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Approximation of Analytic Functions by Shifts of Certain Compositions

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Abstract: In the paper, we obtain universality theorems for compositions of some classes of operators in multidimensional space of analytic functions with a collection of periodic zeta-functions. The used shifts of periodic zeta-functions involve the sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function.

Keywords: non-trivial zeros of the Riemann zeta-function; periodic zeta-function; space of analytic functions; universality

MSC: 11M41; 11M26

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1. Introduction

Let $s = \sigma + it$ be a complex variable, and let

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

be the Riemann zeta-function having the meromorphic continuation to the whole complex plane with the unique simple pole at the point $s = 1$. In [1], Voronin discovered the universality of the function $\zeta(s)$, on the approximation of analytic functions by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. More precisely, he proved that if $0 < r < 1/4$, $f(s)$ is a continuous non-vanishing function on $|s| \leqslant r$, and analytic on $|s| < r$, then, for every $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Various authors, among them Gonek, Reich, Bagchi, Laurinčikas, Matsumoto, Macaitienė, Kačinskaitė, Pańkowski, Steuding and others, improved and extended the above Voronin theorem. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D, and let $H_0(K)$, $K \in \mathcal{K}$, denote the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the modern version of the Voronin theorem, see, for example [2], says that if $K \in \mathcal{K}$, $f(s) \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0,$$

where meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The latter inequality shows that the set of shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in$

$H_0(K)$ has a positive lower density. In [3], it was obtained that the above set has a positive density for all but at most countably many $\varepsilon > 0$. The discrete versions of the mentioned results on the approximation by shifts $\zeta(s + ikh)$, $h > 0$, $k = 0, 1, \ldots$, were studied in [4–7], see also [8].

Our investigation object is the periodic zeta-functions. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers. The periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$. This follows from the representation

$$\zeta(s; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right), \quad \sigma > 1,$$

where $\zeta(s, \alpha)$, $0 < \alpha \leqslant 1$, is the classical Hurwitz zeta-function, and $q \in \mathbb{N}$ is a minimal period of the sequence $\mathfrak{a}$.

Universality of the function $\zeta(s; \mathfrak{a})$, i.e., approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau; \mathfrak{a})$, $\tau \in \mathbb{R}$, was studied by various authors. Among them, Bagchi [5], Steuding [9,10], Kaczorowski [11], and others. In [12], an universality theorem for $\zeta(s; \mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$ has been obtained. We recall that the sequence $\mathfrak{a}$ is called multiplicative if $a_1 = 1$ and $a_{m_1 m_2} = a_{m_1} a_{m_2}$ for all coprime m_1 and m_2. More general is the joint universality for collections of zeta-functions. In this case, a collection of analytic functions simultaneously is approximated by a collection of shifts of zeta-functions. The first joint universality result was also obtained by Voronin in [13] for Dirichlet L-functions with pairwise non-equivalent Dirichlet characters, see also [14]. Joint universality theorems involving the function $\zeta(s, \alpha)$ were studied in [15–20]. The papers [21,22] are devoted to joint approximation of analytic functions by generalized non-linear shifts of periodic zeta-functions. The aim of this paper is universality theorems for compositions of collections of periodic zeta-functions studied in [22].

Let $\widetilde{H}(D)$ denote the space of analytic functions on D endowed with the topology of uniform convergence on compacta. The first universality theorems for compositions $F(\zeta(s))$, where $F : \widetilde{H}(D) \to \widetilde{H}(D)$ is a certain operator, were proved in [23,24]. Later, universality for compositions of other zeta-functions was obtained; for example, the paper [25] is devoted to compositions of zeta-functions of normalized Hecke cusp forms.

