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Gytenis
LILEIKA

Weak approximations of CKLS model by discrete random variables

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mathematics (N 001)

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VILNIAUS UNIVERSITETAS

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CKLS modelio silpnosios aproksimacijos diskrečiaisiais atsitiktiniais dydžiais

DAKTARO DISERTACIJA

Gamtos mokslai,
matematika (N 001)

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Contents

List of Figures

List of Tables

Notation and Abbreviations

Notations	Descriptions
$\mathbb{N}$	The set of positive integers $\{1, 2, \ldots\}$.
$\mathbb{N}_0$	The set of nonnegative integers, $\mathbb{N} \bigcup \{0\}$.
$\mathbb{R}$	The set of real numbers $(-\infty, \infty)$.
$\mathbb{R}_+$	The set of positive real numbers $(0, \infty)$.
$\overline{\mathbb{R}}_+$	The set of nonnegative real numbers $[0, \infty)$.
$\mathbb{D}$	The domain of the solution of SDE. The domain of the solution of the CKLS equation is $\overline{\mathbb{R}}_+$.
$C^\infty(\mathbb{D})$	The set of infinitely differentiable functions $f : \mathbb{D} \to \mathbb{R}$.
$C_0^\infty(\mathbb{D})$	The set of functions $f : \mathbb{D} \to \mathbb{R}$ of class C^∞ with a compact support.
$C_{\mathrm{pol}}^\infty(\mathbb{D})$	The set of functions $f : \mathbb{D} \to \mathbb{R}$ of class C^∞ with all partial derivatives of polynomial growth.
$O(h^n)$	A function of polynomial growth with respect to h^n, i.e., we write $g(x, h) = O(h^n)$ if for some $C > 0$, $k \in \mathbb{N}$, and $h_0 > 0$, $\lvert g(x, h)\rvert \leqslant C(1 + \lvert x\rvert^k)h^n, \quad x \geqslant 0, \quad 0 < h \leqslant h_0$.
$\mathcal{O}(h^n)$	A function of polynomial growth with respect to h^n when the function g is expressed in terms of another function $f \in C_{\mathrm{pol}}^\infty(\mathbb{D})$ and the constants C, h_0, and k depend on a good sequence for f only.
$\lfloor x \rfloor$	The integer part of a number x.
Δ^h	The (equidistant) discretization of a time interval.
$\mathbb{E}X$	The mean of a random variable X.
$\mathcal{N}(a, \sigma^2)$	The normal distribution with mean a and variance σ^2.
B_t, $\tilde{B}_t$	Standard Brownian motions (Wiener processes).
$\hat{X}$	A discretization of a random variable X.

Notations	Descriptions
$f^{(i)}(z)$	The ith derivative of a function f of a real variable z.
$\hat{R}_p, \hat{Q}_p$	Polynomials of pth order.
SDE	Stochastic differential equation.
CIR	Cox–Ingersoll–Ross model.
CEV	Constant elasticity of variance model.
CKLS	Chan–Karolyi–Longstaff–Sanders model.

Chapter 1

Introduction

In this chapter, we present our research topic, aim and applied methods, novelty of main results, the list of published papers, and the list of conferences where our results were presented.

1.1 Research topic

We are interested in first- and second-order weak approximations for the Chan–Karolyi–Longstaff–Sanders (CKLS) model [7]

$$\mathrm{d}X_t = (\theta - \beta\, X_t)\,\mathrm{d}t + \sigma X_t^\gamma\,\mathrm{d}B_t, \quad X_0 = x \geq 0, \tag{1.1.1}$$

with parameters $(\theta, \beta, \sigma, \gamma) \in \overline{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R}_+ \times [1/2, 1)$, where $\mathbb{R}_+ := (0, \infty)$ and $\overline{\mathbb{R}}_+ := [0, \infty)$. In particular, when $\theta = 0$ and $\beta < 0$, we have the constant elasticity of variance (CEV) model [8]; when $\gamma = 1/2$ and $\beta \geq 0$, we have the well-known Cox–Ingersoll–Ross (CIR) model [9]; when $\gamma = 0$, it becomes the Vašíček model; when $\gamma = 0$ and $\theta = 0$, we have a geometric Brownian motion.

It is well known that equation (1.1.1) has a unique nonnegative solution, and if $x > 0$ and $\gamma > 1/2$, then $X_t > 0$ for all $t \geq 0$ almost surely [35].

1.2 Aim and difficulties

The aim of research was to construct simple and effective first- and second-order weak approximations for the solution of the CKLS model which would use only generation of discrete random variables at each approximation step. Also, it was important to provide proofs of accuracy order.

The main problem in developing numerical methods for such a diffusion equation/model is that the diffusion coefficient has unbounded derivatives near zero, and therefore standard methods (see, e.g., Milstein and Tretyakov [36]) are not applicable: discretization schemes that (explicitly or implicitly) involve the derivatives of the coefficients usually lose their accuracy near zero, especially, for large σ. This problem for the CIR processes was solved by modifying the scheme considered by switching near zero to another scheme, which (i) is sufficiently regular and (ii) enough accurate near zero; we refer, for example, to [2, 31, 33] and references therein.

Lan, Hu and Zhang studied the relation between CKLS model and CIR model in [22]. They proved that under a suitable transformation, any CKLS model of order $1/2 < \gamma < 1$ or $\gamma > 1$ corresponds to the CIR model under a new probability measure. Moreover, they get an explicit solution and the precise distribution of the CKLS model at any time t under the new probability measure. However, such a transformation cannot be applied to reducing weak approximations of CKLS processes to those of CIR processes.

1.3 Methods

Methods of calculus, stochastic calculus, probability theory, statistics, and functional analysis are applied in the thesis. Numerical experiments were simulated using the programming language C++. The figures were generated using computing environment Maple. The same software also was used for solving equalities and inequalities.

1.4 Actuality and novelty

Typically, a first-order approximation near zero is constructed by discrete random variables matching two or three moments with those of the solution. Our construction of the same order has an important change in this procedure: there is no need to know these moments.

Our construction method for second-order weak approximations is significantly different from that of the first-order approximation. Another novel feature of the same order weak approximations is that in our schemes, no switching between schemes near zero is used, in contrast to [2, 33]. This simplifies the implementation of approximations.

The same techniques that were used in this research can be used for constructing discretization schemes for other models with singular diffusion coefficients.

1.5 Main results

We managed to construct simple and effective first-order and second-order weak approximations for the solution of the CKLS model. These discretization schemes use only generation of discrete random variables at each approximation step. They are presented in the theorems below.[1]

Theorem 1.1. *Let*

$$D_t^x = D(x,t) := \begin{cases} x\mathrm{e}^{-\beta t} + \frac{\theta}{\beta}(1-\mathrm{e}^{-\beta t}), & \beta \neq 0, \\ x + \theta t, & \beta = 0, \end{cases} \tag{1.5.1}$$

and let the random variables $\hat{S}_h^x$ $(\hat{S}_h^0 = 0)$ *take the values*

$$\begin{cases} x_1 = x + x^{2\gamma-1}\sigma^2 h - \sqrt{(x^{2\gamma} + x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h} > 0, & x > 0, \\ x_2 = x + x^{2\gamma-1}\sigma^2 h + \sqrt{(x^{2\gamma} + x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h} > 0, & x > 0, \end{cases}$$

with probabilities

$$\mathbb{P}\{\hat{S}_h^x = x_{1,2}\} = p_{1,2} = \frac{x}{2\,x_{1,2}}, \quad x > 0.$$

Then the one-step approximation $\hat{X}_h^x$ *defined by the composition*

$$\hat{X}_h^x := D(\hat{S}_h^x, h),\ x \geq 0,\ h > 0,$$

defines a strongly potential first-order discretization scheme for the CKLS equation (1.1.1).

Theorem 1.2. *Let* $D_t^x = D(x,t)$ *be defined in* (1.5.1), *and let the random variables* $\hat{S}_h^x = \hat{S}(x,h)$ $(\hat{S}_h^0 = 0)$ *take the values*

$$\begin{aligned} x_{1,3} =& x + (A + \tfrac{3}{4})\sigma^2 h \mp \sqrt{\big(3x + (A+\tfrac{3}{4})^2\sigma^2 h\big)\sigma^2 h}, \\ x_2 =& x + A\,\sigma^2 h, \quad A \in [3/4, 3/2], \end{aligned}$$

with probabilities

$$\begin{aligned} p_1 &= \frac{m_1 x_2 x_3 - m_2(x_2 + x_3) + m_3}{x_1(x_3 - x_1)(x_2 - x_1)}, \\ p_2 &= \frac{m_2(x_1 + x_3) - m_1 x_1 x_3 - m_3}{x_2(x_2 - x_1)(x_3 - x_2)}, \\ p_3 &= \frac{m_1 x_1 x_2 - m_2(x_1 + x_2) + m_3}{x_3(x_3 - x_2)(x_3 - x_1)}. \end{aligned} \tag{1.5.2}$$

[1] For definitions, see Chapter 3.

