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VILNIUS UNIVERSITY

Gediminas
BAGDONAS

A class of bivariate copula mappings

DOCTORAL DISSERTATION

Natural sciences,
Mathematics (N 001)

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VILNIAUS UNIVERSITETAS

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Dvimačių kopulų transformacijos

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CONTENTS

NOTATION

$C, C_f, C_{\alpha,\beta}, C_\theta$	copulas
C^*	dual function of a copula
$\widehat{C}$	survival copula
$\overline{C}$	survival function corresponding to C
$\mathcal{C}$	the set of (2-dimensional) copulas
∂^2_{xy}	mixed partial derivative operator
∂_x, ∂_y	partial derivative operators
F, G, F_X, G_X	distribution functions
$\phi^{[-1]}$	pseudo-inverse of ϕ
$F^{(-1)}$	quasi-inverse of F
x^+	positive part of x, $x^+ := \max\{x, 0\}$
$x \vee y$	maximum of x and y
$x \wedge y$	minimum of x and y
$\tau, \tau_C, \tau_{X,Y}$	Kendall's tau corresponding to copula C or random variables X and Y
$\rho, \rho_C, \rho_{X,Y}$	Spearman's rho corresponding to copula C or random variables X and Y
λ_U, λ_L	upper, lower tail dependence coefficients
$\mathbb{R}$	real line $(-\infty, \infty)$
$\overline{\mathbb{R}}$	extended real line $[-\infty, \infty]$
Ran	range
V_C, V_H	C-volume, H-volume (or measure) of a set
$\circ$	composition of functions
$\prec$	point-wise or concordance ordering

1 INTRODUCTION

1.1 Research topic and relevance

It is well known that copulas play a central role in modelling multivariate dependence and linking univariate marginal distributions together to form a multivariate distribution. This is the essence of the famous Sklar's theorem (see Theorem 1). Having many choices for bivariate or, more generally, multivariate copulas is hence of fundamental importance in describing probabilistic models of natural phenomena. Interest in copulas, or more generally quasi-copulas, and their construction methods is growing also among researchers working on fuzzy set theory, when modelling preferences and similarities as well as describing aggregation processes, on expert and decision support systems, multicriteria and group decision making. For more in these directions we refer to [13, 26, 25, 34, 1, 14], and references therein.

Over the years, many different methods have been suggested to construct various copulas; one can consult the books by Nelsen [36], Joe [23], Durante and Sempi [12]. Sometimes several new construction methods can be unified under one umbrella, giving rise to unifying characterization results. In this thesis we describe one such case and offer an even wider range of new copulas, parametrized not by one or several parameters, as is often the case, but rather by a family of general univariate functions satisfying predefined criteria. As particular cases we recover several known results given in, for example, [10, 13, 9, 17]. For a more precise definition of a bivariate copula see Definition 1.

Let $\mathcal{C}$ denote the class of bivariate copulas and $\mathcal{G}$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}_+$. Consider $\emptyset \neq \mathcal{C}' \subset \mathcal{C}$ and $\emptyset \neq \mathcal{G}' \subset \mathcal{G}$. In our work we consider the map H defined on $\mathcal{C}' \times \mathcal{G}'$ by $(C, f) \mapsto H_f(C) : [0,1]^2 \to [0,1]$, where

$$H_f(C)(u,v) := C(u,v) f(\overline{C}(u,v)),$$

$\overline{C}$ denotes the survival function[1] corresponding to copula C (see Definition 2). First, we characterize the subset $\mathcal{G}'$ so that

$$H(\mathcal{C}' \times \mathcal{G}') \subset \mathcal{C}$$

whenever $\mathcal{C}' = \mathcal{C}$ (see Theorem 3 in Section 3.1), which, together with known examples from the literature, suggests a classification of functions f used to construct H_f into *eligible*, *conditionally-eligible* and *non-eligible*; see Definition 6. Having described eligible functions, in Section 3.3 we study the properties of the transformation $C \mapsto H_f(C)$. In Theorem 4 we show that it preserves concordance order on the set of copulas, yielding integral bounds for the Kendall's τ, Spearman's ρ, and Gini's γ; moreover, $H_f(C)$ has simple expressions for the tail behaviour indices, provided they exist for the chosen $C \in \mathcal{C}$. Among the properties of a copula C that are not necessarily preserved by the transformation, we mention the TP_2 property and that of being Archimedean (see Proposition 1). Further, the iterations of the map $C \mapsto H_f(C)$ converge in the supremum norm to the unique fixed point of this transformation, namely the countermonotonicity copula $W(u, v)$ (see Theorem 6) for any initial $C \in \mathcal{C}$.

As we see in Chapter 3 for $H_f(C)$ to be a copula for *any* bivariate copula C, quite restrictive conditions on f must be imposed. On the other hand, for a *particular* copula C, these requirements on f can be substantially loosened, yielding a much bigger set of allowable functions f; this set contains many *conditionally eligible* functions, i.e. those that depend on the choice of the initial copula C and are not *eligible*. This problem, for the case of comonotonicity copula $M(u, v)$ (see Theorem 6), was solved by Durante and coworkers [10, 13].

Motivated by their nice results, we have decided to investigate the case of the independence copula $\Pi(u, v) = uv$ since, together with an appropriate function f, it defines several well known families of copulas (see Examples 9, 10 and 11 below). The class of eligible functions unfortunately halves the allowable parameter ranges for the well-known families of Farlie–Gumbel–Morgenstern, Ali–Mikhal–Haq (henceforth denoted as FGM and AMH; see, e.g. [36] for definitions

[1] The function $\overline{C}(u, v)$ is not a copula and should not be confused with the survival copula $\widehat{C}(u, v)$.

and properties) and Çelebioğlu–Cuadras (see [4, 7]) copulas, yielding at the same time many examples of *conditionally eligible* functions f.

Therefore, in Chapter 4 we focus on the problem of finding functions f for which

$$C_f(u,v) = uvf((1-u)(1-v)), \quad u,v \in [0,1],$$

is a bivariate copula. In fact, this question is also related to an open problem raised in a short note by Cuadras [6] where the author considers "canonical" copulas of the form

$$CF_\theta(u,v) = uvH_\theta(Q(u,v)), \quad H_\theta(\rho) = \int_\rho^1 \frac{f_\theta(t)}{t^2}dt + 1, \quad u,v,\rho \in [0,1],$$

where f_θ is the so-called "canonical" correlation function depending on a parameter θ and discussed in [5], and Q is an appropriate bivariate function on the unit square to be characterized. In [6, Examples 4,6 and 8],

$$Q(u,v) = 1-(1-u)(1-v) = \Pi^*(u,v),$$

i.e. the dual of independence copula Π, is taken, in which case our function f has the form

$$f(t) = H_\theta(1-t), \quad t \in [0,1). \tag{1.1}$$

Therefore, our results could be used to shed some light on the properties of a certain subclass of canonical correlation functions and corresponding copulas.

An interesting relation between copulas of the considered form with the TP_2 property and geometric Jensen convexity of the function f is discovered in this chapter. Also we consider the case of twice continuously differentiable functions f for which a much more complicated sufficient condition appears, indicating possible difficulties with a complete solution for the problem under consideration.

1.2 Novelty

The obtained results are new. Most of the results are included in the following publications:

- M. Manstavičius and G. Bagdonas. A class of bivariate copula mappings. *Fuzzy Sets and Systems*, 354:48–62, 2019
- M. Manstavičius and G. Bagdonas. A class of bivariate independence copula transformations. *Fuzzy Sets and Systems*, 2021.

1.3 Citations

Our results were cited in the following publications:

- S. Girard. Transformation of a copula using the associated co-copula. *Dependence Modeling*, 6(1):298–308, 2018
- S. Saminger-Platz, A. Kolesárová, R. Mesiar, and E. P. Klement. The key role of convexity in some copula constructions. *European Journal of Mathematics*, pages 1–28, 2019
- S. Saminger-Platz, A. Kolesárová, A. Šeliga, R. Mesiar, and E. P. Klement. The impact on the properties of the EFGM copulas when extending this family. *Fuzzy Sets and Systems*, 11 2020.
- P. Le and N. K. T. Ngoc. On some bivariate copula transformations. *Thai Journal of Mathematics*, 18(3):1063–1079, 2020
- A. Šeliga, M. Kauers, S. Saminger-Platz, R. Mesiar, A. Kolesárová, and E. P. Klement. Polynomial bivariate copulas of degree five: characterization and some particular inequalities. *Dependence Modeling*, 9(1):13–42, 2021

1.4 Conferences

The results of the thesis were presented in the following conferences:

- Dvimačių kopulų transformacijos. *59th conference of Lithuanian Mathematical Society*, June 18-19, 2018, Kaunas.
- Dvimatės nepriklausomumo kopulos transformacijos, *60th conference of Lithuanian Mathematical Society*, June 19–20, 2019, Vilnius.

- Some transformations of bivariate independence copula. *CFE-CMStatistics*, December 14–16, 2019, London.
- Dvimatės nepriklausomumo kopulos transformacijos II. Būtinosios sąlygos, *61st conference of Lithuanian Mathematical Society*, December 4, 2020, Šiauliai.
- Certain copula transformations: From 2D to 3D. *CFE-CMStatistics*, December 19–21, 2020, London.

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2 | PRELIMINARIES

In this chapter we collect necessary definitions and facts from copula theory that will be used later on. We start by defining copula as a bivariate function and mentioning Sklar's theorem which is considered to be the foundation of copula theory. This theorem first appeared in [40]. The name "copula" was chosen to emphasize the manner in which a copula "couples" a joint distribution function to its univariate margins.

Definition 1. A bivariate copula[1] (a copula, for short) is a function C defined on $[0,1]^2$ with values in $[0,1]$ such that

- $C(u,0) = C(0,u) = 0$ for any $u \in [0,1]$,
- $C(u,1) = C(1,u) = u$ for any $u \in [0,1]$, and
- (2-increasingness) for all $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, denoting $\square := [u_1, u_2] \times [v_1, v_2]$,

$$V_C(\square) = C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0.$$

The first two conditions for C are also known as the boundary conditions. We denote the class of bivariate copulas by $\mathcal{C}$.

Theorem 1. *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all $x, y \in \overline{\mathbb{R}}$,*

$$H(x,y) = C(F(x), G(y)). \tag{2.1}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $\operatorname{Ran} F \times \operatorname{Ran} G$. *Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (2.1) is a joint distribution function with margins F and G.*

Now let us define popular functions involving bivariate copulas.

Definition 2. Let $C \in \mathcal{C}$ and $(u,v) \in [0,1]^2$. Then

[1] One can also consider n-variate copulas for any $n \geq 2$ (see, e.g. [36], [23], or [12]), but we mostly will be concerned with bivariate copulas in this thesis.

- $\overline{C}(u,v) = 1 - u - v + C(u,v)$ is called survival function (of the copula C);
- $C^*(u,v) = u + v - C(u,v)$ is called dual function (of the copula C);
- $\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ is called survival copula;

Further we mention the well-known Fréchet–Hoeffding bounds satisfied by all copulas. The bounds are true in the n-dimensional setting, but as we consider only bivariate copulas in this thesis, the formulation is adapted to the case $n = 2$ (see Figures 2.1 and 2.2 for visual representation).

Theorem 2 ([36, Theorem 2.2.3]). *For any copula $C \in \mathcal{C}$ and any $(u,v) \in [0,1]^2$,*

$$W(u,v) := (u+v-1)^+ \leq C(u,v) \leq M(u,v) := \min\{u,v\}.$$

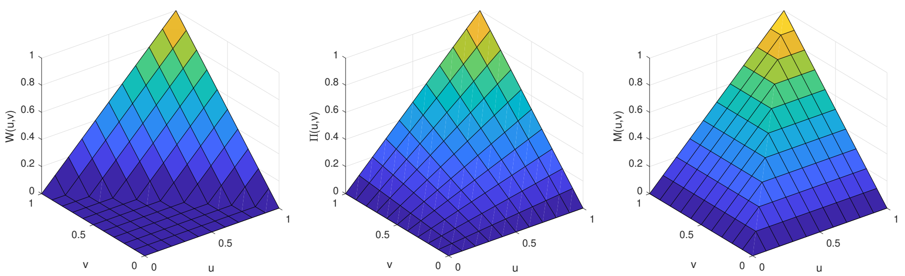


Figure 2.1: Copulas $W(u,v)$, $\Pi(u,v)$ and $M(u,v)$

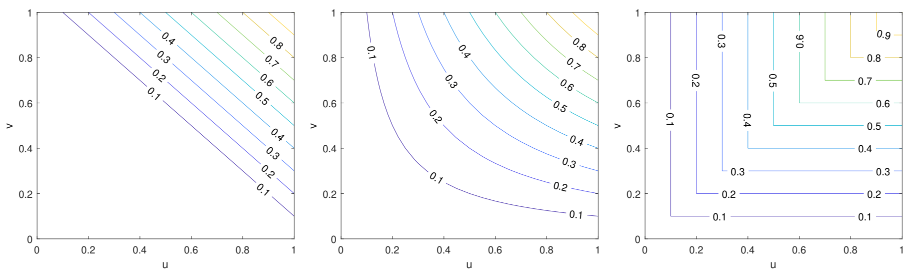


Figure 2.2: Contour maps of copulas $W(u,v)$, $\Pi(u,v)$ and $M(u,v)$

Concordance order and implied dependence properties

In this subsection we mention the (partial) concordance order on the set of copulas C and popular dependence properties.

Definition 3 ([36, Definition 2.8.1])**.** For any $C_1, C_2 \in \mathcal{C}$, we say that C_1 is *smaller* (resp. *larger*) than C_2 and denote it by $C_1 \prec C_2$ (resp. $C_1 \succ C_2$) if $C_1(u,v) \leq C_2(u,v)$ (resp. $C_1(u,v) \geq C_2(u,v)$) for any $(u,v) \in [0,1]^2$.

Then the Fréchet–Hoeffding bounds can be written succinctly as $W \prec C \prec M$ for any $C \in \mathcal{C}$.

A parametric family $\{C_a\}$ is said to be positively (negatively) ordered if $C_a \preceq C_b$ when $a \leq b$ ($a \geq b$). For the class under consideration, this translates to partial ordering of functions $\{f_a\}$, i.e. $\{C_f\}$ is positively (negatively) ordered iff $f_a(t) \leq f_b(t)$, $t \in [0,1]$ when $a \leq b$ ($a \geq b$). Families of copulas in Examples 9, 10, 11 and 15 are all positively ordered.

We now recall some dependence properties.

Definition 4. Let X and Y be random variables. It is said that (see [36, Definitions 5.2.1 and 5.2.3])

- X and Y are positively quadrant dependent (PQD) if

$$\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y), \quad \forall (x,y) \in \mathbb{R}^2.$$

- Y is left tail decreasing in X (denoted $LTD(Y|X)$) if $\mathbb{P}(Y \leq y | X \leq x)$ is a nonincreasing function of x for all y.
- Y is right tail increasing in X (denoted $RTI(Y|X)$) if $\mathbb{P}(Y > y | X > x)$ is a nondecreasing function of x for all y.
- Y is stochastically increasing in X (denoted $SI(Y|X)$) if $\mathbb{P}(Y > y | X = x)$ is a nondecreasing function of x for all y.

It is often desirable to quantify the strength of an association (dependence) of a pair of random variables (X, Y) whose copula is C. Just by looking at the copula C, which provides the dependence, is often not clear if X and Y are strongly/weakly associated (dependent). Many measures of association are known in the literature, and we will mention several of them below. One way to quantify association might be through concordance of random variables. Informally, we say that a random variable X is concordant with Y if “large” values of one of them tend to be associated with “large” values of the other, and likewise “small” values of one with “small” values of the other (see [36, Chap. 5.1]). More precisely, if we have two observations (x_i, y_i) and (x_j, y_j) of continuous random variables (X, Y), then these observations are *concordant* if

$$(x_i - x_j)(y_i - y_j) > 0$$

and *discordant* if

$$(x_i - x_j)(y_i - y_j) < 0.$$

In nonparametric statistics, the sample version of Kendall's τ is extensively used (e.g. to test for independence of X and Y; see [21, Chap. 8]), which is simply the difference between the number of concordant pairs among the sample observations $\{(x_i, y_i), i = 1, \ldots, n\}$ (observations are assumed distinct, and each pair is either concordant or discordant[2]) and the number discordant pairs, normalized by the number of all pairs. Among other measures of association, we mention also Spearman's ρ and Gini's γ. On the population level, for a random vector (X, Y) whose copula is C, the three measures are defined as (see, e.g. [36, Chap. 5.1]):

$$\begin{aligned}
\tau_{X,Y} &= \mathbb{P}((X_1 - X_2)(Y_1 - Y_2) > 0) - \mathbb{P}((X_1 - X_2)(Y_1 - Y_2) < 0); \\
\rho_{X,Y} &= 3(\mathbb{P}((X_1 - X_2)(Y_1 - Y_3) > 0) - \mathbb{P}((X_1 - X_2)(Y_1 - Y_3) < 0)); \\
\gamma_{X,Y} &= 2\int_0^1 \int_0^1 (|u + v - 1| - |u - v|) dC(u, v),
\end{aligned}$$

where (X_i, Y_i), $i = 1, 2, 3$ are independent copies of (X, Y). In terms of the copula C, they are given by the following expressions (see [36,

[2] If ties among x_is or y_is are observed in the data, there are several statistically meaningful ways to extend the definition of concordant/discorcdant pairs to such a case, leading to appropriate modifications of the statistical tests; see [21, Chap. 8].

Theorems 5.1.3, 5.1.6; Corollary 5.1.14]):

$$\tau_{X,Y} = 4 \int_0^1 \int_0^1 C(u,v) dC(u,v) - 1, \tag{2.2}$$

$$\rho_{X,Y} = 12 \int_0^1 \int_0^1 C(u,v) du dv - 3, \tag{2.3}$$

$$\gamma_{X,Y} = 4 \left(\int_0^1 C(u, 1-u) du - \int_0^1 (u - C(u,u)) du \right). \tag{2.4}$$

For any reasonable concordance measure $\kappa_{X,Y}$ in the sense of Scarsini (Kendall's τ, Spearman's ρ and Gini's γ are examples; see [36, Definition 5.1.7]), measuring association between continuous random variables X and Y whose copula is C, an increase of C in concordance order means an increase in $\kappa_{X,Y}$, which justifies the name of the order.

