_{G_0}|\nabla_y\hat{\mathbf{v}}(y)|^2dy,$$
$$\int\limits_{\omega_l}|\nabla_xv_2(x)|^2dx\le\frac{c}{\varphi^2(h_{l-1})}\int\limits_{\omega_l}\Big(|\nabla_x\hat v_2(\mathcal{P}_l(x))|^2+\frac{|\hat v_2(\mathcal{P}_l(x))|^2}{\varphi^2(x_2)}\Big)dx$$
$$\le\frac{c}{\varphi^2(h_{l-1})}\int\limits_{G_0}|\nabla_y\hat{\mathbf{v}}(y)|^2dy.$$

Estimating now $\|\nabla_y \widehat{\mathbf{v}}\|^2_{L^2(G_0)}$ by inequality (2.10), using the expression for the Jacobean and returning to coordinates x, we derive estimate (2.9) with a constant independent of l. ■

Remark 2.1. It is easy to see that Lemmas 2.1–2.4 remain valid if we take the domains $\omega_l^\star$, $l = 2, \dots, k$, or $\widehat{\omega}_k$, $\widehat{\omega}_k^\star$ instead of ω_l.

3 Stokes problem and Stokes operator

3.1 Estimates of solutions to the Stokes problem

In Ω_K consider the Dirichlet boundary value problem for the Stokes system

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}\big|_{\partial\Omega_K} = 0. \end{cases} \tag{3.1}$$

The weak solution $\mathbf{v} \in H(\Omega_K)$ to (3.1) satisfies the integral identity

$$\nu \int_{\Omega_K} \nabla\mathbf{v} \cdot \nabla\boldsymbol{\eta}\, dx = \int_{\Omega_K} \mathbf{f} \cdot \boldsymbol{\eta}\, dx \quad \forall \boldsymbol{\eta} \in H(\Omega_K).$$

Lemma 3.1. *Let* $\mathbf{f} \in L_1^2(\Omega_K)$. *Then problem* (3.1) *has a unique solution* $\mathbf{v} \in H(\Omega_K)$ *and there holds the estimate*

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_K)} \le c\|\mathbf{f}\|_{L_1^2(\Omega_K)} \tag{3.2}$$

with a constant c independent of K.

Proof. By Poincaré's inequality (2.4) with $\kappa = 0$,

$$\Big| \int_{\Omega_K} \mathbf{f} \cdot \boldsymbol{\eta}\, dx \Big| \le c\|\mathbf{f}\|^2_{L_1^2(\Omega_K)} \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega_K)}.$$

Hence, the statement of the lemma follows from Lax–Milgram's theorem. ■

Lemma 3.2. *Let* $\mathbf{f} \in L^2(\Omega_K) \subset L_1^2(\Omega_K)$. *Then the weak solution* $\mathbf{v}$ *of* (3.1) *satisfies the estimate*

$$\int_{\Omega_0} |\nabla\mathbf{v}|^2 dx + \int_{\Omega_K \setminus \Omega_0} \varphi^{-2}(x_2)|\nabla\mathbf{v}|^2 dx \le c \int_{\Omega_K} |\mathbf{f}|^2 dx \tag{3.3}$$

with a constant c independent of K.

Proof. Let

$$\Phi(x_2) = \begin{cases} \frac{1}{\varphi^2(h_l)}, & x \in \Omega_0 \bigcup \Big(\bigcup_{j=1}^{l} \omega_j \Big) = \Omega_l^\sharp, \\ \frac{1}{\varphi^2(x_2)}, & x \in \Big(\bigcup_{j=l+1}^{K-1} \omega_j \Big) \bigcup \widehat{\omega}_K = \Omega_K \setminus \Omega_l^\sharp. \end{cases}$$

Here and below the number l is fixed; we specify it during the proof.
Consider the function $\mathbf{u} = \Phi(x_2)\mathbf{v}$. Then $\operatorname{div}\mathbf{u} = \Phi'(x_2)v_2$. Since $\mathbf{v} \in H(\Omega_K)$, the flux of $\mathbf{v}$ over any section $x_2 = \text{const}$ of G_H is equal to zero, i.e.,

$$\int_{-\varphi(x_2)}^{\varphi(x_2)} \Phi'(x_2)v_2(x_1, x_2)\, dx_1 = 0 \quad \forall x_2 \in (h_K, H].$$

Integrating over x_2 we conclude that

$$\int_{\omega_j} \Phi'(x_2)v_2(x_1, x_2)\, dx = 0, \quad j = l+1, \dots, K-1; \quad \int_{\widehat{\omega}_K} \Phi'(x_2)v_2(x_1, x_2)\, dx = 0.$$

Then by Lemma 2.4, there exist functions $\mathbf{w}_j \in \mathring{W}^{1,2}(\omega_j)$, $j = l+1, \dots, K-1$, $\mathbf{w}_K \in \mathring{W}^{1,2}(\widehat{\omega}_K)$ such that

$$\operatorname{div} \mathbf{w}_j = -\Phi'(x_2) v_2 \quad \text{in} \ \ \omega_j, \ j = l+1, \dots, K-1, \quad \operatorname{div} \mathbf{w}_K = -\Phi'(x_2) v_2 \quad \text{in} \ \ \widehat{\omega}_K.$$

Moreover, the following estimates

$$\begin{gathered} \|\nabla \mathbf{w}_j\|_{L^2(\omega_j)} \le c \|\Phi' v_2\|_{L^2(\omega_j)} \le c \max_{x \in \omega_j} |\varphi'(x_2)| \|\varphi^{-3} v_2\|_{L^2(\omega_j)}, \\ j = l+1, \dots, K-1, \\ \|\nabla \mathbf{w}_K\|_{L^2(\widehat{\omega}_K)} \le c \max_{x \in \widehat{\omega}_K} |\varphi'(x_2)| \|\varphi^{-3} v_2\|_{L^2(\widehat{\omega}_K)} \end{gathered} \tag{3.4}$$

hold with a constant c independent of j and K. Taking into account inequalities (2.1), from (3.4) we obtain

$$\begin{gathered} \|\varphi(x_2) \nabla \mathbf{w}_j\|_{L^2(\omega_j)} \le c \max_{x \in \omega_j} |\varphi'(x_2)| \|\varphi^{-2}(x_2) v_2\|_{L^2(\omega_j)}, \\ j = l+1, \dots, K-1, \\ \|\varphi(x_2) \nabla \mathbf{w}_K\|_{L^2(\widehat{\omega}_K)} \le c \max_{x \in \widehat{\omega}_K} |\varphi'(x_2)| \|\varphi^{-2}(x_2) v_2\|_{L^2(\widehat{\omega}_K)}. \end{gathered} \tag{3.5}$$

Define the function

$$\mathbf{w}(x) = \begin{cases} 0, & x \in \Omega_l^\sharp, \\ \mathbf{w}_j(x), & x \in \omega_j, \ j = l+1, \dots, K-1, \\ \mathbf{w}_K(x), & x \in \widehat{\omega}_K \end{cases}$$

(recall that $\Phi'(x_2) = 0$ in $\Omega_l^\sharp$). Take in the integral identity $\boldsymbol{\eta} = \mathbf{u} + \mathbf{w}$. By construction, $\operatorname{div}(\mathbf{u} + \mathbf{w}) = 0$ and hence, $\mathbf{u} + \mathbf{w} \in H(\Omega_K)$. This yields

$$\begin{aligned} \nu \int_{\Omega_K} \Phi |\nabla \mathbf{v}|^2 dx = -\nu \int_{\Omega_K} \nabla \mathbf{v} \cdot \nabla \Phi \cdot \mathbf{v} dx - \nu \int_{\Omega_K} \nabla \mathbf{v} \cdot \nabla \mathbf{w} dx \\ + \int_{\Omega_K} \mathbf{f} \cdot (\Phi \mathbf{v} + \mathbf{w}) dx = J_1 + J_2 + J_3. \end{aligned}$$

Let us estimate the integrals J_i in the right-hand side of the last relation. Using (2.4) and (3.5) we get

$$\begin{aligned} |J_1| = \Big| \nu \int_{\Omega_K \setminus \Omega_l^\sharp} \nabla \mathbf{v} \cdot \nabla \Phi \cdot \mathbf{v} dx \Big| \le 2\nu \int_{\Omega_K \setminus \Omega_l^\sharp} |\nabla \mathbf{v}| |\mathbf{v}| |\varphi'(x_2)| \varphi^{-3}(x_2) dx \\ \le c \sup_{x \in \Omega_K \setminus \Omega_l^\sharp} |\varphi'(x_2)| \Big(\int_{\Omega_K \setminus \Omega_l^\sharp} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx \Big)^{1/2} \Big(\int_{\Omega_K \setminus \Omega_l^\sharp} \varphi^{-4}(x_2) |\mathbf{v}|^2 dx \Big)^{1/2} \\ \le c \sup_{x \in \Omega_K \setminus \Omega_l^\sharp} |\varphi'(x_2)| \int_{\Omega_K \setminus \Omega_l^\sharp} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx; \end{aligned}$$

$$\begin{aligned} |J_2| \le \nu \sum_{j=l+1}^{K-1} \Big| \int_{\omega_j} \nabla \mathbf{v} \cdot \nabla \mathbf{w}_j dx \Big| + \Big| \int_{\widehat{\omega}_K} \nabla \mathbf{v} \cdot \nabla \mathbf{w}_K dx \Big| \\ \le \varepsilon \sum_{j=l+1}^{K-1} \int_{\omega_j} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx + \varepsilon \int_{\widehat{\omega}_K} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx \\ + c_\varepsilon \sum_{j=l+1}^{K-1} \int_{\omega_j} \varphi^2(x_2) |\nabla \mathbf{w}_j|^2 dx + c_\varepsilon \int_{\widehat{\omega}_K} \varphi^2(x_2) |\nabla \mathbf{w}_K|^2 dx \\ \le \varepsilon \int_{\Omega_K \setminus \Omega_l^\sharp} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx + c_\varepsilon \sup_{x \in \Omega_K \setminus \Omega_l^\sharp} |\varphi'(x_2)|^2 \Big(\sum_{j=l+1}^{K-1} \int_{\omega_j} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx \\ + \int_{\widehat{\omega}_K} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx \Big) \le \big(\varepsilon + c_\varepsilon \sup_{x \in \Omega_K \setminus \Omega_l^\sharp} |\varphi'(x_2)|^2 \big) \int_{\Omega_K \setminus \Omega_l^\sharp} \varphi^{-2}(x_2) |\nabla \mathbf{v}|^2 dx; \end{aligned}$$

$$\begin{aligned}
\Big|\int_{\Omega_K} \mathbf{f}\cdot\Phi\mathbf{v}dx\Big| &\le \frac{1}{\varphi^2(h_l)}\Big|\int_{\Omega_l^\sharp} \mathbf{f}\cdot\mathbf{v}dx\Big| \\
&+\sum_{j=l+1}^{K-1}\Big|\int_{\omega_j}\mathbf{f}\cdot\frac{1}{\varphi^2(x_2)}\mathbf{v}dx\Big| + \Big|\int_{\widehat{\omega}_K}\mathbf{f}\cdot\frac{1}{\varphi^2(x_2)}\mathbf{v}dx\Big| \\
&\le c_\varepsilon\Big(\frac{1}{\varphi^2(h_l)}\int_{\Omega_l^\sharp}|\mathbf{f}|^2dx + \sum_{j=l+1}^{K-1}\int_{\omega_j}|\mathbf{f}|^2dx + \int_{\widehat{\omega}_K}|\mathbf{f}|^2dx\Big) \\
&+\varepsilon\Big(\frac{1}{\varphi^2(h_l)}\int_{\Omega_l^\sharp}|\mathbf{v}|^2dx + \sum_{j=l+1}^{K-1}\int_{\omega_j}\frac{1}{\varphi^4(x_2)}|\mathbf{v}|^2dx + \int_{\widehat{\omega}_K}\frac{1}{\varphi^4(x_2)}|\mathbf{v}|^2dx\Big) \\
&\le c\int_{\Omega_K}|\mathbf{f}|^2dx + \varepsilon\Big(\frac{c}{\varphi^2(h_l)}\int_{\Omega_l^\sharp}|\nabla\mathbf{v}|^2dx + \sum_{j=l+1}^{K-1}\int_{\omega_j}\frac{1}{\varphi^2(x_2)}|\nabla\mathbf{v}|^2dx \\
&+\int_{\widehat{\omega}_K}\frac{1}{\varphi^2(x_2)}|\nabla\mathbf{v}|^2dx\Big) \le c\int_{\Omega_K}|\mathbf{f}|^2dx + c\varepsilon\int_{\Omega_K}\Phi|\nabla\mathbf{v}|^2dx;
\end{aligned}$$

$$\begin{aligned}
\Big|\int_{\Omega_K}\mathbf{f}\cdot\mathbf{w}dx\Big| &= \Big|\int_{\Omega_K\setminus\Omega_l^\sharp}\mathbf{f}\cdot\mathbf{w}dx\Big| \le \sum_{j=l+1}^{K-1}\Big|\int_{\omega_j}\mathbf{f}\cdot\mathbf{w}_jdx\Big| + \Big|\int_{\widehat{\omega}_K}\mathbf{f}\cdot\mathbf{w}_Kdx\Big| \\
&\le c\int_{\Omega_K}|\mathbf{f}|^2dx + \sum_{j=l+1}^{K-1}\int_{\omega_j}|\mathbf{w}_j|^2dx + \int_{\widehat{\omega}_K}|\mathbf{w}_K|^2dx \\
&\le c\int_{\Omega_K}|\mathbf{f}|^2dx + \sum_{j=l+1}^{K-1}\int_{\omega_j}\varphi^2(x_2)|\nabla\mathbf{w}_j|^2dx + \int_{\widehat{\omega}_K}\varphi^2(x_2)|\nabla\mathbf{w}_K|^2dx \\
&\le c\int_{\Omega_K}|\mathbf{f}|^2dx + c\sum_{j=l+1}^{K-1}\sup_{x\in\omega_j}|\varphi'(x_2)|^2\int_{\omega_j}\varphi^{-4}(x_2)|v_2|^2dx \\
&+c\sup_{x\in\widehat{\omega}_K}|\varphi'(x_2)|^2\int_{\widehat{\omega}_K}\varphi^{-4}(x_2)|v_2|^2dx \\
&\le c\int_{\Omega_K}|\mathbf{f}|^2dx + c\sup_{x\in\Omega_K\setminus\Omega_l^\sharp}|\varphi'(x_2)|^2\int_{\Omega_K\setminus\Omega_l^\sharp}\varphi^{-2}(x_2)|\nabla v|^2dx.
\end{aligned}$$