Let the one-step approximations $\hat{X}^x_h$ be defined by the composition

$$\hat{X}^h(x,h) := \begin{cases} D\Big(\hat{S}\big(D(x,h/2),h\big),h/2\Big), & h>0, \\ x, & h=0, \end{cases} \tag{1.5.3}$$

then $\hat{X}^x_h$ defines a strongly potential second-order discretization scheme for the CIR equation.

We have also constructed a one-step approximation $\hat{S}^x_h$ taking four values that defines a potential third-order weak approximation of the stochastic part $\mathrm{d}S^x_t = \sigma\sqrt{S^x_t}\,\mathrm{d}B_t$ of the CIR equation.

Theorem 1.3. *Let $\hat{X}^x_h$ be the one-step discretization scheme defined by composition* (1.5.3), *where $D^x_t = D(x,t)$ is defined in* (1.5.1).
Let $\hat{S}^x_h$ take the values

$$\begin{aligned} x_{1,3} &= x + \tfrac{5}{2}\,x^{1/2}\sigma^2 h + \tfrac{15}{64}\,(\sigma^2 h)^2 \\ &\qquad \mp \sqrt{\big(3\,x^{3/2} + \tfrac{103}{16}\,x\sigma^2 h + \tfrac{75}{64}\,x^{1/2}(\sigma^2 h)^2 + \tfrac{225}{4096}\,(\sigma^2 h)^3\big)\sigma^2 h}, \\ x_2 &= x + \tfrac{11}{8}\,x^{1/2}\sigma^2 h + \tfrac{15}{64}\,(\sigma^2 h)^2 \end{aligned}$$

in the case $\gamma = 3/4$ or

$$\begin{aligned} x_{1,3} &= x + \tfrac{3}{2}\,x^{2/3}\sigma^2 h + \tfrac{485}{816}\,x^{1/3}(\sigma^2 h)^2 + \tfrac{1681}{22032}\,(\sigma^2 h)^3 \\ &\qquad \mp \Big((3\,x^{5/3} + \tfrac{2077}{612}\,x^{4/3}\sigma^2 h + \tfrac{125695}{66096}\,x(\sigma^2 h)^2 + \tfrac{1162907}{1997568}\,x^{2/3}(\sigma^2 h)^3 \\ &\qquad + \tfrac{815285}{8989056}\,x^{1/3}(\sigma^2 h)^4 + \tfrac{2825761}{485409024}(\sigma^2 h)^5)\sigma^2 h\Big)^{1/2}, \\ x_2 &= x + \tfrac{1}{4}\,x^{2/3}\sigma^2 h + \tfrac{5}{72}\,x^{1/3}(\sigma^2 h)^2 + \tfrac{1}{72}(\sigma^2 h)^3 \end{aligned}$$

in the case $\gamma = 5/6$ with probabilities p_1, p_2, and p_3 defined in (1.5.2) *($\hat{S}^0_h = 0$). Then $\hat{X}^x_h$ defines a strongly potential second-order discretization scheme for the CKLS equation with $\gamma = 3/4$ or $\gamma = 5/6$, respectively.*

1.6 Publications

1. G. Lileika and V. Mackevičius, *Weak approximation of CKLS and CEV processes by discrete random variables.* Lithuanian Mathematical Journal 60: 208–224 (2020).

2. G. Lileika and V. Mackevičius, *Second-order weak approximations of CKLS and CEV processes by discrete random variables.* Mathematics, 9(12), 1337 (2021).

1.7 Conferences

The results of the thesis-related studies were presented in the following conferences:

- 12th International Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius, Lithuania, 2018–07–03.

- 58th Conference of Lithuanian Mathematical Society (LMS), Vilnius, Lithuania, 2017–06–21.

- 59th Conference of LMS, Kaunas, Lithuania, 2018–06–19.

- 60th Conference of LMS, Vilnius, Lithuania, 2019–06–19.

- 61st Conference of LMS, Šiauliai, Lithuania, 2020–12–04.

- 62nd Conference of LMS, Vilnius, Lithuania, 2021–06–16.

1.8 Structure of the thesis

The thesis is organized as follows. In Chapter 2, we give an overview of related results obtained by other authors. Preliminaries and definitions are provided in Chapter 3. In Section 4.2, we discuss a general construction for potential first-order approximations of the stochastic part. In Section 4.3, we construct a strongly potential first-order weak approximation of the CKLS model, and in Sections 5.2 and 5.4, we construct a strongly potential second-order weak approximation of the CKLS model. We summarize the constructed algorithms of first- and second-order in Sections 4.4 and 5.5, respectively. In Sections 4.5 and 5.6, we illustrate the accuracy of the first- and second-order schemes by numerical simulation results. We provide conclusions of the thesis in Chapter 6, and in the Appendix (Chapter 7), we provide additional calculations.

1.9 Acknowledgments

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Chapter 2

A short historical overview

In this chapter, we discuss the importance of the CKLS model and various attempts to construct discretization schemes for the solution of the CKLS model, mostly the CIR model.

There are only few works where approximations for the CKLS model were considered, and they consider pathwise approximations only. The first and simplest method uses modifications of the Euler–Maruyama discretization scheme. In the literature, we can find many results and ad hoc methods on weak approximations for the CIR equation. We use some their ideas for construction of weak approximations of the CKLS equations. To the best of our knowledge, for the CKLS equations (with $\gamma > 1/2$), no weak approximations were constructed before.

In this chapter, $\sigma > 0$ is a constant.

2.1 The need to approximate the CKLS equation

Contradicting the intuition, an interest rate is assumed to be a constant in the Black–Scholes model and in many other models. In 1975, one of the first attempts to use a stochastic process modeling asset prices X_t and to make assumptions more realistic, was suggested by Cox [8]:

$$\mathrm{d}X_t = \beta\, X_t\, \mathrm{d}t + \sigma X_t^{\gamma}\, \mathrm{d}B_t, \quad X_0 = x \geq 0,$$

where B_t is a Brownian motion, $\beta \in \mathbb{R}_+$, $\gamma \geq 1/2$, and $\sigma \in \mathbb{R}_+$. The model is called the constant elasticity of variance (CEV) model, and it is capable of reproducing the volatility smile observed in the empirical data: the model incorporates a variance adjustment that causes the absolute level of the variance to decline as the stock price rises and to rise as the stock price declines.

In 1977, another mean-reverting process, where $\mathbb{E}X_t \to \theta$ as $t \to \infty$, was proposed by Vašíček [41] for interest rate modeling:

$$\mathrm{d}X_t = \beta\,(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$$

with β, θ, and $\sigma \in \mathbb{R}_+$. The solution of this equation is called an Ornstein– Uhlenbeck stochastic process. It is known in an explicit form, and this fact attracts researchers to use this model widely. but there is a serious drawback for interest rate modeling: it takes negative values with nonzero probability.

In 1985, John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross presented a square-root model, usually called the CIR model ((1.1.1) with $\gamma = 1/2$),

$$\mathrm{d}X_t = (\theta - \beta\,X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t, \quad X_0 = x \geq 0,$$

$\theta \in \overline{\mathbb{R}}_+$ and β, $\sigma \in \mathbb{R}_+$, which is widely used for modeling of interest rate or as an alternative to geometric Brownian motion that occurs in the Black–Scholes model of dynamic asset pricing in mathematical finance. In this model, the process never becomes negative if it starts from a nonnegative value (different boundary behavior at 0 depending on the values of θ and σ), and it has the main advantage of the Vašíček model that it is a mean-reverting. The process has a unique solution, which is not known in an explicit form.

In 1992, K. C. Chan, G. A. Karolyi, F. A. Longstaff, and A. B. Sanders suggested the generalization of the CIR equation for modeling the behavior of the instantaneous interest rate, the so-called the CKLS model [7]

$$\mathrm{d}X_t = (\theta - \beta\,X_t)\,\mathrm{d}t + \sigma X_t^{\gamma}\,\mathrm{d}B_t, \quad X_0 = x \geq 0,$$

where B is a Brownian motion, $\theta \in \overline{\mathbb{R}}_+$, $\beta \in \mathbb{R}_+$, $\gamma \geq 1/2$, and $\sigma \in \mathbb{R}_+$. Finally, in 1993, Heston [17] extends the CIR model to a stock price model with a second source of randomness and assuming that not only underlying asset, but also its volatility is controlled by the CIR process:

$$\begin{cases} \mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{Y_t}S_t\,\mathrm{d}\tilde{B}_t, \quad S_0 = s \geq 0, \\ \mathrm{d}Y_t = \beta(\theta - Y_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}B_t, \quad Y_0 = y \geq 0, \\ \mathrm{d}B_t\,\mathrm{d}\tilde{B}_t = \rho\,\mathrm{d}t. \end{cases}$$

where B and $\tilde{B}$ are (possibly, dependent, with correlation coefficient ρ) standard Brownian motions, with parameters θ, β, and $\sigma \in \mathbb{R}_+$. In his model, Heston introduces the dynamics of the underlying asset, the asymmetry and excess kurtosis, which are typically observed in financial assets returns. The model was an instant success. One of the main reasons was the fact that European vanilla call (and put) option price is known in the (quasi)closed form.