Tail dependence coefficients

Apart from concordance measures which quantify how large/small values of one variable appear with large/small values of the other, one might also be interested in measuring dependence in the upper right and/or lower left corner of $[0,1]^2$. This dependence can be measured in several ways, for example, by upper and lower dependence coefficients, whenever they exist and belong to $(0,1]$, which are given by the following limits:

$$\lambda_U = \lim_{t \uparrow 1} \mathbb{P}(Y > G^{(-1)}(t) | X > F^{(-1)}(t)),$$

$$\lambda_L = \lim_{t \downarrow 0} \mathbb{P}(Y \leq G^{(-1)}(t) | X \leq F^{(-1)}(t)),$$

where $F^{(-1)}(t) := \inf\{x | F(x) \geq t\}$ is the *quasi-inverse* distribution (quantile) function of a continuous random variable X (see [36, Def. 2.3.6]), and likewise $G^{(-1)}$ is the quasi-inverse distribution function of a continuous random variable Y. By [36, Teorem 5.4.2], in terms of the copula C of the variables X and Y, the tail dependence coefficients can be computed as

$$\lambda_U = 2 - \lim_{t \uparrow 1} \frac{1 - \delta_C(t)}{1 - t} = 2 - \delta_C'(1-); \tag{2.5}$$

$$\lambda_L = \lim_{t \downarrow 0} \frac{\delta_C(t)}{t} = \delta_C'(0+), \tag{2.6}$$

where $\delta_C(t) = C(t,t)$ is the diagonal section of copula C.

In order to measure (left-tail) dependence among two continuous random variables, one can use the notion of *left corner set decreasingness*, which has a tight connection to the so-called TP_2 property of the joint distribution function of those variables, as well as that of the corresponding copula. To be more precise, given a pair of continuous random variables X and Y and following [36, Definition 5.2.13], variables X and Y are said to be *left corner set decreasing* (denoted $LCSD(X, Y)$) if

$$\mathbb{P}(X \leq x, Y \leq y | X \leq x', Y \leq y')$$

is non-increasing in x' and y' for any fixed x and y. By [36, Corollaries 5.2.16 and 5.2.17], the $LCSD(X, Y)$ property is equivalent to the joint distribution H (and also corresponding copula C) of X and Y having the TP_2 property, namely,

$$H(x_1, y_1)H(x_2, y_2) - H(x_1, y_2)H(x_2, y_1) \geq 0, \tag{2.7}$$

for all $0 \leq x_1 \leq x_2 \leq 1,\ 0 \leq y_1 \leq y_2 \leq 1$. Copulas with TP_2 property are sometimes called "TP_2 copulas"'.

Symmetry properties

A pair of random variables (X, Y) can also exhibit various symmetries. Recall (see [36, Definition 2.7.1]) that, given a random pair (X, Y) and a point $(a, b) \in \mathbb{R}^2$,

- (X, Y) is radially symmetric about (a, b) if and only if $(X-a, Y-b)$ and $(a - X, y - Y)$ have the same joint distribution function;
- (X, Y) is jointly symmetric about (a, b) if and only if $(X-a, Y-b)$, $(X - a, b - Y)$, $(a - X, Y - b)$ and $(a - X, b - Y)$ have the same joint distribution function.

It is well known (see [36, Theorem 2.7.3]) that if X and Y are continuous random variables with distributions symmetric about a and b, respectively, and joined by copula C, then radial symmetry of the joint distribution of (X, Y) about (a, b) is equivalent to having $C \equiv \widehat{C}$.

Also note that $C_f(x, y) = C_f(y, x)$, i.e. if X and Y are identically distributed with associated copula C_f, then X and Y are exchangeable (see [36, Theorem 2.7.4]).

Archimedean copulas

We close this chapter by mentioning one special family of copulas, namely Archimedean copulas. Such copulas are very popular among practitioners, due to the easy way to compute them, excellent properties and wide range of dependence structures that can be modelled through this family. To be precise, a copula C is called Archimedean if it can be written as

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)),\ (u,v) \in [0,1]^2$$

for some continuous, strictly decreasing, and convex function $\phi : [0,1] \to [0,+\infty]$, called generator, such that $\phi(1) = 0$, where $\phi^{[-1]}$ denotes any appropriately defined *pseudo-inverse* of ϕ. For example,

Definition 5 ([36, Def. 4.1.1.])**.** Let $\phi : I \to [0,+\infty]$ be a continuous, strictly decreasing function such that $\phi(1) = 0$. The pseudo-inverse of ϕ is the function $\phi^{[-1]} : [0,+\infty] \to I$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & 0 \leq \text{t} \leq \phi(0), \\ 0, & \phi(0) \leq \text{t} \leq \infty. \end{cases}$$

Clearly, any Archimedean copula is symmetric, i.e. $C(u,v) = C(v,u)$ for any $(u,v) \in [0,1]^2$. For more about Archimedean copulas; see, e.g. [36, Chap. 4].

Some popular families of copulas, which will be used in examples in following chapters, are Archimedean, for example, Ali-Mikhail-Haq and Gumbel–Hougaard. Unfortunately, in general, proposed mapping H_f does not preserve Archimedean property, i.e. if C is Archimedian, $H_f(C)$ need not be Archimedian (see Proposition 1).

3 CHARACTERIZATION OF ELIGIBLE FUNCTIONS

In this chapter we try to specify the set of eligible functions, i.e. find functions f for which

$$H_f(C)(u,v) := C(u,v)f(\overline{C}(u,v)), \quad (u,v) \in [0,1]^2$$

is a bivariate copula for any $C \in \mathcal{C}$. The rest of the chapter is organized as follows. In Sect. 3.1 we provide necessary and sufficient conditions for the problem. Then, in Sect. 3.2, we provide some examples of function f and corresponding mappings. Section 3.3 constains some properties of the mapping. Finally, in Sect. 3.4, we discuss some extensions to n-dimensional case and in Sect. 3.5 provide some possible extensions of the considered transformation.

3.1 Main results

This section contains the statement and proof of our main result of this chapter (Theorem 3), as well as describes many known examples of transformations which have inspired our work. To distinguish three possible situations, we suggest the following classification of transformation functions:

Definition 6. A function $f : [0,1] \to \mathbb{R}_+$ is said to be

- **eligible** if $H_f(C) \in \mathcal{C}$ for any $C \in \mathcal{C}$; the set of eligible functions will be denoted by $\mathcal{G}_0$;
- **conditionally eligible** if there are $C_1, C_2 \in \mathcal{C}$ such that $H_f(C_1) \in \mathcal{C}$ but $H_f(C_2) \notin \mathcal{C}$;
- **non-eligible** if $H_f(C) \notin \mathcal{C}$ for any $C \in \mathcal{C}$.

For example, any function $f : [0,1] \to \mathbb{R}_+$, for which $f(0) \neq 1$, is non-eligible. Also note that f cannot assume negative values if $H_f(C)$ is to be a copula for all copulas C. Furthermore, any function f such that $f(0) = 1$ is a candidate to be eligible/conditionally-eligible since $H_f(W) = Wf(\overline{W}) = W$ as $W(u,v) = 0$ on

$$\{(u,v) \in [0,1]^2 : u + v \leq 1\}$$

while $\overline{W} = 0$ on

$$\{(u,v) \in [0,1]^2 : u + v \geq 1\}.$$

Many examples of eligible/conditionally-eligible functions are also known. Given the setup described in the Introduction, they can be described as follows:

- F. Durante and coworkers (see [10, Theorem 1] or [13, Theorem 3] where an extension to n-variate copulas is given) characterized $\mathcal{G}'$ when $\mathcal{C}' = \{M\}$, where $M(u,v) = \min\{u,v\}$ is the comonotonicity copula (upper Fréchet–Hoeffding bound). Indeed,[1]

  $$H_f(M)(u,v) = \min\{u,v\} f(1 - \max\{u,v\})$$

  is a copula for a continuous[2] $f : [0,1] \to [0,1]$ iff

  (a) $f(0) = 1$,
  (b) f is non-increasing on $[0,1]$, and
  (c) $t \mapsto f(t)/(1-t)$ is non-decreasing on $[0,1)$.

  This result will prove the necessity of several conditions for a function f to be eligible. Combined with our characterization of eligible functions, it will also give an indication why certain functions f are only conditionally-eligible.

- A. Dolati and M. Úbeda-Flores [9] and A. Kolesárová et al. [26, 27] considered $\mathcal{C}' = \mathcal{C}$ and

  $$\mathcal{G}' = \{\delta_\alpha(u) = 1 - \alpha u : u \in [0,1], \alpha \in [0,1]\}.$$

  This transformation provides a rare (in this context) example of known probabilistic characterization on the level of random variables. Indeed, if X and Y are random variables distributed

[1] To fit current setting, Durante's $f(t)$ is our $f(1-t)$ for $t \in [0,1]$.

[2] Theorem 1 in [10] contained additional differentiability assumption on f which was subsequently removed in [13, Theorem 5].

uniformly on $[0,1]$ whose copula is C, then $H_{\delta_\alpha}(C)$ is the copula of the pair

$$(Z_1, Z_2) := \begin{cases} (\min\{X_1, X_2\}, \max\{Y_1, Y_2\}) & \text{with prob. } \alpha/2; \\ (\max\{X_1, X_2\}, \min\{Y_1, Y_2\}), & \text{with prob. } \alpha/2; \\ (X_1, Y_1), & \text{with prob. } 1-\alpha, \end{cases}$$

where $(X_i, Y_i), i = 1, 2$ are independent copies of (X, Y). Note that $H_{\delta_\alpha}(\Pi)$ for $\alpha \in [0,1]$ gives part of the FGM copula family, the other part corresponding to $\alpha \in [-1, 0)$. This fact, combined with our results, implies that δ_α is eligible iff $\alpha \in [0,1]$ and conditionally-eligible if $\alpha \in [-1, 0)$. Whether δ_α is conditionally-eligible or non-eligible for $\alpha < -1$ remains to be seen; the values $\alpha > 1$ lead to negative values of δ_α so are clearly unacceptable.

- A. Dolati and M. Úbeda-Flores [9] also suggested $\mathcal{C}' = \{\Pi\}$, $\Pi(x,y) = xy$, or $\mathcal{C}' = \{M\}$ and

  $$\mathcal{G}' = \{g_\delta(u) = \exp\{\delta u\} : u \in [0,1], \delta \in \mathbb{R}\}.$$

  The authors have shown that $H_{g_\delta}(\Pi)$ is a copula iff $\delta \in [-1, 1]$, whereas no conditions on δ have been provided for the $H_{g_\delta}(C)$ to be a copula for a general C. Our main results (Theorem 3) shows that g_δ is eligible iff $\delta \in [-1, 0]$. Moreover, g_δ is conditionally-eligible if $\delta \in (0, 1]$, while it is not known if g_δ is conditionally-eligible or non-eligible for $\delta \notin [-1, 1]$.

- F. Durante et al. [17] considered $\mathcal{C}' = \mathcal{C}$ and

  $$\mathcal{G}' = \{f_\lambda(u) = (1 + \lambda u)^{-1} : u \in [0,1], \lambda \in [0,1]\}.$$

  Our results imply that f_λ is eligible iff $\lambda \in [0,1]$. It is also known that for $\lambda \in [-1, 0)$, $H_{f_\lambda}(\Pi)$ is a member of the Ali–Mikhail–Haq family of copulas, hence f_λ is conditionally-eligible for $\lambda \in [-1, 0)$. For $\lambda < -1$, f_λ is clearly non-eligible as it can attain negative when u is close to 1, while it is not known, to the best of our knowledge, if f_λ is conditionally-eligible or non-eligible for $\lambda > 1$.

Now we state our main result of this chapter:

Theorem 3. *A function $f : [0,1] \to \mathbb{R}_+$ is eligible if and only if* *(i)* f *is non-increasing,*

(ii) $f(0) = 1$*,*

(iii) $f(x) \geq 1 - x$ *for any* $x \in [0, 1]$*, and*

(iv) f *is convex.*

Remark 1. Note that conditions (i)–(iii) restrict the range of eligible functions to the interval $[0, 1]$. If one were to replace $\mathbb{R}_+$ with $[0, 1]$, then condition (ii) would become superfluous due to (iii) used with $x = 0$. As condition (ii) is explicitly referred to in the proofs, we chose to keep it in the formulation. Also note that in [10, Theorem 1] condition (c) does not imply condition (a), even if f takes $[0, 1]$ to $[0, 1]$.

To prove the 2-increasingness property of $H_f(C)$ the following result will be crucial.

Lemma 1. *Let* $C \in \mathcal{C}$ *and let* f *satisfy conditions (i)–(iv) of Theorem 3. Then for any* $(u_i, v_i) \in [0, 1]^2, i = 1, 2$ *and such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$*,*

$$V_{f \circ \overline{C}}([u_1, u_2] \times [v_1, v_2]) \geq -V_C([u_1, u_2] \times [v_1, v_2]),$$

where for any bivariate function $g : [0, 1]^2 \to \mathbb{R}$*,*

$$V_g([u_1, u_2] \times [v_1, v_2]) = g(u_2, v_2) - g(u_1, v_2) - g(u_2, v_1) + g(u_1, v_1).$$

Remark 2. The bound of Lemma 1 is sharp, as for $f(x) = 1 - x$ the lower bound is attained.

Proof of Lemma 1. Denote

$$z_1 := \overline{C}(u_2, v_2), \qquad z_4 := \overline{C}(u_1, v_1),$$
$$z_2 := \overline{C}(u_2, v_1) \wedge \overline{C}(u_1, v_2), \qquad z_3 := \overline{C}(u_2, v_1) \vee \overline{C}(u_1, v_2).$$

Here for $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Since any copula C is 1-Lipschitz, $\overline{C}$ is non-increasing in each of its arguments and so $z_1 \leq z_2 \leq z_3 \leq z_4$. Thus, since f is also non-increasing (by (i) of Theorem 3),

$$f(z_4) \leq f(z_3) \leq f(z_2) \leq f(z_1)$$

and

$$V_{f\circ\overline{C}}([x_1,x_2]\times[y_1,y_2]) = f(z_1) - f(z_2) - f(z_3) + f(z_4) =: \Delta.$$

Now consider several cases:

Case 1. $0 \le z_1 < z_2 \le z_3 < z_4 \le 1$. Then

$$\Delta = -\frac{f(z_2)-f(z_1)}{z_2-z_1}(z_2-z_1) + \frac{f(z_4)-f(z_3)}{z_4-z_3}(z_4-z_3),$$

and for the considered f we have

$$-1 \overset{(iii)}{\le} \frac{f(z_2)-1}{z_2} \overset{(ii),(iv)}{\le} \frac{f(z_2)-f(z_1)}{z_2-z_1} \overset{(iv)}{\le} \frac{f(z_4)-f(z_3)}{z_4-z_3} \overset{(i)}{\le} 0, \tag{3.1}$$

yielding

$$\begin{aligned}\Delta &\ge \frac{f(z_4)-f(z_3)}{z_4-z_3}(z_4-z_3-z_2+z_1)\\ &= \frac{f(z_4)-f(z_3)}{z_4-z_3}V_{\overline{C}}([u_1,u_2]\times[v_1,v_2])\\ &= \frac{f(z_4)-f(z_3)}{z_4-z_3}V_{C}([u_1,u_2]\times[v_1,v_2])\\ &\overset{(3.1)}{\ge} -V_C([u_1,u_2]\times[v_1,v_2]).\end{aligned}$$

Case 2. $0 \le z_1 = z_2 \le z_3 < z_4$. Similar to Case 1, we get

$$\Delta = \frac{f(z_4)-f(z_3)}{z_4-z_3}(z_4-z_3-z_2+z_1) \overset{(3.1)}{\ge} -V_C([u_1,u_2]\times[v_1,v_2]).$$

Case 3. $0 \le z_1 < z_2 \le z_3 = z_4$. Analogous to Case 2.

Case 4. $0 \le z_1 = z_2 \le z_3 = z_4$. In this case, $\Delta = 0$ trivially, and likewise

$$V_{\overline{C}}([u_1,u_2]\times[v_1,v_2]) = V_C([u_1,u_2]\times[v_1,v_2]) = 0.$$

□

Now we can prove our main theorem of this chapter.

Proof of Theorem 3. (Sufficiency) Using condition (ii) and the fact that $C \in \mathcal{C}$ is a copula, we easily verify that $H_f(C)$ satisfies the boundary conditions of a copula:

$$H_f(C)(u,1) = uf(0) = u, \qquad H_f(C)(1,v) = vf(0) = v,$$
$$H_f(C)(u,0) = H_f(0,v) = 0, \qquad \forall u,v \in [0,1].$$

It remains to show that $H_f(C)$ is 2-increasing for any $C \in \mathcal{C}$ and f satisfying (i)–(iv), that is,

$$\begin{aligned} 0 \leq V_{H_f(C)}(\Box) = H_f(C)(u_1,v_1) - H_f(C)(u_1,v_2) \\ - H_f(C)(u_2,v_1) + H_f(C)(u_2,v_2), \end{aligned}$$

where we denote $\Box := [u_1,u_2] \times [v_1,v_2]$ for $u_1,u_2,v_1,V_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Observe that

$$\begin{aligned} V_{H_f(C)}(\Box) &= C(u_1,v_1)V_{f\circ\overline{C}}(\Box) + V_C(\Box) f\circ\overline{C}(u_2,v_2) \\ &\quad + (C(u_1,v_2) - C(u_1,v_1))(f\circ\overline{C}(u_2,v_2) - f\circ\overline{C}(u_1,v_2)) \\ &\quad + (C(u_2,v_1) - C(u_1,v_1))(f\circ\overline{C}(u_2,v_2) - f\circ\overline{C}(u_2,v_1)) \\ &\geq C(u_1,v_1)V_{f\circ\overline{C}}(\Box) + V_C(\Box) f\circ\overline{C}(u_2,v_2), \end{aligned}$$

since C and $f\circ\overline{C}$ are non-decreasing in each argument (here we used condition (i)). Now by Lemma 1 and condition (iii),

$$\begin{aligned} V_{H_f(C)}(\Box) &\geq V_C(\Box)(f\circ\overline{C}(u_2,v_2) - C(u_1,v_1)) \\ &\geq V_C(\Box)(1 - \overline{C}(u_2,v_2) - C(u_1,v_1)) \\ &= V_C(\Box)(u_2 + v_2 - C(u_2,v_2) - C(u_1,v_1)) \\ &\geq V_C(\Box)(C(u_2,1) - C(u_2,v_2) + C(1,v_1) - C(u_1,v_1)) \geq 0, \end{aligned}$$

since being a copula C is non-decreasing in each argument.

(Necessity) Conditions (i)–(iii) are necessary for f to be eligible. This follows from Theorem 1 in [10]. As for condition (iv), assume that f is eligible and suppose on the contrary that f is not convex. We will construct a (diagonal) copula C_δ such that $H_f(C_\delta)$ is not a copula.