Collecting the obtained estimates yields

$$\begin{aligned}
\nu\int_{\Omega_K}\Phi|\nabla\mathbf{v}|^2dx &\le c_1\int_{\Omega_K}|\mathbf{f}|^2dx \\
&+c_2\big(\varepsilon + \sup_{x\in\Omega_K\setminus\Omega_l^\sharp}|\varphi'(x_2)|\big)\int_{\Omega_K}\Phi|\nabla v|^2dx,
\end{aligned}\tag{3.6}$$

where the constant c_1 is independent of K (but c_1 depends on l) and c_2 is independent of K and l. The function $\varphi'(x_2)$ is monotonically decreasing and tends to zero as $x_2 \to 0$. Hence $\sup_{x\in\Omega_K\setminus\Omega_l^\sharp}|\varphi'(x_2)| = |\varphi'(h_l)|$ and $\lim_{l\to\infty}\varphi'(h_l) = 0$. We choose and fix l such that $|\varphi'(h_l)| \le \nu/(4c_2)$ and take $\varepsilon = \nu/(4c_2)$. Then from (3.6) follows the inequality

$$\int_{\Omega_K}\Phi|\nabla\mathbf{v}|^2dx \le c_1\int_{\Omega_K}|\mathbf{f}|^2dx$$

which is equivalent to (3.3). ∎

Lemma 3.3. *For sufficiently large K the weak solution $\mathbf{v}$ of problem* (3.1) *satisfies the estimate*

$$\|\nabla^2\mathbf{v}\|_{L^2(\Omega_K)} \le c\|\mathbf{f}\|_{L^2(\Omega_K)}\tag{3.7}$$

with a constant c independent of K.

Proof. Consider the solution $(\mathbf{v}, p)$ of problem (3.1) in the domain $\omega_l^\star$, $l = 2, \dots, K-2$. Changing the variables $y = \mathcal{P}_l(x)$ (see (2.2)) we rewrite problem (3.1) in coordinate y in the domain G_2:

$$\begin{cases}
-\nu\Delta_y\widehat{\mathbf{v}}(y) + \nabla_y\widehat{q}(y)\big) = \frac{\varphi^2(h_{l-1})}{4L^2}\widehat{\mathbf{f}}(y) + \widehat{\mathbf{H}}(y), \\
\operatorname{div}_y\tilde{\mathbf{v}}(y) = \widehat{g}(y), \\
\tilde{\mathbf{v}}\big|_{y_1=\pm 2L} = 0,
\end{cases}\tag{3.8}$$

where $\widehat{\mathbf{v}}(y)=\mathbf{v}(\mathcal{P}_l^{-1}y)$, etc., $\widehat{q}(y)=\frac{\varphi(h_{l-1})}{2L}\widehat{p}(y)$,

$$
\begin{gathered}
\widehat{\mathbf{H}}(y)=\alpha_{11}(y)\frac{\partial^2\widehat{\mathbf{v}}}{\partial y_1^2}+\alpha_{12}(y)\frac{\partial^2\widehat{\mathbf{v}}}{\partial y_1\partial y_2}+\beta_1(y)\frac{\partial\widehat{\mathbf{v}}}{\partial y_1}+\gamma(y)\frac{\partial\widehat{q}}{\partial y_1},\\
\alpha_{11}(y)=\nu\Big(\frac{(\varphi^2(h_{l-1})-\varphi^2(x_2))}{\varphi^2(x_2)}+\frac{x_1^2\varphi'(x_2)\varphi^2(h_{l-1})}{\varphi^4(x_2)}\Big)\Big|_{x=\mathcal{P}_l^{-1}(y)},\\
\alpha_{12}(y)=-\nu\frac{2x_1\varphi'(x_2)\varphi(h_{l-1})}{\varphi^2(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)},\\
\beta_1(y)=-\nu\frac{x_1\big(\varphi''(x_2)\varphi(x_2)-2\varphi'^2(x_2)\big)\varphi^2(h_{l-1})}{2L\varphi^3(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)},\\
\gamma(y)=(\gamma_1(y),\gamma_2(y))^T,\\
\gamma_1(y)=-\frac{\varphi(h_{l-1})-\varphi(x_2)}{\varphi(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)},\ \gamma_2(y)=-\frac{x_1\varphi'(x_2)\varphi(h_{l-1})}{\varphi^2(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)},\\
\widehat{g}(y)=\mu_1(y)\frac{\partial\widehat{v}_1}{\partial y_1}+\mu_2(y)\frac{\partial\widehat{v}_2}{\partial y_1},\\
\mu_1(y)=\frac{\varphi(x_2)-\varphi(h_{l-1})}{\varphi(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)},\ \mu_2(y)=\frac{x_1\varphi'(x_2)\varphi(h_{l-1})}{\varphi^2(x_2)}\Big|_{x=\mathcal{P}_l^{-1}(y)}.
\end{gathered}
\tag{3.9}
$$

Applying the usual local ADN-estimates for elliptic problems (see [19, 20]) in the pair of domains $G_0\subset G_2$, we obtain the estimate

$$
\begin{aligned}
&\|\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y^2\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{q}\|^2_{L^2(G_0)}\\
&\le c\Big(\varphi^4(h_{l-1})\|\widehat{\mathbf{f}}\|^2_{L^2(G_2)}+\|\widehat{\mathbf{H}}\|^2_{L^2(G_2)}+\|\widehat{g}\|^2_{W^{1,2}(G_2)}\\
&\quad+\|\widehat{\mathbf{v}}\|^2_{L^2(G_2)}+\|\widehat{q}-\bar{q}\|^2_{L^2(G_2)}\Big),
\end{aligned}
\tag{3.10}
$$

where $\bar{q}=\frac{1}{|G_2|}\int\limits_{G_2}\widehat{q}(y)dy$. Since $\int\limits_{G_2}(\widehat{q}(y)-\bar{q})dy=0$, there exists $\mathbf{w}\in\overset{\circ}{W}{}^{1,2}(G_2)$ such that $\operatorname{div}_y\mathbf{w}=\widehat{q}(y)-\bar{q}$ in G_2 and

$$\|\nabla_y\mathbf{w}\|_{L^2(G_2)}\le c\|\widehat{q}-\bar{q}\|_{L^2(G_2)}$$

(see [17]). Multiplying (3.8) by $\mathbf{w}$ and integrating by parts yields

$$
\begin{aligned}
\|\widehat{q}-\bar{q}\|^2_{L^2(G_2)}&=\int\limits_{G_2}\widehat{q}(y)(\widehat{q}(y)-\bar{q})dy=\int\limits_{G_2}\widehat{q}(y)\operatorname{div}_y\mathbf{w}dy\\
&=\nu\int\limits_{G_2}\nabla_y\widehat{\mathbf{v}}\cdot\nabla_y\mathbf{w}dy-\frac{\varphi^2(h_{l-1})}{4L^2}\int\limits_{G_2}\widehat{\mathbf{f}}\cdot\mathbf{w}dy-\int_{G_2}\widehat{\mathbf{H}}\cdot\mathbf{w}dy\\
&\le\nu\|\nabla_y\widehat{\mathbf{v}}\|_{L^2(G_2)}\|\nabla_y\mathbf{w}\|_{L^2(G_2)}+\frac{\varphi^2(h_{l-1})}{4L^2}\|\widehat{\mathbf{f}}\|_{L^2(G_2)}\|\mathbf{w}\|_{L^2(G_2)}+\|\widehat{\mathbf{H}}\|_{L^2(G_2)}\|w\|_{L^2(G_2)}\\
&\le c\|\nabla_y\widehat{\mathbf{v}}\|_{L^2(G_2)}\|\widehat{q}-\bar{q}\|_{L^2(G_2)}+c\varphi^2(h_{l-1})\|\widehat{\mathbf{f}}\|_{L^2G_2)}\|\widehat{q}-\bar{q}\|_{L^2(G_2)}+c\|\widehat{\mathbf{H}}\|_{L^2(G_2)}\|\hat{q}-\bar{q}\|_{L^2(G_2)}.
\end{aligned}
$$

Therefore,

$$\|\widehat{q}-\bar{q}\|_{L^2(G_2)}\le c\Big(\|\nabla_y\widehat{\mathbf{v}}\|_{L^2(G_2)}+\varphi^2(h_{l-1})\|\widehat{\mathbf{f}}\|_{L^2(G_2)}+\|\widehat{\mathbf{H}}\|_{L^2(G_2)}\Big).\tag{3.11}$$

From (3.10) using (3.11) and Poincaré's inequality (2.3) we derive

$$
\begin{aligned}
&\|\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y^2\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{q}\|^2_{L^2(G_0)}\\
&\le c\Big(\varphi^4(h_{l-1})\|\widehat{\mathbf{f}}\|^2_{L^2(G_2)}+\|\widehat{\mathbf{H}}\|^2_{L^2(G_2)}+\|\widehat{g}\|^2_{W^{1,2}(G_2)}+\|\nabla_y\widehat{\mathbf{v}}\|^2_{L^2(G_2)}\Big).
\end{aligned}
\tag{3.12}
$$

By definition $\omega_l^\star=\{x:|x_1|\le\varphi(x_2),\ h_{l-1}-\frac{\varphi(h_{l-1})}{L}<x_2<h_{l-1}+\frac{\varphi(h_{l-1})}{2L}\}$ and the following inequality

$$|\varphi(h_{l-1})-\varphi(x_2)|\le\max_{x_2\in\omega_l^\star}|\varphi'(x_2)||h_{l-1}-x_2|\le c\max_{x_2\in\omega_l^\star}|\varphi'(x_2)|\varphi(h_{l-1})$$

holds. For $y=\mathcal{P}_l(x),\ x\in\omega_l^\star$, using this inequality, (2.1) and the definition $\varphi(x_2)=\gamma_0x_2^\lambda,\ \lambda>1$, we obtain

$$|\alpha_{11}(y)|+|\alpha_{12}(y)|+|\beta_1(y)|+|\gamma_1(y)|+|\gamma_1(y)|+|\mu_1(y)|+|\mu_2(y)|\le\varepsilon(l),$$

where $\lim\limits_{l\to\infty}\varepsilon(l)=0$ (see (3.9)). Therefore, from (3.12) follows the estimate

$$
\begin{aligned}
&\|\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y^2\widehat{\mathbf{v}}\|^2_{L^2(G_0)}+\|\nabla_y\widehat{q}\|^2_{L^2(G_0)}\\
&\le c\Big(\varphi^4(h_{l-1})\|\widehat{\mathbf{f}}\|^2_{L^2(G_2)}+\|\nabla_y\widehat{\mathbf{v}}\|^2_{L^2(G_2)}\Big)\\
&\quad+c\varepsilon(l)\Big(\|\nabla_y^2\widehat{\mathbf{v}}\|^2_{L^2(G_2)}+\|\nabla_y\widehat{q}\|^2_{L^2(G_2)}\Big).
\end{aligned}
$$

Passing to coordinates x and using the same arguments we derive

$$\begin{aligned}&\tfrac{1}{\varphi^2(h_{l-1})}\|\mathbf{v}\|^2_{L^2(\omega_l)}+\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_l)}+\varphi^2(h_{l-1})\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_l)}\\&+\|\nabla_x q\|^2_{L^2(\omega_l)}\le c\Big(\varphi^2(h_{l-1})\|\mathbf{f}\|^2_{L^2(\omega_l^*)}+\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_l^*)}\Big)\\&+c\varepsilon(l)\Big(\varphi^2(h_{l-1})\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_l^*)}+\|\nabla_x q\|^2_{L^2(\omega_l^*)}\Big),\ l=1,\dots,K-2.\end{aligned}\tag{3.13}$$

The constant c in (3.13) is independent of l. Multiplying (3.13) by $\frac{1}{\varphi^2(h_{l-1})}$ yields

$$\begin{aligned}&\tfrac{1}{\varphi^4(h_{l-1})}\|\mathbf{v}\|^2_{L^2(\omega_l)}+\tfrac{1}{\varphi^2(h_{l-1})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_l)}+\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_l)}\\&+\|\nabla_x p\|^2_{L^2(\omega_l)}\le c\Big(\|\mathbf{f}\|^2_{L^2(\omega_l^*)}+\tfrac{1}{\varphi^2(h_{l-1})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_l^*)}\Big)\\&+c\varepsilon(l)\Big(\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_l^*)}+\|\nabla_x p\|^2_{L^2(\omega_l^*)}\Big),\ l=1,\dots,K-2.\end{aligned}\tag{3.14}$$

By the same ADN-estimate together with the properties of the domain $\widehat{\omega}_K$, we get the inequalities

$$\begin{aligned}&\tfrac{1}{\varphi^4(h_{K-2})}\|\mathbf{v}\|^2_{L^2(\omega_{K-1})}+\tfrac{1}{\varphi^2(h_{K-2})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_{K-1})}+\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_{K-1})}\\&+\|\nabla_x p\|^2_{L^2(\omega_{K-1})}\le c\Big(\|\mathbf{f}\|^2_{L^2(\omega_{K-1}^*)}+\tfrac{1}{\varphi^2(h_{K-2})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_{K-1}^*)}\Big)\\&+c\varepsilon(K)\Big(\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_{K-1}^*)}+\|\nabla_x p\|^2_{L^2(\omega_{K-1}^*)}\Big)\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}&\tfrac{1}{\varphi^4(h_{K-1})}\|\mathbf{v}\|^2_{L^2(\omega_K)}+\tfrac{1}{\varphi^2(h_{K-1})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_k)}+\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_K)}\\&+\|\nabla_x p\|^2_{L^2(\omega_K)}\le c\Big(\|\mathbf{f}\|^2_{L^2(\omega_K^*)}+\tfrac{1}{\varphi^2(h_{K-1})}\|\nabla_x\mathbf{v}\|^2_{L^2(\omega_K^*)}\Big)\\&+c\varepsilon(K)\Big(\|\nabla_x^2\mathbf{v}\|^2_{L^2(\omega_K^*)}+\|\nabla_x p\|^2_{L^2(\omega_K^*)}\Big),\end{aligned}\tag{3.16}$$

with constants independent of K.