Applications of numerical methods for the Heston model possesses the same difficulties as approximations of the CIR equation have: the volatility in the Heston model is controlled by the CIR process. In the Heston model, square roots are used not only controlling the volatility but also in the expression of volatility itself.[1] For more information about problems and their solutions applying Heston model, see [3, 10, 14, 23–25, 39].

2.2 One-step approximations

Modifications of the Euler–Maruyama scheme

Having a fixed time interval $[0, T]$, consider an equidistant time discretization $\Delta^h = \{h_i = ih, i = 0, 1, \ldots, n := \lfloor T/h \rfloor, h \in (0, T]\}$, where $\lfloor a \rfloor$ is the integer part of a. The first and simplest idea is to use the Euler–Maruyama scheme for CKLS (1.1.1), where $\gamma \in [1/2, 1)$ and $\beta \geq 0$. The scheme has form

$$\hat{X}^n_{h_{i+1}} = \hat{X}^n_{h_i} + (\theta - \beta\,\hat{X}^n_{h_i})\frac{T}{n} + \sigma(\hat{X}^n_{h_i})^\gamma(B_{h_{i+1}} - B_{h_i})$$

[1] For more detail, see [26]

with $\hat{X}^n_{h_0} = x \geq 0$ and B_{h_i} is a value of Brownian motion at the time moment h_i. Unfortunately, although the process itself is guaranteed to be nonnegative, this discretization scheme has a nonzero probability of becoming negative in the next time step, regardless of the size of the time step. Practitioners solved this problem by either setting the process equal to 0 when it obtains a negative value or by reflecting it to the origin and starting again from this point. Such fixes are often referred to as absorption and reflection respectively, see e.g. Gatheral [15]. Lord, Koekkoek and Van Dijk [28] note that this terminology is somewhat at odds with the terminology used to classify the boundary behavior of stochastic processes, see Karlin and Taylor [20]. They think that in that respect the absorption fix is much more similar to reflection in the origin for a continuous stochastic process, whereas absorption as a boundary classification means that the process stays in the absorbed state for the rest of time.

Deelstra and Delbaen [11] (Glasserman [16] uses this scheme for the CIR process) proposed to modify this model for $\gamma = 1/2$ and suggested partial truncation

$$\hat{X}^n_{h_{i+1}} = \hat{X}^n_{h_i} + (\theta - \beta\,\hat{X}^n_{h_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{h_i} 1_{\hat{X}^n_{h_i}>0}}(B_{h_{i+1}} - B_{h_i}).$$

Lord et al. [28] proposed full truncation modification for $\gamma = 1/2$

$$\hat{X}^n_{h_{i+1}} = \hat{X}^n_{h_i} + (\theta - \beta\hat{X}^n_{h_i} 1_{\hat{X}^n_{h_i}>0})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{h_i} 1_{\hat{X}^n_{h_i}>0}}(B_{h_{i+1}} - B_{h_i}).$$

Diop [12] proposed another modification for $\gamma = 1/2$

$$\hat{X}^n_{h_{i+1}} = \left|\hat{X}^n_{h_i} + (\theta - \beta\,\hat{X}^n_{h_i})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{h_i}}(B_{h_{i+1}} - B_{h_i})\right|.$$

Bossy and Diop [5] proposed, and Berkaoui, Bossy, and Diop [4] analyzed

$$\hat{X}^n_{h_{i+1}} = \left|\hat{X}^n_{h_i} + (\theta - \beta\,\hat{X}^n_{h_i})\frac{T}{n} + \sigma(\hat{X}^n_{h_i})^{\gamma}(B_{h_{i+1}} - B_{h_i})\right|, \quad \gamma \in [1/2, 1).$$

Higham and Mao [18] suggested for $\gamma = 1/2$

$$\hat{X}^n_{h_{i+1}} = \hat{X}^n_{h_i} + (\theta - \beta\,\hat{X}^n_{h_i})\frac{T}{n} + \sigma\sqrt{\left|\hat{X}^n_{h_i}\right|}(B_{h_{i+1}} - B_{h_i}).$$

Lord et al. [28] unifies all of these Euler–Maruyama schemes in a single general framework:

$$\widetilde{X}^n_{h_{i+1}} = f_1\left(\widetilde{X}^n_{h_i}\right) - \left(f_2\left(\widetilde{X}^n_{h_i}\right) - \theta\right)\beta\,\frac{T}{n} + \sigma f_3\left(\widetilde{X}^n_{h_i}\right)^{\gamma}\left(B_{h_{i+1}} - B_{h_i}\right),$$
$$X^n_{h_{i+1}} = f_3\left(\widetilde{X}^n_{h_{i+1}}\right)$$

with $X^n_{h_0} = \widetilde{X}^n_{h_0}$ and the functions f_i, $i = 1, 2, 3$, satisfying:

- $f_i(x) = x$ for $x \geq 0$ and $i = 1, 2, 3$;
- $f_i(x) \geq 0$ for $x \in \mathbb{R}_+$ and $i = 1, 2, 3$.

The schemes considered so far then in the literature are summarized in Table 2.1 (for $\gamma = 1/2$). Lord et al. [28] also considered the behavior

Table 2.1: Overview of Euler schemes known in the literature.

Scheme	**Paper**	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption	Unknown	$1_{\{x>0\}}$	$1_{\{x>0\}}$	$1_{\{x>0\}}$
Reflection	Diop [12], Bossy and Diop [5], Berkaoui et al. [4]	$\lvert x\rvert$	$\lvert x\rvert$	$\lvert x\rvert$
Higham and Mao	Higham and Mao [18]	x	x	$\lvert x\rvert$
Partial truncation	Deelstra and Delbaen [11]	x	x	$1_{\{x>0\}}$
Full truncation	Lord, Koekkoek, and Van Dijk [28]	x	$1_{\{x>0\}}$	$1_{\{x>0\}}$

of schemes fixing negative variances. The origin of the true process is strongly reflecting: if it is obtained, in the sense that when the trajectory touches 0, it leaves the origin again immediately. They considered the case where an Euler–Maruyama discretisation causes the variance to go negative, $\hat{X}^n_{h_i} = -\delta < 0$, whereas the true process would stay positive and close to zero, $X^n_{h_i} = \epsilon \geq 0$. In Table 2.2 we provide results of their analysis: what would be the new starting point $f_1\left(\widetilde{X}^n_{h_i}\right)$, the effective variance (by this they mean the instantaneous variance of the stock price) $f_3\left(\widetilde{X}^n_{h_i}\right)$, and the drift for all fixes as well for the true process.

Table 2.2: Analysis of the dynamics when discretization equals $\hat{X}^n_{h_{i+1}} = -\delta < 0$ but $X^n_{h_{i+1}} = \epsilon \geq 0$.

Scheme	**New starting point**	**Effective variance**	**Drift**
True process	ϵ	ϵ	$\beta\,(\theta - \epsilon)$
Absorption	0	0	$\beta\,\theta$
Reflection	δ	δ	$\beta\,(\theta - \delta)$
Higham and Mao	-δ	δ	$\beta\,(\theta + \delta)$
Partial truncation	-δ	0	$\beta\,(\theta + \delta)$
Full truncation	-δ	0	$\beta\,\theta$

Brigo and Alfonsi [6] define, at least when the time step is small enough, for $\gamma = 1/2$,

$$\hat{X}^n_{h_{i+1}} = \hat{X}^n_{h_i} + (\theta - \frac{\sigma^2}{2} - \beta\,\hat{X}^n_{h_{i+1}})\frac{T}{n} + \sigma\sqrt{\hat{X}^n_{h_{i+1}}}(B_{h_{i+1}} - B_{h_i}).$$

If $\hat{X}^n_{h_i} \geq 0$ and $\frac{T}{n} \leq 1/\beta^-$, where $y^- = \max\{-y, 0\}$, they define

$$\hat{X}^n_{h_{i+1}} = \left(\left(\sqrt{\sigma^2\left(B_{h_{i+1}} - B_{h_i}\right)^2 + 4\left(\hat{X}^n_{h_i} + \left(\theta - \frac{\sigma^2}{2}\right)\frac{T}{n}\right)\left(1 + \beta\frac{T}{n}\right)}\right.\right.$$
$$\left.\left. + \sigma\left(B_{h_{i+1}} - B_{h_i}\right)\right) \Big/ 2\left(1 + \beta\frac{T}{n}\right)\right)^2.$$

Schemes employing the properties of variance

The Ninomiya and Victoir scheme [39] for the CIR equation described in [2] $\hat{X}^x_h = \varphi(x, h, \sqrt{h}N)$, where $N \sim \mathcal{N}(0,1)$, and

$$\varphi(x, h, \omega) = \mathrm{e}^{-\frac{\beta h}{2}}\left(\sqrt{(\theta - \sigma^2/4)\,\psi_\beta(h/2) + \mathrm{e}^{-\frac{\beta h}{2}}x} + \frac{\sigma}{2}\omega\right)^2 + \left(\theta - \sigma^2/4\right)\psi_\beta(h/2), \tag{2.2.1}$$

$$\psi_\beta(h) = \frac{1 - \mathrm{e}^{-\beta h}}{\beta}, \quad \beta \neq 0, \text{ and } \psi_0(h) = h, \quad \beta = 0.$$

Unfortunately, the scheme only solves the problem of negative values for $\sigma^2 \leq 4\,\theta$.