Now we recall the main result of [22]. For $j = 1, \ldots, r$, let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequences of complex numbers, and let $\zeta(s; \mathfrak{a}_j)$ be the corresponding periodic zeta-function. In [22], for shifts of $\zeta(s; \mathfrak{a}_j)$, the sequence $\{\gamma_k : k \in \mathbb{N},\ \gamma_k > 0\}$ of imaginary parts of non-trivial zeros of the Riemann zeta-function is used. Moreover, it was required that the estimate

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T, \quad T \to \infty, \tag{1}$$

with $c > 0$ should be satisfied. (Note that (1) follows from the Montgomery pair correlation hypothesis [26]). Then the main result of [22] is the following statement. Let $\#A$ denote the cardinality of the set A, and N runs over the set of natural numbers $\mathbb{N}$.

Theorem 1. *Suppose that the estimate* (1) *is true, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $\mathfrak{a}_j$ be multiplicative, $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \#\left\{1 \leqslant k \leqslant N : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + ih_j\gamma_k; \mathfrak{a}_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

The sequence $\{\gamma_k\}$ satisfying estimate (1) was used for the first time in the theory of universality in [27] in the case of the Riemann zeta-function. Recall that the Riemann hypothesis (RH) asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$. A similar result under RH was obtained in [28] by using moment estimates of [29]. Universality of the Hurwitz zeta-function with the sequence $\{\gamma_k\}$ satisfying (1) was considered in [30,31]. A version of the Mishou theorem with the sequence $\{\gamma_k\}$ satisfying estimate (1) was proved in [32].

Let

$$\widetilde{H}^r(D) = \underbrace{\widetilde{H}(D) \times \cdots \times \widetilde{H}(D)}_{r}.$$

Define some classes of operators $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$. For brevity, denote $\underline{g} = (g_1, \ldots, g_r) \in \widetilde{H}^r(D)$, and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{R}^+)^r$, where $\mathbb{R}^+$ is the set of all positive real numbers.

We say that the operator $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ belongs to the class $Lip(\underline{\alpha})$ if:

1° For every polynomial $p = p(s)$ and sets $K_1, \ldots, K_r \in \mathcal{K}$, there exists an element $\underline{g} \in F^{-1}\{p\} \subset \widetilde{H}^r(D)$ such that $g_j(s) \neq 0$ on K_j, $j = 1, \ldots, r$.

2° For every $K \subset \mathcal{K}$, there exists a constant $c > 0$ and the sets $K_1, \ldots, K_r \in \mathcal{K}$ such that

$$\sup_{s \in K} \left| F(\underline{g}_1) - F(\underline{g}_2) \right| \leqslant c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_{1j}(s) - g_{2j}(s) \right|^{\alpha_j}$$

for all $\underline{g}_1, \underline{g}_2 \in \widetilde{H}^r(D)$.

For brevity, we say that the statement $A(\underline{\mathfrak{a}}, \underline{h}, (1))$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_r)$, $\underline{h} = (h_1, \ldots, h_r)$, holds if the sequences $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are multiplicative, $h_1, \ldots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate (1) is valid. Moreover, let

$$\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) = (\zeta(s + ih_1\gamma_k; \mathfrak{a}_1), \ldots, \zeta(s + ih_r\gamma_k; \mathfrak{a}_r)).$$

For example, we may take $\underline{\mathfrak{a}} = (\chi_1(m), \ldots, \chi_r(m))$, where $\chi_1(m), \ldots, \chi_r(m)$ are Dirichlet characters modulo q, and $\underline{h} = (\sqrt{2}, \sqrt[3]{2}, \ldots, \sqrt[r+1]{2})$ because it is well known that Dirichlet characters are periodic and multiplicative, and the algebraic numbers $\sqrt{2}, \sqrt[3]{2}, \ldots, \sqrt[r+1]{2}$ are linearly independent over $\mathbb{Q}$.

Denote by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K that are analytic in the interior of K.