Let $l_0 < K-2$ be a positive natural number (l_0 be fixed later). Arguing as above we can prove the following local estimate for the pair of domains $\Omega^\sharp_{l_0+1}\subset\Omega^\sharp_{l_0+2}$:

$$\|\mathbf{v}\|^2_{W^{2,2}(\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega^\sharp_{l_0+1})}\le c\Big(\|\mathbf{f}\|^2_{L^2(\Omega^\sharp_{l_0+2})}+\|\nabla_x\mathbf{v}\|^2_{L^2(\Omega^\sharp_{l_0+2})}\Big).\tag{3.17}$$

Summing inequalities (3.14) from l_0 to $K-2$, adding (3.15)-(3.17) and taking into account that $\varphi(h_{l-1})\sim\varphi(x_2)$ in ω_l and ω_l^* (see (2.1)), we get

$$\begin{aligned}&\|\mathbf{v}\|^2_{W^{2,2}(\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega^\sharp_{l_0+1})}+\|\varphi^{-2}\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}\\&+\|\varphi^{-1}\nabla\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}+\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}\\&\le c_1\Big(\|\mathbf{f}\|^2_{L^2(\Omega_K)}+\|\nabla\mathbf{v}\|^2_{L^2(\Omega^\sharp_{l_0+2})}+\|\varphi^{-1}\nabla\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big)\\&+c_2\varepsilon(l_0)\Big(\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}+\|\nabla p\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big).\end{aligned}\tag{3.18}$$

Since $\lim\limits_{l\to\infty}\varepsilon(l)=0$, we can chose l_0 to satisfy $c_2\varepsilon(l_0)\le\kappa$, where κ is such that

$$\begin{aligned}\kappa\Big(\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}+\|\nabla p\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big)\le\tfrac12\Big(&\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega_K)}\\&+\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big).\end{aligned}$$

Then estimate (3.18) takes the form

$$\begin{aligned}&\|\mathbf{v}\|^2_{W^{2,2}(\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega^\sharp_{l_0+1})}+\|\varphi^{-2}\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}\\&+\|\varphi^{-1}\nabla\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}+\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}+\|\nabla p\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0+1})}\\&\le c_1\Big(\|\mathbf{f}\|^2_{L^2(\Omega_K)}+\|\nabla\mathbf{v}\|^2_{L^2(\Omega^\sharp_{l_0+2})}+\|\varphi^{-1}\nabla\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big).\end{aligned}$$

In particular,

$$\|\nabla^2\mathbf{v}\|^2_{L^2(\Omega_K)} \le c_1\Big(\|\mathbf{f}\|^2_{L^2(\Omega_K)} + \|\nabla\mathbf{v}\|^2_{L^2(\Omega^\sharp_{l_0+2})} + \|\varphi^{-1}\nabla\mathbf{v}\|^2_{L^2(\Omega_K\setminus\Omega^\sharp_{l_0})}\Big). \tag{3.19}$$

Estimating the last two term in the right-hand side of (3.19) by (3.2) and (3.3) we obtain (3.7). ■

3.2 Stokes operator

The most results we present in this subsection are standard (e.g., [21]). Problem (3.1) can be rewritten in the operator form (without loss of generality we suppose that $\mathbf{f} \in J_0(\Omega_K)$[1], adding the gradient part to the pressure)

$$\widetilde{\Delta}\mathbf{v} = \mathbf{f},$$

where $\widetilde{\Delta} = P\Delta : H(\Omega_K) \cap W^{2,2}(\Omega_K) \mapsto J_0(\Omega_K)$ is an unbounded operator with the domain $H(\Omega_K) \cap W^{2,2}(\Omega_K)$, where P is the projector from $L^2(\Omega_K)$ onto $J_0(\Omega_K)$ (Leray's projector). For given $\mathbf{w} \in H(\Omega_K) \cap W^{2,2}(\Omega_K)$ the operator $\widetilde{\Delta}\mathbf{w}$ is defined by

$$-\int_{\Omega_K} \widetilde{\Delta}\mathbf{w}\cdot\mathbf{v}dx = \int_{\Omega_K}\big(-\nu\Delta\mathbf{w} + \nabla p\big)\cdot\mathbf{v}dx = -\nu\int_{\Omega_K}\Delta\mathbf{w}\cdot\mathbf{v}dx = \nu\int_{\Omega_K}\nabla\mathbf{w}\cdot\nabla\mathbf{v}dx \quad \forall\mathbf{v}\in J_0^\infty(\Omega_K)$$

(for $\mathbf{v} \in J_0^\infty(\Omega_K)$ holds $\operatorname{div}\mathbf{v} = 0$). By density argument,

$$-\int_{\Omega_K}\widetilde{\Delta}\mathbf{w}\cdot\mathbf{v}dx = -\nu\int_{\Omega_K}\Delta\mathbf{w}\cdot\mathbf{v}dx \quad \forall\mathbf{v}\in J_0(\Omega_K). \tag{3.20}$$

Hence,

$$-\int_{\Omega_K}|\widetilde{\Delta}\mathbf{w}|^2dx = -\nu\int_{\Omega_K}\Delta\mathbf{w}\cdot\widetilde{\Delta}\mathbf{w}dx. \tag{3.21}$$

From (3.20) also follows the estimate

$$\|\widetilde{\Delta}\mathbf{w}\|_{(H(\Omega_K))^*} \le \|\nabla\mathbf{w}\|_{L^2(\Omega_K)} = \|\mathbf{w}\|_{H(\Omega_K)}.$$

Since $H(\Omega_K) \cap W^{2,2}(\Omega_K)$ is dense in $H(\Omega_K)$, there exists a unique extension of the operator $\widetilde{\Delta}$ (denoted again by $\widetilde{\Delta}$) from $H(\Omega_K)\cap W^{2,2}(\Omega_K)$ to the whole space $H(\Omega_K)$. Moreover, the extension $\widetilde{\Delta} : H(\Omega_K) \mapsto (H(\Omega_K))^*$ is a bijection. $\widetilde{\Delta}$ is called the *Stokes operator*.

It is known (see, e.g., [21, 22]) that

(i) The Stokes operator has a discrete spectrum:

$$-\widetilde{\Delta}\mathbf{w} = \lambda\mathbf{w},\ \ \mathbf{w}\in H(\Omega_K),\quad \mathbf{w}\ne 0;$$

$\lambda_i > 0$, $\lim\limits_{i\to\infty}\lambda_i \to +\infty$.

(ii) The set $\{\mathbf{w}_l\}$ of eigenfunctions of $\widetilde{\Delta}$ is an orthogonal basis in $J_0(\Omega_K)$ and $H(\Omega_K)$, $\|\nabla\mathbf{w}_l\|_{L^2(\Omega_K)} = \sqrt{\lambda_k}$, $\|\mathbf{w}_l\|_{L^2(\Omega_K)} = 1$. Since $\partial\Omega_K$ is smooth, we have $\mathbf{w}_l \in H(\Omega_K)\cap W^{2,2}(\Omega_K)$[2].

Relation (3.21) yields

$$\|\widetilde{\Delta}\mathbf{w}\|_{L^2(\Omega_K)} \le c\|\Delta\mathbf{w}\|_{L^2(\Omega_K)} \le c\|\nabla^2\mathbf{w}\|_{L^2(\Omega_K)}. \tag{3.22}$$

From (3.3) follows the estimate

$$\int_{\Omega_0}|\nabla\mathbf{v}|^2dx + \int_{\Omega_K\setminus\Omega_0}\varphi^{-2}(x_2)|\nabla\mathbf{v}|^2dx \le c\int_{\Omega_K}|\tilde{\Delta}\mathbf{v}|^2dx, \tag{3.23}$$

1 Recall that $J_0(G)$ is the closure of the set $J_0^\infty(G) = \{\mathbf{v}\in C_0^\infty(G) : \operatorname{div}\mathbf{v} = 0\}$ in $L^2(G)$-norm.

2 The eigenfunctions $\mathbf{w}_l$ depend also on K but we will not mark this in the notation.

and from (3.7) we get the inequality

$$\|\nabla^2 \mathbf{w}\|_{L^2(\Omega_K)} \le c\|\tilde{\Delta}\mathbf{w}\|_{L^2(\Omega_K)},$$

which together with (3.22) implies

$$c_1\|\nabla^2 \mathbf{w}\|_{L^2(\Omega_K)} \le \|\tilde{\Delta}\mathbf{w}\|^2_{L^2(\Omega_K)} \le c_2\|\nabla^2 \mathbf{w}\|_{L^2(\Omega_K)}. \tag{3.24}$$

Note that constants in (3.22),(3.24) are independent of K.

4 Solvability of Navier–Stokes problem

4.1 Construction of the extension of boundary data

Consider problem (1.1)–(1.4). Suppose that $\mathbf{a} \in L^2(0,T; W^{3/2,2}_{loc}(\partial\Omega \setminus \{O\}))$, $\mathbf{a}_t \in L^2(0,T; W^{1/2,2}_{loc}(\partial\Omega \setminus \{O\}))$, $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_1^\sharp$.

First we consider the linear extension operator E in the domain $\Omega_3^\sharp$, $E: \ W^{3/2,2}(\partial\Omega_3^\sharp) \mapsto W^{2,2}(\Omega_3^\sharp)$ given by $E\mathbf{a} = \mathbf{w}^{(1)}$, where $\mathbf{w}^{(1)}|_{\partial\Omega_3^\sharp} = \mathbf{a}$. Since the boundary $\partial\Omega \cap \partial\Omega_3^\sharp$ is smooth and $\operatorname{supp}\mathbf{a} \subset \partial\Omega_0 \cap \partial\Omega \subset \partial\Omega_1^\sharp$, the linear operator E is bounded:

$$\|E\mathbf{a}\|^2_{W^{2,2}(\Omega_3^\sharp)} = \|\mathbf{w}^{(1)}\|^2_{W^{2,2}(\Omega_3^\sharp)} \le c\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)}. \tag{4.1}$$

Moreover, $\mathbf{w}^{(1)}$ can be constructed so that $\operatorname{supp}\mathbf{w}^{(1)} \subset \overline{\Omega}_2^\sharp$ (see, e.g., [16]).

If $\mathbf{a} = \mathbf{a}(x,t)$ and $\mathbf{a}_t(\cdot,t) \in W^{1/2,2}(\partial\Omega)$, then, due to the fact that the operator E is linear, we have $E\mathbf{a}_t = \mathbf{w}_t^{(1)}$ and

$$\|E\mathbf{a}_t\|^2_{W^{1,2}(\Omega_3^\sharp)} \le c\|\mathbf{a}_t\|^2_{W^{1/2,2}(\partial\Omega)}. \tag{4.2}$$

Moreover, if $\mathbf{a} \in L^2(0,T; W^{3/2,2}(\partial\Omega))$, $\mathbf{a}_t \in L^2(0,T; W^{1/2,2}(\partial\Omega))$, then integrating (4.1), (4.2) by t yields

$$\begin{aligned} \|\mathbf{w}^{(1)}\|^2_{L^2(0,T;W^{2,2}(\Omega_3^\sharp))} &\le c\|\mathbf{a}\|^2_{L^2(0,T;W^{3/2,2}(\partial\Omega))}, \\ \|\mathbf{w}_t^{(1)}\|^2_{L^2(0,T;W^{1,2}(\Omega_3^\sharp))} &\le c\|\mathbf{a}_t\|^2_{L^2(0,T;W^{1/2,2}(\partial\Omega))}. \end{aligned} \tag{4.3}$$

Let $\mathbf{U}^{[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t, \tau\right)$ be the formal asymptotic decomposition of the velocity component near the cusp point O constructed in [1] . Recall that

$$\mathbf{U}^{[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t, \tau\right) = \mathbf{U}^{O,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, t\right) + \mathbf{U}^{B,[J]}\left(\frac{x_1}{x_2^\lambda}, x_2, \frac{t}{x_2^{2\lambda}}\right),$$

where $\mathbf{U}^{O,[J]}$ is the outer asymptotic expansion and $\mathbf{U}^{B,[J]}$ is the boundary layer expansion (see also formulas (1.5) in Introduction). In order to insure the existence of $\mathbf{U}^{[J]}$, the following regularity requirements for the boundary value $\mathbf{a}$ are needed:

$$\frac{\partial^l \mathbf{a}}{\partial t^l} \in L^2(0,T; W^{1/2,2}(\partial\Omega)), \quad l = 1,2,\dots,J+1.$$

It is proved (see inequality (4.15) in [1]) that the vector field $\mathbf{U}^{[J]}$ satisfies the following estimates

$$\begin{aligned} &\sup_{t\in[0,T]} \|\mathbf{U}^{[J]}(\cdot,y_2,t)\|^2_{W^{1,2}(\Upsilon)} + \|\mathbf{U}^{[J]}\|^2_{L^2(0,T;W^{2,2}(\Upsilon))} \\ &\quad + \|\mathbf{U}_t^{[J]}\|^2_{L^2(0,T;L^2(\Upsilon))} \le \frac{c}{\varphi^2(y_2)} \int_0^T |||F|||^2_{J+1}\, dt, \\ &\sup_{t\in[0,T]} \|\mathbf{U}^{[J]}(\cdot,y_2,t)\|^2_{W^{2,2}(\Upsilon)} + \sup_{t\in[0,T]} \|\mathbf{U}_t^{[J]}\|^2_{L^2(\Upsilon)} \\ &\quad + \|\nabla\mathbf{U}_t^{[J]}\|^2_{L^2(0,\infty;L^2(\Upsilon))} \le \frac{c}{\varphi^2(y_2)} \int_0^T |||F|||^2_{J+2}\, dt, \\ &\sup_{t\in[0,T]} \left\|\frac{\partial \mathbf{U}^{[J]}(\cdot,y_2,t)}{\partial y_2}\right\|^2_{W^{1,2}(\Upsilon)} \le \frac{c}{\varphi^4(y_2)} \int_0^T |||F|||^2_{J+1}\, dt, \end{aligned} \tag{4.4}$$

where $y_1 = \frac{x_1}{x_2^\lambda}$, $y_2 = x_2$, $\varphi(y_2) = \gamma_0 y_2^\lambda$, $\lambda > 1$, $\Upsilon = (-\gamma_0, \gamma_0)$, $|||F|||_J^2 = \sum_{k=0}^{J} \left|\frac{\partial^k F(t)}{\partial t^k}\right|^2$.