Alfonsi in [2, Theorem 2.8.] modifies the previous scheme and suggests

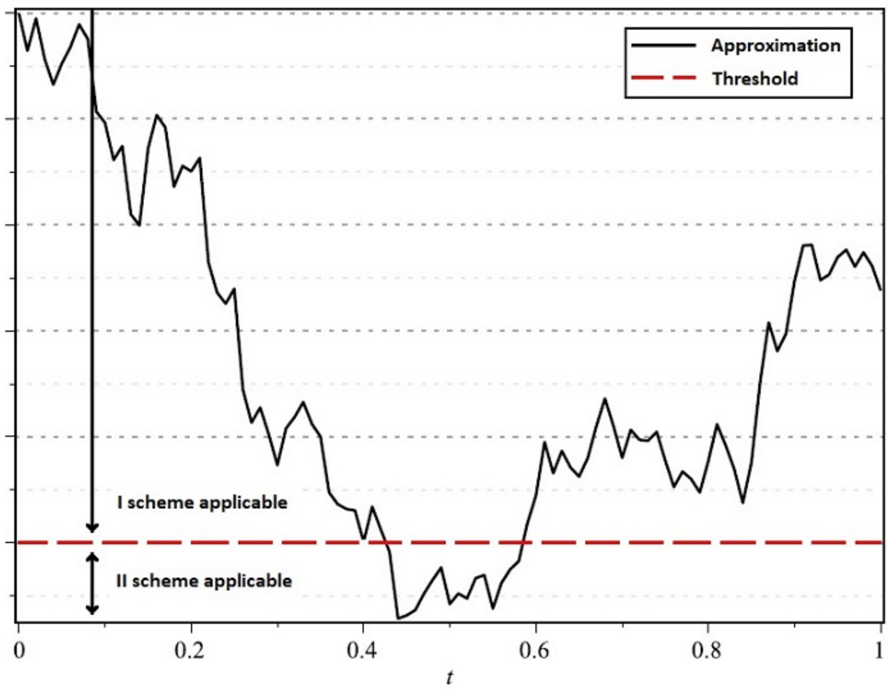

Figure 2.1: Scheme switching near zero.

a second-order weak approximation that is well defined without restriction on the parameters. Problem of unbounded derivatives near zero: discretization scheme loses its accuracy near zero and might obtain negative values, especially, for large σ, was solved by modifying the Ninomiya and Victoir scheme by switching near zero to another scheme, which (1) is sufficiently regular and (2) sufficiently accurate near zero. Visual representation of this idea is given in Figure 2.1. The algorithm of discretization scheme is presented in Figure 2.2, and the threshold $\mathbb{K}_2(h)$, three-valued random variable Y, and the first two moments $\tilde{u}_1(h,x)$, $\tilde{u}_2(h,x)$ of the CIR equation are defined as

$$\mathbb{K}_2(h) = \mathbf{1}_{\{\sigma^2>4\theta\}} \mathrm{e}^{\frac{\beta h}{2}} \Bigg(\left(\frac{\sigma^2}{4} - \theta \right) \psi_\beta(h/2) + \left[\sqrt{\mathrm{e}^{\frac{\beta h}{2}} \left[\left(\sigma^2/4 - \theta \right) \psi_\beta(h/2) \right] } + \frac{\sigma}{2} \sqrt{3\,h} \right]^2 \Bigg),$$

function CIR_02 (x):
if $(x \geq \mathbf{K}_2(t))$ $x \leftarrow \varphi(x, t, \sqrt{t}Y)$
else $\pi \leftarrow \frac{1-\sqrt{1-\tilde{u}_1(t,x)^2/\tilde{u}_2(t,x)}}{2}$ **if** $(U < \pi)$ $x \leftarrow \frac{\tilde{u}_1(t,x)}{2\pi}$
else $x \leftarrow \frac{\tilde{u}_1(t,x)}{2(1-\pi)}$

Figure 2.2: Algorithm computing the value at the next time-step of the 2nd-order scheme of the CIR with a timestep $t = h$, U (resp., Y) being sampled uniformly from [0, 1].

function CIR_03(x):
if $(x \geq \mathbf{K}_3(t))$ { $x \leftarrow \hat{X}_h^x$ }
else { $s \leftarrow \frac{\tilde{u}_3(t,x)-\tilde{u}_1(t,x)\tilde{u}_2(t,x)}{\tilde{u}_2(t,x)-\tilde{u}_1(t,x)^2}$, $p \leftarrow \frac{\tilde{u}_1(t,x)\tilde{u}_3(t,x)-\tilde{u}_2(t,x)^2}{\tilde{u}_2(t,x)-\tilde{u}_1(t,x)^2}$,
$\delta = \sqrt{s^2 - 4p}$, $\pi \leftarrow \frac{\tilde{u}_1-(s-\delta)/2}{\delta}$
if $(U < \pi)$ $x \leftarrow (s+\delta)/2$ **else** $x \leftarrow (s-\delta)/2$ }

Figure 2.3: Algorithm computing the 3rd-order scheme next value, starting from x with a timestep $t = h$. Here U is sampled uniformly from [0, 1].

$$\mathbb{P}(Y = \pm\sqrt{3}) = \tfrac{1}{6}, \quad \mathbb{P}(Y = 0) = \tfrac{2}{3},$$

$$\tilde{u}_1(h, x) = x\mathrm{e}^{-\beta h} + \theta\psi_\beta(h), \tag{2.2.2}$$

$$\tilde{u}_2(h, x) = \tilde{u}_1(h, x)^2 + \sigma\psi_\beta(h)[x\mathrm{e}^{-\beta h} + \theta\psi_\beta(h)/2], \tag{2.2.3}$$

and $\varphi(x, h, \sqrt{h}Y)$ is defined in (2.2.1).

Alfonsi [2, Theorem 3.7.] also suggests a third-order weak approximation that is well defined without restriction on the parameters. The algorithm of discretization scheme is presented in Figure 2.3, and the threshold $\mathbb{K}_3(h)$, four-valued random variable Y, discretization scheme $\hat{X}_h^{x,\beta=0}$, and the third moment $\tilde{u}_3(h, x)$ of the CIR equation are defined as

$$\mathbb{K}_3(h) = \Bigg[\mathbf{1}_{\{4\theta/3<\sigma^2<4\theta\}}\left(\sqrt{\frac{\sigma^2}{4}-\theta+\frac{\sigma}{\sqrt{2}}\sqrt{\theta-\frac{\sigma^2}{4}}}+\frac{\sigma}{2}\sqrt{3+\sqrt{6}}\right)^2$$
$$+\mathbf{1}_{\{\sigma^2\le 4\theta/3\}}\frac{\sigma}{\sqrt{2}}\sqrt{\theta-\frac{\sigma^2}{4}}$$
$$+\mathbf{1}_{\{4\theta<\sigma^2\}}\left[\frac{\sigma^2}{4}-\theta+\left(\sqrt{\frac{\sigma}{\sqrt{2}}\sqrt{\frac{\sigma^2}{4}-\theta}}+\frac{\sigma}{2}\sqrt{3+\sqrt{6}}\right)^2\right]\Bigg]$$
$$\times\,\psi_{-\beta}(h),$$

$$\mathbb{P}\left(Y=\pm\sqrt{3+\sqrt{6}}\right)=\tfrac{\sqrt{6}-2}{4\sqrt{6}},\quad \mathbb{P}\left(Y=\pm\sqrt{3-\sqrt{6}}\right)=\tfrac{1}{2}-\tfrac{\sqrt{6}-2}{4\sqrt{6}},$$

$$\hat{X}_h^x = \mathrm{e}^{-\beta h}\hat{X}^{x,\beta=0}_{\psi_{-\beta}(h)},$$

where for $\sigma^2 \le 4\theta$ ($\sigma^2 > 4\theta$),

$$\hat{X}_h^{x,\beta=0} = \begin{cases} \tilde{X}(\epsilon h, X_0^{\mathrm{CIR}}(h, X_1^{\mathrm{CIR}}(\sqrt{h}Y, x))) \\ (\text{resp. } \tilde{X}(\epsilon h, X_1^{\mathrm{CIR}}(\sqrt{h}Y, X_0^{\mathrm{CIR}}(h,x)))) \text{ if } \zeta = 1, \\ X_0^{\mathrm{CIR}}(h, \tilde{X}(\epsilon h, X_1^{\mathrm{CIR}}(\sqrt{h}Y, x))) \\ (\text{resp. } X_1^{\mathrm{CIR}}(\sqrt{h}Y, \tilde{X}(\epsilon h, X_0^{\mathrm{CIR}}(h,x)))) \text{ if } \zeta = 2, \\ X_0^{\mathrm{CIR}}(h, X_1^{\mathrm{CIR}}(\sqrt{h}Y, \tilde{X}(h,x))) \\ (\text{resp. } X_1^{\mathrm{CIR}}(\sqrt{h}Y, X_0^{\mathrm{CIR}}(\epsilon h, \tilde{X}(\epsilon h,x)))) \text{ if } \zeta = 3, \end{cases}$$