To this end, first observe that, being continuous, such f cannot be convex in the Jensen sense (see [35, Section 1.4.3]), i.e. there are points $x_0, y_0 \in [0,1]$ such that $x_0 < y_0$ and

$$f\left(\frac{x_0+y_0}{2}\right) > \frac{f(x_0)+f(y_0)}{2}. \tag{3.2}$$

Without loss of generality, we may assume that $y_0 < 1$. If this were not the case, we could consider $y_0' < y_0$ such that (3.2) holds with y_0' in place of y_0 as the function

$$g(y) := f\left(\frac{x_0 + y}{2}\right) - \frac{f(x_0) + f(y)}{2}$$

is continuous and $g(y_0) > 0$.

Now for $y_0 < 1$ we can find $n_0 \in \mathbb{N}$ such that $y_0 < 1 - 2^{-n_0}$. Next define the function

$$\delta(t) := \begin{cases} \alpha t, & \text{if } 0 \le t \le v_1 := (1 + 2^{-n_0} - y_0)/2; \\ 2^{-n_0}, & \text{if } v_1 < t \le v_2 := (1 + 2^{-n_0} - x_0)/2; \\ 1 + \beta(t-1), & \text{if } v_2 < t \le 1, \end{cases}$$

where $2^{-n_0} < v_1 < v_2 \le (1 + 2^{-n_0})/2$ and

$$\alpha = 2^{-n_0}/v_1 < 1, \quad \text{and} \quad \beta = (1 - 2^{-n_0})/(1 - v_2) \le 2.$$

Clearly, such a function $\delta(t) \le t$ is continuous (piecewise linear), non-decreasing, β-Lipschitz, and, moreover, $\delta(0) = 0$, $\delta(1) = 1$. By [36, Theorem 3.2.12], there is a diagonal copula, namely

$$C_\delta(u, v) = \min\left\{u, v, \frac{\delta(u) + \delta(v)}{2}\right\}, \tag{3.3}$$

with $\delta(t)$ as its diagonal section. Now if $\square = [v_1, v_2] \times [v_1, v_2]$, then

$$\begin{aligned} V_{H_f(C_\delta)}(\square) &= C_\delta(v_2, v_2) f(\overline{C_\delta}(v_2, v_2)) - C_\delta(v_1, v_2) f(\overline{C_\delta}(v_1, v_2)) \\ &\quad - C_\delta(v_2, v_1) f(\overline{C_\delta}(v_2, v_1)) + C_\delta(v_1, v_1) f(\overline{C_\delta}(v_1, v_1)) \\ &= \delta(v_2) f(x_0) - 2C_\delta(v_1, v_2) f\left(\frac{x_0 + y_0}{2}\right) + \delta(v_1) f(y_0) \\ &= 2^{-n_0}\left(f(x_0) - 2f\left(\frac{x_0 + y_0}{2}\right) + f(y_0)\right) < 0, \end{aligned}$$

since

$$\begin{aligned} &\overline{C_\delta}(v_1, v_1) = 1 - 2v_1 + \delta(v_1) = y_0, \quad \overline{C_\delta}(v_2, v_2) = 1 - 2v_2 + \delta(v_2) = x_0, \\ &\overline{C_\delta}(v_1, v_2) = \overline{C_\delta}(v_2, v_1) = 1 - v_1 - v_2 + C_\delta(v_1, v_2) = \frac{x_0 + y_0}{2}. \end{aligned}$$

This shows that $H_f(C_\delta)$ is not a copula, contrary to our assumption, and completes the proof of the theorem. □

Remark 3. Comparing our Theorem 3 with Durante's [10, Theorem 1], we note the following:

- Condition (iii) follows from a stronger condition (c) by using $t = 0$ and condition (a), which is also condition (ii).

- Conditions (i), (iii), and (iv) together imply condition (c). Indeed, to show (c), by [10, Lemma 1] (see also [13, Lemma 2]), it is enough to show that for $0 \le v < u \le 1$ one has

  $$(1-u)f(u) + (1-v)f(v) - 2(1-u)f(v) \ge 0,$$

  which can be equivalently written as

  $$(1-u)\frac{f(u)-f(v)}{u-v} + f(v) \ge 0.$$

  But the latter follows from the following inequalities which are implied by conditions (i), (iii) and (iv):

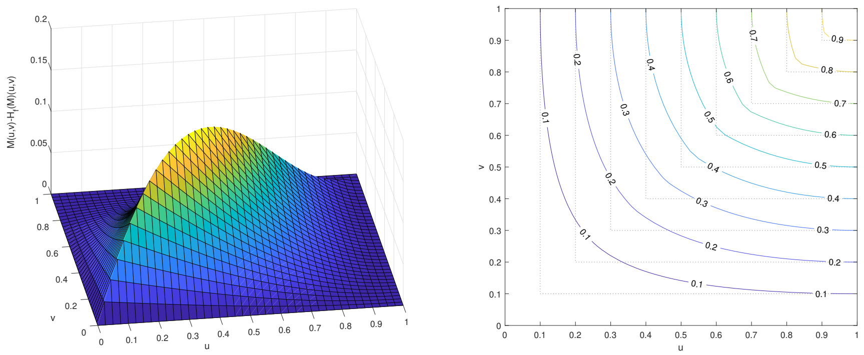
  $$-1 \le \frac{f(u)-1}{u} \le \frac{f(u)-f(v)}{u-v} \le 0.$$

- Some concave functions satisfy condition (c). For example, consider simple concave functions $f_\theta(x) = 1 \wedge \frac{1-x}{1-\theta}$ for $\theta \in [(5-\sqrt{13})/8, 1/4) \approx [0.1743, 0.25)$, satisfying (i)–(iii) and even condition (c), which produce a "ripple" in the graph of $H_{f_\theta}(\Pi)$ making it a non-copula, since

  $$V_{H_{f_\theta}(\Pi)}([1/2,1/2] \times [1-2\theta, 1-2\theta]) = \frac{3}{16(1-\theta)} - 2\theta(1-2\theta) < 0.$$

3.2 Additional examples

In this section we present a few more examples of the copulas $H_f(C)$ for different f and C. We begin with an illustration of the counterexample constructed in the proof of Theorem 3.



(a) A difference $M - H_f(M)$ (b) contour map of copula $H_f(M)$

Figure 3.1: Copula $H_f(M)$ whith $f(t) = \sin(\pi(1-t)/2)$

Example 1. Let $f(t) = \sin(\pi(1-t)/2)$ (see [10, Example 5]). It was shown that $H_f(M)$ is a copula (see Figure 3.1). Yet, f is concave, so our construction from the necessity part proof of Theorem 3 applies. Indeed, take $x_0 = 1/4$, $y_0 = 3/4$, $n_0 = 3$, $v_1 = 3/16$, $v_2 = 7/16$, $\alpha = 2/3$, $\beta = 14/9$, and

$$\delta(t) = \begin{cases} 2t/3, & \text{if } 0 \le t \le 3/16; \\ 1/8, & \text{if } 3/16 < t \le 7/16; \\ 1 + 14(t-1)/9, & \text{if } 7/16 < t \le 1. \end{cases}$$

Then for $\square = [3/16, 3/16] \times [7/16, 7/16]$ and C_δ given in (3.3),

$$V_{H_f(C_\delta)}(\square) = \frac{1}{16}\left(\sqrt{2+\sqrt{2}} - 2\sqrt{2} + \sqrt{2-\sqrt{2}}\right) \approx -0.01346.$$

A couple of known copula families that fit our setting are the following:

Example 2.

- Let $f_{\theta,\phi}(t) = 1 + \theta t + \frac{1}{2}\theta\phi t^2$, $\theta, \phi \in [-1, 0]$. Then $H_{f_{\theta,\phi}}(\Pi)$ is known as part of Lin's iterated FGM copula family

  $$H_{f_{\theta,\phi}}(\Pi)(u,v) = uv\left[1 + \theta(1-u)(1-v)\left\{1 + \frac{\phi}{2}(1-u)(1-v)\right\}\right].$$

  For more on Lin's iterated FGM copulas, see [30].

- Let $f_\alpha(t) = \exp\{((1-t)^\alpha - 1)/\alpha\}$ for $\alpha > 0$ (see [10, Example 4]). Again, $H_{f_\alpha}(M)$ was shown to be a copula. Yet f_α

satisfies conditions (i)–(iv) of Theorem 3 if and only if $\alpha \geq 1$. For $\alpha \in (0,1)$, f_α is only conditionally eligible.

Here are a couple of additional examples:

Example 3. Let $(a,b) \in \{(s,t) \in [0,1]^2 : t \geq 1-s\}$ and $f_{(a,b)}(x) := b \vee \left(1 - \frac{1-b}{a}x\right)$. Then setting $C_{a,b} := H_{f_{(a,b)}}(\Pi)$ we have

$$C_{a,b}(u,v) = uv\left[b \vee \left(1 - \frac{1-b}{a}(1-u)(1-v)\right)\right]$$
$$= \begin{cases} uv\left(1 - \frac{1-b}{a}(1-u)(1-v)\right), & 0 \leq (1-u)(1-v) \leq a; \\ buv, & a < (1-u)(1-v) \leq 1, \end{cases}$$

which, by setting $a = 1$, recovers part of FGM copula family with parameter $\theta = 1 - b \geq 0$.

Notice that $H_f(\Pi)$ is a function of $\Pi(u,v) = uv$ and

$$\overline{\Pi}(u,v) = (1-u)(1-v).$$

One can get new copulas from Π by composing H_f with H_g for different $f, g \in \mathcal{G}_0$:

Example 4.

- Let $f_{\alpha_i}(t) = 1 - \alpha_i t$, $i = 1,2$ and $\alpha_i \in [0,1]$. Then

$$\begin{aligned} C_{\alpha_1,\alpha_2}(u,v) &:= \left(H_{f_{\alpha_1}} \circ H_{f_{\alpha_2}}\right)(\Pi)(u,v) \\ &= [H_{f_{\alpha_2}}(\Pi)(1 - \alpha_1 \overline{H_{f_{\alpha_2}}(\Pi)})](u,v) \\ &= uv(1 - \alpha_2(1-u)(1-v)) \\ &\quad \times [1 - \alpha_1(1-u)(1-v)(1-\alpha_2 uv)], \end{aligned}$$

  which differs from iterated FGM copulas of (a) Kotz and Johnson [28] and (b) Lin [30]:

  (a) $C_{\theta,\phi}(u,v) = uv\{1 + \theta(1-u)(1-v)[1 + \phi uv]\}$;

  (b) $C_{\theta,\phi}(u,v) = uv\{1 + \theta(1-u)(1-v)[1 + \phi(1-u)(1-v)]\}$;

  where $\theta, \phi \in [-1,1]$.

- For $f_\alpha(x) = 1 - \alpha x$ and $g_\lambda(x) = (1 + \lambda x)^{-1}$ with $\alpha, \lambda \in [0,1]$ as considered by Dolati and Úbeda-Flores [9], composing in different order one respectively gets:

$$C_{\lambda,\alpha}(u,v) := (H_{g_\lambda} \circ H_{f_\alpha})(\Pi)(u,v) = \frac{H_{f_\alpha}(\Pi)}{(1+\lambda\overline{H_{f_\alpha}(\Pi)})}(u,v)$$
$$= \frac{uv(1-\alpha(1-u)(1-v))}{1+\lambda(1-u)(1-v)[1-\alpha uv]}$$

and

$$\begin{aligned} C_{\alpha,\lambda}(u,v) &:= (H_{f_\alpha} \circ H_{g_\lambda})(\Pi)(u,v) \\ &= [H_{g_\lambda}(\Pi)(1-\alpha\overline{H_{g_\lambda}(\Pi)})](u,v) \\ &= \frac{uv}{(1+\lambda(1-u)(1-v))}\left[1 - \frac{\alpha(1-u)(1-v)[1+\lambda(1-u-v)]}{1+\lambda(1-u)(1-v)}\right] \\ &= \frac{uv\,[1+(1-u)(1-v)[\lambda-\alpha-\lambda\alpha(1-u-v)]}{(1+\lambda(1-u)(1-v))^2}. \end{aligned}$$

Also by considering $f_\alpha(x)$ we get

$$H_{f_\alpha}(C)(u,v) = -\alpha C^2(u,v) + \alpha u C(u,v) + \alpha v C(u,v) + (1-\alpha)C(u,v)$$

which partially recovers copula mapping suggested in [27] (with parameter $d=0$ in their setting).

- If $f_{\lambda_i}(x) = (1+\lambda_i x)^{-1}, \lambda \in [0,1]$ and $i=1,2$, then

$$C_{\lambda_1,\lambda_2}(u,v) := \left(H_{g_{\lambda_1}} \circ H_{g_{\lambda_2}}\right)(\Pi)(u,v)$$
$$= \frac{uv}{(1+\lambda_1(1-u-v))(1+\lambda_2(1-u)(1-v))+\lambda_1 uv}.$$

Note that in this example, all listed two-parameter copulas are polynomial or rational functions of Π and $\overline{\Pi}$ since

$$1-u-v = \overline{\Pi}(u,v) - \Pi(u,v).$$

3.3 Properties of transformation H

The mapping H possesses several nice properties. Recall that $W(u,v) = (u+v-1)^+$ is the lower Fréchet–Hoeffding bound, i.e. for any $C \in \mathcal{C}$, $C \succ W$. Then we have:

Theorem 4. *For any $f \in \mathcal{G}_0$ and any $C, C_1, C_2 \in \mathcal{C}$, the mapping $C \mapsto H_f(C)$*

(i) has a unique fixed point, W, for all $f \not\equiv 1$, i.e. $H_f(C) = C$ if and only if $C = W$;

(ii) is injective, i.e. $H_f(C_1) = H_f(C_2)$ if and only if $C_1 = C_2$;

(iii) preserves concordance order, i.e. if $C_1 \prec C_2$ then $H_f(C_1) \prec H_f(C_2)$ (see Definition 3);

(iv) decreases upper and lower tail dependence indices, whenever they exist, i.e. if a copula C has upper and lower tail indices, $\lambda_U = \lambda_U(C)$ and $\lambda_L = \lambda_L(C)$, respectively, then

$$\lambda_U(H_f(C)) = (1+f'(0+))\lambda_U(C) \quad \text{and} \quad \lambda_L(H_f(C)) = f(1)\lambda_L(C).$$

Moreover,

(v) if $C \in \mathcal{C}$ is symmetric (that is, $C(u,v) = C(v,u)$ for any $(u,v) \in [0,1]^2$), then $H_f(C)$ is also symmetric; and

(vi) if C is radially symmetric (that is, $C = \widehat{C}$) and $C > 0$ on $(0,1]^2$ then $H_f(C) = \widehat{H_f(C)}$ iff $f(t) = 1 - \alpha t$, $\alpha \in [0,1]$.

Proof. (i) Having $H_f(C) = Cf(\overline{C}) = C$ is equivalent to having $\overline{C} = 0$ on the set

$$A := \{(u,v) \in [0,1]^2 : C(u,v) > 0\},$$

since $f(0) = 1$, $f \not\equiv 1$, f is non-increasing and convex (see (i), (ii) and (iv) of Theorem 3). So for any $(x,y) \in A$,

$$\overline{C}(u,v) = 0 = 1 - u - v + C(u,v)$$

implies that

$$0 < C(u,v) = u + v - 1 = W(u,v).$$

On the complement $[0,1]^2 \setminus A$, both C and W are zero, so also equal.

(ii) Sufficiency is trivial. To prove necessity, omitting arguments to simplify notation, observe that having $H_f(C_1) = H_f(C_2)$ is equivalent to

$$
\begin{aligned}
0 &= C_1 f(\overline{C}_1) - C_2 f(\overline{C}_2) \\
&= C_1 (f(\overline{C}_1) - f(\overline{C}_2)) + (C_1 - C_2) f(\overline{C}_2) := B
\end{aligned}
\tag{3.4}
$$

on the whole square $[0,1]^2$. Suppose there exists a point $(u_0, v_0) \in [0,1]$ such that $C_1(u_0, v_0) > C_2(u_0, v_0)$. Then also

$$
z_2 := \overline{C}_1(u_0, v_0) > z_1 := \overline{C}_2(u_0, v_0)
$$

and

$$
0 \le f(z_1) - f(z_2) \le z_2 - z_1, \tag{3.5}
$$

by the first two inequalities in (3.1). Now using (iii) of Theorem 3,

$$
f(z_1) \ge u_0 + v_0 - C_2(u_0, v_0) > u_0 + v_0 - C_1(u_0, v_0) \ge C_1(u_0, v_0),
$$

where the last inequality follows from the upper Fréchet–Hoeffding bound (see Theorem 2):

$$
C_1(u_0, v_0) \le M(u_0, v_0) = x_0 \wedge y_0 \le \frac{u_0 + v_0}{2}.
$$

Putting together the obtained inequalities yields

$$
\begin{aligned}
B(u_0, v_0) &= C_1(u_0, v_0)(f(z_2) - f(z_1)) + (C_1(u_0, v_0) - C_2(u_0, v_0)) f(z_1) \\
&\ge (C_1(u_0, v_0) - C_2(u_0, v_0))(f(z_1) - C_1(u_0, v_0)) > 0,
\end{aligned}
$$

which contradicts (3.4).

(iii) Suppose $C_1 \ge C_2$ on the whole $[0,1]^2$. Then also $\overline{C}_1 \ge \overline{C}_2$ on the same square and so $f(\overline{C}_1) \le f(\overline{C}_2)$ for any eligible f by (i) of Theorem 3. Using (3.5), on I^2 then

$$
0 \le f(\overline{C}_2) - f(\overline{C}_1) \le \overline{C}_1 - \overline{C}_2 = C_1 - C_2.
$$

Therefore we can write

$$\begin{aligned} H_f(C_1) - H_f(C_2) &= C_1 f(\overline{C}_1) - C_2 f(\overline{C}_2) \\ &= f(\overline{C}_1)(C_1 - C_2) - (f(\overline{C}_2) - f(\overline{C}_1))C_2 \\ &\geq f(\overline{C}_1)(C_1 - C_2) - (C_1 - C_2)C_2 \\ &= (C_1 - C_2)(f(\overline{C}_1) - C_2) \\ &\geq (C_1 - C_2)(1 - \overline{C}_1 - C_2) \\ &\geq 0, \end{aligned}$$

since for any $(u, v) \in [0,1]^2$,

$$1 - \overline{C}_1(u,v) = u + v - C_1(u,v) \geq u + v - u \wedge v = u \vee v \geq u \wedge v \geq C_2(u,v).$$

(iv) Using (2.5), together with the facts that

$$\delta_{H_f(C)}(t) = \delta_C(t) f(1 - 2t + \delta_C(t)),$$

f is convex and $f(0) = 1$ (by Theorem 3), yields

$$\begin{aligned} \lambda_U(H_f(C)) &= 2 - \lim_{t \uparrow 1} \frac{1 - \delta_{H_f(C)}(t)}{1 - t} \\ &= 2 - \lim_{t \uparrow 1} \left[\frac{1 - \delta_C(t)}{1 - t} - \delta_C(t) \frac{1 - f(1 - 2t + \delta_C(t))}{0 - (1 - 2t + \delta_C(t))} \frac{1 - 2t + \delta_C(t)}{1 - t} \right] \\ &= \lambda_U(C) + \delta_C(1) f'(0+) \lambda_U(C), \end{aligned}$$

proving the first formula since $\delta_C(1) = 1$. To get the second, we use (2.6) and obtain

$$\lambda_L(H_f(C)) = \lim_{t \downarrow 0} \frac{\delta_{H_f(C)}(t)}{t} = \lim_{t \downarrow 0} \frac{\delta_C(t) f(1 - 2t + \delta_C(t))}{t} = \lambda_L(C) f(1),$$

since $\delta_C(0) = 0$.