Since $W^{1,2}(\Upsilon) \subset C(\Upsilon)$, we have

$$\begin{aligned}
&\sup_{\substack{t\in(0,T)\\ y_1\in\Upsilon}} \left(|\mathbf{U}^{[J]}(y_1, y_2, t)|^2 + \left|\frac{\partial \mathbf{U}^{[J]}(y_1,y_2,t)}{\partial y_1}\right|^2\right)\\
&\leq c \sup_{t\in(0,T)} \|\mathbf{U}^{[J]}(\cdot, y_2, t)\|^2_{W^{2,2}(\Upsilon)} \leq \frac{c}{\varphi^2(y_2)} \int_0^T |||F|||^2_{J+2}\, dt,\\
&\sup_{\substack{t\in(0,T)\\ y_1\in\Upsilon}} \left|\frac{\partial \mathbf{U}^{[J]}(y_1,y_2,t)}{\partial y_2}\right|^2 dt \leq c \sup_{t\in(0,T)} \left\|\frac{\partial \mathbf{U}^{[J]}(\cdot,y_2,t)}{\partial y_2}\right\|^2_{W^{1,2}(\Upsilon)}\\
&\leq \frac{c}{\varphi^4(y_2)} \int_0^T |||F|||^2_{J+1}\, dt.
\end{aligned} \tag{4.5}$$

Passing to coordinates x we obtain

$$\begin{aligned}
&\sup_{\substack{t\in(0,T)\\ x_1\in(-\varphi(x_2),\varphi(x_2))}} |\mathbf{U}^{[J]}(x_1, x_2, t)|^2 dt \leq \frac{c}{\varphi^2(x_2)} \int_0^T |||F|||^2_{J+1}\, dt,\\
&\sup_{\substack{t\in(0,T)\\ x_1\in(-\varphi(x_2),\varphi(x_2))}} |\nabla_x \mathbf{U}^{[J]}(x_1, x_2, t)|^2 \leq \frac{c}{\varphi^4(x_2)} \int_0^T |||F|||^2_{J+1}\, dt.
\end{aligned} \tag{4.6}$$

Notice that $F(t) = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS$, and hence,

$$\int_0^T |||F|||_J^2 dt \leq c \int_0^T |||\mathbf{a}|||_J^2 dt, \quad \sup_{t\in(0,T)} |||F|||_J^2 dt \leq c \int_0^T |||\mathbf{a}|||_{J+1}^2 dt, \tag{4.7}$$

where $|||\mathbf{a}|||_J^2 = \sum_{k=0}^{J} \left\|\frac{\partial^k \mathbf{a}(\cdot,t)}{\partial t^k}\right\|^2_{W^{1/2,2}(\partial\Omega)}$.

Consider the function $\mathbf{B} = \mathbf{w}^{(1)} + \zeta\mathbf{U}^{[J]}$, where $\zeta = \zeta(x_2)$ is a smooth cut-off function equal to one in $\Omega\setminus\Omega_2^\sharp$ and equal to zero in $\Omega_1^\sharp$. Obviously, $\mathbf{B}|_{\partial\Omega} = \mathbf{a}$, however, $\mathbf{B}$ is not solenoidal, $\operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{w}^{(1)} + \nabla\zeta\cdot\mathbf{U}^{[J]} := h$. Notice that

$$\begin{aligned}
\int_{\Omega_2^\sharp} h dx &= \int_{\partial\Omega_2^\sharp} (\mathbf{w}^{(1)} + \zeta\mathbf{U}^{[J]})\cdot\mathbf{n} dS = \int_{\partial\Omega_0\cap\partial\Omega} \mathbf{a}\cdot\mathbf{n} dS + \int_{\partial\Omega_2^\sharp\setminus\partial\Omega} \mathbf{U}^{[J]}\cdot\mathbf{n} dS\\
&= F(t) - F(t) = 0.
\end{aligned}$$

Since $\operatorname{supp} h \subset \overline{\Omega_2^\sharp}$ and the boundary $\partial\Omega_3^\sharp \cap \partial\Omega$ is smooth, there exist a function $\mathbf{w}^{(2)} \in W^{2,2}(\Omega_3^\sharp)$ such that $\operatorname{supp}\mathbf{w}^{(2)} \subset \overline{\Omega_3^\sharp}$, $\mathbf{w}^{(2)} = 0$ in the neighbourhood of $\partial\Omega_3^\sharp\setminus\partial\Omega$ and

$$\begin{cases} \operatorname{div}\mathbf{w}^{(2)} = h \ \text{ in } \ \Omega_3^\sharp,\\ \mathbf{w}|^{(2)}_{\partial\Omega_3^\sharp} = 0. \end{cases} \tag{4.8}$$

Moreover,

$$\begin{aligned}
&\|\mathbf{w}^{(2)}(\cdot,t)\|^2_{W^{2,2}(\Omega_3^\sharp)} \leq c\|h(\cdot,t)\|^2_{W^{1,2}(\Omega_3^\sharp)}\\
&\leq c\left(\|\mathbf{w}^{(1)}(\cdot,t)\|^2_{W^{2,2}(\Omega_3^\sharp)} + \|\mathbf{U}^{[J]}(\cdot,t)\|^2_{W^{1,2}(\Omega_3^\sharp)}\right)
\end{aligned} \tag{4.9}$$

(see [23]). By construction in [23], it follows that the operator $\mathcal{D}$ of problem (4.8), $\mathcal{D} : \mathbf{w}^{(2)}(\cdot,t) \in W^{2,2}(\Omega_3^\sharp) \mapsto \operatorname{div}\mathbf{w}^{(2)}(\cdot,t) = h(\cdot,t) \in W^{1,2}(\Omega_3^\sharp)$ is linear and the inverse operator $\mathcal{D}^{-1}$ defined on functions $h \in W^{1,2}(\Omega_3^\sharp)$, satisfying the condition $\int_{\Omega_3^\sharp} h\, dx = 0$, is bounded. Moreover, equations (4.8) can be differentiated with respect to t:

$$\begin{cases} \operatorname{div}\mathbf{w}_t^{(2)} = h_t \ \text{ in } \ \Omega_3^\sharp,\\ \mathbf{w}_t|^{(2)}_{\partial\Omega_3^\sharp} = 0, \end{cases}$$

and

$$\begin{aligned} \|\mathbf{w}_t^{(2)}(\cdot,t)\|^2_{W^{1,2}(\Omega_3^\sharp)} &\le c\|h_t(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} \\ &\le c\Big(\|\mathbf{w}_t^{(1)}(\cdot,t)\|^2_{W^{1,2}(\Omega_3^\sharp)} + \|\mathbf{U}_t^{[J]}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)}\Big). \end{aligned} \tag{4.10}$$

Integrating inequalities (4.9), (4.10) with respect to t and using estimates (4.3), (4.4) and (4.7) we obtain

$$\begin{aligned} \|\mathbf{w}^{(2)}\|^2_{L^2(0,T;W^{2,2}(\Omega_3^\sharp))} &\le c\int_0^T \Big(\|\mathbf{a}(\cdot,t)\|^2_{W^{3/2,2}(\partial\Omega)} + |||\mathbf{a}(\cdot,t)|||^2_{J+1} \\ &+|||F(t)|||^2_{J+1}\Big)dt \le c\int_0^T \Big(\|\mathbf{a}(\cdot,t)\|^2_{W^{3/2,2}(\partial\Omega)} + |||\mathbf{a}(\cdot,t)|||^2_{J+1}\Big)dt \\ &= c\int_0^T \langle\mathbf{a}\rangle^2_{J+1}dt, \\ \|\mathbf{w}_t^{(2)}\|^2_{L^2(0,T;W^{1,2}(\Omega_3^\sharp))} &\le c\int_0^T \Big(\|\mathbf{a}_t(\cdot,t)\|^2_{W^{1/2,2}(\partial\Omega)} + |||F(t)|||^2_{J+1}\Big)dt \\ &\le \int_0^T |||\mathbf{a}(\cdot,t)|||^2_{J+1}dt, \end{aligned} \tag{4.11}$$

where $\langle\mathbf{a}\rangle^2_{J+1} = \|\mathbf{a}(\cdot,t)\|^2_{W^{3/2,2}(\partial\Omega)} + |||\mathbf{a}(\cdot,t)|||_{J+1}$.

Define

$$\mathbf{W} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}, \quad \mathbf{V} = \mathbf{W} + \zeta\mathbf{U}^{[J]},$$

where ζ is a smooth cut-off function defined above. By construction $\operatorname{div}\mathbf{V} = 0$, $\mathbf{V}|_{\partial\Omega} = \mathbf{a}$ and $\mathbf{V} = \mathbf{U}^{[J]}$ for $x \in \Omega\setminus\Omega_3^\sharp$. Therefore, for $x \in \Omega\setminus\Omega_3^\sharp$ the function $\mathbf{V}$ satisfies estimates (4.4), (4.6), while for $x \in \Omega_3^\sharp$ from (4.11) and (4.4) it follows that

$$\begin{aligned} &\int_0^T \|\mathbf{V}(\cdot,t)\|^2_{W^{2,2}(\Omega_3^\sharp)}dt + \int_0^T \|\mathbf{V}_t(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)}dt \\ &\le c\int_0^T \Big(\|\mathbf{a}\|^2_{W^{3/2,2}(\partial\Omega)} + |||\mathbf{a}|||^2_{J+1} + |||F|||^2_{J+1}\Big)dt \\ &\le c\int_0^T \langle\mathbf{a}\rangle^2_{J+1}dt. \end{aligned} \tag{4.12}$$

We look for the solution $(\mathbf{u}, p)$ of problem (1.1) in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{V}, \quad p = q + \zeta P^{[J]}.$$

Then for $(\mathbf{v}, q)$ we obtain the following problem

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{V}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{V} + \nabla q = \widehat{\mathbf{f}}, \\ \operatorname{div}\mathbf{v} = 0, \\ \mathbf{v}|_{\partial\Omega\setminus O} = 0, \\ \mathbf{v}(x,0) = \widehat{\mathbf{u}}_0(x), \end{cases} \tag{4.13}$$

where $\widehat{\mathbf{f}} = \mathbf{f} - \mathbf{f}_1$, $\mathbf{f}_1 = \mathbf{V}_t - \nu\Delta\mathbf{V} + (\mathbf{V}\cdot\nabla)(\mathbf{V}) + \nabla(\zeta P^{[J]})$, $\widehat{\mathbf{u}}_0 = \mathbf{b} - \mathbf{W}|_{t=0}$ (since $\mathbf{U}^{[J]}(x,0) = 0$). Recall that the number J was chosen such that

$$\mathbf{U}_t^{[J]} - \nu\Delta\mathbf{U}^{[J]} + (\mathbf{U}^{[J]}\cdot\nabla)\mathbf{U}^{[J]} + \nabla P^{[J]} = \mathbf{H}^{(J)} \in L^2(0,T;L^2(G_H)).$$

(see [1]). Therefore, taking into account that $\mathbf{W}$ has compact support in $\overline{\Omega}_3^\sharp$, we conclude

$$\widehat{\mathbf{f}} \in L^2(0,T;L^2(\Omega)), \quad \widehat{\mathbf{u}}_0 \in \mathring{W}^{1,2}(\Omega).$$

Moreover, using results obtained in [1] we get (see estimate (4.19) in [1])

$$\begin{aligned} \|\widehat{\mathbf{f}}\|^2_{L^2(0,T;L^2(\Omega))} &\le c\Big(\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^T \langle\mathbf{a}\rangle^2_{J+1}dt.\Big), \\ \|\widehat{\mathbf{u}}_0\|^2_{W^{1,2}(\Omega)} &\le c\Big(\|\mathbf{b}\|^2_{W^{1,2}(\Omega)} + \int_0^T \langle\mathbf{a}\rangle^2_{J+1}dt\Big). \end{aligned} \tag{4.14}$$

In the next subsections we construct the sequence of weak solutions $\mathbf{v}_K$ to the Navier–Stokes equations in regular domains Ω_K and prove the uniform (with respect to K) estimates for them. The solution of problem (4.13) is then found as a limit of $\{\mathbf{v}_K\}$.

4.2 Existence of the solution in Ω_K

Consider in Ω_K the following boundary value problem

$$\begin{cases} \mathbf{v}_{Kt} - \nu\Delta\mathbf{v}_K + (\mathbf{v}_K\cdot\nabla)\mathbf{v}_K + (\mathbf{V}\cdot\nabla)\mathbf{v}_K \\ +(\mathbf{v}_K\cdot\nabla)\mathbf{V} + \nabla q_K = \widehat{\mathbf{f}}, \\ \operatorname{div}\mathbf{v}_K = 0, \\ \mathbf{v}_K|_{\partial\Omega_K} = 0, \\ \mathbf{v}_K(x,0) = \widehat{\mathbf{u}}_{K0}(x), \end{cases} \tag{4.15}$$

where $\widehat{\mathbf{u}}_{K0} \in \overset{\circ}{W}{}^{1,2}(\Omega_K)$, $\operatorname{div}\widehat{\mathbf{u}}_{K0} = 0$ and $\|\widehat{\mathbf{u}}_{K0} - \widehat{\mathbf{u}}_0\|_{W^{1,2}(\Omega)} \to 0$ as $K \to +\infty$ (here we suppose that $\widehat{\mathbf{u}}_{K0}$ is extended by zero into $\Omega \setminus \Omega_K$).