Theorem 2. *Suppose that* $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ *is valid, and the operator* $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ *belongs to the class* $Lip(\underline{\alpha})$*. Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$*. Then, for every* $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} \left| F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \right) - f(s) \right| < \varepsilon \right\} > 0. \tag{2}$$

For example, the operator

$$F(g_1, \ldots, g_r) = c_1 g_1 + \cdots + c_r g_r, \quad g_1, \ldots, g_r \in \widetilde{H}(D),$$

with complex $c_j \neq 0$, $j = 1, \ldots, r$, belongs to the class $Lip(\underline{1})$. Actually, if $p(s)$ is a polynomial and $K_1, \ldots, K_r \in \mathcal{K}$, then there exists $a \in \mathbb{C}$ such that $p(s) - a - (c_1 + \cdots + c_{r-2}) \neq 0$ on K_r. Therefore taking

$$g_1(s) = 1, \ldots, g_{r-2}(s) = 1, g_{r-1}(s) = \frac{a}{c_{r-1}}, g_r(s) = \frac{p(s) - a - (c_1 + \cdots + c_{r-2})}{c_r},$$

we obtain that $F(g_1, \dots, g_r) = p(s)$. Thus, hypothesis 1° of the class $Lip(\underline{1})$ is satisfied. Hypothesis 2° follows from the integral Cauchy formula.

Now we state universality theorems for other classes of operators $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$. In their definitions, the set

$$S \stackrel{def}{=} \{g \in \widetilde{H}(D) : g(s) \neq 0 \text{ for all } s \in D \text{ or } g(s) \equiv 0\}$$

is involved.

Theorem 3. *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ is valid, and $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ is a continuous operator such that, for every open set $G \subset \widetilde{H}(D)$, the intersection $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the inequality (2) is valid. Moreover, limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s\in K} \left| F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\right) - f(s) \right| < \varepsilon \right\} > 0 \tag{3}$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 3 can be applied for the following statement with a modified hypothesis $(F^{-1}G) \cap S^r \neq \varnothing$.

Theorem 4. *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ is valid, and $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the inequalities (2) and (3) are valid.*

For some classes of approximated function, the set $K \in \mathcal{K}$ can be replaced by arbitrary compact set.

Theorem 5. *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ is valid, and $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ is a continuous operator. Let $K \subset D$ be a compact set, and $f(s) \in F(S^r)$. Then the assertion of Theorem 4 is true.*

It is not easy to deal with the set $F(S^r)$. The problem becomes more complicated when it is known a certain simple set lying in $F(S^r)$. For distinct complex numbers $c_1, \dots, c_m$, define the set

$$\widetilde{H}_{c_1,\dots,c_m}(D) = \{g \in H(D) : g(s) \neq c_j \text{ for all } s \in D,\ j = 1, \dots, m\}.$$

Theorem 6. *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ is valid, and $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ is a continuous operator such that $\widetilde{H}_{c_1,\dots,c_m}(D) \subset F(S^r)$. For $m = 1$, let $K \subset \mathcal{K}$, $f(s) \in H(K)$ and $f(s) - c_1 \in H_0(K)$. For $m \geqslant 2$, let $K \subset D$ be arbitrary compact set, and $f(s) \in \widetilde{H}_{c_1,\dots,c_m}(D)$. Then the assertion of Theorem 4 is true.*

We give an example. Let

$$F(g_1, \dots, g_r) = b_1 g_1 + \cdots + b_r g_r, \quad g_1, \dots, g_r \in \widetilde{H}(D),$$

and non-zero complex numbers $b_1, \dots, b_r$. Then we have the inclusion $\widetilde{H}_{c_1}(D) \subset F(S^r)$. Actually, if $g \in \widetilde{H}_{c_1}(D)$, then $(g - c_1)/b_1 \in S$. Consequently, by the definition of F,

$$F\left(\frac{g - c_1}{b_1}, \frac{c_1}{b_2}, 0, \dots, 0\right) = g.$$

This shows that $g \in F(S^r)$.

Similarly we obtain that $H_0(D) \subset F(S^r)$, where

$$F(g_1, \dots, g_r) = (g_1 + \dots + g_r)^n$$

with $n \in \mathbb{N}$.

2. Proof of Theorem 2

We will derive Theorem 2 from Theorem 1 and the Mergelyan theorem on the approximation of analytic functions by polynomials [33]. For convenience, we state the latter theorem as the following lemma.