(v) Obvious from the definition of $H_f(C)$.

(vi) If $C = \widehat{C}$ and $f(t) = 1 - \alpha t$ for $\alpha \in [0,1]$, then $H_f(C) = \widehat{H_f(C)}$ by [9, Theorem 3.9].

The proof of necessity mimics the proof of Theorem 3(b) in [10]. Indeed, consider an $f \in \mathcal{G}_0$, let $C = \widehat{C}$ be such that $C > 0$ on $(0,1]^2$, and suppose that $H_f(C) = \widehat{H_f(C)}$. Then

$$\overline{C}(u,v) = C(1-u, 1-v), \quad \forall (u,v) \in [0,1]^2$$

and

$$\begin{aligned}
0 &= \overline{H_f(C)}(u,v) - H_f(C)(1-u,1-v) \\
&= 1-u-v+C(u,v)f(\overline{C}(u,v)) - C(1-u,1-v)f(\overline{C}(1-u,1-v)) \\
&= (\overline{C}(u,v) - C(u,v)) + C(u,v)f(C(1-u,1-v)) \\
&\quad - C(1-u,1-v)f(C(u,v)),
\end{aligned}$$

which after rearrangement for $(u,v) \in (0,1)^2$ becomes

$$\frac{1-f(C(u,v))}{C(u,v)} = \frac{1-f(C(1-u,1-v))}{C(1-u,1-v)}.$$

Letting $v \uparrow 1$ for a fixed $u \in (0,1)$, we get $(1-f(u))/u$ on the left-hand side, while the right-hand side tends to $-f'(0+)$. Hence,

$$f(u) = 1 + f'(0+)u, \quad \forall u \in (0,1).$$

By being eligible, f is continuous on $[0,1]$, hence $f(x) = 1 - \alpha x$ for any $x \in [0,1]$ with $\alpha = -f'(0+) \in (0,1]$. The case $f \equiv 1$ corresponds to $\alpha = f'(0+) = 0$, in which case $H_f(C) = C$ and there's nothing to prove. □

Remark 4. Condition $C > 0$ on $(0,1]^2$ of Theorem 4(v) is satisfied, e.g. by Frank, Gauss and Student copulas which are known to be radially symmetric. This condition eliminates the trivial case $C = W = H_f(C)$ for any eligible f and also helps avoid situations where the function f can be uniquely specified only on a subinterval of $[0,1]$. To illustrate this possibility, consider the diagonal copula corresponding to W, namely

$$C(u,v) = C_{\delta_W}(u,v) = \min\{u, v, (u-1/2)^+ + (v-1/2)^+\}$$

and consider $\tilde{f}_\alpha(t) = f_{(1/2, 1-\alpha/2)}(t) = (1-\alpha t) \vee (1-\alpha/2)$, $\alpha \in (0,1]$, $t \in [0,1]$ (see Example 3). It is straightforward to check that both C

and $H_{\tilde{f}_\alpha}(C)$ are symmetric and radially symmetric. Indeed,

$$C(u,v) = \overline{C}(1-u,1-v) = \begin{cases} 0, & 0 \le u,v \le \frac{1}{2}; \\ u \wedge \left(v - \frac{1}{2}\right), & 0 \le u \le \frac{1}{2} < v \le 1; \\ v \wedge \left(u - \frac{1}{2}\right), & 0 \le v \le \frac{1}{2} < u \le 1; \\ u+v-1, & \frac{1}{2} < u,v \le 1, \end{cases}$$

and

$$\begin{aligned} &\overline{H_{\tilde{f}_\alpha}(C)}(u,v) = H_{\tilde{f}_\alpha}(C)(1-u,1-v) \\ &= \begin{cases} 1-u-v, & 0 \le u,v \le \frac{1}{2}; \\ \left(1-\alpha\left(u \wedge \left(v-\frac{1}{2}\right)\right)\right)\left(\frac{1}{2} - u \vee \left(v - \frac{1}{2}\right)\right), & 0 \le u \le \frac{1}{2} < y \le 1; \\ \left(1-\alpha\left(v \wedge \left(u-\frac{1}{2}\right)\right)\right)\left(\frac{1}{2} - v \vee \left(u - \frac{1}{2}\right)\right), & 0 \le v \le \frac{1}{2} < u \le 1; \\ 0, & \frac{1}{2} < u,v \le 1. \end{cases} \end{aligned}$$

As a corollary of Theorem 4 we immediately obtain the following result.

Corollary 5. *For any $f \in \mathcal{G}_0$, $C \in \mathcal{C}$, and any continuous random variables X and Y, having copula $H_f(C)$, we have the following inequalities:*

$$-1 \le \tau_{X,Y}(H_f(C)) \le 4\int_0^1 (1-x)f^2(x)dx - 1, \tag{3.6}$$

$$-1 \le \rho_{X,Y}(H_f(C)) \le 12\int_0^1 (1-x)^2 f(x)dx - 3, \tag{3.7}$$

$$-1 \le \gamma_{X,Y}(H_f(C)) \le 4\int_0^{1/2} \left(x(f(x)+f(1-x)) + f(x)\right)dx - 2. \tag{3.8}$$

Proof. By Theorem 4 (i) and (iii), $W = H_f(W) \prec H_f(C) \prec H_f(M)$ for any $C \in \mathcal{C}$ and any $f \in \mathcal{G}$, and due to [36, Theorems 5.1.9 and 5.1.13], for Kendall's τ, Spearman's ρ, and Gini's γ, we have

$$-1 = \tau_{X,Y}(W) \le \tau_{X,Y}(H_f(C)) \le \tau_{X,Y}(H_f(M)), \tag{3.9}$$

$$-1 = \rho_{X,Y}(W) \le \rho_{X,Y}(H_f(C)) \le \rho_{X,Y}(H_f(M)), \tag{3.10}$$

$$-1 = \gamma_{X,Y}(W) \le \gamma_{X,Y}(H_f(C)) \le \gamma_{X,Y}(H_f(M)). \tag{3.11}$$

Now using [10, Theorem 4], the fact that our $f(t)$ is Durante's $f(1-t)$, and a simple change of variables, we immediately get the claimed inequalities. □

The next proposition contains some negative facts about the mapping H_f, including a few important properties of a copula that are not preserved, in general.

Proposition 1. *The following is true about H_f:*

(a) *Given a general $f \in \mathcal{G}_0$, the mapping $H_f : \mathcal{C} \to \mathcal{C}$ is not necessarily surjective.*

(b) *Given a general $f \in \mathcal{G}_0$ and a copula $C \in \mathcal{C}$ that has the TP_2 property, i.e. for all $0 \leq u_1 \leq u_2 \leq 1,\ 0 \leq v_1 \leq v_2 \leq 1$,*

$$C(u_1, v_1)C(u_2, v_2) - C(u_1, v_2)C(u_2, v_1) \geq 0,$$

the image $H_f(C)$ does not necessarily have this property.

(c) *Given a general $f \in \mathcal{G}_0$ and an Archimedean copula $C \in \mathcal{C}$, the image $H_f(C)$ need not be Archimedean.*

Proof. **(a)** Let $f(t) = 1 - t$ and see [26, Example 2] where a Bertino copula B_δ for a specific function δ is constructed, which has no preimage under the map H_f taking C to $C(1 - \overline{C})$.

(b) Both M and $H_f(M)$ are known to be TP_2 [15, Example 3.7]. On the other hand, $H_f(\Pi)$ belongs to the Farlie–Gumbel–Morgenstern (FGM) copula family when

$$f(t) = 1 - \alpha t, \quad \alpha \in (0, 1],$$

which is easily checked to be RR_2 (*reverse regular of order two*, i.e. the inequality of TP_2 property is reversed), so TP_2 property of Π is not preserved.

(c) The independence copula Π is clearly Archimedean with generator $\phi(t) = -\ln t$. Then, on the one hand, the members of the FGM copula family, $H_f(\Pi)$ for

$$f(t) = 1 - \alpha t, \quad \alpha \in (0, 1],$$

are not Archimedean; see [36, Example 4.7]. On the other hand, the members of the Ali–Mikhail–Haq (AMH) family, namely $H_f(\Pi)$ for

$$f(t) = (1 + \alpha t)^{-1}, \quad \alpha \in (0, 1]$$

are Archimedean; see [36, Example 4.8].

□

Let

$$H_f^0 := id \text{ (identity mapping on } \mathcal{C}), \quad H_f^k := H_f \circ H_f^{k-1}, \; k \geq 1.$$

Excluding the trivial case $f \equiv 1$ which implies $H_f^k = id$ for any $k \geq 0$, we have

Theorem 6. *If f is eligible and $f \not\equiv 1$, then for each $C \in \mathcal{C}$,*

$$\lim_{k \to \infty} \left\| H_f^k(C) - W \right\|_\infty = 0.$$

Proof. As pointwise convergence of copulas to a copula implies uniform convergence (due to equicontinuity of the family of copulas), it suffices to show for any $(u, v) \in [0, 1]^2$ that

$$\lim_{k \to \infty} H_f^k(C)(u, v) - W(u, v) = 0. \tag{3.12}$$

Now observe that for any $(u, v) \in [0, 1]^2$, since $f(t) \leq 1$ for all $t \in [0, 1]$ and $k \geq 1$

$$H_f^k(C)(u, v) = H_f^{k-1}(C)(u, v) f(\overline{H_f^{k-1}(C)}(u, v)) \leq H_f^{k-1}(C)(u, v).$$

Therefore the sequence of copulas $H^k(C), k \geq 0$ is point-wise monotonic and hence it converges to some copula which must be H_f-invariant. By using Theorem 4 we get that the limit equals W. □

We close this section by considering another function that is often encountered in applications, namely that of a copula density. It is required when trying to fit various copula models to given data using maximum likelihood or Bayesian approach. So a question of how a density of an absolutely continuous copula is transformed under the map $C \mapsto H_f(C)$ arises. The answer is provided by the following simple proposition where to simplify notation we let $\partial_u = \frac{\partial}{\partial u}$, $\partial_v = \frac{\partial}{\partial v}$ and $\partial^2_{uv} = \frac{\partial^2}{\partial u \partial v}$.

Proposition 2. *If f is eligible and such that f'' exists on $(0,1)$ and C is an absolutely continuous bivariate copula, then (dropping the arguments on both sides) the density of $H_f(C)$ is given by*

$$\begin{aligned}\partial^2_{uv} H_f(C) = f'(\overline{C}) \cdot [\partial_u C \cdot (\partial_v C - 1) + \partial_v C \cdot (\partial_u C - 1)] \\ + C \cdot f''(\overline{C}) \cdot (\partial_u C - 1) \cdot (\partial_v C - 1) \\ + \partial^2_{uv} C \cdot \left[f(\overline{C}) + C \cdot f'(\overline{C}) \right].\end{aligned}$$

Proof. Differentiating with respect to u yields

$$\partial_u H_f(C) = \partial_u C \cdot f(\overline{C}) + C \cdot f'(\overline{C}) \cdot (\partial_u C - 1).$$

Now differentiating with respect to v, we get

$$\begin{aligned}\partial^2_{uv} H_f(C) &= \partial^2_{uv} C \cdot f(\overline{C}) + \partial_u C \cdot f'(\overline{C}) \cdot (\partial_v C - 1) \\ &\quad + \partial_v C \cdot f'(\overline{C}) \cdot (\partial_u C - 1) \\ &\quad + C \cdot \left[f''(\overline{C}) \cdot (\partial_u C - 1) \cdot (\partial_v C - 1) + f'(\overline{C}) \cdot \partial^2_{uv} C \right] \\ &= \partial_u C \cdot f'(\overline{C}) \cdot (\partial_v C - 1) + \partial_v C \cdot f'(\overline{C}) \cdot (\partial_u C - 1) \\ &\quad + C \cdot f''(\overline{C}) \cdot (\partial_u C - 1) \cdot (\partial_v C - 1) \\ &\quad + \partial^2_{uv} C \cdot \left[f(\overline{C}) + C \cdot f'(\overline{C}) \right],\end{aligned}$$

which after rearrangement yields the claim. □

3.4 Possible extensions to the multivariate case

While working with binary copulas, one is always inclined to extend his/her work to higher-dimensional copulas. In this case, however, this proves to be a difficult task. Following one possible n-dimensional extension of FGM copula (see [12, Section 6.3.], [24]), i.e.

$$C_\theta(u_1, \ldots, u_n) = \prod_{i=1}^n u_i \left(1 + \theta \prod_{i=1}^n (1 - u_i) \right),$$

let us define

$$H^n_f(C)(u_1, \ldots, u_n) := C(u_1, ..., u_n) f(\overline{C}(u_1, ..., u_n)). \tag{3.13}$$

Here $\overline{C}$ is n-dimensional survival function, i.e. suppose $(X_1, \dots, X_n)$ is random vector with copula C and marginal distribution functions $\{F_i : i = 1, \dots, n\}$, then

$$\overline{C}(u_1, \dots, u_n) = \mathbb{P}(X_1 > F_1^{(-1)}(x_1), \dots, X_n > F_n^{(-1)}(x_n)).$$

One could hope that similar conditions on f exist for it to be eligible. However, it is not the case. Let us take

$$\begin{aligned} C(u_1, \dots, u_n) = M(u_1, \dots, u_n) &= \min\{u_1, \dots, u_n\}, \\ \overline{M}(u_1, \dots, u_n) &= 1 - \max\{u_1, \dots, u_n\} \end{aligned}$$

and hypercubes

$$\begin{aligned} B_1 &:= [a, 1] \times [a, 1] \times \underbrace{[0, a] \times \cdots \times [0, a]}_{n-2}, \\ B_2 &:= [a, 1] \times \underbrace{[0, a] \times \cdots \times [0, a]}_{n-1}, \end{aligned}$$

where $a \in [0, 1]$. Then, after simple calculations, for $n \geq 3$, we have that

$$V_{H_f^n(M)}(B_1) = a(f(1-a) - 1), \qquad V_{H_f^n(M)}(B_2) = a(1 - f(1-a)).$$

Since, for $H_f^n(M)$ to be a copula, $V_{H_f^n(M)}(B_1)$ and $V_{H_f^n(M)}(B_2)$ must be greater than or equal to zero, we get that function f must be equal to 1 on the interval $[0, 1]$. Of course, this case is not interesting since $H_f^n(C) = C$, when $f(u) = 1$.

This shows that for the mapping $H_f^n(C)$ defined by (3.13) the set of eligible functions is trivial. However, if one considers conditional eligibilty, it may well be possible to find some useful results. In particular, the case of independence copula, i.e. $C = \Pi$ is of special interest since it would generalize multivariate AMH and FGM families of copulas. Altough it looks promising, no results in this regard, have been proven as of yet.

Another aproach would be to define $H_f^n(C)$ differently. One example would be

$$H_f^n(C)(u_1, \dots, u_n) = C(u_1, \dots, u_n) f\left(n - 1 - \sum_{i=1}^{n} u_i + C(u_1, \dots, u_n)\right),$$

which for $n = 2$, also recovers mapping H_f and, given $f(0) = 1$, also satisfies boundary conditions.

3.5 Some generalizations and related work

In [38] an interesting extension to the results of this chapter is suggested. Consider a function $g : [0,1] \to [0,1]$, some binary copula $D \in \mathcal{C}$ and mapping $D(C, g(C^*)) : [0,1]^2 \to [0,1]$ given by

$$D(C, g(C^*))(u,v) = D(C(u,v), g(C^*(u,v))).$$

This clearly extends our construction, since if we take $D = \Pi$ and $g(t) = f(1-t)$, we recover mapping $H_f(C)$. In [38], the sufficient conditions on "outer" copula D (ultramodularity and Schur concavity) and f such that, for each copula $C \in \mathcal{C}$, $D(C, f(C^*))$ is also a copula, are provided. However these conditions are not necessary (contrary to our results in Theorem 3). More information on this topic can be found in [38].

4 CONDITIONAL ELIGIBILITY

In this chapter we focus our attention to conditional eligibility for the case of independence copula $\Pi(u, v)$, i.e. to finding functions f for which

$$C_f(u, v) = uvf((1-u)(1-v)), \quad u, v \in [0, 1], \tag{4.1}$$

is a bivariate copula. As mentioned in introduction, this construction allows us to recover the well-known families of FGM, AMH and Çelebioğlu–Cuadras copulas.

The rest of the chapter is organized as follows. In Sect. 4.1, we give several necessary conditions on f for C_f in Eq. (4.1) to be a copula. Then, in Sect. 4.2, we state and prove our main results of this chapter giving sufficient conditions on the function f. Section 4.3 contains applications and further examples of copulas which show how our results can be applied in copula constructions.

4.1 Necessary conditions

In this section we present necessary conditions for a function f so that C_f in (4.1) is a bivariate copula. In what follows, $\partial_u, \partial_v, \ldots$ will denote the partial derivative with respect to $u, v, \ldots$, respectively.

Proposition 3. *Suppose C_f given in (4.1) is a bivariate copula on $[0, 1]^2$. Then*

(i) f is nonnegative, continuous on $[0, 1)$ and $f(0) = 1$;

(ii) For Lebesgue-almost all $t \in (0, 1)$, $f'(t)$ and $f''(t)$ exist;

(iii) The function $t \mapsto (1-t)f(t)$ is nonincreasing on $[0, 1)$. Moreover, f must satisfy the following inequalities:

$$\frac{\max\{1 - 2\sqrt{t}, 0\}}{(1-\sqrt{t})^2} \leq f(t) \leq \frac{1}{1-t}, \quad t \in [0, 1).$$

The upper bound is achievable.