In this subsection we omit the subscript K in notation of the solution $\mathbf{v}_K$.

By a weak solution of problem (4.15) we mean the function $\mathbf{v} \in L^2(0,T;H(\Omega_K))$ with $\mathbf{v}_t, \nabla^2\mathbf{v} \in L^2(0,T;L^2(\Omega_K))$ satisfying the initial condition $\mathbf{v}(x,0) = \widehat{\mathbf{u}}_{K0}(x)$, and for all $t \in [0,T]$ satisfying the integral identity

$$\begin{aligned} &\int_0^T\int_{\Omega_K} \big(\mathbf{v}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta} - \big((\mathbf{v}+\mathbf{V})\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{v} - \big(\mathbf{v}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{V}\big)\,dxdt \\ &= \int_0^T\int_{\Omega_K} \widehat{\mathbf{f}}\cdot\boldsymbol{\eta}\,dxdt \end{aligned} \tag{4.16}$$

for any test function $\boldsymbol{\eta} \in L^2(0,T;H(\Omega_K))$ with $\boldsymbol{\eta}_t \in L^2(0,T;L^2(\Omega_K))$.

Lemma 4.1. *Let $\widehat{\mathbf{u}}_0 \in \overset{\circ}{W}{}^{1,2}(\Omega)$. There exists a sequence $\widehat{\mathbf{u}}_{K0} \in \overset{\circ}{W}{}^{1,2}(\Omega_K)$ such that* $\operatorname{div}\widehat{\mathbf{u}}_{K0} = 0$ *and* $\lim\limits_{K\to\infty}\|\widehat{\mathbf{u}}_{K0} - \widehat{\mathbf{u}}_0\|_{W^{1,2}(\Omega)} = 0$. *Moreover there hods the estimate*

$$\|\widehat{\mathbf{u}}_{K0}\|^2_{W^{1,2}(\Omega)} \le c\Big(\|\mathbf{b}\|^2_{W^{1,2}(\Omega)} + \int_0^T \langle\mathbf{a}\rangle^2_{J+1}\,dt\Big) \tag{4.17}$$

with the constant c independent of K.

Proof. Let $\chi_K(x_2)$ be a smooth cut-off function such that $\chi_K(x_2) = 0$ for $x_2 \ge h_{K-1}$, $\chi_K(x_2) = 1$ for $x_2 \le h_{K-2}$ and $|\nabla\chi(x_2)| \le \frac{c}{\varphi(h_{K-1})}$. Consider the sequence of functions $\widehat{\mathbf{u}}_{K0} = \chi_K(x_2)\widehat{\mathbf{u}}_0 + \widehat{\mathbf{w}}_K$, where $\widehat{\mathbf{w}}_K$ is a solution of the problem

$$\begin{cases} \operatorname{div}\widehat{\mathbf{w}}_K = -\nabla\chi_K\cdot\widehat{\mathbf{u}}_0, & x \in \omega_{K-1}, \\ \widehat{\mathbf{w}}_K = 0, & x \in \partial\omega_{K-1}. \end{cases}$$

Obviously, $\int_{\omega_{K-1}} \nabla\chi_K\cdot\widehat{\mathbf{u}}_0\,dx = 0$. Therefore, by Lemma 2.4, there exists $\widehat{\mathbf{w}}_K \in \overset{\circ}{W}{}^{1,2}(\omega_{K-1})$ satisfying the estimate

$$\begin{aligned} \|\nabla\widehat{\mathbf{w}}_K\|_{L^2(\omega_{K-1})} &\le c\|\nabla\chi_K\cdot\widehat{\mathbf{u}}_0\|_{L^2(\omega_{K-1})} \\ &\le c\|\varphi^{-1}\widehat{\mathbf{u}}_0\|_{L^2(\omega_{K-1})} \le c\|\nabla\widehat{\mathbf{u}}_0\|_{L^2(\omega_{K-1})} \end{aligned} \tag{4.18}$$

with the constant c independent of K. By construction $\operatorname{div}\widehat{\mathbf{u}}_{K0} = 0$ and

$$\|\widehat{\mathbf{u}}_{K0} - \widehat{\mathbf{u}}_0\|_{W^{1,2}(\Omega)} \le \|(1-\chi_K)\widehat{\mathbf{u}}_0\|_{W^{1,2}(\omega_{K-1})} + \|\widehat{\mathbf{w}}_K\|_{W^{1,2}(\omega_{K-1})} \le c\|\widehat{\mathbf{u}}_0\|_{W^{1,2}(\omega_{K-1})}.$$

Since the right-hand side of this inequality tend to zero as $K \to \infty$, we get $\lim\limits_{K\to\infty}\|\widehat{\mathbf{u}}_{K0} - \widehat{\mathbf{u}}_0\|_{W^{1,2}(\Omega)} = 0$. Estimate (4.17) follows from (4.14). ∎

Theorem 4.1. *Suppose that* $\mathbf{f} \in L^2(0, T; \Omega)$, $\mathbf{b} \in W^{1,2}(\Omega)$, *the boundary value* $\mathbf{a}$ *has the finite norm* $\int_0^T \langle \mathbf{a} \rangle^2_{J+2} dt$ *and let there hold the compatibility conditions* $\operatorname{div} \mathbf{b} = 0$, $\mathbf{b}|_{\partial\Omega} = \mathbf{a}(x, 0)$.*There exists a positive constant* κ_0 *such that if the flux* $F(t)$ *satisfies the inequality* $\int_0^T |||F|||^2_{J+2} dt \le \kappa_0$, *then problem* (4.15) *admits at least one weak solution* $\mathbf{v}$. *The following estimates*

$$\sup_{t\in[0,T]} \|\mathbf{v}(\cdot, t)\|^2_{L^2(\Omega_K)} + \int_0^T \|\nabla\mathbf{v}(\cdot, t)\|^2_{L^2(\Omega_K)} dt \le cB_1 \tag{4.19}$$

$$\sup_{t\in[0,T]} \|\nabla\mathbf{v}(\cdot, t)\|^2_{L^2(\Omega_K)} \le ce^{B_2}(A_0 + A_1 + B_1) := cB_3; \tag{4.20}$$

$$\int_0^T \|\nabla^2\mathbf{v}(\cdot, t)\|^2_{L^2(\Omega_K)} dt \le cB_4 \tag{4.21}$$

$$\int_0^T \|\mathbf{v}_t(\cdot, t)\|^2_{L^2(\Omega_K)} dt \le cB_5 \tag{4.22}$$

hold. The constants in estimates (4.19)–(4.22) *are independent of K and*

$$\begin{gathered}
A_0 = \|\mathbf{b}\|^2_{W^{1,2}(\Omega)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))}, \quad A_1 = \int_0^T \langle \mathbf{a} \rangle^2_{J+1} dt, \\
B_1 = (1 + e^{c_4 A_1} A_1)(A_0 + A_1), \quad B_2 = c_9 A_1 + c_{10} B_1^2, \\
B_3 = e^{B_2}(A_0 + A_1 + B_1), \\
B_4 = A_0 + A_1 + A_1 B_3 + B_3 B_1^2 + B_1, \\
B_5 = A_0 + A_1 + (1 + A_1)B_4 + (1 + B_1 B_3)B_1 + B_3 A_1.
\end{gathered}$$

The constants c_4, c_9 *and* c_{10} *are defined in the proof of the theorem.*

Proof. We follow the scheme of O.A. Ladyzhenskaya book [22] (see also [21, 24]) where the solvability of problem (4.15) is proved by the Galerkin method. Let $\{\psi_l\}_{l=1}^\infty$ be an orthogonal basis in the space $H(\Omega_K)$. Consider Galerkin approximations $\mathbf{v}^{(N)}(x, t) = \sum_{l=1}^N \gamma_l^{(N)}(t)\psi_l(x)$ of the solution $\mathbf{v}$ of problem (4.15) which are defined by the following system of ordinary differential equations (with respect to functions $\gamma_l^{(N)}(t)$, $l = 1, \dots, N$):

$$\begin{gathered}
\int_{\Omega_K} \Big(\mathbf{v}_t^{(N)} \cdot \psi_l + \nu\nabla\mathbf{v}^{(N)} \cdot \nabla\psi_l - \big((\mathbf{v}^{(N)} + \mathbf{V}) \cdot \nabla\big)\psi_l \cdot \mathbf{v}^{(N)} \\
-\big(\mathbf{v}^{(N)} \cdot \nabla\big)\psi_l \cdot \mathbf{V}\Big) dx = -\int_{\Omega_K} \widehat{\mathbf{f}} \cdot \psi_l dx, \quad l = 1, \dots, N, \\
\gamma_l^{(N)}(0) = \alpha_l, \quad l = 1, \dots, N,
\end{gathered} \tag{4.23}$$

where α_l are the coefficients of the initial function $\widehat{\mathbf{u}}_{K0}$ in the basis $\{\psi_k\}_{k=1}^\infty$, i.e., $\widehat{\mathbf{u}}_{K0} = \sum_{l=1}^\infty \alpha_l\psi_l(x)$. Multiplying (4.23) by $\gamma_l^{(N)}(t)$ and summing by l from 1 to N we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\Omega_K} |\mathbf{v}^{(N)}|^2 dx + \nu \int_{\Omega_K} |\nabla\mathbf{v}^{(N)}|^2 dx \\
&= \int_{\Omega_K} \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{v}^{(N)} \cdot \mathbf{V} dx + \int_{\Omega_K} \widehat{\mathbf{f}} \cdot \mathbf{v}^{(N)} dx.
\end{aligned}$$

Consider the integral

$$\begin{aligned}
\Big|\int_{\Omega_K} \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{v}^{(N)} \cdot \mathbf{V} dx\Big| &\le \Big|\int_{\Omega_3^\sharp} \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{V} \cdot \mathbf{v}^{(N)} dx\Big| \\
&+ \Big|\int_{\Omega_K \setminus \Omega_3^\sharp} \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{V} \cdot \mathbf{v}^{(N)} dx\Big| = I_1 + I_2.
\end{aligned}$$

For I_1 we have the estimate

$$|I_1| \le \|\nabla \mathbf{V}\|_{L^2(\Omega_3^\sharp)} \|\mathbf{v}^{(N)}\|^2_{L^4(\Omega_3^\sharp)}$$

$$\le c\|\nabla \mathbf{V}\|_{L^2(\Omega_3^\sharp)} \|\mathbf{v}^{(N)}\|_{L^2(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|_{L^2(\Omega_3^\sharp)}$$

$$\le \frac{\nu}{4} \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx + c\|\nabla \mathbf{V}\|^2_{L^2(\Omega_3^\sharp)} \int\limits_{\Omega_K} |\mathbf{v}^{(N)}|^2 dx.$$

Consider I_2. By Poincaré inequality (2.4) and (4.6),

$$\begin{aligned}\Big|\int\limits_{\Omega_K\setminus\Omega_3^\sharp} (\mathbf{v}^{(N)}\cdot\nabla)\mathbf{V}\cdot\mathbf{v}^{(N)}dx\Big| &\le \int\limits_{\Omega_K\setminus\Omega_3^\sharp} |\mathbf{v}^{(N)}|^2 |\nabla \mathbf{U}^{[J]}| dx \\ &\le c\Big(\int\limits_0^T |||F|||^2_{J+2} dt\Big)^{1/2} \int\limits_{\Omega_K\setminus\Omega_3^\sharp} \frac{|\mathbf{v}^{(N)}(x)|^2}{\varphi^2(x_2)} dx \\ &\le c_1\Big(\int\limits_0^T |||F|||^2_{J+2} dt\Big)^{1/2} \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx.\end{aligned}$$

Further,

$$\begin{aligned}\Big|\int\limits_{\Omega_K} \widehat{\mathbf{f}}\cdot\mathbf{v}^{(N)}dx\Big| &\le c(\varepsilon)\Big(\int\limits_{\Omega_3^\sharp} |\widehat{\mathbf{f}}|^2 dx + \int\limits_{\Omega_K\setminus\Omega_3^\sharp} \varphi^2(x_2)|\widehat{\mathbf{f}}|^2 dx\Big) \\ &+\varepsilon\Big(\int\limits_{\Omega_K\setminus\Omega_3^\sharp} \frac{|\mathbf{v}^{(N)}|^2}{\varphi^2(x_2)} dx + \int\limits_{\Omega_3^\sharp} |\mathbf{v}^{(N)}|^2 dx\Big) \\ &\le c(\varepsilon)\int\limits_{\Omega_K} |\widehat{\mathbf{f}}|^2 dx + c_2\varepsilon \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx.\end{aligned}$$

Thus, for $F(t)$ such that $\Big(\int\limits_0^T |||F|||^2_{J+2} dt\Big)^{1/2} \le \frac{\nu}{4c_1}$ and $\varepsilon = \frac{\nu}{4c_2}$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_K} |\mathbf{v}^{(N)}|^2 dx + \frac{\nu}{4}\int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx \\ &\le c_3\|\nabla \mathbf{V}\|^2_{L^2(\Omega_3^\sharp)} \int\limits_{\Omega_K} |\mathbf{v}^{(N)}|^2 dx + c\int\limits_{\Omega_K} |\widehat{\mathbf{f}}|^2 dx.\end{aligned} \tag{4.24}$$