$$\tilde{X}(h,x) = x + h\frac{\sigma}{\sqrt{2}}\sqrt{\left|\theta-\frac{\sigma^2}{4}\right|},$$

$$X_0^{\mathrm{CIR}}(h,x) = x\mathrm{e}^{-\beta h} + \left(\theta - \frac{\sigma^2}{4}\right)\psi_\beta(h),$$

$$X_1^{\mathrm{CIR}}(h,x) = \left(\left(\sqrt{x}-\frac{\sigma}{2}h\right)^+\right)^2,$$

$$\tilde{u}_3(h,x) = \tilde{u}_1(h,x)\tilde{u}_2(h,x)$$
$$+\sigma^2\psi_\beta(h)[2\,x^2\mathrm{e}^{-2\beta h}+\psi_\beta(h)(\theta+\frac{\sigma^2}{2})(3\,x\mathrm{e}^{-\beta h}+\theta\psi_\beta(h))],$$

where ϵ and ζ are respectively independent uniform random variables on $\{-1,1\}$ and $\{1,2,3\}$. The first two moments $\tilde{u}_1(h,x)$ and $\tilde{u}_2(h,x)$ are defined in (2.2.2) and (2.2.3), respectively.

Split-step schemes

Mackevičius [33, Theorem 3] proves that the Alfonsi scheme [2, Proposition 2.7], used for approximation of the CIR equation ((1.1.1), $\gamma = 1/2$) in a neighborhood of 0 is actually a first-order approximation on $[0, +\infty)$ for $f \in C^\infty_{\mathrm{pol}}(\mathbb{R})$ without scheme switch:

$$\hat{X}^h(x,h) = D\Big(\hat{S}(x,h), h\Big), \quad h > 0, \quad \hat{X}^h(x,0) = x, \quad a := \sigma^2,$$

where

$$D^x_t = D(x,t) := \begin{cases} x\mathrm{e}^{-\beta t} + \frac{\theta}{\beta}(1 - \mathrm{e}^{-\beta t}), & \beta \neq 0, \\ x + \theta t, & \beta = 0, \end{cases} \tag{2.2.4}$$

$$\hat{S}^x_h = \hat{S}(x,h), \quad \hat{S}^0_h := 0, \tag{2.2.5}$$

$$x_{1,2} = x_{1,2}(x,h) = x + ah \mp \sqrt{(x+ah)ah} > 0, \quad x \geq 0,$$
$$\mathbb{P}\{\hat{S}^x_h = x_{1,2}\} = p_{1,2} = \frac{x}{2\,x_{1,2}}, \quad x > 0.$$

In the same paper, Mackevičius (see Theorem 4) suggested a second-order weak approximation for the CIR equation:

$$\hat{X}^h(x,h) = D\Big(\hat{S}\big(D(x,h/2), h\big), h/2\Big),\ h > 0,\ \hat{X}^h(x,0) = x,\ a := \sigma^2,$$

where D^x_t is defined in (2.2.4), and the values of the random variables $\hat{S}^x_h = \hat{S}(x,h)$ are defined as follows.
If $x \geq 2ah$, then

$$x_{1,2} := x + \frac{s \mp \sqrt{\Delta}}{2}, \quad x_0 = x, \quad p_{1,2} := \frac{2xah}{\sqrt{\Delta}(\sqrt{\Delta} \mp s)},$$

where

$$p_0 = 1 - p_1 - p_2, \quad s = \frac{3\,ah}{2}, \quad \Delta = \frac{21}{4}(ah)^2 + 12xah.$$

If $0 \leq x < 2ah$, then

$$x_{1,2} = \frac{s \mp \sqrt{\Delta}}{2}, \quad x_0 = 0, \quad p_{1,2} = \frac{x(2\,ah + 2\,x - s \mp \sqrt{\Delta})}{\sqrt{\Delta}(\sqrt{\Delta} \mp s)},$$

where

$$p_0 = 1 - p_1 - p_2, \quad s = \frac{4\,x^2 + 9\,xah + 3\,(ah)^2}{2\,x + ah},$$

$$\Delta = \frac{ah\,(16\,x^3 + 33\,x^2ah + 18\,x(ah)^2 + 3\,(ah)^3)}{(2\,x + ah)^2}.$$

If $0 \le x < 2ah$, and we substitute expressions s and Δ into $x_{1,2}$ and $p_{1,2}$, then we get

$$\begin{aligned} x_{1,2} = & \frac{3\,(ah)^2 + 9\,ahx + 4x^2}{2\,(ah + 2\,x)} \\ & \mp \frac{\sqrt{ah(3\,(ah)^3 + 18\,x(ah)^2 + 33\,x^2ah + 16\,x^3)}}{2\,(ah + 2\,x)}, \end{aligned}$$

$$x_0 = 0, \quad p_1 = \frac{x(x_2 - (ah + x))}{x_1(x_2 - x_1)}, p_2 = \frac{x((ah + x) - x_1)}{x_2(x_2 - x_1)},$$
$$p_0 = 1 - p_1 - p_2.$$

If we accept that $\mathbb{E}(\hat{S}_h^x)^i - \mathbb{E}(S_h^x)^i = O(h^3)$, $i = 1, 2, 3, 4$, then the expressions of $x_{1,2}$ can be significantly simplified:

$$x_{1,2} = x + \frac{7}{4}\,ah \mp \frac{1}{4}\sqrt{(16\,x + 17\,ah)\,ah}.$$

Chapter 3

Preliminaries

In this chapter, we provide all definitions and describe techniques used to construct discretization schemes.

3.1 Preliminaries and definitions

In this section, we give some definitions for the general one-dimensional stochastic differential equation

$$X_t^x = x + \int_0^t b(X_s^x)\,\mathrm{d}s + \int_0^t \tilde{\sigma}(X_s^x)\,\mathrm{d}B_s, \quad t \geq 0, \quad x \in \mathbb{D} \subset \mathbb{R}. \quad (3.1.1)$$

To avoid ambiguity, we indicate functions with the supplementary symbol ˜ if the same letter is used for a function and a constant, for example, we denote by $\tilde{\sigma}$ the diffusion coefficient in the general equation (3.1.1) and by σ the constant in the CKLS equation (1.1.1).

We assume that the equation has a unique weak solution X_t^x such that $\mathbb{P}(X_t^x \in \mathbb{D}, t \geq 0) = 1$ for all $x \in \mathbb{D}$. For example, for Eq. (1.1.1), we can take $\mathbb{D} = \overline{\mathbb{R}}_+$.

Having a fixed time interval $[0, T]$, consider an equidistant time discretization $\Delta^h = \{ih, i = 0, 1, \ldots, \lfloor T/h \rfloor, h \in (0, T]\}$, where $\lfloor a \rfloor$ is the integer part of a. By a discretization scheme of Eq. (3.1.1) we mean a family of discrete-time homogeneous Markov chains $\hat{X}^h = \{\hat{X}^h(x, t), x \in \mathbb{D}, t \in \Delta^h\}$ with initial values $\hat{X}^h(x, 0) = x$ and one-step transition probabilities $p^h(x, \mathrm{d}z)$, $x \in \mathbb{D}$. For convenience, we only consider steps

$h = T/n$, $n \in \mathbb{N}$. We shortly write $\hat{X}_t^x$ or $\hat{X}(x,t)$ instead of $\hat{X}^h(x,t)$. Note that because of the Markovity, a one-step approximation $\hat{X}_h^x$ of the scheme completely defines the distribution of the whole discretization scheme $\hat{X}_t^x$, so that we only need to construct the former.

We denote by $C^\infty(\mathbb{D})$ the space of C^∞ functions $f : \mathbb{D} \to \mathbb{R}$, by $C_0^\infty(\mathbb{D})$ the functions $f \in C^\infty(\mathbb{D})$ with compact support in $\mathbb{D}$, and by $C_{\mathrm{pol}}^\infty(\mathbb{D})$ the functions $f \in C^\infty(\mathbb{D})$ such that

$$|f^{(n)}(x)| \le C_n(1 + |x|^{k_n}), x \in \mathbb{D}, n \in \mathbb{N}_0 := \{0, 1, 2, \dots\},$$

for some sequence $(C_n, k_n) \in \mathbb{R}_+ \times \mathbb{N}_0$. Following Alfonsi [2], we say that such a sequence $\{(C_n, k_n), n \in \mathbb{N}_0\}$ is a *good* sequence for f.