(iv) The following limit is true:

$$\lim_{t\downarrow 0} \frac{1-(1+t)f(t)}{\sqrt{t}} = 0; \tag{4.2}$$

(v) For any $t \in [0,1)$, denoting the diagonal of C_f by δ, i.e. $\delta(t) = C_f(t,t)$, we have

$$f(t^2) = \frac{\delta(1-t)}{(1-t)^2} = \frac{\partial_v C_f(1-t^2,v)\big|_{v=0}}{1-t^2}; \tag{4.3}$$

(vi) For any $t \in [0,1/4]$,

$$tf(t) = C_f\Big(\frac{1}{2}+\sqrt{\frac{1}{4}-t}, \frac{1}{2}-\sqrt{\frac{1}{4}-t}\Big).$$

Proof. (i) Since $C_f(1,v) = v$ for any $v \in [0,1]$, we must have $f(0) = 1$. Also for any $u, v \in [0,1)$, we can write

$$f(uv) = \frac{C_f(1-u,1-v)}{(1-u)(1-v)}, \tag{4.4}$$

which is a ratio of two nonnegative continuous functions. In particular, setting $u = v = \sqrt{z}$, we get

$$f(z) = \frac{C_f(1-\sqrt{z},1-\sqrt{z})}{(1-\sqrt{z})^2} = \frac{\delta(1-\sqrt{z})}{(1-\sqrt{z})^2}, \quad \forall z \in [0,1).$$

Hence, f is continuous on $[0,1)$.

(ii) Since C_f is assumed to be a copula, partial derivatives $\partial_u C_f(u,v)$ exist (see, e.g. [36, Theorem 2.2.7]) for almost all $u \in [0,1]$ and any $v \in [0,1]$ (similarly for $\partial_v C_f(u,v)$), and we get that $f(uv)$ in (4.4) is differentiable for almost all $u \in (0,1)$ and all $v \in (0,1)$ so that

$$f(t) = \frac{C_f(1-t/v,1-v)}{(1-t/v)(1-v)}$$

is differentiable for almost all $t \in (0,v)$ for any fixed $v \in (0,1)$.

Now given any $v \in (0,1)$, let

$$E_v := \{t \in (0,v) : f'(t) \text{ does not exist}\}$$

and consider

$$E := \bigcup_{i=2}^{\infty} E_{1-1/i} = \lim_{i\to\infty} E_{1-1/i}.$$

Since $E_{1-1/i} \subset E_{1-1/(i+1)}$ for all $i \geq 2$, and each E_v has zero Lebesgue measure, so does set E by the continuity of Lebesgue measure. Therefore, on the set $E^c = (0,1) \setminus E$, which has Lebesgue measure of 1, f' exists, proving the first part of this claim.

Then for any $v \in (0,1)$ such that $uv \in E^c$, differentiating both sides of (4.4) with respect to u, using the chain rule, and then rearranging gives

$$f'(t)\big|_{t=uv} = \frac{C_f(1-u, 1-v) - (1-u)\partial_z C_f(z, 1-v)\big|_{z=1-u}}{v(1-v)(1-u)^2}.$$

The right-hand side of the latter equality is differentiable with respect to v for Lebesgue almost all $v \in (0,1)$ (let this set be D^c) since C_f is assumed to be a copula. And so $f'(uv)$ is differentiable with respect to v for almost all $v \in (0,1)$ such that $uv \in E^c$. Mapping each $t = uv \in E^c$ onto a point on the main diagonal of the unit square, namely $(\sqrt{t}, \sqrt{t})$ and considering a part of hyperbola $uv = t$ for $(u,v) \in [0,1]^2$, we can discard those points (u,v) on it where $v \in D$, still leaving a set of positive one-dimensional Lebesgue measure of points (u,v) such that $uv = t$. Hence

$$\{t : f''(t) \text{ exists}\} \supset E^c,$$

so that f'' exists Lebesgue almost everywhere on $(0,1)$.

(iii) Since a copula must be nondecreasing with respect to each argument, we get

$$0 \leq C_f(u_2, v) - C_f(u_1, v) = v(u_2 f((1-u_2)(1-v)) - u_1 f((1-u_1)(1-v))),$$

for all $0 \leq u_1 \leq u_2 \leq 1$ and all $v \in [0,1]$. So if $v \neq 0$, the function

$$t \mapsto tf((1-t)(1-u))$$

must be nondecreasing for $t \in [0,1]$ and any $v \in (0,1]$. The limit of such functions as $v \downarrow 0$ must also be nondecreasing, i.e. $t \mapsto tf(1-t)$ is nondecreasing. The first claim of part (iii) follows by a simple change of variable.

Furthermore, since the aforementioned function is nondecreasing, we get that the upper bound is $tf(1-t) \le f(0) = 1$, so $(1-s)f(s) \le 1$ where $s = 1-t$, and the stated upper bound follows for all $t \in [0,1)$. Also note that for $f(t) = 1/(1-t)$,

$$C_f(u,v) = \frac{uv}{1-(1-u)(1-v)} = \mathrm{AMH}(1)(u,v), \quad u,v \in [0,1], \quad (4.5)$$

is the Ali–Mikhail–Haq copula with $\lambda = 1$ (see Example 11) and therefore the stated upper bound on f is achievable.

For the lower bound, apply the lower Fréchet–Hoeffding bound to the expression in (4.4) and substitute $t = uv$ and $s = u+v$. Then

$$f(t) \ge \frac{\max\{1-s,0\}}{1-s+t}, \quad \forall t \in [0,1], \forall s \in [2\sqrt{t}, 1+t).$$

Observe that if $t \ge 1/4$, then $s \ge 1$, so the right-hand side of the above inequality is trivially zero, hence we need only to consider $0 \le t < 1/4$. In this case, taking the maximum with respect to $s \in [2\sqrt{t}, 1]$ of the right-hand side of this inequality, we get

$$f(t) \ge \frac{\max\{1-2\sqrt{t},0\}}{1-2\sqrt{t}+t} = \frac{\max\{1-2\sqrt{t},0\}}{(1-\sqrt{t})^2} =: \widehat{f}(t).$$

Unfortunately, the lower bound $\hat{f}$ does not yield a copula. Indeed, by considering $S_1 = [0.5, 0.85] \times [0.5, 0.85]$, one gets

$$V_{C_{\widehat{f}}}(S_1) = C_{\widehat{f}}(0.85, 0.85) - 2C_{\widehat{f}}(0.85, 0.5) + C_{\widehat{f}}(0.5, 0.5) \approx -0.0291 < 0.$$

Even more can be said: $C_{\widehat{f}}$ does satisfy the boundary conditions, but it is not even a proper quasi-copula. Indeed, for $S_2 = [0.5, 1] \times [0.5, 0.85]$,

$$V_{C_{\widehat{f}}}(S_2) = 0.35 - C_{\widehat{f}}(0.5, 0.85) \approx -0.0145 < 0,$$

so that the condition of [19, Proposition 3] fails.

(iv) Suppose on the contrary that (4.2) is not true. Then there is an $\varepsilon \in (0,1)$ such that

$$\varlimsup_{t \downarrow 0} \frac{1-(1+t)f(t)}{\sqrt{t}} \ge \varepsilon.$$

The latter implies the existence of a sequence $\{t_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$t_n \to 0$ as $n \to \infty$ and

$$\frac{1-(1+t_n^2)f(t_n^2)}{t_n} \geq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Now consider a square $\square := [u_1, u_2] \times [v_1, v_2] = [1-t_n, 1] \times [1-t_n, 1]$ and estimate the C_f-volume of it.

$$\begin{aligned}
V_{C_f}(\square) &:= C_f(u_2, v_2) - C_f(u_1, v_2) - C_f(u_2, v_1) + C_f(u_1, v_1) \\
&= 1 - 2(1-t_n) + (1-t_n)^2 f(t_n^2) \\
&\leq 1 - 2(1-t_n) + (1-t_n)^2 \frac{1-\varepsilon t_n}{1+t_n^2} \\
&\leq 1 - 2(1-t_n) + (1-t_n)^2 \frac{1-\varepsilon t_n}{1-t_n^2} \\
&= \frac{t_n}{1+t_n}\big(t_n(2+\varepsilon) - \varepsilon\big) < 0,
\end{aligned}$$

as soon as $t_n < \varepsilon/(2+\varepsilon)$ which happens eventually since $t_n \to 0$, contradicting the fact that C_f is a copula.

(v) The first equality follows easily from (4.4) by taking $u = v = t$. To get the second, observe that for any $u \in [0, 1)$,

$$f(u) = \lim_{v \uparrow 1} f(uv) = \lim_{v \uparrow 1} \frac{C_f(1-u, 1-v)}{(1-u)(1-v)} = \frac{\partial_v C_f(1-u, v)\big|_{v=0}}{1-u}.$$

Hence, taking $u = t^2$ we obtain the desired expression. Note that the limit exists as f is continuous on $[0, 1)$ by part (i).

(vi) Taking $v = 1-u$ in the expression for C_f, we get

$$C_f(u, 1-u) = u(1-u)f(u(1-u)), \quad \forall x \in [0, 1].$$

Thus after substituting $t = u(1-u) \in [0, 1/4]$, we obtain $u = \frac{1}{2} \pm \sqrt{\frac{1}{4} - t}$ which, due to the symmetry of C_f, gives

$$tf(t) = C_f\Big(\frac{1}{2} + \sqrt{\frac{1}{4} - t}, \frac{1}{2} - \sqrt{\frac{1}{4} - t}\Big),$$

completing the proof. $\square$

Remark 5. From the second equality in (4.3) with $1 - t^2 = s$, one has

$$sf(1-s) = \partial_v C_f(s, v)\big|_{v=0},$$

that is, the requirement in (ii) restricts only the behaviour of partial derivatives of C_f on a part of the boundary of the unit square. Since partial derivatives (or Dini derivatives, in general; see [11, Theorem 2.3]) of a copula must be nondecreasing also for other points v, not just $v = 0$, this implies that (ii) of Proposition 3 cannot be sufficient to always make C_f a copula.

Even though (4.3) seems to be a rather restrictive condition on f, it is still not a sufficient condition for C_f to be a copula. To illustrate we provide two examples:

Example 5. Suppose that $f \not\equiv 1$ and $\delta(t) = t^\beta$ for $\beta \in [1, 2)$. Then there is an Archimedean copula with such a diagonal, namely a member of the Gumbel–Hougaard family (see [36, family (4.2.4)]) with generator $\phi(s) = (-\ln s)^\theta$ where $s \in [0, 1]$ and $\theta = \ln 2/\ln \beta$:

$$C(u, v) = \exp\Big\{ -((-\ln u)^\theta + (-\ln v)^\theta)^{1/\theta}\Big\}.$$

Yet, for such a diagonal, the function f as given by (4.3) does not yield a copula, that is,

$$C_f(u, v) = xy(1 - \sqrt{(1-u)(1-v)})^{\beta-2}$$

fails to be a copula as $\partial^2_{uv} C_f(u, v) < 0$ for (u, v) near $(1, 0)$ or $(0, 1)$. Indeed, by taking any $0 < \varepsilon < 1$ and $u = \varepsilon$, $v = 1 - \varepsilon$, we can show that

$$\partial^2_{uv} C_f(\varepsilon, 1-\varepsilon) = \frac{(1-z)^{\beta-4}}{4z}\Big[\beta^2 z^3 + (2-5\beta)z^2 + (8-2\beta)z + (2\beta-4)\Big],$$

where $z = \sqrt{\varepsilon(1-\varepsilon)}$, which, clearly, becomes negative as $z \downarrow 0$ as $\beta < 2$.

Including the partial derivative on the boundary into consideration allows for the following example:

Example 6. Consider $f(t) = 1 + \mu t^2$. It is known that (see [2]; or Example 15 below)

$$C_f(u, v) = uv(1 + \mu(1-u)^2(1-v)^2), \quad u, v \in [0, 1],$$

is a bivariate copula for any $\mu \in [-1, 3]$. Here we consider only $\mu \in (0, 3]$. Then the diagonal $\delta_\mu(t) = C_f(t, t) = t^2(1 + \mu(1-t)^4)$ satisfies Condition

(3.4) of [14, Corollary 5], i.e.

$$\delta(t) \leq t\delta'(t) \leq 2\delta(t)$$

and so the function

$$S_{\delta_\mu}(u,v) = \min\{u,v\} \max\left\{\frac{\delta_\mu(u)}{u}, \frac{\delta_\mu(v)}{v}\right\} = uv\Big(1 + \mu(1 - \max(u,v))^4\Big)$$

is also a (semilinear) copula, different from C_f, which satisfies (4.3). This shows that even if (4.3) holds, a copula need not be of the form given by (4.1).

4.2 Sufficient conditions – main results

We begin with a known result showing the implications of geometric Jensen convexity. A similar fact was stated without proof in [33, Remark 3.1, p. 629], but with the domain of f being $(0,+\infty)$. Equivalence of parts (i) and (ii) was also mentioned in [37]. For part (iii) with slightly different notation, see [18, Theorem 1(iv)].

Lemma 2. *Let $0 \leq a < b < \infty$ and $J = (a,b)$. Consider a continuous function $f : J \to (0,+\infty)$. Then the following statements are equivalent:*

(i) The function f is geometrically Jensen convex on J, i.e.

$$f(\sqrt{xy}) \leq \sqrt{f(x)f(y)}, \quad x,y \in J. \tag{4.6}$$

(ii) The function $g := \ln \circ f \circ \exp : (\ln a, \ln b) \to [0,+\infty)$ is convex (assuming $\ln a = -\infty$ if $a = 0$).

(iii) For any $\lambda \in (1, b/a)$ (taking $\frac{b}{0} := +\infty$), the function $h(t) = f(\lambda t)/f(t)$ is nondecreasing for $t \in \left(a, \frac{b}{\lambda}\right)$.

Proof. (The proof is presented for the reader's convenience.) "(i) $\Longrightarrow$ (ii)" Consider any $x,y \in (\ln a, \ln b)$. Then, by (i),

$$g\left(\frac{x+y}{2}\right) = \ln\left(f(e^{x/2}e^{y/2})\right) \leq \ln\left(\sqrt{f(e^x)f(e^y)}\right) = \frac{g(x)+g(y)}{2},$$

hence, g is Jensen convex and, by construction, also continuous. Therefore, g is convex (see, e.g., [35, Section 1.4.2, p. 14]).

"(ii) $\Longrightarrow$ (i)" If g is convex, then, for any $x, y \in (\ln a, \ln b)$ and $t \in [0,1]$,

$$f\left(e^{xt} e^{(1-t)y}\right) \leq f(e^x)^t f(e^y)^{1-t},$$

i.e. $f \circ \exp$ is geometrically convex, and hence geometrically Jensen convex (by taking $t = 1/2$). After the transformation of the domain $(\ln a, \ln b)$ into (a, b), we get that f is geometrically Jensen convex.

"(ii) $\Longleftrightarrow$ (iii)" Consider any $\ln a \leq z_1 < z_2 < \ln b$ and any $\xi \in (0, \ln b - z_2)$. Let $t_1 = e^{z_1} < t_2 = e^{z_2}$ and $\lambda = e^{\xi} \in (1, b/t_2)$. Then

$$h(t_1) = \frac{f(\lambda t_1)}{f(t_1)} \leq \frac{f(\lambda t_2)}{f(t_2)} = h(t_2)$$

if and only if

$$g(\xi + z_1) - g(z_1) \leq g(\xi + z_2) - g(z_2).$$

The latter is equivalent to the convexity of g on $(\ln a, \ln b)$. □

Remark 6. As the pointwise supremum of any family of proper convex functions defined on the same domain is also convex and since $\ln(x)$ is continuous and increasing, from Lemma 2(ii) we easily get that the pointwise supremum of any family of geometrically Jensen convex functions defined on the same domain is also geometrically Jensen convex. This fact will be useful when considering maxima or suprema of certain copulas.

Example 7. Consider $f : [0,1] \to [1, 1+\ln 2]$ given by $f(x) = 1 + \ln(1 + x)$ and let $g := \ln \circ f \circ \exp : [-\infty, 0] \to [0, \ln(1 + \ln 2)]$. Then it is easy to check that for all $x \in [-\infty, 0]$

$$g'(x) = \frac{e^x}{(1+e^x)(1+\ln(1+e^x))},$$
$$g''(x) = \frac{e^x(1-e^x+\ln(1+e^x))}{(1+e^x)^2(1+\ln(1+e^x))^2} \geq 0.$$

Thus, by Lemma 2(ii), the function f is geometrically Jensen convex. Checking this property by definition would be rather cumbersome. Also note that f is concave!

Theorem 7. *Let* $f : [0,1] \to [1, +\infty]$ *be continuous on* $[0,1)$, *with*

$f(0) = 1$, *and let*

$$C_f(u, v) := uvf((1-u)(1-v)), \quad u, v \in [0, 1].$$

The following statements are equivalent:

(a) The function C_f is a bivariate copula, with the TP_2 property (see Eq. (2.7)),

(b) The function f has the following two properties:

(i) $t \mapsto tf(1-t)$ is nondecreasing on $(0, 1)$,

(ii) f is geometrically Jensen convex on $[0, 1)$ (see Eq. (4.6)).

Proof. "$(b) \implies (a)$" Since $f(0) = 1$, C_f clearly satisfies the boundary conditions of a copula:

$$\begin{aligned} &C_f(0, v) = C_f(u, 0) = 0, \\ &C_f(1, v) = v, \quad C_f(u, 1) = u, \quad u, v \in [0, 1]. \end{aligned} \tag{4.7}$$

First, we show that C_f is nondecreasing in each argument. Indeed, due to obvious symmetry, it is enough to check nondecreasingness in the first argument. So consider any $0 \le u_1 < u_2 \le 1$ and $v \in (0, 1)$. Then

$$\Delta_1 := C_f(u_2, v) - C_f(u_1, v) = y[u_2 f((1-v)(1-u_2)) - u_1 f((1-v)(1-u_1))], \tag{4.8}$$

where letting $u_i := 1 - (1 - z_i)/(1 - v)$, $i = 1, 2$, we get $0 \le 1 - z_2 < 1 - z_1 \le 1 - v$ and

$$\begin{aligned} \Delta_1 &= v\left[\left(1 - \frac{1-z_2}{1-v}\right) f(1-z_2) - \left(1 - \frac{1-z_1}{1-v}\right) f(1-z_1)\right] \\ &= \frac{v}{1-v}\left[\left(1 - \frac{v}{z_2}\right) z_2 f(1-z_2) - \left(1 - \frac{v}{z_1}\right) z_1 f(1-z_1)\right] \\ &> \frac{v(1-v/z_1)}{1-v}(z_2 f(1-z_2) - z_1 f(1-z_1)] \ge 0, \end{aligned} \tag{4.9}$$

due to condition (i). Furthermore, if $v = 0$ or 1, then $\Delta_1 = 0$ or $\Delta_1 = u_2 - u_1 \ge 0$, respectively.

Now we check that C_f satisfies the TP_2 property, which by [15, Lemma 3.1] will imply that C_f is 2-nondecreasing, and hence a copula.