Lemma 1. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and $g(s)$ is a continuous on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s \in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

Proof of Theorem 2. The function $f(s)$ is continuous on $K \subset \mathcal{K}$ and analytic in the interior of K. Therefore, by Lemma 1, there exists a polynomial $p_\varepsilon = p_\varepsilon(s)$ such that

$$\sup_{s \in K} |f(s) - p_\varepsilon(s)| < \frac{\varepsilon}{2}. \tag{4}$$

Now we will apply the properties of the class $Lip(\underline{\alpha})$. In view of hypothesis 1°, we find an element $\underline{g} \in F^{-1}\{p_\varepsilon\}$ such that $g_j(s) \neq 0$ on a given set $K_j \in \mathcal{K}$, $j = 1, \dots, r$. Let $\alpha = \min_{1 \leqslant j \leqslant r} \alpha_j$, and the sets $K_1, \dots, K_r \in \mathcal{K}$ correspond the set K in hypothesis 2°. Suppose that $k \in \mathbb{N}$ satisfies the inequality

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| \zeta(s + ih_j\gamma_k; \mathfrak{a}_j) - g_j(s) \right| < c^{-1/\alpha} \left(\frac{\varepsilon}{2} \right)^{1/\alpha}.$$

Then, by hypothesis 2° of the class $Lip(\underline{\alpha})$, for such k,

$$\begin{aligned} \sup_{s \in K} \left| F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \right) - p_\varepsilon(s) \right| &= \sup_{s \in K} \left| F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \right) - F(g_1(s), \dots, g_r(s)) \right| \\ &\leqslant c \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| \zeta(s + ih_j\gamma_k; \mathfrak{a}_j) - g_j(s) \right|^{\alpha_j} \\ &\leqslant c\, c^{-1} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

In view of Theorem 1, the set of $k \in \mathbb{N}$ satisfying the above inequality has a positive lower density. Therefore, the set of $k \in \mathbb{N}$ satisfying the inequality

$$\sup_{s \in K} \left| F\left(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}) \right) - p_\varepsilon(s) \right| < \frac{\varepsilon}{2}$$

has a positive lower density as well. Thus, taking into account inequality (4), we obtain the assertion of Theorem 2. □

Unfortunately, Theorem 1 does not imply a version of Theorem 2 with "lim" in place of "lim inf".

For the proof of Theorems 3–6, a limit theorem in the space $\widetilde{H}(D)$ plays a crucial role.

3. Probabilistic Background

We start with a limit theorem for probability measures in the space $\widetilde{H}^r(D)$. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by $\mathbb{P}$ the set of all prime numbers. Define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Then Ω^r is a compact topological Abelian group, therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r exists, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $\omega_j(p)$ the pth component of the element $\omega_j \in \Omega_j$, $p \in \mathbb{P}$, $j = 1, \ldots, r$, and by $\omega = (\omega_1, \ldots, \omega_r)$ the elements of Ω^r. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$ define the $\widetilde{H}^r(D)$-valued random element

$$\underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) = (\zeta(s, \omega_1; \mathfrak{a}_1), \ldots, \zeta(s, \omega_r; \mathfrak{a}_r)),$$

where

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{\omega_j^l(p) a_{p^l}}{p^{ls}}\right), \quad j = 1, \ldots, r.$$

Let $P_{\underline{\zeta}}$ be the distribution of $\underline{\zeta}(s, \omega; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H^r \Big\{\omega \in \Omega^r : \underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) \in A\Big\}, \quad A \in \mathcal{B}(\widetilde{H}^r(D)).$$

For $A \in \mathcal{B}(\widetilde{H}^r(D))$, define

$$P_N(A) = \frac{1}{N} \# \Big\{1 \leqslant k \leqslant N : \underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}}\} \in A\Big\}.$$

Then Theorem 6 of [22] is the following statement.