By Gronwall's inequality, (4.24) yields

$$\begin{aligned}\int\limits_{\Omega_K} |\mathbf{v}^{(N)}(x,t)|^2 dx &\le e^{c_3\int_0^t \|\nabla \mathbf{V}(\cdot,s)\|^2_{L^2(\Omega_3^\sharp)} ds} \int\limits_{\Omega_K} |\mathbf{v}^{(N)}(x,0)|^2 dx \\ &+ c e^{c_3\int_0^t \|\nabla \mathbf{V}(\cdot,s)\|^2_{L^2(\Omega_3^\sharp)} ds} \int\limits_0^t e^{-c_3\int_0^\tau \|\nabla \mathbf{V}(\cdot,s)\|^2_{L^2(\Omega_3^\sharp)} ds} \|\widehat{\mathbf{f}}(\cdot,\tau)\|^2_{L^2(\Omega_K)} d\tau \\ &\le e^{c_3\int_0^T \|\nabla \mathbf{V}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} dt}\Big(\int\limits_{\Omega_K} |\widehat{\mathbf{u}}_{K0}(x)|^2 dx + c\int\limits_0^T \|\widehat{\mathbf{f}}(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau\Big).\end{aligned} \tag{4.25}$$

Inequalities (4.24), (4.25) imply

$$\begin{aligned}&\sup_{t\in[0,T]} \|\mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)} + \nu\int\limits_0^T\int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}(x,t)|^2 dxdt \\ &\le c\Big(\int\limits_0^T \|\widehat{\mathbf{f}}(\cdot,t)\|^2_{L^2(\Omega_K)} dt + \int\limits_{\Omega} |\widehat{\mathbf{u}}_{K0}|^2 dx\Big) \\ &+ c\int\limits_0^T \Big(\|\nabla \mathbf{V}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} \int\limits_{\Omega_K} \|\mathbf{u}^{(N)}(x,t)|^2 dx\Big) dt \\ &\le c\Big(1 + e^{c_3\int_0^T \|\nabla \mathbf{V}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} dt} \int\limits_0^T \|\nabla \mathbf{V}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} dt\Big)\widehat{A},\end{aligned}$$

where

$$\widehat{A} = \int_{\Omega_K} |\widehat{\mathbf{u}}_{K0}(x)|^2 dx + \int_0^T \|\widehat{\mathbf{f}}(\cdot, t)\|^2_{L^2(\Omega)} dt$$
$$\le c\Big(\|\mathbf{b}\|^2_{W^{1,2}(\Omega)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega))}\Big) + c\int_0^T \langle \mathbf{a}\rangle^2_{J+1} dt := c\big(A_0 + A_1\big)$$

(see (4.14), (4.17)). By (4.12),

$$\int_0^T \|\mathbf{V}(\cdot, t)\|^2_{W^{2,2}(\Omega_3^\sharp)} dt \le c\int_0^T \langle \mathbf{a}\rangle^2_{J+1} dt. \tag{4.26}$$

Therefore, using (4.17) we obtain

$$\begin{aligned}\sup_{t\in[0,T]} \|\mathbf{v}^{(N)}(\cdot, t)\|^2_{L^2(\Omega_K)} + \nu\int_0^T\int_{\Omega_K} |\nabla \mathbf{v}^{(N)}(x, t)|^2 dxdt \\ \le c_5\Big(1 + e^{c_4 A_1} A_1\Big)\big(A_0 + A_1\big) := c_5 B_1.\end{aligned} \tag{4.27}$$

Estimate (4.27) guaranties that the Cauchy problem (4.23) admits a unique solution for each fixed N. Now we derive a number of a priori estimates for Galerkin approximations $\mathbf{v}^{(N)}$. Estimate (4.27) is valid for Galerkin approximations constructed using an arbitrary orthogonal basis. In order to estimate the higher derivatives of $\mathbf{v}^{(N)}$, as a basis we shall use the eigenfunctions of the Stokes operator.

Taking in (4.23) $\psi_l = \mathbf{w}_l$, where $\mathbf{w}_l$ are eigenfunctions of the Stokes operator, i.e., $-\tilde{\Delta}\mathbf{w}_l = \lambda_l \mathbf{w}_l$, multiplying the obtained relations by $\lambda_l \gamma_l^{(N)}(t)$ and summing by l from 1 to N we obtain

$$\begin{aligned}\int_{\Omega_K} \Big(\mathbf{v}_t^{(N)} \cdot \tilde{\Delta}\mathbf{v}^{(N)} - \nu\Delta \mathbf{v}^{(N)} \cdot \tilde{\Delta}\mathbf{v}^{(N)} + \big((\mathbf{v}^{(N)} + \mathbf{V}) \cdot \nabla\big)\mathbf{v}^{(N)} \cdot \tilde{\Delta}\mathbf{v}^{(N)} \\ + \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{V} \cdot \tilde{\Delta}\mathbf{v}^{(N)}\Big) dx = \int_{\Omega_K} \widehat{\mathbf{f}} \cdot \tilde{\Delta}\mathbf{v} dx.\end{aligned}$$

This is equivalent to (see the properties of the Stokes operator)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx + \int_{\Omega_K} |\tilde{\Delta}\mathbf{v}^{(N)}|^2 dx \\ = -\int_{\Omega_K} \big((\mathbf{v}^{(N)} + \mathbf{V}) \cdot \nabla\big)\mathbf{v}^{(N)} \cdot \tilde{\Delta}\mathbf{v}^{(N)} dx \\ -\int_{\Omega_K} \big(\mathbf{v}^{(N)} \cdot \nabla\big)\mathbf{V} \cdot \tilde{\Delta}\mathbf{v}^{(N)} dx + \int_{\Omega_K} \widehat{\mathbf{f}} \cdot \tilde{\Delta}\mathbf{v}^{(N)} dx = \sum_{i=1}^3 J_i.\end{aligned} \tag{4.28}$$

Let us estimate the right-hand side of (4.28). By Young's inequality,

$$|J_3| \le \varepsilon \int_{\Omega_K} |\tilde{\Delta}\mathbf{v}^{(N)}|^2 dx + c_\varepsilon \int_{\Omega_K} |\widehat{\mathbf{f}}|^2 dx. \tag{4.29}$$

Further, by (4.6), (2.4) and (3.23),

$$
\begin{aligned}
|J_2| \le & \Big\{ \|\nabla \mathbf{V}\|_{L^4(\Omega_3^\sharp)} \|\mathbf{v}^{(N)}\|_{L^4(\Omega_3^\sharp)} \\
&+ \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} |\mathbf{v}^{(N)}|^2 |\nabla \mathbf{U}^{[J]}|^2 dx \Big)^{1/2} \Big\} \times \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & \varepsilon \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c_\varepsilon \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)} \\
&+ \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} |\mathbf{v}^{(N)}|^2 |\nabla \mathbf{U}^{(J)}|^2 dx \Big)^{1/2} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & \varepsilon \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c_\varepsilon \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)} \\
&+ \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} \frac{|\mathbf{v}^{(N)}|^2}{\varphi^4(x_2)} dx \Big)^{1/2} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & \varepsilon \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c_\varepsilon \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)} \\
&+ \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} \frac{|\nabla \mathbf{v}^{(N)}|^2}{\varphi^2(x_2)} dx \Big)^{1/2} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & \varepsilon \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c_\varepsilon \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)} \\
&+ \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx.
\end{aligned}
\tag{4.30}
$$

Similarly ($J_1 = J_{11} + J_{12}$) we obtain the estimates

$$
\begin{aligned}
|J_{11}| = & \Big| \int\limits_{\Omega_K} (\mathbf{V} \cdot \nabla) \mathbf{v}^{(N)} \cdot \tilde{\Delta} \mathbf{v}^{(N)} dx \Big| \\
\le & \|\mathbf{V}\|_{L^4(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|_{L^4(\Omega_3^\sharp)} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
&+ c \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} \frac{|\nabla \mathbf{v}^{(N)}|^2}{\varphi^2(x_2)} dx \Big)^{1/2} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & c \|\mathbf{V}\|_{W^{1,2}(\Omega_3^\sharp)} \|\mathbf{v}^{(N)}\|_{W^{2,2}(\Omega_3^\sharp)} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
&+ c \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \Big(\int\limits_{\Omega_K \setminus \Omega_3^\sharp} \frac{|\nabla \mathbf{v}^{(N)}|^2}{\varphi^2(x_2)} dx \Big)^{1/2} \Big(\int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \Big)^{1/2} \\
\le & c \Big(\varepsilon + \Big(\int\limits_0^T |||F|||^2_{J+2} dt \Big)^{1/2} \Big) | \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \\
&+ c_\varepsilon \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)},
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
|J_{12}| = & \Big| \int\limits_{\Omega_K} (\mathbf{v}^{(N)} \cdot \nabla) \mathbf{v}^{(N)} \cdot \tilde{\Delta} \mathbf{v}^{(N)} dx \Big| \le \varepsilon \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx \\
&+ c_\varepsilon \Big(\|\mathbf{v}^{(N)}\|^2_{L^4(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^4(\Omega_3^\sharp)} + \sum_{l=1}^{K-1} \|\mathbf{v}^{(N)}\|^2_{L^4(\omega_l)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^4(\omega_l)} \\
&+ \|\mathbf{v}^{(N)}\|^2_{L^4(\widehat{\omega}_K)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^4(\widehat{\omega}_K)} \Big).
\end{aligned}
\tag{4.32}
$$

Applying inequalities (2.6) and (2.5) we get

$$
\begin{aligned}
& \|\mathbf{v}^{(N)}\|^2_{L^4(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^4(\Omega_3^\sharp)} \le \\
& c \|\mathbf{v}^{(N)}\|_{L^2(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)} \Big(\|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)} + \|\nabla^2 \mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)} \Big)^{1/2} \\
& \le c_\varepsilon \Big(\|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)} + \|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)} \|\nabla \mathbf{v}^{(N)}\|^4_{L^2(\Omega_3^\sharp)} \Big) \\
& + \varepsilon \|\nabla^2 \mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)};
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
&\|\mathbf{v}^{(N)}\|^2_{L^4(\omega_l)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^4(\omega_l)}\\
&\le c\varphi^{-1}(h_{l-1})\|\mathbf{v}^{(N)}\|_{L^2(\omega_l)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}\Big(\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}\\
&+\varphi^2(h_{l-1})\|\nabla^2\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}\Big)^{1/2} \le c_\varepsilon\|\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\omega_l)}\\
&+\varepsilon\Big(\|\nabla^2\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}+\varphi^{-2}(h_{l-1})\|\nabla\mathbf{v}^{(N)}\|^2\Big);
\end{aligned}\tag{4.34}
$$

$$
\begin{aligned}
&\|\mathbf{v}^{(N)}\|^2_{L^4(\widehat{\omega}_K)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^4(\widehat{\omega}_K)} \le c_\varepsilon\|\mathbf{v}^{(N)}\|^2_{L^2(\widehat{\omega}_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\widehat{\omega}_K)}\\
&+\varepsilon\Big(\|\nabla^2\mathbf{v}^{(N)}\|^2_{L^2(\widehat{\omega}_K)}+\varphi^{-2}(h_{K-1})\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\widehat{\omega}_K)}\Big).
\end{aligned}\tag{4.35}
$$

Thus, (4.32)–(4.35) and (3.23), (3.24) imply

$$
\begin{aligned}
|J_{12}| &\le c\varepsilon\Big(\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx+\int\limits_{\Omega_K}|\nabla^2\mathbf{v}^{(N)}|^2dx+\int\limits_{\Omega_K\setminus\Omega_3^\sharp}\frac{|\nabla\mathbf{v}^{(N)}|^2}{\varphi^2(x_2)}dx\Big)\\
&+c\Big(\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)}+\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_3^\sharp)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\Omega_3^\sharp)}\Big)\\
&+c\sum_{l=1}^{K-1}\|\mathbf{v}^{(N)}\|^2_{L^2(\omega_l)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\omega_l)}+c\|\mathbf{v}^{(N)}\|^2_{L^2(\widehat{\omega}_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\widehat{\omega}_K)}\\
&\le c\varepsilon\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx+c\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\\
&+\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\\
&\times\Big(\int\limits_{\Omega_3^\sharp}|\nabla\mathbf{v}^{(N)}|^2dx+\sum_{l=1}^{K-1}\int\limits_{\omega_l}|\nabla\mathbf{v}^{(N)}|^2dx+\int\limits_{\widehat{\omega}_K}|\nabla\mathbf{v}^{(N)}|^2dx\Big)\\
&\le c\varepsilon\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx+c\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\Omega_K)}\\
&+c\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}.
\end{aligned}\tag{4.36}
$$

Substituting (4.29)–(4.31), (4.36) into (4.28) yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\limits_{\Omega_K}|\nabla\mathbf{v}^{(N)}|^2dx+\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx\\
&\le c_6\Big(\varepsilon+\Big(\int\limits_0^T|||F|||^2_{J+2}dt\Big)^{1/2}\Big)\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{u}^{(N)}|^2dx\\
&+c\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}+c\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\Omega_K)}\\
&+c\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}+c\int\limits_{\Omega_K}|\widehat{\mathbf{f}}|^2dx.
\end{aligned}\tag{4.37}
$$

Taking in (4.37) $\varepsilon=\frac{1}{4c_6}$ and assuming that $\Big(\int\limits_0^T|||F|||^2_{J+2}dt\Big)^{1/2}\le\frac{1}{4c_6}$ we derive

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\Omega_K}|\nabla\mathbf{v}^{(N)}|^2dx+\int\limits_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx\le c_7\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\\
&+c_8\Big(\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\Omega_K)}+\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\Big)+c_9\int\limits_{\Omega_K}|\widehat{\mathbf{f}}|^2dx
\end{aligned}\tag{4.38}
$$