First, consider any $0 < u_1 < u_2 < 1$ and $0 < v_1 < v_2 < 1$ and compute

$$\begin{aligned} D &:= C_f(u_2, v_2)C_f(u_1, v_1) - C_f(u_2, v_1)C_f(u_1, v_2) \\ &= u_1v_1u_2v_2[f(s)f(\lambda t) - f(s)f(\lambda t)] \\ &= u_1v_1u_2v_2\Delta_2, \end{aligned} \tag{4.10}$$

where

$$\lambda := (1 - v_1)/(1 - v_2) > 1,$$
$$s = (1 - u_2)(1 - v_2), \quad t = (1 - u_1)(1 - v_2).$$

Observe that $0 < s < t < 1$ and for fixed v_1 and v_2 (and hence fixed λ),

$$\Delta_2 = f(s)f(\lambda t) - f(t)f(\lambda s) \geq 0,$$

by condition (ii) and Lemma 2(iii).

If $u_1 = 0$ or $v_1 = 0$, both sides of Eq. (4.10) are zero. And if $u_2 = 1$ or $v_2 = 1$, then the expression in Eq. (4.10) is nonnegative since C_f is nondecreasing in each argument as already shown above. Thus in all cases $D \geq 0$ as required.

"$(a) \implies (b)$" If C_f is a TP_2 copula, then D in Eq. (4.10) is nonnegative for any $0 \leq u_1 < u_2 \leq 1$ and $0 \leq y_1 < y_2 \leq 1$, which implies that $\Delta_2 \geq 0$ for any $0 < u_1 < u_2 < 1$ and $0 < v_1 < v_2 < 1$. This means that $t \mapsto f(\lambda t)/f(t)$ is nondecreasing for any $\lambda \in (0, 1 - v_1)$. So by Lemma 2(iii), f is geometrically Jensen convex on $(0, 1 - v_1)$, for any $v_1 \in (0, 1)$, and hence on $(0, 1)$. So f satisfies property (ii) on $(0, 1)$ and trivially at 0.

To check that f also satisfies property (i), observe that Δ_1 in Eq. (4.8) is nonnegative for any $0 \leq u_1 < u_2 \leq 1$ and any $v \in [0, 1]$, since any copula is nondecreasing in each argument. Thus for any $0 < u_1 < u_2 \leq 1$ and $v \in (0, 1)$, we obtain

$$u_2 f((1 - v)(1 - u_2)) - u_1 f((1 - v)(1 - u_1)) \geq 0.$$

Rearranging gives

$$\frac{u_1}{u_2} \leq \frac{f((1 - v)(1 - u_2))}{f((1 - v)(1 - u_1))}.$$

Treating u_1 and u_2 as fixed, letting $v \downarrow 0$, and using continuity of f, we have

$$\frac{u_1}{u_2} \le \lim_{v \downarrow 0} \frac{f((1-v)(1-u_2))}{f((1-v)(1-u_1))} = \frac{f(1-u_2)}{f(1-u_1)}, \quad 0 < u_1 < u_2 < 1.$$

The latter shows property (i) for function f, completing the proof. □

Combining with [15, Corollary 3.5], we get the following corollary:

Corollary 8. *Let $f : [0,1] \to \mathbb{R}_+$ be a continuous, geometrically Jensen convex function such that $f(0) = 1$. Let $\gamma > 0$ be such that $t \mapsto t^{1/\gamma} f(1-t)$ is nondecreasing on $(0,1]$. Then for any $\alpha > 0$ the function*

$$C_{\alpha,\gamma}(u,v) = uv\Big(f\big((1-u^{1/\alpha})(1-v^{1/\alpha})\big)\Big)^{\gamma\alpha}, \quad u,v \in [0,1], \tag{4.11}$$

is a bivariate copula with the TP_2 property.

Proof. As geometric (Jensen) convexity is preserved when taking positive powers of functions, under the stated assumptions, the function $f^{\gamma}(t)$ satisfies conditions of Theorem 7(b), and so the function $C_{1,\gamma}$ (see Eq. (4.11)) is a bivariate copula with the TP_2 property. Now an application of [15, Corollary 3.5] gives, for $\phi(t) = t^{\alpha}$, $\alpha > 0$, that

$$C_{\alpha,\gamma}(u,v) = C_{1,\gamma}^{\alpha}(u^{1/\alpha}, v^{1/\alpha})$$

is again a TP_2 copula, finishing the proof. □

It is known that the maximum of two bivariate copulas C_1 and C_2, i.e. $C_3(u,v) := \max\{C_1(u,v), C_2(u,v)\}$, need not be a copula (it must only be a quasi-copula; see [36, Example 6.3 and Theorem 6.2.5]), but if C_1 and C_2 are of the form (4.1) and have the TP_2 property, then C_3 will always be a TP_2 copula. Indeed, we have

Corollary 9. *Let $\mathcal{C} = \{C_f, f \in \mathcal{F}\}$ be a family of bivariate copulas as in Theorem 7. Then*

$$C_{\sup}(u,v) = \sup_{C \in \mathcal{C}} C(u,v) = uv \sup_{f \in \mathcal{F}} f((1-u)(1-v))$$

is also a TP_2 copula.

Proof. Let $f_{\sup}(t) := \sup_{f\in\mathcal{F}} f(t),\ t \in [0,1)$. Since each $f \in \mathcal{F}$ is continuous on $[0,1)$, $f(0) = 1$, and conditions (i) and (ii) of Theorem 7(b) hold, the same conditions hold also for $f_{\sup}$ (recall Remark 6) so then $C_{\sup}$ is a copula by Theorem 7(a). □

Example 8. To illustrate Corollary 9, consider

$$\begin{aligned} C_{\alpha,\beta}(u,v) &:= \max\{\mathrm{AMH}(\beta)(u,v), \mathrm{FGM}(\alpha)(u,v)\} \\ &= uvf_{\alpha,\beta}((1-u)(1-v)), \end{aligned}$$

where

$$f_{\alpha,\beta}(t) := \max\{(1-\beta t)^{-1}, 1+\alpha t\}, \quad \beta \in (0,1),\ \alpha \in (\beta, \beta/(1-\beta)).$$

Such α is taken to make the graphs of the two functions, whose maximum is considered, cross inside the unit interval (another crossing point $t = 0$ is not interesting). Then such $C_{\alpha,\beta}$ form a family of TP_2 copulas.

To present the second main result of this chapter, for any twice differentiable function $g : [0,1) \to \mathbb{R}_+$ let

$$\begin{aligned} A_1(g) &:= \{t \in (0,1) : (tg(t))'' \geq 0,\ g(t) + (t-1)g'(t) \geq 0\}, \\ A_2(g) &:= \{t \in (0,1) : (tg(t))'' < 0, \\ &\qquad g(t) + (1-\sqrt{t})((1-3\sqrt{t})g'(t) + t(1-\sqrt{t})g''(t)) \geq 0\}. \end{aligned}$$

Observe that $A_1(g) \cap A_2(g) = \emptyset$.

The following theorem provides a different set of conditions characterizing suitable twice continuously differentiable functions f for C_f to be a copula.

Theorem 10. *Let $f : [0,1] \to [0,+\infty]$ be twice differentiable, with f' absolutely continuous, on $(0,1)$ such that*

$$f(0) = 1, \quad \lim_{t\downarrow 0} tf'(t) = 0, \quad \textit{and} \quad (1-t)f(t) \leq 1 \textit{ for all } t \in [0,1). \tag{4.12}$$

Then $C_f(u,v) = uvf((1-u)(1-v))$ defined for $u, v \in [0,1]$ is a bivariate absolutely continuous copula if $A_1(f) \cup A_2(f) = (0,1)$. If, in addition, f'' is assumed continuous, then the above conditions are also necessary.

Proof. For convenience, we make a change of variables. Recall that a function $C(u,v)$ on $[0,1]^2$ is a bivariate copula iff such is

$$\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

Hence, it suffices to check if

$$\widehat{C}_f(u,v) = u + v - 1 + (1-u)(1-v)f(uv)$$

is a copula under specified conditions. Since $f(0) = 1$, both C_f and $\widehat{C}_f$ satisfy boundary conditions (see Eq. (4.7)). So we only need to check the 2-nondecreasingness of $\widehat{C}_f$. Since f is twice differentiable, it suffices to show that $\partial^2_{uv}\widehat{C}_f \geq 0$ on $(0,1)^2$. Then a straightforward application of [8, Theorem 3.2] (for its formulation and application details, see the Appendix) will yield that $\widehat{C}_f$ is an absolutely continuous bivariate copula. See also Remark 9 for additional comments about the necessity of the restrictions on f.

A simple computation gives

$$\begin{aligned}\partial^2_{uv}\widehat{C}_f(u,v) &= f(uv) + [1 - 2u - 2v + 3uv]f'(uv) \\ &\quad + uv(1-u)(1-v)f''(uv).\end{aligned} \tag{4.13}$$

Now consider another change of coordinates:

$$t = uv, \quad s = u + v,$$

so that the right-hand side of (4.13) becomes

$$\begin{aligned}K(s,t) &:= f(t) + [1 - 2s + 3t]f'(t) + t(1 - s + t)f''(t) \\ &= \{f(t) + (1+3t)f'(t) + t(1+t)f''(t)\} - s\{2f'(t) + tf''(t)\},\end{aligned}$$

where $t \in [0,1]$, $2\sqrt{t} \leq s \leq t+1$. Here we consider 2 cases:

(i) If $t \in A_1(f)$, then

$$\begin{aligned}K(s,t) &\geq \big[f(t) + (1+3t)f'(t) + t(1+t)f''(t)\big] \\ &\quad - (t+1)\big[2f'(t) + tf''(t)\big] \\ &= f(t) + (t-1)f'(t) \geq 0,\end{aligned}$$

that is, the mixed derivative in (4.13) is nonnegative as needed.

(ii) If, on the other hand, $t \in A_2(f)$, then

$$\begin{aligned}K(s,t) &\geq \{f(t) + (1+3t)f'(t) + t(1+t)f''(t)\} \\ &\quad - 2\sqrt{t}\{2f'(t) + tf''(t)\} \\ &= f(t) + (1-\sqrt{t})((1-3\sqrt{t})f'(t) + t(1-\sqrt{t})f''(t)) \geq 0,\end{aligned}$$

i.e. we again have a nonnegative mixed-derivative in (4.13).

Since $A_1(f) \cup A_2(f) = (0,1)$, we have shown that C_f is a bivariate absolutely continuous copula.

On the other hand, if $t \in (0,1)$ is such that

$$(tf(t))'' = 2f'(t) + tf''(t) \geq 0, \quad f(t) + (t-1)f'(t) < 0,$$

then $K(t+1,t) < 0$, which, due to continuity of K, implies that in some neighbourhood of points $(u,v) = (t,1)$ and $(u,v) = (1,t)$, $\partial^2_{uv}\widehat{C}_f(u,v) < 0$, which cannot happen if C_f is an absolutely continuous bivariate copula, since by picking a rectangle inside such a neighbourhood, we would get a negative $\widehat{C}_f$-volume of it.

Similarly, if $t \in (0,1)$ is such that

$$(tf(t))'' < 0, \quad f(t) + (1-\sqrt{t})((1-3\sqrt{t})f'(t) + t(1-\sqrt{t})f''(t)) < 0,$$

then $K(2\sqrt{t},t) < 0$, which implies that in a neighbourhood of the point $(u,v) = (\sqrt{t},\sqrt{t})$, $\partial^2_{uv}\widehat{C}_f(u,v) < 0$, which is again a contradiction if C_f is an absolutely continuous bivariate copula. □

Proposition 4. *Let $f : [0,1] \to [0,+\infty]$ be twice differentiable on $(0,1)$ such that*

$$f(0) = 1 \quad \textit{and} \quad \lim_{t\downarrow 0} tf'(t) = 0. \tag{4.14}$$

Then

(i) if $A_1(f) = (0,1)$ then the function f is nondecreasing;

(ii) if $A_2(f) = (0,1)$ then the function f is decreasing.

Proof. (i) Since $(tf(t))'' \geq 0$ on $(0,1)$, the function $g(t) := tf(t)$ is convex, hence the slopes of secant lines are nondecreasing and so we get

$$f(t_1) = \frac{t_1 f(t_1) - 0f(0)}{t_1 - 0} \leq \frac{t_2 f(t_2) - t_1 f(t_1)}{t_2 - t_1}, \quad \forall 0 < t_1 < t_2 < 1, \tag{4.15}$$

which after rearrangement gives $f(t_2) - f(t_1) \geq 0$ for all $0 < t_1 < t_2 < 1$, i.e. $f(t)$ is nondecreasing.

Part (ii) follows in the same way by a change of appropriate inequality signs. □

Note that the function f needs not be convex or satisfy the lower bound $f(t) \geq 1 - t$ as is required in Theorem 3 (see Examples 15 and 16, respectively).

4.3 Applications and more examples

Example 9. Let $f_\delta(t) = e^{\delta t}$ where $\delta > 0$. Then, due to the arithmetic–geometric mean inequality, f_δ is geometrically Jensen convex, and being continuous also geometrically convex. Condition (i) of Theorem 7(b) is satisfied for $0 < \delta \leq 1$. Hence we recover the known fact that

$$C_{f_\delta}(u, v) = uve^{\delta(1-u)(1-v)}, \quad u, v \in [0, 1]$$

is a bivariate copula, which is also known as Çelebioğlu–Cuadras copula [4, 7]. Moreover, we have proved that this copula, in particular, has the TP_2 property.

Example 10. Let $f_\alpha(t) = 1 + \alpha t$, where $\alpha \in [0, 1]$. Again, by the arithmetic–geometric mean inequality, f_α can be easily shown to be geometrically Jensen convex, and so it is geometrically convex, too. Condition (i) of Theorem 7(b) holds also for any $\alpha \in (0, 1]$. So by Theorem 7,

$$C_{f_\alpha}(u, v) = xy(1 + \alpha(1 - u)(1 - v)),$$

is a bivariate copula, also known as Farlie–Gumbel–Morgenstern FGM(α) copula, with the TP_2 property.

If $\gamma \in (0, 1/\alpha]$, then $t^{1/\gamma}(1 + \alpha(1 - t))$ is nondecreasing on $(0, 1)$, so by Corollary 8,

$$C_{\alpha,\beta,\gamma}(u, v) = uv\Big(1 + \alpha(1 - u^{1/\beta})(1 - v^{1/\beta})\Big)^{\beta\gamma}, \quad u, v \in [0, 1]; \ \beta > 0$$

is a TP_2 copula. In particular, taking $\gamma = 1$, we recover a result of Durante et al. [15, Example 3.6], while if $\gamma = 1/\alpha \geq 1/\beta$, we have a copula family considered by Huang and Kotz [22]. When $\beta\gamma \in \mathbb{N}$, we get a family of copulas considered by Bekrizadeh et al. [3].

Example 11. Let $f_\lambda(t) = (1 - \lambda t)^{-1}$, where $\lambda \in [0, 1]$. Once more, by the arithmetic–geometric mean inequality, f_λ can be easily shown to be geometrically Jensen convex, and so it is geometrically convex, too. Condition (i) of Theorem 7(b) is easily checked to hold for any $\lambda \in (0, 1]$. So once more by Theorem 7,

$$C_{f_\lambda}(u, v) = \frac{uv}{1 - \lambda(1 - u)(1 - v)},$$

is a bivariate copula, also known as Ali–Mikhail–Haq AMH(λ) copula, with the TP_2 property.

If $\gamma \in (0, 1/\lambda]$, then $t^{1/\gamma}/(1 - \lambda(1 - t))$ is nondecreasing on $(0, 1)$, so by Corollary 8,

$$C_{\lambda,\beta,\gamma}(u, v) = \frac{uv}{\left(1 - \lambda\left(1 - u^{1/\beta}\right)\left(1 - v^{1/\beta}\right)\right)^{\beta\gamma}}, \quad u, v \in [0, 1]; \ \beta > 0$$

is a TP_2 copula. So we have a generalized AMH copula family.

Example 12. Consider the function f from Example 7, namely

$$f(x) = 1 + \ln(1 + x).$$

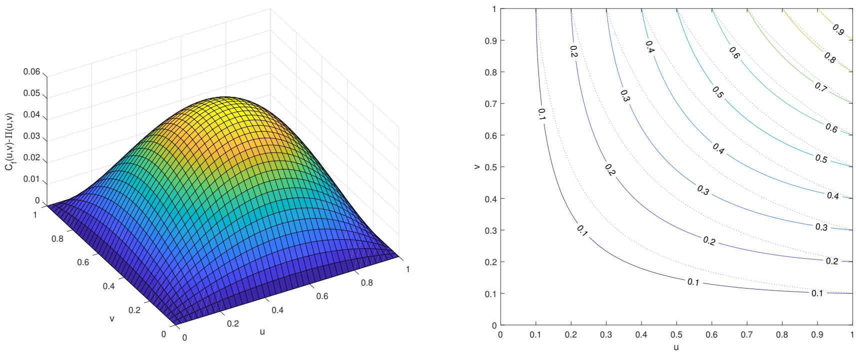
We already know that it is geometrically Jensen convex. As for Condition (i) of Theorem 7(b), we have

$$tf(1-t) = t(1+\ln(2-t)) \quad \text{and} \quad (tf(1-t))' = \frac{2(1-t)}{2-t} + \ln(2-t) \geq 0,$$

for all $t \in [0, 1]$, whence this condition is satisfied as well. Therefore, applying Theorem 7, we obtain that

$$C_f(u, v) = uv\big(1 + \ln(1 + (1 - u)(1 - v))\big) \tag{4.16}$$

is a bivariate copula with the TP_2 property (see Figure 4.1).



(a) A difference $C_f - \Pi$ (b) contour map of copula C_f

Figure 4.1: Copula C_f whith $f(x) = 1 + \ln(1 + x)$

To illustrate Theorem 10, we present the following example:

Example 13. Recall the function f from Example 7, namely

$$f(t) = 1 + \ln(1 + t).$$

It is straightforward to check that for all $t \in [0, 1]$,

$$\begin{aligned} 2f'(t) + tf''(t) &= \frac{2+t}{(1+t)^2} > 0, \\ f(t) + (t-1)f'(t) &= \frac{2t}{1+t} + \ln(1+t) \geq 0, \end{aligned}$$

that is, $A_1(f) = (0, 1)$, so by Theorem 10, $C_f(u, v)$ given in (4.16) is a bivariate copula, in agreement with the result of Example 12.