Lemma 2. *Suppose that $A(\underline{\mathfrak{a}}, \underline{h}, (1))$ is valid. Then P_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Let P be a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and $u : \mathbb{X} \to \mathbb{Y}$ be a $(\mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{Y}))$-measurable mapping. Then the measure P induces on $(\mathbb{X}, \mathcal{B}(\mathbb{Y}))$ the unique probability measure Pu^{-1} defined by

$$Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(\mathbb{Y}).$$

Now we recall an useful lemma on a preservation of weak convergence under continuous mappings [34].

Lemma 3. *Suppose that P and P_n, $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, P_n converges weakly to P as $n \to \infty$, and $u : \mathbb{X} \to \mathbb{Y}$ is a continuous mapping. Then $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$.*

For $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$, define

$$P_{N,F}(A) = \frac{1}{N} \# \Big\{1 \leqslant k \leqslant N : F\Big(\underline{\zeta}(s + i\underline{h}\gamma_k; \underline{\mathfrak{a}})\Big) \in A\Big\}, \quad A \in \mathcal{B}(\widetilde{H}(D)).$$

Then Lemmas 2 and 3 imply the following statement.

Lemma 4. *Suppose that $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ is a continuous operator, and that $A(\mathfrak{a}, \underline{h}, (1))$ is valid. Then $P_{N,F}$ converges weakly to $P_{\underline{\zeta}} F^{-1}$ as $N \to \infty$.*

For the proof of universality, the support of limit measures in limit theorems in the space of analytic functions plays a crucial role: it defines the class of approximated functions. We recall that the support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, where $\mathbb{X}$ is a separable space, is a minimal closed set S_P such that $P(S_P) = 1$. The set S_P consists of all elements $x \in \mathbb{X}$ such that, for every open neighborhood G of x, the inequality $P(G) > 0$ is satisfied.

Lemma 5. *Suppose that $A(\mathfrak{a}, \underline{h}, (1))$ is valid. Then the support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

Proof of the lemma is given in [22], Lemma 9.

4. Proof of Theorems 3–6

For convenience, we recall the equivalents of weak convergence of probability measures that will be used in the proofs of universality theorems.

Lemma 6. *Let P and P_n, $n \in \mathbb{N}$, be the probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following assertions are equivalent:*

(i) P_n converges weakly to P as $n \to \infty$;

(ii) For every open set $G \subset \mathbb{X}$,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

(iii) For every continuity set A of P ($P(\partial A) = 0$, where ∂A is the boundary of A),

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 from [34].

Proof of Theorem 3. First of all, we will show that, under hypotheses of Theorem 3, the support of the measure $P_{\underline{\zeta}} F^{-1}$ is the whole space of $\widetilde{H}(D)$.

Let g be an arbitrary element of $\widetilde{H}(D)$, and G is an open neighborhood of g. Then the set $F^{-1}G$ is open as well. By the hypothesis $(F^{-1}G) \cap S^r \neq \varnothing$, there exists an element $g_1 \in F^{-1}G$ lying in S^r. Therefore, by Lemma 5, the set $F^{-1}G$ is an open neighborhood of an element of the support of the measure $P_{\underline{\zeta}}$. Hence, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Therefore,

$$P_{\underline{\zeta},F}(G) = P_{\underline{\zeta}} F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Since g and G are arbitrary, we have that the support of $P_{\underline{\zeta}} F^{-1}$ is the space $\widetilde{H}(D)$.

For a polynomial $p(s)$, define the set

$$G_\varepsilon = \left\{ g \in \widetilde{H}(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since $p(s) \in \widetilde{H}(D)$, the set G_ε is an open neighborhood of the support of the measure $P_{\underline{\zeta},F}$. Therefore, by a property of a support,

$$P_{\underline{\zeta},F}(G_\varepsilon) > 0. \tag{5}$$

Thus, by Lemmas 4 and 6 ((i) and (ii)), we have

$$\liminf_{N\to\infty} P_{N,F}(G_\varepsilon) \geqslant P_{\underline{\zeta},F}(G_\varepsilon) > 0.$$

Hence, the definitions of $P_{N,F}$ and G_ε yield

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \sup_{s\in K}\left|F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\mathfrak{a})\right) - p(s)\right| < \frac{\varepsilon}{2}\right\}. \tag{6}$$