Denoting

$$
\begin{aligned}
Y(t)=\int\limits_{\Omega_K}|\nabla\mathbf{v}^{(N)}(x,t)|^2dx,\quad \mathcal{A}(t)=c_9\int\limits_{\Omega_K}|\widehat{\mathbf{f}}|^2dx+c_8\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)},\\
\mathcal{B}(t)=c_7\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}+c_8\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)},
\end{aligned}
$$

we rewrite (4.38) as

$$
Y'(t)\le\mathcal{B}(t)Y(t)+\mathcal{A}(t).
$$

By Gronwall's lemma,

$$
Y(t)\le e^{\int_0^t\mathcal{B}(\tau)d\tau}\Big(Y(0)+\int\limits_0^t e^{-\int_0^s B(\tau)d\tau}\mathcal{A}(s)ds\Big)\tag{4.39}
$$

$$\leq e^{\int_0^T \mathcal{B}(\tau)d\tau}\Big(Y(0)+\int_0^T \mathcal{A}(s)ds\Big).$$

Estimates (4.26), (4.27) yield

$$\begin{aligned}\int_0^T \mathcal{B}(t)dt &= c_7\int_0^T \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}dt + c_8\int_0^T \|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}dt\\ &\leq c\int_0^T \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}dt + c\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)}\int_0^T\int_{\Omega_K}|\nabla\mathbf{v}^{(N)}|^2dxdt\\ &\leq c_9A_1+c_{10}B_1^2 := B_2.\end{aligned}$$

Therefore, from (4.39) and (4.27) we have

$$\begin{aligned}&\sup_{t\in(0,T)]}\|\nabla\mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)}\\ &\leq ce^{B_2}\Big(\|\nabla\widehat{\mathbf{u}}_{k0}\|^2_{L^2(\Omega_K)}+\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(0,T;L^2(\Omega_K))}\\ &+\|\widehat{\mathbf{f}}\|^2_{L^2(0,T;L^2(\Omega))}\Big)\leq c_{11}e^{B_2}\big(A_0+A_1+B_1\big)=c_{11}B_3.\end{aligned} \tag{4.40}$$

Substituting (4.40) into (4.38) and integrating over t yield

$$\begin{aligned}&\int_0^T\int_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dxdt \leq \int_{\Omega_K}|\nabla\widehat{\mathbf{u}}_{0k}|^2dx + c\int_0^T\int_{\Omega}|\widehat{\mathbf{f}}|^2dxdt\\ &+c\int_0^T\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}dt+\int_0^T\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}dt\\ &+c\int_0^T\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\|\nabla\mathbf{v}^{(N)}\|^4_{L^2(\Omega_K)}dt\\ &\leq c\big(A_0+A_1\big)+c\sup_{t\in(0,T)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\int_0^T\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}dt\\ &+c\sup_{t\in(0,T)}\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\sup_{t\in(0,T)}\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\int_0^T\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}dt\\ &+c\int_0^T\|\nabla\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)dt}\\ &\leq c_{12}\Big[A_0+A_1+B_3A_1+B_3B_1^2+B_1\Big]:=c_{12}B_4.\end{aligned} \tag{4.41}$$

Let us estimate the norm of $\mathbf{u}_t^{(N)}$. Multiplying (4.23) by $\frac{d}{dt}\gamma_l^{(N)}(t)$ and summing by l from $l=1$ to $l=N$ we obtain

$$\begin{aligned}&\int_{\Omega_K}|\mathbf{v}_t^{(N)}|^2dx+\frac{\nu}{2}\frac{d}{dt}\int_{\Omega_K}|\nabla\mathbf{v}^{(N)}|^2dx\\ &=-\int_{\Omega_K}\big((\mathbf{v}^{(N)}+\mathbf{V})\cdot\nabla\big)\mathbf{v}^{(N)}\cdot\mathbf{v}_t^{(N)}dx\\ &-\int_{\Omega_K}\big(\mathbf{v}^{(N)}\cdot\nabla\big)\mathbf{V}\cdot\mathbf{v}_t^{(N)}dx+\int_{\Omega_K}\mathbf{f}\cdot\mathbf{v}_t^{(N)}dx\\ &\leq\frac12\int_{\Omega_K}|\mathbf{v}_t^{(N)}|^2dx+c\int_{\Omega}|\widehat{\mathbf{f}}|^2dx+c\int_{\Omega_K}\big(|\mathbf{v}^{(N)}|^2+|\mathbf{V}|^2\big)|\nabla\mathbf{v}^{(N)}|^2dx\\ &+c\int_{\Omega_K}|\mathbf{v}^{(N)}|^2|\nabla\mathbf{V}|^2dx.\end{aligned} \tag{4.42}$$

Let us estimate the last two terms in the right-hand side of (4.42). By the same argument as before,

$$\begin{aligned}&\int_{\Omega_K}|\mathbf{v}^{(N)}|^2|\nabla\mathbf{V}|^2dx\\ &\leq c\Big(\big(\int_{\Omega_3^\sharp}|\mathbf{v}^{(N)}|^4dx\big)^{1/2}\big(\int_{\Omega_3^\sharp}|\nabla\mathbf{V}|^4dx\big)^{1/2}+\int_0^T|||F|||^2_{J+2}dt\int_{\Omega_K\setminus\Omega_3^\sharp}\frac{|\mathbf{v}^{(N)}|^2}{\varphi^4(x_2)}dx\Big)\\ &\leq c\Big(\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)}\int_{\Omega_K}|\nabla\mathbf{v}^{(N)}|^2dx+\int_0^T|||F|||^2_{J+2}dt\int_{\Omega_K}|\tilde{\Delta}\mathbf{v}^{(N)}|^2dx\Big);\end{aligned}$$

$$\begin{aligned}&\int\limits_{\Omega_K} |\mathbf{V}|^2 |\nabla \mathbf{v}^{(N)}|^2 dx\\
&\le c\Big(\sup_{x\in\Omega_3^\sharp} |\mathbf{V}(x,t)|^2 \int\limits_{\Omega_3^\sharp} |\nabla \mathbf{v}^{(N)}|^2 dx + \int\limits_0^T |||F|||_{J+2}^2 dt \int\limits_{\Omega_K\setminus\Omega_3^\sharp} \frac{|\nabla \mathbf{v}^{(N)}|^2}{\varphi^2(x_2)} dx\Big)\\
&\le c\Big(\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx + \int\limits_0^T |||F|||_{J+2}^2 dt \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx\Big).\end{aligned}$$

To estimate the integral $\int\limits_{\Omega_K} |\mathbf{v}^{(N)}|^2 |\nabla \mathbf{v}^{(N)}|^2 dx$, we apply inequalities (4.33)-(4.34) and argue as in the proof of (4.36). In virtue of (2.5) applied to $\nabla \mathbf{v}^{(N)}$, we obtain

$$\begin{aligned}&\int\limits_{\Omega_K} |\mathbf{v}^{(N)}|^2 |\nabla \mathbf{v}^{(N)}|^2 dx \le c \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c\|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\\
&+c\|\mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)} \|\nabla \mathbf{v}^{(N)}\|^4_{L^2(\Omega_K)} \le c \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c\|\nabla \mathbf{v}^{(N)}\|^2_{L^2(\Omega_K)}\\
&+c \sup_{t\in[0,T]} \|\mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)} \sup_{t\in[0,T]} \|\nabla \mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)}\big) \|\nabla \mathbf{v}^{(N)}|^2_{L^2(\Omega_K)}\\
&\le c \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c(1 + B_1 B_3) \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx.\end{aligned}$$

Substituting the obtained inequalities into (4.42) yields

$$\begin{aligned}&\frac{1}{2} \int\limits_{\Omega_K} |\mathbf{v}_t^{(N)}|^2 dx + \frac{\nu}{2} \frac{d}{dt} \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx \le c \int\limits_{\Omega} |\widehat{\mathbf{f}}|^2 dx\\
&+c\big(1 + \int\limits_0^T |||F|||_{J+2}^2 dt\big) \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dx + c\big(1 + B_1 B_3\big) \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx\\
&+c\|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dx.\end{aligned}$$

Integrating this inequality over $[0,T]$ and using estimates (4.27), (4.41) we derive

$$\begin{aligned}&\int\limits_0^T \int\limits_{\Omega_K} |\mathbf{v}_t^{(N)}|^2 dxdt \le c\Big(\int\limits_{\Omega_K} |\nabla \widehat{\mathbf{u}}_{0K}|^2 dx + c \int\limits_{\Omega} |\widehat{\mathbf{f}}|^2 dx\Big)\\
&+c\big(1 + \sup_{t\in(0,T)} \int\limits_0^T |||F|||_{J+2}^2 dt\big) \int\limits_0^T \int\limits_{\Omega_K} |\tilde{\Delta} \mathbf{v}^{(N)}|^2 dxdt\\
&+c\big(1 + B_1 B_3\big) \int\limits_0^T \int\limits_{\Omega_K} |\nabla \mathbf{v}^{(N)}|^2 dxdt\\
&+ \sup_{t\in(0,T)} \|\nabla \mathbf{v}^{(N)}(\cdot,t)\|^2_{L^2(\Omega_K)} \int\limits_0^T \|\mathbf{V}\|^2_{W^{2,2}(\Omega_3^\sharp)} dt\\
&\le c_{13}\Big[A_0 + A_1 + (1 + A_1) B_4 + (1 + B_1 B_3) B_1 + B_3 A_1\Big] := c_{13} B_5.\end{aligned} \tag{4.43}$$

Estimates (4.27), (4.41) and (4.43) ensure that there exist a subsequence $\{\mathbf{v}^{(N_l)}\}$ such that $\{\mathbf{v}^{(N_l)}\}$, $\{\nabla \mathbf{v}^{(N_l)}\}$, $\{\nabla^2 \mathbf{v}^{(N_l)}\}$, $\{\mathbf{v}_t^{(N_l)}\}$ are weakly convergent in $L^2(0,T;L^2(\Omega_K))$. The limit $\mathbf{v}$ of this subsequence satisfies the integral identity (4.16) and thus, is a weak solution of (4.15). This part of the proof is standard (see [21, 22, 24]) and we omit the details. Remind that in this section we have omitted the subscript K for notation of the solution and Galerkin approximations, so $\mathbf{v}^{(N)} = \mathbf{v}_K^{(N)}$ and $\mathbf{v} = \mathbf{v}_K$. ■

4.3 Existence and uniqueness of the solution to problem (4.13)

By a weak solution of problem (4.13) in the cusp domain Ω we mean the function $\mathbf{v} \in L^2(0,T;H(\Omega))$ with $\mathbf{v}_t, \nabla^2 \mathbf{v} \in L^2(0,T;L^2(\Omega))$ satisfying the initial condition $\mathbf{v}(x,0) = \widehat{\mathbf{u}}_0(x)$, and for all $t \in [0,T]$ satisfying the integral identity

$$\begin{aligned}&\int\limits_0^t \int\limits_{\Omega} \big(\mathbf{v}_t \cdot \boldsymbol{\eta} + \nu \nabla \mathbf{v} \cdot \nabla \boldsymbol{\eta} - \big((\mathbf{v} + \mathbf{V}) \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{v} - \big(\mathbf{v} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{V}\big) dxdt\\
&= \int\limits_0^t \int\limits_{\Omega} \widehat{\mathbf{f}} \cdot \boldsymbol{\eta}\, dxdt\end{aligned} \tag{4.44}$$

for any test function $\boldsymbol{\eta} \in L^2(0,T;H(\Omega))$, $\boldsymbol{\eta}_t \in L^2(0,T;L^2(\Omega))$ having compact support in $\bar{\Omega} \setminus \{O\}$ (i.e., the support of $\boldsymbol{\eta}$ is separated from the cusp point O)

Theorem 4.2. *Assume that the conditions of Theorem 4.1 are valid. There exists a number $\kappa_1 > 0$ such that if $\int\limits_0^T |||F|||_{J+2}^2 dt \le \kappa_1$, then there exists a unique weak solution $\mathbf{v}$ of problem* (4.13).

Proof. Let $\int\limits_0^T |||F|||_{J+2}^2 dt \le \kappa_0$, where κ_0 is a number from Theorem 4.1. Then, due to estimates (4.19)–(4.22) for $\mathbf{v}_K$, we can extract a subsequence $\{\mathbf{v}_{K_l}\}$ such that $\{\mathbf{v}_{K_l}\}$, $\{\nabla\mathbf{v}_{K_l}\}$, $\{\nabla^2\mathbf{v}_{K_l}\}$ and $\{\mathbf{v}_{K_l,t}\}$ are weakly convergent in $L^2(0,T;L^2(\Omega))$ as $K_l \to \infty$ Taking in (4.16) a test function $\boldsymbol{\eta}$ with the compact support and passing K_l to infinity we obtain for the limit $\mathbf{v}$ integral identity (4.44). Moreover, obviously $\mathbf{v}(x,0) = \widehat{\mathbf{u}}_0(x)$ and thus, $\mathbf{v}$ is a weak solution of problem (4.13). For $\mathbf{v}$ remain valid estimates (4.19)–(4.22).
Let us prove the uniqueness. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of problem (1.1) having the same representations: $\mathbf{u}_1 = \mathbf{V} + \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{V} + \mathbf{v}_2$, where $\mathbf{v}_1$, $\mathbf{v}_2$ are solutions of problem (4.13). The difference $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2 := \mathbf{v} \in \mathring{W}^{1,2}(\Omega)$ satisfies zero initial condition $\mathbf{v}(x,0) = 0$ and the integral identity

$$\begin{aligned}&\int\limits_0^t\int\limits_\Omega \big(\mathbf{v}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta}\big)dxdt - \int\limits_0^t\int\limits_\Omega (\mathbf{V}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}dxdt\\ &-\int\limits_0^t\int\limits_\Omega \big(\mathbf{v}\cdot\nabla\big)\boldsymbol{\eta}\cdot\mathbf{V}dxdt - \int\limits_0^t\int\limits_\Omega (\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{v}_1dxdt = 0.\end{aligned} \tag{4.45}$$