We will write $g(x,h) = O(h^n)$ if for some $C > 0$, $k \in \mathbb{N}$, and $h_0 > 0$,

$$|g(x,h)| \le C(1 + |x|^k)h^n, \quad x \ge 0, \quad 0 < h \le h_0.$$

If, in particular, the function g is expressed in terms of another function $f \in \mathrm{C}_{\mathrm{pol}}^\infty(\mathbb{R})$ and the constants C, k, and h_0 only depend on a *good* sequence for f, then we will write, instead, $g(x,h) = \mathcal{O}(h^n)$.

Definition 3.1. A discretization scheme $\hat{X}^h$ is a weak νth-order approximation for the solution $(X_t^x, t \in [0,T])$ of Eq. (3.1.1) if for every $f \in C_0^\infty(\mathbb{D})$, there exists $C > 0$ such that

$$|\mathbb{E}f(X_T^x) - \mathbb{E}f(\hat{X}_T^x)| \le Ch^\nu,\ h > 0.$$

Definition 3.2. Let $Lf = bf' + \frac{1}{2}\tilde{\sigma}^2 f''$ be the generator of the solution of Eq. (3.1.1). Suppose $Lf \in C_{\mathrm{pol}}^\infty(\mathbb{D})$ for all $f \in C_{\mathrm{pol}}^\infty(\mathbb{D})$, that is, $b, \tilde{\sigma}^2 \in C_{\mathrm{pol}}^\infty(\mathbb{D})$. The νth-order remainder of a discretization scheme $\hat{X}_t^x$ for X_t^x is the operator $R_\nu^h : C_{\mathrm{pol}}^\infty(\mathbb{D}) \to C(\mathbb{D})$ defined by

$$R_\nu^h f(x) := \mathbb{E}f(\hat{X}_h^x) - \left[f(x) + \sum_{k=1}^{\nu} \frac{L^k f(x)}{k!} h^k\right],\ x \in \mathbb{D},\ h > 0. \quad (3.1.2)$$

A discretization scheme $\hat{X}_t^x$ is a local νth-order weak approximation of Eq. (3.1.1) if

$$R_\nu^h f(x) = O(h^{\nu+1}),\ h \to 0,$$

for all $f \in C_{\mathrm{pol}}^\infty(\mathbb{D})$ and $x \in \mathbb{D}$.

Remark 3.1. Iterating the Dynkin formula

$$\mathbb{E}f(X_h^x) = f(x) + \int_0^h \mathbb{E}Lf(X_s^x)\mathrm{d}s,$$

we have

$$\mathbb{E}f(X_h^x) = f(x) + \sum_{k=1}^{\nu} \frac{L^k f(x)}{k!} h^k$$
$$+ \int_0^h \int_0^{s_1} \cdots \int_0^{s_\nu} \mathbb{E}L^{\nu+1} f(X_{s_{\nu+1}}^x)\mathrm{d}s_{\nu+1} \cdots \mathrm{d}s_2 \mathrm{d}s_1, \tag{3.1.3}$$

which motivates Definition 3.2: If $L^{\nu+1}f$ behaves "well" (e.g., $b, \tilde{\sigma}^2, f \in C_0^\infty(\mathbb{D})$, and $\mathbb{E}L^{\nu+1}f$ is bounded), then for the "one-step" νth-order weak approximation scheme $\hat{X}_h^x$, we have

$$|\mathbb{E}f(X_h^x) - \mathbb{E}f(\hat{X}_h^x)| = O(h^{\nu+1}),\ h \to 0. \tag{3.1.4}$$

We may expect that in "good" cases, a local νth-order weak discretization scheme is a νth-order (global) approximation. Rigorous statements require certain uniformity of (3.1.4) with respect to f and regularity of L.

Definition 3.3. A discretization scheme $\hat{X}_t^x$ is a potential νth-order weak approximation for Eq. (3.1.1) if for every $f \in C_{\mathrm{pol}}^\infty(\mathbb{D})$,

$$|R_\nu^h f(x)| = \mathcal{O}(h^{\nu+1}).$$

Definition 3.4. A discretization scheme $\hat{X}_t^x = \hat{X}^h(x,t)$, $h > 0$, has uniformly bounded moments if there exists $h_0 > 0$ such that

$$\sup_{0<h\le h_0} \sup_{t\in\Delta^h} \mathbb{E}(|\hat{X}^h(x,t)|^n) < +\infty,\ n \in \mathbb{N},\ x \in \mathbb{D}.$$

We say that a potential νth-order weak approximation is a *strongly* potential νth-order weak approximation if it has uniformly bounded moments.

Remark 3.2. Typically, a strongly potential νth-order discretization is a νth-order weak approximation in the sense of Definition 3.1. At least, we do not know any counterexample. A rigorous proof for the CIR equation is given by Alfonsi [1] (see also [34]).

3.2 Split-step technique for SDE

The general idea of a split-step (also called a splitting-step) method, which is based on the idea in [40] (see also [13, 37]), is described by Moro and Schurz [38]. Their approach heavily relies on the exploitation of the specific structure of the original system (3.1.1), which allows a natural splitting into explicitly known parts of the underlying dynamics with well-known boundary behavior and the remaining parts to be treated numerically with naturally incorporated boundary behavior (or even without restrictions). The method is especially efficient at approximating solution of SDE (3.1.1), where boundary conditions are naturally inherent. Suppose that the equation has form

$$\mathrm{d}X_t^x = [\alpha(X_t^x) + \tilde{\beta}(X_t^x)]\mathrm{d}t + \tilde{\sigma}(X_t^x)\,\mathrm{d}B_t. \tag{3.2.1}$$

They decompose the above equation into the two equations

$$\mathrm{d}S_t^x = \alpha(S_t^x)\mathrm{d}t + \tilde{\sigma}(S_t^x)\,\mathrm{d}B_t, \tag{3.2.2}$$

$$\mathrm{d}D_t^x = \tilde{\beta}(D_t^x)\mathrm{d}t, \tag{3.2.3}$$

where the splitting is done assuming that one knows the exact strong solution for S_t^x or the conditional probability $\mathbb{P}(S_t^x)$. They suggest approximating the solution of (3.2.1) by a stochastic process $\hat{X}_t^x$ along time intervals $[t,\, t+h]$ for each h using two-step algorithm:

1. Knowing the value of x they obtain an intermediate value S_h^x through the exact integration of (3.2.2).

2. Then S_h^x is used as the initial condition for (3.2.3) which is now integrated using any converging deterministic numerical algorithm to get $D_h^{S_h^x}$ (at least of deterministic order 1). Then $\hat{X}_{t+h}^x = D_h^{S_h^x}$.

The advantage of this split-step technique for SDE subject to boundary conditions is that if (3.2.2) is simple enough and we know the solution S_h^x of (3.2.2), then the stochastic part of the problem can be handled correctly.

A different kind of splitting technique has been suggested by Higham et al. [19] and reviewed in [38]. Their algorithm, called the split-step Euler method, is related to another subclass of splitting of SDE and their resulting split-step algorithm is of lower order 0.5 of mean square convergence, which is restricted by their use of the (drift-implicit) backward Euler method. Their method indirectly refers to the splitting

$$\begin{aligned} \mathrm{d}D_t^x &= [\alpha(D_t^x) + \tilde{\beta}(D_t^x)]\mathrm{d}t, \\ \mathrm{d}S_t^x &= \tilde{\sigma}(S_t^x)\,\mathrm{d}B_t, \end{aligned}$$

where both equations for D and S are numerically integrated in a separated fashion.

3.3 Split-step technique for the CKLS model

We use approach described in [38] and decompose Eq. (1.1.1) into equations

$$\mathrm{d}D_t^x = (\theta - \beta D_t^x)\mathrm{d}t,\ \ D_0^x = x \geq 0,$$

and

$$\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma\,\mathrm{d}B_t, \quad S_0^x = x \geq 0, \tag{3.3.1}$$

and assign names the deterministic part and the stochastic part, respectively, with some loss of accuracy.

The solution of the deterministic part is positive for all $(x,t) \in \overline{\mathbb{R}}_+ \times (0,T]$, namely:

$$D_t^x = D(x,t) = \begin{cases} x\mathrm{e}^{-\beta t} + \frac{\theta}{\beta}(1 - \mathrm{e}^{-\beta t}), & \beta \neq 0, \\ x + \theta t, & \beta = 0. \end{cases} \tag{3.3.2}$$

The solution of the stochastic part is not explicitly known. The following theorem allows us to reduce the construction of a weak second-order approximation to that of the stochastic part. Let $\hat{S}_t^x = \hat{S}(x,t)$ be a discretization scheme for the stochastic part (3.3.1).

Theorem 3.3. (see [2, Thm. 1.17]) *Let* $\hat{S}_t^x$ *be a potential first-order weak approximation of the stochastic part* (3.3.1) *of Eq.* (1.1.1). *Then the* split-step *composition*

$$\hat{X}_h^x := D(\hat{S}_h^x, h), \ x \geq 0, \ h > 0, \tag{3.3.3}$$

defines a potential first-order weak approximation of Eq. (1.1.1).