On the other hand, if we take $f(t) = 1 + \sin(\pi t/4)$, then for all $t \in [0, 1]$,

$$2f'(t) + tf''(t) = \frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right)\left(2 - \frac{\pi t}{4}\tan\left(\frac{\pi t}{4}\right)\right) \geq \frac{\pi\sqrt{2}}{8}\left(2 - \frac{\pi}{4}\right) > 0$$

and

$$\begin{aligned} f(t) + (t-1)f'(t) &= 1 + \sin\left(\frac{\pi t}{4}\right) + (t-1)\frac{\pi}{4}\cos\left(\frac{\pi t}{4}\right) \\ &> \sin\left(\frac{\pi t}{4}\right) + \frac{\pi t}{4}\cos\left(\frac{\pi t}{4}\right) \geq 0. \end{aligned}$$

So again $A_1(f) = (0,1)$ and, by Theorem 10,

$$C_f(u,v) = uv\Big(1 + \sin\Big(\frac{\pi}{4}(1-u)(1-v)\Big)\Big)$$

is a bivariate copula. Note also that in this case Theorem 7 is not applicable as

$$g(t) = (\ln \circ f \circ \exp)(t) = \ln\left(1 + \sin\left(\pi e^t/4\right)\right)$$

is concave at least for $t \in [-0.05, 0]$, i.e. f is not geometrically Jensen convex by Lemma 2.

Example 14. Recall the function from Example 10, i.e. $f_\alpha(t) = 1 + \alpha t$, but now take $\alpha \in [-1, 0)$. Then

$$2f'(t) + tf''(t) = 2\alpha < 0$$

and

$$\begin{aligned} &f(t) + (1-\sqrt{t})((1-3\sqrt{t})f'(t) + t(1-\sqrt{t})f''(t)) \\ &\quad = 1 + \alpha t + (1-\sqrt{t})(1-3\sqrt{t})\alpha \\ &\quad = 1 + \alpha - 4\alpha(\sqrt{t} - t) \geq 0. \end{aligned}$$

Therefore, $A_2(f) = (0,1)$ and, by Theorem 10,

$$C_f(u,v) = uv\Big(1 + \alpha(1-u)(1-v)\Big)$$

is a bivariate copula. Together with Example 10, this fully covers the case of the Farlie–Gumbel–Morgenstern copula family.

Similarly, it can be shown that $f_\delta(t) = e^{\delta t}$ from Example 9, where $\delta \in [-1, 0)$, and $f_\lambda(t) = (1-\lambda t)^{-1}$ from Example 11, where $\lambda \in [-1, 0)$, are such that $A_2(f) = (0,1)$.

Example 15. Consider $f(t) := f_\mu(t) = 1 + \mu t^2$, $t \in [0,1]$, $\mu \in \mathbb{R}$. We will apply Theorem 10 to find μ, such that $\widehat{C}_{f_\mu}$ (and hence also C_{f_μ}) is a bivariate absolutely continuous copula. It is clear that

$$2f'(t) + tf''(t) = 6\mu t \begin{cases} \geq 0, & \text{if } \mu \geq 0; \\ < 0, & \text{if } \mu < 0. \end{cases}$$

In the case $\mu \geq 0$, to have $t \in A_1(f)$, we also need to check

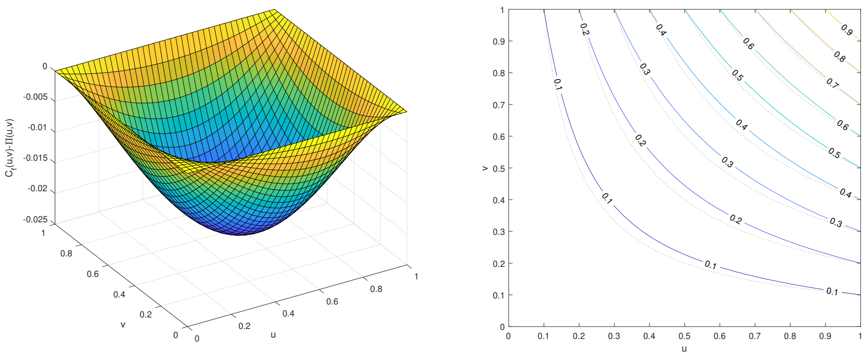
$$f(t) + (t-1)f'(t) = 1 - 2\mu t + 3\mu t^2 \geq 0,$$

which is true for all $t \in [0,1]$ iff $\mu \leq 3$.

On the other hand, if $\mu < 0$, then to have $t \in A_2(f)$, we check that

$$\begin{aligned} f(t) - (1-\sqrt{t})((1-3\sqrt{t})f'(t) + t(1-\sqrt{t})f''(t)) \\ = 1 + \mu t^2 + 4\mu t(1 - 3\sqrt{t} + 2t) \geq 0, \end{aligned}$$

which is true for all $t \in [0,1]$ iff $\mu \geq -1$.



(a) A difference $C_f - \Pi$ (b) contour map of copula C_f

Figure 4.2: Copula C_f whith $f(t) = 1 - t/2 + t^2/2$

Combining both cases, we get that if $\mu \in [-1,3]$, then C_{f_μ} is a bivariate absolutely continuous copula. On the other hand, if we take $\mu > 3$ or $\mu < -1$, then there exist at least one point $t \in (0,1) \setminus (A_1(f) \cup A_2(f))$ (and hence even a small interval of such points). Since $f''(t) = 2\mu$ is continuous, by Theorem 10, we get that C_f is not a copula. Therefore C_{f_μ} is a bivariate absolutely continuous copula iff $\mu \in [-1,3]$, which is in agreement with the findings of [2].

To get an example of a function f for which $A_i(f) \neq (0,1)$ for $i = 1,2$, consider

$$f(t) = 1 - t/2 + t^2/2 = (1-t)/2 + (1+t^2)/2,$$

i.e. a convex combination of two admissible functions. For such $f(t)$, $2f'(t) + tf''(t)$ is nonnegative for $0 \leq t \leq a$ and negative for $a < t \leq 1$

where $a \approx 0.33$. See Figure 4.2 for illustration.

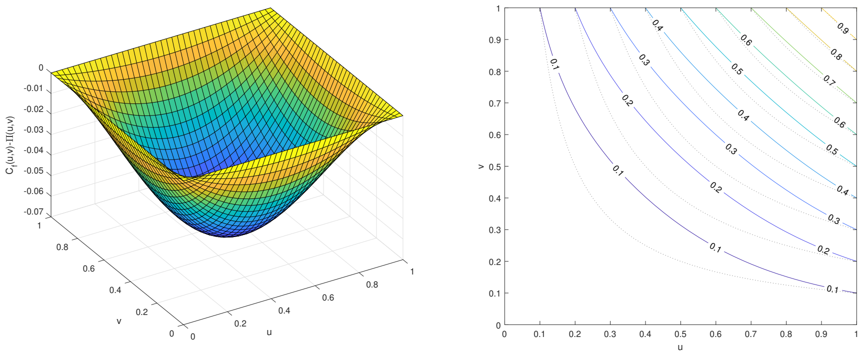
Example 16. Let $f(t) = 1 - t - \frac{t^2}{4} + \frac{t^3}{4}$. Then

$$f(t) < 1 - t, \quad x \in (0, 1),$$

so, in particular, it fails Condition (iii) of Theorem 3, yet after some calculation it could be shown that $A_2(f) = (0, 1)$. Therefore, by Theorem 10,

$$C_f(u, v) = uv\Big(1 - (1-u)(1-v) - \frac{1}{4}(1-u)^2(1-v)^2 + \frac{1}{4}(1-u)^3(1-v)^3\Big),$$

is a bivariate copula (see Figure 4.3).



(a) A difference $C_f - \Pi$ (b) contour map of copula C_f

Figure 4.3: Copula C_f whith $f(t) = 1 - t - \frac{t^2}{4} + \frac{t^3}{4}$

4.4 Properties of the considered copulas

We now investigate some properties of copulas C_f given in (4.1). We start with a partial order relation, then describe tail dependence coefficients, and finally discuss symmetry properties.

Concordance order and implied dependence properties

It is well known that given two copulas C and D, D is said to be more concordant (or more positively quadrant dependent) than C (see [36, p. 223]), we write $C \preceq D$, if $C(u, v) \leq D(u, v)$ for all $(u, v) \in [0, 1]^2$. The following result is easily proved.

Proposition 5. *Let C_f and C_g be two copulas as in (4.1). Then $C_f \preceq C_g$ if and only if $f(t) \leq g(t)$ for all $t \in [0,1]$.*

In particular, for $f(t) \geq 1$, $t \in [0,1]$ (for example, if $A_1(f) = (0,1)$) we get $C_f \succeq \Pi$ and similarly, for $g(t) \leq 1$, $t \in [0,1]$ we get $C_g \preceq \Pi$.

Now recall dependence properties defined in Chapter 2. Then we can prove the follwing proposition.

Proposition 6. *Let (X,Y) be a pair of continuous random variables with copula C_f. Then we have:*

(i) $LTD(Y|X)$ if and only if $f(t)$ is nondecreasing in $t \in [0,1)$;

(ii) $RTI(Y|X)$ if and only if

$$f(t) \geq 1 + t(1-t)\max\{f'(t), 0\}, \quad \textit{for a.e. } t \in (0,1);$$

(iii) $SI(Y|X)$ if and only if for any $a \in [0,1]$, the function

$$t \mapsto tf(a(1-t))$$

is concave on $[0,1]$.

Proof. (i) It suffices to observe that

$$\mathbb{P}(Y \leq y | X \leq x) = \frac{C_f(x,y)}{x} = yf((1-x)(1-y))$$

is nonincreasing in x on $(0,1]$ for all $y \in [0,1]$ if and only if $f(1-x)$ is nonincreasing in x (since f is continuous on $[0,1)$ due to Proposition 3(i)), or equivalently, $f(t)$ is nondecreasing in t.

(ii) By [36, Corollary 5.2.6], $RTI(Y|X)$ holds if and only if for any $v \in [0,1]$

$$\partial_u C_f(u,v) \geq \frac{v - C_f(u,v)}{1-u}$$

for all $u \in A \subset [0,1]$ where A has Lebesgue measure of 1. Rearranging, for all $y \in (0,1]$, we get

$$f((1-u)(1-v)) - u(1-u)(1-v)f'((1-u)(1-v)) - 1 \geq 0. \quad (4.17)$$

Now let $t = (1-u)(1-v)$ and note that $0 \leq t < 1-u$. Then for (4.17) to hold it is necessary (and sufficient) that for Lebesgue almost all $t \in [0,1]$,

$$\begin{cases} f'(t) \geq 0, \\ f(t) - (1-t)tf'(t) - 1 \geq 0 \quad \text{(taking } y \downarrow 0\text{);} \end{cases}$$

$$\text{or} \quad \begin{cases} f'(t) < 0, \\ f(t) \geq 1 \quad \text{(letting } u \downarrow 0 \text{ along a sequence from } A\text{).} \end{cases}$$

Combining both cases and rearranging, we get that for almost all $t \in [0,1]$,

$$f(t) \geq \max\{1, 1 + t(1-t)f'(t)\} = 1 + t(1-t)\max\{f'(t), 0\}.$$

(iii) By [36, Corollary 5.2.11], $SI(Y|X)$ holds if and only if $C_f(u,v)$ is concave in u. This is equivalent to having secant slopes of $C_f(\cdot, v)$ nonincreasing for any $v \in [0,1]$, that is,

$$\frac{C_f(u_2,v) - C_f(u_1,v)}{u_2 - u_1} \geq \frac{C_f(u_3,v) - C_f(u_2,v)}{u_3 - u_2},$$

for any $0 \leq u_1 < u_2 < u_3 \leq 1$ and any $v \in [0,1]$. For $v = 0$ the above inequality is trivial, so consider only $v \in (0,1]$. Then after rearrangement, the above inequality is equivalent to

$$\begin{aligned} &\frac{u_2 f((1-u_2)(1-v)) - u_1 f((1-u_1)(1-v))}{u_2 - u_1} \\ &\geq \frac{u_3 f((1-u_3)(1-v)) - u_2 f((1-u_2)(1-v))}{u_3 - u_2}, \end{aligned} \tag{4.18}$$

which is to say that for any $a \in [0,1)$, the function

$$t \mapsto tf(a(1-t)), \quad t \in [0,1]$$

is concave. Letting $y \downarrow 0$ in (4.18) also yields that $t \mapsto tf(1-t)$ is concave (also nondecreasing by Proposition 3(iii)), so we can allow even $a = 1$.

□

Remark 7. Observe that if X and Y are continuous random variables with copula C_f, then by [36, Corollary 5.2.16] we have that $LCSD(X,Y)$ holds if and only if C_f is TP_2, and therefore necessary and sufficient conditions on f for $LCSD(X,Y)$ to hold are already given by Theorem 7.

Let us now recall some popular measures of association, namely Kendall's tau and Spearman's rho. Using Proposition 3(iii) and τ and ρ integral representations for copulas, one can show the following result.

Proposition 7. *Let (X, Y) be a pair of continuous random variables with copula C_f. Then*

$$-0.8636 \approx C_1 \leq \rho \leq 4\pi^2 - 39 \approx 0.4784$$

and

$$\frac{2\pi^2 - 21}{3} + 4(J_2 - J_1) \leq \tau \leq \frac{1}{3},$$

where

$$J_1 := \int_0^1 \int_0^1 u^2 v^2 (1-u)(1-v)(f'((1-u)(1-v)))^2 du dv$$
$$J_2 := \int_0^1 \int_0^1 C_f(u,v)[u + v - 2uv] f'((1-u)(1-v)) du dv.$$

Proof. Due to (4.5), the upper bounds for τ and ρ follow from the known values of the AMH(1) copula (see, e.g. [36, Exercise 5.10]), as required.

To obtain the stated lower bound for τ, we observe that by Proposition 3(ii), $f'(t)$ exists for almost all $t \in (0, 1)$ and hence

$$\begin{aligned}\partial_u C_f(u,v) &= v f((1-u)(1-v)) - uv(1-v) f'((1-u)(1-v)), \\ &\qquad \text{for all } v \in (0,1) \text{ and almost all } u \in (0,1); \\ \partial_v C_f(u,v) &= u f((1-u)(1-v)) - uv(1-u) f'((1-u)(1-v)), \\ &\qquad \text{for all } u \in (0,1) \text{ and almost all } v \in (0,1).\end{aligned}$$

Thus for almost all $(u, v) \in (0, 1)^2$,

$$\begin{aligned}&\partial_u C_f(u,v) \partial_v C_f(u,v) \\ &= uv f^2((1-u)(1-v)) + u^2 v^2 (1-u)(1-v)(f'((1-u)(1-v)))^2 \\ &\quad - uv[y(1-u) + u(1-v)] f((1-u)(1-v)) f'((1-u)(1-v)).\end{aligned}$$

Now using [36, Eq. (5.1.12)], we get

$$\tau = 1 - 4 \int_0^1 \int_0^1 \partial_u C_f(u,v) \partial_v C_f(u,v) du dv = 1 - 4(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} I_1 &:= \int_0^1 \int_0^1 uvf^2((1-u)(1-v))dudv \\ &\leq \int_0^1 \int_0^1 \frac{uv}{(1-(1-u)(1-v))^2} dudv = 2 - \frac{\pi^2}{6}, \\ I_2 &:= \int_0^1 \int_0^1 u^2v^2(1-u)(1-v)(f'((1-u)(1-v)))^2 dudv = J_1, \end{aligned}$$

and

$$\begin{aligned} I_3 &:= -\int_0^1 \int_0^1 uv[u+v-2uv]f((1-u)(1-v))f'((1-u)(1-v))dudv \\ &= -\int_0^1 \int_0^1 C_f(u,v)[u+v-2uv]f'((1-u)(1-v))dudv = -J_2. \end{aligned}$$

Thus,

$$\tau \geq \frac{2\pi^2 - 21}{3} + 4(J_2 - J_1).$$

As for Spearman's ρ lower bound, we simply use [36, Eq. (5.1.15c)] and Proposition 3(iii) to get

$$\begin{aligned} \rho &= 12 \int_0^1 \int_0^1 C_f(u,v)dudv - 3 \\ &\geq 12 \int_0^1 \int_0^1 \frac{uv \max\{1 - 2\sqrt{(1-u)(1-v)}, 0\}}{(1 - \sqrt{(1-u)(1-v)})^2} dudv - 3 \approx -0.8636, \end{aligned}$$

where the approximation was obtained by Maple 2018.1 software. □

Remark 8. As can be observed from the proof of Proposition 7, the computation of the lower bound for Kendall's τ requires information on the bounds of f'. If one knows, for example, that f is increasing so that $f' > 0$, then J_2 can be dropped while J_1 can be bounded above by $\frac{5}{3} - \frac{\pi^2}{6}$ since $f'(t) \leq 1/(1-t)^2$ due to the fact that $(1-t)f(t)$ is nonincreasing on $(0,1)$ and $f(t) \leq 1/(1-t)$. On the other hand, if f' can be negative, we have no tight lower bound. Trying to exploit the fact that $\delta(1-t) = (1-t)^2 f(t^2)$ where $\delta(t) = C_f(t,t)$ and that δ is nondecreasing and 2-Lipschitz (so that $\delta'(t) \in [0,2]$ for almost all $t \in [0,1]$), unfortunately, gives a looser lower bound for f', resulting in a worse estimate for J_1 than in the case of $f' > 0$.

Also Proposition 7 shows the limitations of the copula construction method under investigation. Indeed, a very strong (positive) association cannot be modelled using the considered copulas.

Tail dependence coefficients

Another interesting property of copulas is tail dependence. This dependence can be measured in several ways, for example, by upper and lower tail dependence parameters, which are calculated in the following proposition.

Proposition 8. *Let C_f be a copula as in* (4.1). *Then the upper tail dependence coefficient of C_f is zero, i.e. $\lambda_U = 0$, while for the lower tail coefficient we have*

$$\lambda_L = \lim_{t\uparrow 1}(1-\sqrt{t})f(t) = \frac{1}{2}\lim_{t\uparrow 1}(1-t)f(t) \in [0,1/2],$$

where both limits always exist.

Proof. Notice that $\delta_{C_f}(s) = s^2 f((1-s)^2)$ and, therefore, using $t=(1-s)^2$,

$$\begin{aligned}\lambda_U &= 2 - \lim_{s\uparrow 1}\frac{1-s^2f((1-s)^2)}{1-s}\\ &= 2-\lim_{t\downarrow 0}\frac{1-(1+t)f(t)+2\sqrt{t}f(t)}{\sqrt{t}} = 0,\end{aligned}$$

due to (4.2). Similarly, and denoting $t=(1-s)^2$, we get

$$0 \le 2\frac{C_f(s,s)}{s} = 2\frac{s^2f((1-s)^2)}{s} = 2(1-\sqrt{t})f(t) = 2\frac{(1-t)f(t)}{1+\sqrt{t}}.$$

Now letting $s\downarrow 0$, or equivalently, $t\uparrow 1$, and using Proposition 3, (i) and (iii), we see that the limit on the right-hand side exists and is equal to $\lim_{t\uparrow 1}(1-t)f(t)\in[0,1]$, while the limit of the left-hand side is $2\lambda_L$. □

Example 17. In this example we show that the full range of λ_L, namely $[0,1/2]$, can be attained. To see this, consider

$$f_\beta(t) = \frac{1-\beta t}{1-t}, \quad t\in[0,1),\ \beta\in[0,1].$$

To check that $C_{f_\beta}(u,v) = uvf_\beta((1-u)(1-v))$ is a bivariate copula for any $\beta\in[0,1]$, we use Theorem 7 and show that f_β is geometrically

Jensen convex. For this we only need to show that for any $u, v \in [0,1)$ and $\beta \in [0,1]$,

$$\left(\frac{1-\beta\sqrt{uv}}{1-\sqrt{uv}}\right)^2 \leq \frac{(1-\beta u)(1-\beta v)}{(1-u)(1-v)}.$$

This amounts to checking that

$$\begin{aligned} 0 &\leq (1-\sqrt{uv})^2(1-\beta u)(1-\beta v) - (1-\beta\sqrt{uv})^2(1-u)(1-v) \\ &= (1-\beta)(\sqrt{u}-\sqrt{v})^2(1-\beta uv), \end{aligned}$$

which is true for all $u, v \in [0,1)$ provided $\beta \in [0,1]$.