Now, using Lemma 1, we choose the polynomial $p(s)$ satisfying

$$\sup_{s\in K}|f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{7}$$

Suppose that $k \in \mathbb{N}$ satisfies

$$\sup_{s\in K}\left|F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\mathfrak{a})\right) - p(s)\right| < \frac{\varepsilon}{2}.$$

Then, in view of (7), for such k,

$$\sup_{s\in K}\left|F\left(\underline{\zeta}(s+i\underline{h}\gamma_k;\mathfrak{a})\right) - f(s)\right| < \varepsilon.$$

This remark together with (6) proves the inequality (2).
To prove inequality (3), define the set

$$\mathcal{G}_\varepsilon = \left\{g \in \widetilde{H}(D) : \sup_{s\in K}|g(s) - f(s)| < \varepsilon\right\}.$$

Then the boundary $\partial\mathcal{G}_\varepsilon$ lies in the set

$$\left\{g \in \widetilde{H}(D) : \sup_{s\in K}|g(s) - f(s)| = \varepsilon\right\},$$

therefore, $\mathcal{G}_{\varepsilon_1} \cap \mathcal{G}_{\varepsilon_2} = \varnothing$ for positive $\varepsilon_1 \neq \varepsilon_2$. Hence it follows that $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta},F}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 4 and 6 ((i) and (iii)), it follows that

$$\lim_{N\to\infty} P_{N,F}(\mathcal{G}_\varepsilon) = P_{\underline{\zeta},F}(\mathcal{G}_\varepsilon) \tag{8}$$

for all but at most countably many $\varepsilon > 0$. Therefore, by (5), the right-hand side of (8) is strictly positive. The theorem is proved. □

Proof of Theorem 4. We will show that the hypothesis of the theorem $(F^{-1}\{p\}) \cap S^r \neq \varnothing$ for every polynomial $p = p(s)$ implies that of Theorem 3. Let G be an arbitrary non-empty open set of $\widetilde{H}(D)$. Then, by Lemma 1, there exists a polynomial $p = p(s)$ lying in G. Thus, $F^{-1}\{p\} \subset F^{-1}G$. Therefore, $(F^{-1}G) \cap S^r \supset (F^{-1}\{p\}) \cap S^r \neq \varnothing$. □

Proof of Theorem 5. It is not difficult to see that the support of the measure $P_{\underline{\zeta},F}$ is the set $F(S^r)$. Actually, let g be an arbitrary element of $F(S^r)$ and G be its any open neighborhood. Then $F^{-1}\{g\} \in S^r$, and lies in the open set $F^{-1}G$. Thus, by Lemma 5, $P_{\underline{\zeta}}(F^{-1}G) > 0$. Hence,

$$P_{\underline{\zeta},F}(G) = P_{\underline{\zeta}}F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Moreover,

$$P_{\underline{\zeta},F}(F(S^r)) = P_{\underline{\zeta}}F^{-1}(F(S^r)) = P_{\underline{\zeta}}(F^{-1}F(S^r)) = P_{\underline{\zeta}}(S^r) = 1.$$

Since g is an arbitrary element of $F(S^r)$, we have that the support of $P_{\underline{\zeta},F}$ is the set $F(S^r)$.

Let $\mathcal{G}_\varepsilon$ be the same set as in the proof of Theorem 3. Since $f(s) \in F(S^r)$, by the above remark, $P_{\underline{\zeta},F}(\mathcal{G}_\varepsilon) > 0$. Therefore, by Lemmas 4 and 6 ((i) and (ii)), we have

$$\lim_{N\to\infty} P_{N,F}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\zeta},F}(\mathcal{G}_\varepsilon) > 0,$$

and the definitions of $P_{N,F}$ and $\mathcal{G}_\varepsilon$ give inequality (2).