Theorem 3.4. (see [2, Thm. 1.17]) *Let* $\hat{S}_t^x$ *be a potential second-order weak approximation of the stochastic part* (3.3.1) *of Eq.* (1.1.1). *Then the (split-step) composition*

$$\hat{X}^h(x,h) := \begin{cases} D\Big(\hat{S}\big(D(x,h/2),h\big),h/2\Big), & h > 0, \\ x, & h = 0, \end{cases} \tag{3.3.4}$$

defines a potential second-order weak approximation of Eq. (1.1.1).

Corollary 3.5. *If* $\hat{S}_t^x$ *is a strongly potential first-order weak approximation of the stochastic part* (3.3.1) *of Eq.* (1.1.1), *then composition* (3.3.3) *is a strongly potential first-order weak approximation of Eq.* (1.1.1).

Corollary 3.6. *If* $\hat{S}_t^x$ *is a strongly potential second-order weak approximation of the stochastic part* (3.3.1) *of Eq.* (1.1.1), *then composition* (3.3.4) *is a strongly potential second-order weak approximation of Eq.* (1.1.1).

The theorems and corollaries allow us to restrict ourselves, without loss of generality, on the strongly potential first-order and second-order weak approximations of the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma \, \mathrm{d}B_t$ of Eq. (1.1.1).

3.4 Moment matching technique for the CKLS model

Let $\hat{S}_h^x$ be any discretization scheme. Using Taylor's formula for $f \in C^4(\mathbb{R})$, we have

$$\mathbb{E}f(\hat{S}_h^x) = f(x) + f'(x)\mathbb{E}(\hat{S}_h^x - x) + \frac{f''(x)}{2}\mathbb{E}(\hat{S}_h^x - x)^2$$

$$+\frac{f'''(x)}{6}\mathbb{E}(\hat{S}_h^x - x)^3 + \frac{1}{6}\mathbb{E}\int_x^{\hat{S}_h^x} f^{(4)}(s)(\hat{S}_h^x - s)^3 \mathrm{d}s.$$

It is worth noting that further technical calculations were mainly made by using MAPLE software.

For brevity, we denote $z := ah = \sigma^2 h$. Since the generator of the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma \mathrm{d}B_t$ is (see Definition (3.2))

$$L_0 f(x) = \frac{1}{2}\, a x^{2\gamma} f''(x),$$

where the subscript 0 indicates that $b = 0$, the first-order remainder for $\hat{S}_h^x$ is

$$\begin{aligned} R_1^h f(x) &= \mathbb{E} f(\hat{S}_h^x) - \Big[f(x) + L_0 f(x) h \Big] \\ &= f'(x)\mathbb{E}(\hat{S}_h^x - x) + \frac{f''(x)}{2}[\mathbb{E}(\hat{S}_h^x - x)^2 - x^{2\gamma} z] \qquad (3.4.1) \\ &\quad + \frac{f'''(x)}{6}\mathbb{E}(\hat{S}_h^x - x)^3 + r_1(x,h), \quad x \geq 0, \quad h > 0, \end{aligned}$$

where

$$\begin{aligned} |r_1(x,h)| &= \frac{1}{6}\left|\mathbb{E}\int_x^{\hat{S}_h^x} f^{(4)}(s)(\hat{S}_h^x - s)^3 \mathrm{d}s\right| \\ &\leq \frac{1}{24}\mathbb{E}\left[\max_{0 \leq s \leq \hat{S}_h^x} |f^{(4)}(s)|(\hat{S}_h^x - x)^4\right]. \end{aligned}$$

By the above expression of the remainder $R_1^h f(x)$ the discretization scheme $\hat{S}_h^x$ is a potential first-order approximation of the stochastic part (3.3.1) if

$$\mathbb{E}(\hat{S}_h^x - x) = O(h^2),\ x \geq 0, \qquad (3.4.2)$$

$$\mathbb{E}(\hat{S}_h^x - x)^2 = x^{2\gamma} z + O(h^2),\ x \geq 0, \qquad (3.4.3)$$

$$|\mathbb{E}(\hat{S}_h^x - x)^3| = O(h^2),\ x \geq 0, \qquad (3.4.4)$$

$$\mathbb{E}\Big[\max_{0 \leq s \leq \hat{S}_h^x} |f^{(4)}(s)|(\hat{S}_h^x - x)^4\Big] = \mathcal{O}(h^2). \qquad (3.4.5)$$

We easily convert conditions (3.4.2)–(3.4.3) for the central moments of $\hat{S}_h^x$ into conditions for the noncentral moments:

$$\mathbb{E}(\hat{S}_h^x)^i = {}_1\hat{m}_i + O(h^2),\ i = 1, 2, \tag{3.4.6}$$

where the "moments" (further we call them approximate moments; to indicate the accuracy of an approximate moment, we will write a subscript k: $\mathbb{E}(\hat{S}_h^x)^i = {}_k\hat{m}_i + O(h^{k+1}),\ k, i \in \mathbb{N}$) ${}_1\hat{m}_i = {}_1\hat{m}_i(x, h),\ x \geq 0, h > 0$, $i = 1, 2$, are defined as

$$\begin{aligned} {}_1\hat{m}_1 =& x, \\ {}_1\hat{m}_2 =& x^{2\gamma} z + x^2. \end{aligned} \tag{3.4.7}$$

Using Taylor's formula for $f \in C^6(\mathbb{R})$, we get

$$\begin{aligned} \mathbb{E}f(\hat{S}_h^x) = f(x) + f'(x)\mathbb{E}(\hat{S}_h^x - x) + \frac{f''(x)}{2}\mathbb{E}(\hat{S}_h^x - x)^2 \\ + \frac{f'''(x)}{6}\mathbb{E}(\hat{S}_h^x - x)^3 + \frac{f^{(4)}(x)}{4!}\mathbb{E}(\hat{S}_h^x - x)^4 \\ + \frac{f^{(5)}(x)}{5!}\mathbb{E}(\hat{S}_h^x - x)^5 + \frac{1}{5!}\mathbb{E}\int_x^{\hat{S}_h^x} f^{(6)}(s)(\hat{S}_h^x - s)^5 \mathrm{d}s. \end{aligned}$$

Since the square of the generator of stochastic part (3.3.1) is (see Definition 3.2)

$$\begin{aligned} L_0^2 f(x) = \tfrac{1}{2}\, a x^{2\gamma} \left(\tfrac{1}{2}\, a x^{2\gamma} f''(x)\right)'' = \tfrac{1}{4}\, a^2 x^{2\gamma} \left(2\,\gamma x^{2\gamma-1} f''(x) + x^{2\gamma} f'''(x)\right)' \\ = \gamma(\gamma - \tfrac{1}{2})\, a^2 x^{4\gamma-2} f''(x) + \gamma a^2 x^{4\gamma-1} f'''(x) + \tfrac{1}{4}\, a^2 x^{4\gamma} f^{(4)}(x), \end{aligned}$$

the second-order remainder for $\hat{S}_h^x$ is

$$\begin{aligned} R_2^h f(x) = \mathbb{E}f(\hat{S}_h^x) - \left[f(x) + L_0 f(x) h + L_0^2 f(x) \frac{h^2}{2} \right] \\ = f'(x)\mathbb{E}(\hat{S}_h^x - x) \\ + \frac{f''(x)}{2}[\mathbb{E}(\hat{S}_h^x - x)^2 - (1 + \gamma(\gamma - \frac{1}{2})\, x^{2(\gamma-1)} z)\, x^{2\gamma} z] \\ + \frac{f'''(x)}{6}[\mathbb{E}(\hat{S}_h^x - x)^3 - 3\,\gamma x^{4\gamma-1} z^2] \\ + \frac{f^{(4)}(x)}{4!}[\mathbb{E}(\hat{S}_h^x - x)^4 - 3\, x^{4\gamma} z^2] \end{aligned}$$

$$+\frac{f^{(5)}(x)}{5!}\mathbb{E}(\hat{S}_h^x - x)^5 + r_2(x,h),\ x \geq 0, \quad h > 0,$$

where

$$\begin{aligned} |r_2(x,h)| &= \frac{1}{5!}\left|\mathbb{E}\int_x^{\hat{S}_h^x} f^{(6)}(s)(\hat{S}_h^x - s)^5 \mathrm{d}s\right| \\ &\leq \frac{1}{6!}\mathbb{E}\left[\max_{0\leq s\leq \hat{S}_h^x} |f^{(6)}(s)|(\hat{S}_h^x - x)^6\right]. \end{aligned}$$

By the above expression of the remainder $R_2^h f(x)$ the discretization scheme $\hat{S}_h^x$ is a potential second-order approximation of the stochastic part (3.3.1) if

$$\mathbb{E}(\hat{S}_h^x - x) = O(h^3),\ x \geq 0, \tag{3.4.8}$$

$$\mathbb{E}(\hat{S}_h^x - x)^2 = (1 + \gamma(\gamma - \frac{1}{2})x^{2(\gamma-1)}z)x^{2\gamma}z + O(h^3),\ x \geq 0, \tag{3.4.9}$$

$$\mathbb{E}(\hat{S}_h^x - x)^3 = 3\gamma x^{4\gamma-1}z^2 + O(h^3),\ x \geq 0, \tag{3.4.10}$$

$$\mathbb{E}(\hat{S}_h^x - x)^4 = 3x^{4\gamma}z^2 + O(h^3),\ x \geq 0, \tag{3.4.11}$$

$$|\mathbb{E}(\hat{S}_h^x - x)^5| = O(h^3),\ x \geq 0, \tag{3.4.12}$$

$$\mathbb{E}\Big[\max_{0\leq s\leq \hat{S}_h^x} |f^{(6)}(s)|(\hat{S}_h^x - x)^6\Big] = \mathcal{O}(h^3). \tag{3.4.13}$$

Initially, for constructing our approximations, instead of (3.4.13), we will require a slightly weaker condition

$$\mathbb{E}(\hat{S}_h^x - x)^6 = O(h^3). \tag{3.4.13a}$$

Later we will see that, actually, all our approximations satisfy the required stronger condition (3.4.13).