Let us take in (4.45) $\boldsymbol{\eta} = \chi_K(x_2)\mathbf{v} + \mathbf{w}_K$, where χ_K is defined in the proof of Lemma 4.1 and $\mathbf{w}_K$ is a solution of the problem

$$\begin{cases}\operatorname{div}\mathbf{w}_K = -\nabla\chi_K\cdot\mathbf{v}, & x\in\omega_{K-1},\\ \mathbf{w}_K = 0, & x\in\partial\omega_{K-1},\end{cases}$$

satisfying the estimate

$$\begin{aligned}&\|\nabla\mathbf{w}_K\|_{L^2(\omega_{K-1})} \le c\|\nabla\chi_K\cdot\mathbf{v}\|_{L^2(\omega_{K-1})}\\ &\le c\|\varphi^{-1}\mathbf{v}\|_{L^2(\omega_{K-1})} \le c\|\nabla\mathbf{v}\|_{L^2(\omega_{K-1})}.\end{aligned} \tag{4.46}$$

This gives

$$\begin{aligned}\frac{1}{2}\int\limits_0^t\frac{d}{dt}\int\limits_\Omega \chi_K|\mathbf{v}|^2dxdt + \nu\int\limits_0^t\int\limits_\Omega\chi_K|\nabla\mathbf{v}|^2dxdt &= -\int\limits_0^t\int\limits_\Omega \mathbf{v}_t\cdot\mathbf{w}_Kdxdt\\ &-\nu\int\limits_0^t\int\limits_\Omega\nabla\mathbf{v}\cdot\nabla\chi_K\cdot\mathbf{v}dxdt - \nu\int\limits_0^t\int\limits_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{w}_Kdxdt\\ &+\int\limits_0^t\int\limits_\Omega(\mathbf{V}\cdot\nabla)(\chi_K\mathbf{v}+\mathbf{w}_K)\cdot\mathbf{v}dxdt\\ &+\int\limits_0^t\int\limits_\Omega\big(\mathbf{v}\cdot\nabla\big)(\chi_K\mathbf{v}+\mathbf{w}_K)\cdot\mathbf{V}dxdt\\ &+\int\limits_0^t\int\limits_\Omega(\mathbf{v}\cdot\nabla)(\chi_K\mathbf{v}+\mathbf{w}_K)\cdot\mathbf{v}_1dxdt := \sum_{i=1}^6 J_i.\end{aligned} \tag{4.47}$$

Let us estimate the right-hand side of (4.47). Using the properties of χ_K and $\mathbf{w}_K$ we obtain

$$\begin{aligned}|J_1| &\le \int\limits_0^t \|\mathbf{v}_t\|_{L^2(\omega_{K-1})}\|\mathbf{w}_K\|_{L^2(\omega_{K-1})}dt\\ &\le c\int\limits_0^t\Big(\|\mathbf{v}_t\|^2_{L^2(\omega_{K-1})} + \|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}\Big)dt;\\ |J_2| + |J_3| &\le \int\limits_0^t\|\nabla\mathbf{v}\|_{L^2(\omega_{K-1})}\|\varphi^{-1}\mathbf{v}\|_{L^2(\omega_{K-1})}dt\\ &+\int\limits_0^t\|\nabla\mathbf{v}\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{w}_K\|_{L^2(\omega_{K-1})}dt \le c\int\limits_0^t\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt;\end{aligned} \tag{4.48}$$

Integrating by parts yields

$$\int_0^t \int_\Omega (\mathbf{V}\cdot\nabla)(\chi_K \mathbf{v})\cdot \mathbf{v}\,dxdt = \frac{1}{2}\int_0^t \int_{\omega_{K-1}} (\mathbf{V}\cdot\nabla)\chi_K |\mathbf{v}|^2\,dxdt.$$

Therefore, by (4.6), (4.46),

$$\begin{aligned}|J_4| &\le \Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}\int_0^t \Big(\|\varphi^{-1}\mathbf{v}\|^2_{L^2(\omega_{K-1})}\\ &\quad+\|\varphi^{-1}\mathbf{v}\|_{L^2(\omega_{K-1})}\|\nabla \mathbf{w}_K\|_{L^2(\omega_{K-1})}\Big)dt\\ &\le \Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}\int_0^t \|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt.\end{aligned}\tag{4.49}$$

Moreover, using (4.12), (4.6), (2.5) we get

$$\begin{aligned}|J_5| &\le \Big|\int_0^t\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{V}\cdot(\chi_K\mathbf{v}+\mathbf{w}_K)\,dxdt\Big|\\ &\le \Big|\int_0^t\int_{\Omega_3^\sharp} (\mathbf{v}\cdot\nabla)\mathbf{V}\cdot\mathbf{v}\,dxdt\Big|\\ &\quad+c\Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}\int_0^t\int_{\Omega\setminus\Omega_3^\sharp}\chi_K\varphi^{-2}|\mathbf{v}|^2\,dxdt+\Big|\int_0^t\int_{\omega_{K-1}}(\mathbf{v}\cdot\nabla)\mathbf{V}\cdot\mathbf{w}_K\,dxdt\Big|\\ &\le c\Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}\int_0^t\int_{\Omega\setminus\Omega_3^\sharp}\chi_K|\nabla\mathbf{v}|^2\,dxdt\\ &\quad+c\int_0^t\Big(\Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}\|\varphi^{-1}\mathbf{v}\|_{L^2(\omega_{K-1})}\|\varphi^{-1}\mathbf{w}_K\|_{L^4(\omega_{K-1})}\\ &\quad+\|\nabla\mathbf{V}\|_{L^2(\Omega_3^\sharp)}\|\mathbf{v}\|^2_{L^4(\Omega_3^\sharp)}\Big)dt\\ &\le c\Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}_{J+1}\int_0^t\int_{\Omega\setminus\Omega_3^\sharp}\chi_K|\nabla\mathbf{v}|^2\,dxdt\\ &\quad+c\int_0^t\|\nabla\mathbf{V}\|_{L^2(\Omega_3^\sharp)}\|\mathbf{v}\|_{L^2(\Omega_3^\sharp)}\|\nabla\mathbf{v}\|_{L^2(\Omega_3^\sharp)}dt+c\int_0^t\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt\\ &\le c\Big(\Big(\int_0^T |||F|||_{J+2}^2\,dt\Big)^{1/2}+\varepsilon\Big)\int_0^t\int_\Omega\chi_K|\nabla\mathbf{v}|^2\,dxdt+c_\varepsilon\int_0^t\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt+c\int_0^t\|\nabla\mathbf{V}\|^2_{L^2(\Omega_3^\sharp)}\|\chi_K\mathbf{v}\|^2_{L^2(\Omega)}dt\end{aligned}\tag{4.50}$$

and

$$\begin{aligned}|J_6| &\le \Big|\int_0^t\int_\Omega\chi_K(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}_1\,dxdt\Big|+\Big|\int_0^t\int_{\omega_{K-1}}(\mathbf{v}\cdot\nabla)\chi_K(\mathbf{v}\cdot\mathbf{v}_1)\,dxdt\Big|\\ &\quad+\Big|\int_0^t\int_{\omega_{K-1}}(\mathbf{v}\cdot\nabla)\mathbf{w}_K\cdot\mathbf{v}_1\,dxdt\Big|\le\varepsilon\int_0^t\int_\Omega\chi_K|\nabla\mathbf{v}|^2\,dxdt\\ &\quad+c_\varepsilon\int_0^t\|\chi_K\mathbf{v}\|^2_{L^4(\Omega)}\|\mathbf{v}_1\|^2_{L^4(\Omega)}dt\\ &\quad+c\int_0^t\|\varphi^{-1}\mathbf{v}\|_{L^2(\omega_{K-1})}\|\mathbf{v}\|_{L^4(\omega_{K-1})}\|\mathbf{v}_1\|_{L^4(\omega_{K-1})}dt\\ &\quad+c\int_0^t\|\nabla\mathbf{w}_K\|_{L^2(\omega_{K-1})}\|\mathbf{v}\|_{L^4(\omega_{K-1})}\|\mathbf{v}_1\|_{L^4(\omega_{K-1})}dt\\ &\le\varepsilon\int_0^t\int_\Omega\chi_K|\nabla\mathbf{v}|^2\,dxdt\\ &\quad+c\int_0^t\|\chi_K\mathbf{v}\|_{L^2(\Omega)}\|\nabla(\chi_K\mathbf{v})\|_{L^2(\Omega)}\|\nabla\mathbf{v}_1\|^2_{L^2(\Omega)}dt\\ &\quad+c\int_0^t\|\nabla\mathbf{v}_1\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt\\ &\le c\varepsilon\int_0^t\int_\Omega\chi_K|\nabla\mathbf{v}|^2\,dxdt+c_\varepsilon\int_0^t\|\nabla\mathbf{v}_1\|^4_{L^2(\Omega)}\|\chi_K\mathbf{v}\|^2_{L^2(\Omega)}dt\\ &\quad+c\int_0^t\|\nabla\mathbf{v}_1\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}dt.\end{aligned}\tag{4.51}$$

Substituting estimates (4.48)–(4.51) into (4.47) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{dt}\int_\Omega \chi_K|\mathbf{v}|^2 dxdt + \nu\int_0^t\int_\Omega \chi_K|\nabla\mathbf{v}|^2 dxdt\\
&\le c_{14}\Big(\Big(\int_0^T |||F|||^2_{\mathfrak{J}+2}dt\Big)^{1/2} + \varepsilon\Big)\int_0^t\int_\Omega \chi_K|\nabla\mathbf{v}|^2 dxdt\\
&+c\int_0^t \big(\|\nabla\mathbf{V}\|^2_{L^2(\Omega_3^\sharp)} + \|\nabla\mathbf{v}_1\|^4_{L^2(\Omega)}\big)\|\chi_K\mathbf{v}\|^2_{L^2(\Omega)}dt\\
&+c\int_0^t \Big[\|\mathbf{v}_t\|_{L^2(\omega_{K-1})} + \|\nabla\mathbf{v}_1\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}\Big]dt.
\end{aligned} \tag{4.52}$$

Taking $\varepsilon = \frac{\nu}{4c_{14}}$ in (4.52) and assuming that $c_{14}\Big(\int_0^T |||F|||^2_{\mathfrak{J}+2}dt\Big)^{1/2} \le \frac{\nu}{4}$ we get

$$\begin{aligned}
&\int_\Omega \chi_K|\mathbf{v}(x,t)|^2 dx\\
&\le c\int_0^t \big(\|\nabla\mathbf{V}\|^2_{L^2(\Omega_3^\sharp)} + \|\nabla\mathbf{v}_1\|^4_{L^2(\Omega)}\big)\int_\Omega \chi_K|\mathbf{v}(x,t)|^2 dxdt\\
&+c\int_0^t \Big[\|\mathbf{v}_t\|_{L^2(\omega_{K-1})} + \|\nabla\mathbf{v}_1\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}\Big]dt.
\end{aligned} \tag{4.53}$$

Introduce the notation:

$$Z(t) = \int_\Omega \chi_K|\mathbf{v}(x,t)|^2 dx, \quad \beta(t) = \big(\|\nabla\mathbf{V}(\cdot,t)\|^2_{L^2(\Omega_3^\sharp)} + \|\nabla\mathbf{v}_1(\cdot,t)\|^4_{L^2(\Omega)}\big),$$
$$\alpha(t) = \int_0^t \Big[\|\mathbf{v}_t\|_{L^2(\omega_{K-1})} + \|\nabla\mathbf{v}_1\|_{L^2(\omega_{K-1})}\|\nabla\mathbf{v}\|^2_{L^2(\omega_{K-1})}\Big]dt.$$

Then (4.53) can be written as

$$Z(t) \le c_{15}\alpha(t) + c_{16}\int_0^t \beta(\tau)Z(\tau)d\tau.$$

By Gronwall lemma, the last inequality yields

$$Z(t) \le c_{15}\alpha(t)\exp\Big(c_{16}\int_0^t \beta(\tau)d\tau\Big). \tag{4.54}$$

Estimates (4.19), (4.20) for the solution $\mathbf{v}_1$ and estimate (4.26) imply (see Theorem 4.1)

$$\int_0^T \|\nabla\mathbf{v}_1\|^2_{L^2(\Omega)}dt \le cB_1, \quad \sup_{t\in(0,T)} \|\nabla\mathbf{v}_1\|^2_{L^2(\Omega)} \le cB_3, \quad \int_0^T \|\nabla\mathbf{V}\|^2_{L^2(\Omega_3^\sharp)}dt \le cA_1.$$

This together with (4.54) yields

$$\begin{aligned}
\int_0^T\int_{\Omega^\sharp_{K-2}} |\mathbf{v}(x,t)|^2 dxdt &\le \int_0^T Z(t)dt \le c_{15}\int_0^T \alpha(t)dt\exp\Big(c_{16}\int_0^T \beta(t)dt\Big)\\
&\le ce^{c(A_1+B_1B_3)}\int_0^T\int_{\omega_{K-1}} \big(|\mathbf{v}_t|^2 + |\nabla\mathbf{v}|^2\big)dxdt.
\end{aligned} \tag{4.55}$$

Obviously, the right-hand side of (4.55) vanishes as $K\to\infty$. Therefore, passing in (4.55) to the limit, we obtain $\int_0^T\int_\Omega |\mathbf{v}(x,t)|^2 dxdt = 0$, and hence $\mathbf{v}(x,t) = \mathbf{v}_1(x,t) - \mathbf{v}_2(x,t) = 0$. ■

Remark 4.1. The solution $\mathbf{u}$ of problem (1.1) considered in Theorem 4.2 has the representation $\mathbf{u} = \mathbf{V} + \mathbf{v}$, where $\mathbf{V}$ is a singular part coinciding near the cusp point with the formal asymptotic decomposition of the solution, and $\mathbf{v}$ is a regular part having finite energy norm. Theorem 4.2 states only the uniqueness of regular part $\mathbf{v}$. We do not prove the uniqueness of general singular solution of problem (1.1) having a source or sink in the cusp point.

Remark 4.2. The "smallness" assumption of Theorems 4.1 and 4.2 concerns only the smallness of fluxes $F(t) = \int_{\partial\Omega} \mathbf{a}(x,t)\cdot\mathbf{n}(x)dS$, i.e. of the magnitude of $\int_0^T |||F|||_{J+2}dt$. We do not suppose that the norms of the boundary value **a**, initial condition **b** or the right-hand side **f** are small.

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