Inequality (3) is obtained in the same way as in the proof of Theorem 3. □

Proof of Theorem 6. By a proof of Theorem 5 and the inclusion $F(S^r) \supset \widetilde{H}_{c_1,\dots,c_m}(D)$, we have that the support of $P_{\underline{\zeta},F}$ contains the set $\widetilde{H}_{c_1,\dots,c_m}(D)$. Since the support is a closed set, hence, the support of $P_{\underline{\zeta},F}$ contains the closure of $\widetilde{H}_{c_1,\dots,c_m}(D)$.

We consider two cases.

(1) $m = 1$. Since the function $f(s) \neq c_1$ on K, the function $f_1(s) = f(s) - c_1 \neq 0$ on K. Therefore, the principal branch of logarithm $\log f(s)$ satisfies on K the hypotheses of Lemma 1. Thus, for every $\varepsilon_1 > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}|\log f_1(s) - p(s)| < \varepsilon_1.$$

Hence, after a corresponding choosing of $\varepsilon_1 = \varepsilon_1(\varepsilon)$ we find

$$\sup_{s\in K}\left|f_1(s) - \mathrm{e}^{p(s)}\right| = \sup_{s\in K}\left|\mathrm{e}^{\log f(s)} - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{9}$$

Obviously, $f_2(s) = c_1 + \mathrm{e}^{p(s)} \in \widetilde{H}_{c_1}(D)$. Therefore, by the above remark, $f_2(s)$ is an element of the support of the measure $P_{\underline{\zeta},F}$. Hence, putting

$$\hat{G}_\varepsilon = \left\{ g \in \widetilde{H}(D) : \sup_{s\in K}|g(s) - f_2(s)| < \frac{\varepsilon}{2} \right\},$$

we have

$$P_{\underline{\zeta},F}(\hat{G}_\varepsilon) > 0. \tag{10}$$

Moreover, for $g \in \hat{G}_\varepsilon$,

$$\sup_{s\in K}|g(s) - f(s)| \leqslant \sup_{s\in K}|g(s) - f_2(s)| + \sup_{s\in K}\left|f_1(s) - \mathrm{e}^{p(s)}\right| < \varepsilon.$$

Thus, $\hat{G}_\varepsilon \subset \mathcal{G}_\varepsilon$, where $\mathcal{G}_\varepsilon$ is the same as in the proof of Theorem 3. This, Lemmas 4 and 6 ((i) and (ii)), and (10) prove inequality (2). Inequality (3) is obtained analogically as in the proof of Theorem 3.

(2) Suppose that $m \geqslant 2$. Since $f(s) \in \widetilde{H}_{c_1,\dots,c_m}(D)$, we have that $f(s)$ is an element of the support of the measure $P_{\underline{\zeta},F}$. Therefore, $P_{\underline{\zeta},F}(\mathcal{G}_\varepsilon) > 0$, and it remains to apply Lemmas 4 and 6. □

5. Conclusions

By the Linnik-Ibragimov conjecture, see for example [10], all functions defined in some half-plane by Dirichlet series and having a natural growth of their analytic continuation are universal in the sense of approximating of analytic functions. Unfortunately, this conjecture is very general and, at this moment, the authors are able to consider the universality of some classes of Dirichlet series only. On the other hand, the universality, as a phenomenon of Dirichlet series, goes beyond the Linnik-Ibragimov conjecture. This is also confirmed by using of generalized shifts, in particular, involving the very mysterious sequence $\{\gamma_k\}$. The universality of certain compositions $F(\zeta(s;\mathfrak{a}))$ for some classes of continuous operators $F : \widetilde{H}^r(D) \to \widetilde{H}(D)$ obtained in the paper, shows that the class of universal functions is quite wide. For example, Theorem 6 implies the universality of $F(g_1, \dots, g_r) = \cos(g_1 + \cdots + g_r)$. Actually, if $f \in \widetilde{H}_{-1,1}(D)$, then the equation

$$\frac{\mathrm{e}^{ig} + \mathrm{e}^{-ig}}{2} = f$$

shows that $g \in S$. Hence, $F(g, 0, \dots, 0) = f$, thus, $f \in F(S^r)$, and $\widetilde{H}_{-1,1}(D) \subset F(S^r)$.

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