Now by Proposition 8, the lower tail dependence coefficient of C_{f_β} is

$$\lambda_L = \frac{1}{2}\lim_{t\uparrow 1}(1-t)f_\beta(t) = \frac{1-\beta}{2},$$

which fills the whole interval $[0, 1/2]$ as β changes in $[0,1]$.

Symmetry properties

The following proposition provides a characterization of symmetry of a pair of random variables with associated copula C_f.

Proposition 9. *Let (X, Y) be a pair of random variables with associated copula C_f.*

(i) If X is symmetric about a and Y is symmetric about b, then (X, Y) is radially symmetric about (a, b) if and only if

$$f(t) = 1 - \alpha t,\ \alpha \in [-1, 1].$$

(ii) If X is symmetric about a and Y is symmetric about b, then (X, Y) is jointly symmetric about (a, b) if and only if $f(t) = 1$, i.e. $C_f = \Pi$.

Proof. (i) Using [36, Theorem 2.7.3] (X, Y) is radially symmetric if and only if C_f satisfies

$$C_f(u, v) = u + v - 1 + C_f(1-u, 1-v), \quad \forall x, y \in [0,1].$$

Recalling (4.1) and denoting $t = uv$ and $s = u + v$, we get for $s \neq 1$,

$$tf(1 - s + t) = s - 1 + (1 - s + t)f(t),$$

or equivalently,

$$\frac{f(1 - s + t) - f(t)}{1 - s} = \frac{f(t) - 1}{t},$$

which implies that both sides must be constant in u, say, α. Hence $f(t) = 1 + \alpha t$. From Examples 10 and 14, we get that $\alpha \in [-1, 1]$.

(ii) By [36, Eq. (2.8.1)], (X, Y) is jointly symmetric about (a, b) if and only if, for all $x, y \in [0, 1]$

$$C_f(u, v) = u - C_f(u, 1 - v) \quad \text{and} \quad C_f(u, v) = v - C_f(1 - u, v).$$

Due to the symmetry of C_f, it is enough to verify the first equality, which is equivalent to (for $x \neq 0$)

$$vf((1 - u)(1 - v)) = 1 - (1 - v)f((1 - u)v). \tag{4.19}$$

Since it must be valid for all $v \in [0, 1]$, take $v = 1/2$. Then

$$\frac{1}{2}f\Big(\frac{1 - u}{2}\Big) + \frac{1}{2}f\Big(\frac{1 - u}{2}\Big) = 1$$

and, therefore, $f(t) = 1$, $t \in [0, 1/2)$. On the other hand, if we let $u \downarrow 0$ in (4.19) then

$$vf(1 - v) = 1 - (1 - v)f(v),$$

which for $v = 1/2$ implies $f(1/2) = 1$ and for $v > 1/2$, we get

$$v + (1 - v)f(v) = 1,$$

and therefore $f(t) = 1$, $t \in [0, 1]$.

□

5 | CONCLUSIONS

In Chapter 3 we have provided a characterization of eligible functions f so that

$$H_f(C)(u,v) = C(u,v)f(\overline{C}(u,v))$$

is a copula for any $C \in \mathcal{C}$. Many considered examples in that chapter, especially involving $C = \Pi$, have shown that $H_f(C_0)$ can be a copula for a specific $C_0 \in \mathcal{C}$ even if f is not eligible, i.e. f is conditionally eligible. Therefore in Chapter 4 we focused on this particular case, i.e. we have attempted to characterize all functions f such that C_f in Eq. (4.1) is a bivariate copula. This class contains not only *eligible* functions, but also many more *conditionally eligible* functions, and has proved to be more difficult to characterize succinctly as accomplished by Durante and coworkers in [10, 13] in the case of transformations of the comonotonicity copula M instead of the independence copula Π considered in this chapter. So far we have succeeded in fully characterizing a subclass of such copulas, namely those which have the so-called TP_2 property, as well as those where the function f is twice continuously differentiable. In general, the function f is only twice differentiable Lebesgue almost everywhere on $(0,1)$, so there is a little gap to be filled between necessary and sufficient conditions for C_f to be a copula. An example of such a case is provided in Example 8 which contains only a piecewise smooth function f. This example can be generalized by taking the maximum of a bigger number of appropriate functions, thus potentially distorting differentiability at various points on the unit interval.

Directions for further research

As mentioned in Section 3.4, extensions of the obtained results to multivariate case is still an open problem.

Another direction for further research includes investigation of the relationship between the function f and the "generator" function in Cuadras terminology, given in Eq. (1.1). It would be interesting to see what our necessary and sufficient conditions say about the canonical correlation function in the definition of H_θ. Also what if $Q(u,v) = \Pi^*$ is replaced by some other function in Cuadras' construction? This would further enrich our knowledge about many popular copula families.

Yet another interesting direction for future investigation is the question of finding examples and, possibly, a characterization of all functions f such that

$$C_f(u,v) = uvf((1-u)(1-v)), \quad (u,v) \in [0,1]^2$$

is a (proper) quasi-copula. We have tried our best to find at least one proper quasi-copula among the functions we have considered or those in the literature, but so far failed to find at least one of the required form. Many examples that we tried do satisfy the boundary conditions for a copula/quasi-copula, but failed the other defining condition for a quasi-copula (2-increasingness on special rectangles, or 1-Lipschitz condition). So in our opinion, the posed question deserves a deeper investigation, and a likely separate paper, provided one succeeds in finding at least one proper quasi-copula of the considered form. Many examples from the literature involve, for example, a separable (wavy) perturbation of the independence copula (i.e., $uv + g_1(u)g_2(v)$ for some functions g_1 and g_2, see, e.g. [19, Eq. (5)], [16, Example 3.5]), but not of the form $uvg((1-u)(1-v))$.

Furthermore, in view of several examples considered in this thesis, one can wonder if there is any relationship between the copulas under investigation and copulas with quadratic or cubic sections. Do our results generalise or are particular cases of the known results?

Indeed, many examples with polynomial sections can be obtained from the considered construction by taking a polynomial function f, but the list cannot be exhausted by such construction due to the necessarily symmetric form of

$$C_f(u,v) = uvf((1-u)(1-v)).$$

Moreover, this construction can only give copulas with polynomial sections that have a factor $\Pi(x,y) = xy$. Some other examples can be obtained from other copulas C with polynomial sections and an eligible polynomial f as in

$$H_f(C)(u,v) = C(u,v)f(\overline{C}(u,v))$$

that we considered in in Chapter 3. Again, such an $H_f(C)$ has a special factor in its expression.

As copulas of the form C_f are always symmetric, putting additional restrictions of having polynomial sections would drastically reduce their form. For example, the only copulas of the form C_f with quadratic sections are those of the FGM family, while [36, Theorem 3.2.4] allows for more freedom. On the other hand, if we look for copulas of the form C_f having cubic sections, then due to symmetry, form of the copula and degree restrictions, the only such copulas are easily seen to be of the form considered in Example 2, namely

$$C_\mu(u,v) = uv(1 + \mu(1-u)^2(1-v)^2), \quad \mu \in [-1,3].$$

Again, we clearly do not obtain all copulas with cubic sections as described, e.g. in [36, Theorem 3.2.6].

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APPENDIX

For the reader's convenience, in this appendix we recall a characterization of absolutely continuous n-variate copulas and also give technical details how it is applied in the setting of Theorem 10.

Let $V_1, \ldots, V_n$, $n = 2, 3, \ldots$, denote independent random variables, uniformly distributed on $[0,1]$.

Theorem 11 (Theorem 3.2, [8])**.** *A function $C : [0,1]^n \to [0,1]$ is an absolutely continuous n-dimensional copula if and only if there exist functions $\tilde{g}_{i_1,\ldots,i_c} : \mathbb{R}^c \to \mathbb{R}$, $1 \le i_1 < \cdots < i_c \le n$, $c = 2, \ldots, n$, satisfying the conditions:*

- *(integrability) For any $1 \le i_1 < \cdots < i_c \le n$, $c = 2, \ldots, n$,*

$$\int_0^1 \cdots \int_0^1 |\tilde{g}_{i_1,\ldots,i_c}(t_{i_1}, \ldots, t_{i_c})| dt_{i_1} \cdots dt_{i_c} < \infty;$$

- *(degeneracy) For any $1 \le i_1 < \cdots < i_c \le n$, $k = 1, 2, \ldots, c$, $c = 2, \ldots, n$,*

$$\begin{aligned}&\mathbb{E}(\tilde{g}_{i_1,\ldots,i_c}(V_{i_1}, \ldots, V_{i_c}) \,|\, V_{i_1}, \ldots, V_{i_{k-1}}, V_{i_{k+1}}, \ldots, V_{i_c})\\ &= \int_0^1 \tilde{g}_{i_1,\ldots,i_c}(V_{i_1}, \ldots, V_{i_{k-1}}, t_{i_k}, V_{i_{k+1}}, \ldots, V_{i_c}) dt_{i_k} = 0 \quad (a.s.);\end{aligned}$$

- *(positive definiteness)*

$$\tilde{U}_n(V_1, \ldots, V_n) := \sum_{c=2}^{n} \sum_{1 \le i_1 < \cdots < i_c \le n} \tilde{g}(V_{i_1}, \ldots, V_{i_c}) \ge -1 \quad (a.s.)$$

and such that

$$C(u_1, \ldots, u_n) = \int_0^{u_1} \cdots \int_0^{u_n} (1 + \tilde{U}_n(t_1, \ldots, t_n)) dt_1 \cdots dt_n.$$

As an application of Theorem 11 in the case $n = 2$, we have the following technical, but rather straightforward result.

Lemma 3. *Under the conditions of Theorem 10, that is, if $f : [0,1] \to [0, +\infty]$ is twice differentiable, with f' absolutely continuous, on $(0,1)$*

such that

$$f(0) = 1, \quad \lim_{t \downarrow 0} t f'(t) = 0, \quad \text{and} \quad (1-t) f(t) \le 1 \text{ for all } t \in [0,1), \quad (5.1)$$

and, in addition, for

$$\widehat{C}_f(x,y) = x + y - 1 + (1-x)(1-y) f(xy), \quad (x,y) \in [0,1]^2,$$

one has $\frac{\partial^2 \widehat{C}_f}{\partial x \partial y} \ge 0$ *on* $(0,1)^2$*, then* $\widehat{C}_f$ *is an absolutely continuous bivariate copula.*

Proof. We simply check the conditions of Theorem 11. Indeed, when $n = 2$, we have $c = 2$ and can take

$$\tilde{g}_{1,2}(x,y) = \tilde{U}_2(x,y) = \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y) - 1, \quad (x,y) \in (0,1)^2.$$

Observe that

$$\begin{aligned} \frac{\partial \widehat{C}_f}{\partial x}(x,y) &= 1 - (1-y) f(xy) + y(1-x)(1-y) f'(t)\big|_{t=xy}, \\ \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y) &= f(xy) + [1 - 2x - 2y + 3xy] f'(t)\big|_{t=xy} \\ &\quad + xy(1-x)(1-y) f''(t)\big|_{t=xy}. \end{aligned}$$

Since f is assumed twice differentiable on $(0,1)$, the function $(x,y) \mapsto \frac{\partial \widehat{C}_f}{\partial x}(x,y)$ is jointly continuous (hence Lebesgue measurable) on $(0,1)^2$, therefore its partial derivative with respect to y, namely $\frac{\partial^2 \widehat{C}_f}{\partial x \partial y}$, is also Lebesgue measurable on $(0,1)^2$ as the limit of measurable functions.

As the function f, together with its derivatives, might explode at the boundary of the unit square (e.g. for AMH(1) copula, $f(t) = 1/(1-t)$, so it and its derivatives tend to $+\infty$ as $t \uparrow 1$), we consider any integer $m \ge 1$ and a sequence of nonnegative restrictions

$$g^{(m)}(x,y) := \begin{cases} \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y), & \text{if } (x,y) \in [1/m, 1-1/m]^2; \\ 0, & \text{if } (x,y) \in [0,1]^2 \setminus [1/m, 1-1/m]^2, \end{cases}$$

which clearly monotonically increases to $\frac{\partial^2 \widehat{C}_f}{\partial x \partial y}$ pointwise on $(0,1)^2$.

Now to establish Lebesgue integrability of $\tilde{g}_{1,2}$ on $[0,1]^2$, we need to prove Lebesgue integrability of $g^{(m)}$ for any $m \geq 1$, then apply the monotone convergence theorem to get the integrability of $\frac{\partial^2 \widehat{C}_f}{\partial x \partial y}$, and finally use the triangle inequality.

As f' is assumed absolutely continuous on $(0,1)$ and since $y \mapsto xy$ is strictly increasing for any fixed $x \in (0,1)$, one gets that $y \mapsto \frac{\partial \widehat{C}_f}{\partial x}(x,y)$ is also absolutely continuous on any interval $[1/m, 1-1/m]$, $m \geq 1$, for any fixed $x \in (0,1)$ as compositions of absolutely continuous functions with strictly increasing ones, as well as products and sums are again absolutely continuous. Thus, by the fundamental theorem of calculus for Lebesgue integrals, we have that for all $0 < 1/m \leq y \leq 1 - 1/m < 1$

$$\frac{\partial \widehat{C}_f}{\partial x}(x,y) = \frac{\partial \widehat{C}_f}{\partial x}(x,1/m) + \int_{1/m}^{y} \frac{\partial^2 \widehat{C}_f}{\partial x \partial t}(x,t)dt.$$

Moreover, by the same theorem $y \mapsto \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y)$ is Lebesgue integrable on $[1/m, 1-1/m]$ for any $x \in (0,1)$ and $m \geq 1$. By Tonelli theorem then the function $(x,y) \mapsto \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y)$ is Lebesgue integrable on $[1/m, 1-1/m]^2$ for any $m \geq 1$.

Therefore, for any $m \geq 1$, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 g^{(m)}(x,y)dxdy \\
&= \int_{1/m}^{1-1/m} \int_{1/m}^{1-1/m} \frac{\partial^2 \widehat{C}_f}{\partial x \partial y}(x,y)dxdy \\
&= \widehat{C}_f(1-1/m, 1-1/m) - 2\widehat{C}_f(1/m, 1-1/m) + \widehat{C}_f(1/m, 1/m) \\
&= \Big(\frac{1}{m}\Big)^2 f\Big(\Big(1-\frac{1}{m}\Big)^2\Big) - 2\frac{1}{m}\Big(1-\frac{1}{m}\Big) f\Big(\frac{1}{m}\Big(1-\frac{1}{m}\Big)\Big) \\
&\quad + \Big(1-\frac{1}{m}\Big)^2 f\Big(\Big(\frac{1}{m}\Big)^2\Big) \\
&\leq \frac{1}{2m-1} + \Big(1-\frac{1}{m}\Big)^2 f\Big(\Big(\frac{1}{m}\Big)^2\Big),
\end{aligned} \tag{5.2}$$

where to obtain the last inequality we have used the bound (see (5.1))

$$(1-z)f(z) \leq 1, \quad z \in [0,1).$$

Therefore, using the monotone convergence theorem, (5.2) and the assumption $f(0)=1$,

$$\begin{aligned}\int_0^1\int_0^1 |\tilde{g}_{1,2}(x,y)|dxdy &= \lim_{m\to\infty}\int_0^1\int_0^1 |g^{(m)}-1|(x,y)dxdy\\ &\le \lim_{m\to\infty}\Big(1+\frac{1}{2m-1}+\Big(1-\frac{1}{m}\Big)^2 f\Big(\Big(\frac{1}{m}\Big)^2\Big)\Big)\\ &= 2<\infty.\end{aligned}$$

Hence $\tilde{g}_{1,2}$ is Lebesgue-integrable on $[0,1]^2$.

As for the degeneracy condition of Theorem 11, by Lebesgue dominated convergence theorem, for any $y\in(0,1)$, we get

$$\begin{aligned}&\int_0^1 \tilde{g}_{1,2}(x,y)dx = \lim_{m\to\infty}\int_{1/m}^{1-1/m}\tilde{g}_{1,2}(x,y)dx\\ &=\lim_{m\to\infty}\Big[\frac{\partial\widehat{C}_f}{\partial y}\Big(1-\frac{1}{m},y\Big)-\frac{\partial\widehat{C}_f}{\partial y}\Big(\frac{1}{m},y\Big)\Big]-1\\ &=\lim_{m\to\infty}\Big[-\frac{1}{m}f\Big(y\Big(1-\frac{1}{m}\Big)\Big)\\ &\quad+\frac{1}{m}\Big(1-\frac{1}{m}\Big)(1-y)f'\Big(y\Big(1-\frac{1}{m}\Big)\Big)\\ &\quad+\Big(1-\frac{1}{m}\Big)f\Big(\frac{y}{m}\Big)-\frac{1}{m}\Big(1-\frac{1}{m}\Big)(1-y)f'\Big(\frac{y}{m}\Big)\Big]-1\\ &=0+0+1-\frac{1-y}{y}\lim_{m\to\infty}\Big(\frac{y}{m}\Big)f'\Big(\frac{y}{m}\Big)-1=0,\end{aligned}\tag{5.3}$$

due to condition (5.1). Plugging-in $y=V_2$, where V_2 is a uniformly on $[0,1]$ distributed random variable, we check that the first of the two required degeneracy conditions hold. The other (with y replaced by x) holds by symmetry.

Finally, the choice of $\tilde{g}_{1,2}$ clearly fulfills the required positive definiteness condition of Theorem 11 and provides the needed expression of $\widehat{C}_f$ via its density. □

Remark 9. Note that if $\widehat{C}_f$ is an absolutely continuous bivariate copula, then the second condition in (5.1) is also necessary as seen from equation (5.3). The other two requirements of (5.1) are necessary by Proposition 3, parts (i) and (iii).

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