We convert conditions (3.4.8)–(3.4.12) and (3.4.13a) for the central moments of $\hat{S}_h^x$ into conditions for the noncentral moments:

$$\mathbb{E}(\hat{S}_h^x)^i = {}_2\hat{m}_i + O(h^3),\ i = 1, 2, \ldots, 6, \tag{3.4.14}$$

where the approximate moments ${}_2\hat{m}_i = {}_2\hat{m}_i(x,h),\ x \geq 0, h > 0,\ i = 1,2,\ldots,6$, are defined as

$$
\begin{aligned}
{}_2\hat{m}_1 =& x,\\
{}_2\hat{m}_2 =& \gamma(\gamma - \tfrac{1}{2})x^{2(2\gamma-1)}z^2 + x^{2\gamma}z + x^2,\\
{}_2\hat{m}_3 =& \tfrac{3}{2}\gamma(1+2\gamma)x^{4\gamma-1}z^2 + 3x^{1+2\gamma}z + x^3,\\
{}_2\hat{m}_4 =& 3(1+\gamma)(1+2\gamma)x^{4\gamma}z^2 + 6x^{2(1+\gamma)}z + x^4,\\
{}_2\hat{m}_5 =& 5(3+2\gamma)(1+\gamma)x^{1+4\gamma}z^2 + 10x^{3+2\gamma}z + x^5,\\
{}_2\hat{m}_6 =& \tfrac{15}{2}(2+\gamma)(3+2\gamma)x^{2(1+2\gamma)}z^2 + 15x^{2(2+1\gamma)}z + x^6.
\end{aligned}
\tag{3.4.15}
$$

Using Taylor's formula for $f \in C^8(\mathbb{R})$, we get

$$
\begin{aligned}
\mathbb{E}f(\hat{S}_h^x) = f(x) &+ f'(x)\mathbb{E}(\hat{S}_h^x - x) + \frac{f''(x)}{2}\mathbb{E}(\hat{S}_h^x - x)^2\\
&+ \frac{f'''(x)}{6}\mathbb{E}(\hat{S}_h^x - x)^3 + \frac{f^{(4)}(x)}{4!}\mathbb{E}(\hat{S}_h^x - x)^4\\
&+ \frac{f^{(5)}(x)}{5!}\mathbb{E}(\hat{S}_h^x - x)^5 + \frac{f^{(6)}(x)}{6!}\mathbb{E}(\hat{S}_h^x - x)^6\\
&+ \frac{f^{(7)}(x)}{7!}\mathbb{E}(\hat{S}_h^x - x)^7 + \frac{1}{7!}\mathbb{E}\int_x^{\hat{S}_h^x} f^{(8)}(s)(\hat{S}_h^x - s)^7 \mathrm{d}s.
\end{aligned}
$$

Since the third power of the generator of stochastic part (3.3.1) is (see Definition 3.2)

$$
\begin{aligned}
L_0^3 f(x) = L_0 L_0^2 f(x) &= \tfrac{1}{2}\, a x^{2\gamma} \left(\tfrac{1}{2}\, a x^{2\gamma} \left(\tfrac{1}{2}\, a x^{2\gamma} f''(x)\right)''\right)''\\
&= \tfrac{1}{2}\,\gamma a^3 x^{2\gamma} \left((\gamma - \tfrac{1}{2})\, x^{4\gamma-2} f''(x) + x^{4\gamma-1} f'''(x)\right)''\\
&\quad + \tfrac{1}{8}\, a^3 x^{2\gamma} \left(x^{4\gamma} f^{(4)}(x)\right)''\\
&= \tfrac{1}{2}\,\gamma a^3 x^{6\,\gamma-3} \left((2\,\gamma-1)^2\, f''(x) + \tfrac{1}{2}\ (10\,\gamma-3)\, x f'''(x)\right)\\
&\quad + \tfrac{1}{2}\, a^3 x^{2\gamma} \left(2\,\gamma x^{4\,\gamma-1} f^{(4)}(x) + \tfrac{1}{4}\, x^{4\,\gamma} f^{(5)}(x)\right)'\\
&= \tfrac{1}{2}\,\gamma(4\gamma-3)(2\gamma-1)^2\, a^3 x^{2(3\gamma-2)} f''(x)\\
&\quad + 2\gamma(3\gamma-1)(2\gamma-1)\, a^3 x^{3(2\gamma-1)} f'''(x)\\
&\quad + \tfrac{1}{4}\,\gamma(26\gamma-7)\, a^3 x^{2(3\gamma-1)} f^{(4)}(x) + \tfrac{3}{2}\,\gamma\, a^3 x^{6\gamma-1} f^{(5)}(x)\\
&\quad + \tfrac{1}{8}\, a^3 x^{6\gamma} f^{(6)}(x),
\end{aligned}
$$

the third-order remainder for $\hat{S}_h^x$ is

$$\begin{aligned}
R_3^h f(x) &= \mathbb{E} f(\hat{S}_h^x) - \left[f(x) + L_0 f(x) h + L_0^2 f(x) \frac{h^2}{2} + L_0^3 f(x) \frac{h^3}{6} \right] \\
&= f'(x) \mathbb{E}(\hat{S}_h^x - x) \\
&\quad + \frac{f''(x)}{2} [\mathbb{E}(\hat{S}_h^x - x)^2 - (1 + \gamma(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z \\
&\quad \times (1 + \tfrac{8}{3}(\gamma - \tfrac{3}{4})(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z)) x^{2\gamma} z] \\
&\quad + \frac{f'''(x)}{6} [\mathbb{E}(\hat{S}_h^x - x)^3 \\
&\quad - 3\gamma(1 + 4(\gamma - \tfrac{1}{3})(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z) x^{4\gamma-1} z^2] \\
&\quad + \frac{f^{(4)}(x)}{4!} [\mathbb{E}(\hat{S}_h^x - x)^4 - (3 + \gamma(26\gamma - 7) x^{2(\gamma-1)} z) x^{4\gamma} z^2] \\
&\quad + \frac{f^{(5)}(x)}{5!} [\mathbb{E}(\hat{S}_h^x - x)^5 - 30\gamma x^{6\gamma-1} z^3] \\
&\quad + \frac{f^{(6)}(x)}{6!} [\mathbb{E}(\hat{S}_h^x - x)^6 - 15 x^{6\gamma} z^3] \\
&\quad + \frac{f^{(7)}(x)}{7!} \mathbb{E}(\hat{S}_h^x - x)^7 + r_3(x,h), \ x \geq 0, \quad h > 0,
\end{aligned}$$

where

$$\begin{aligned}
|r_3(x,h)| &= \frac{1}{7!} \left| \mathbb{E} \int_x^{\hat{S}_h^x} f^{(8)}(s)(\hat{S}_h^x - s)^7 \mathrm{d}s \right| \\
&\leq \frac{1}{8!} \mathbb{E} \left[\max_{0 \leq s \leq \hat{S}_h^x} |f^{(8)}(s)| (\hat{S}_h^x - x)^8 \right].
\end{aligned}$$

By the above expression of the remainder $R_3^h f(x)$ the discretization scheme $\hat{S}_h^x$ is a potential third-order approximation of the stochastic part (3.3.1) if

$$\mathbb{E}(\hat{S}_h^x - x) = O(h^4), \ x \geq 0, \tag{3.4.16}$$

$$\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^2 &= (1 + \gamma(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z \\
&\quad \times (1 + \tfrac{8}{3}(\gamma - \tfrac{3}{4})(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z)) x^{2\gamma} z \\
&\quad + O(h^4), \ x \geq 0,
\end{aligned} \tag{3.4.17}$$

$$\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^3 &= 3\gamma(1 + 4(\gamma - \tfrac{1}{3})(\gamma - \tfrac{1}{2}) x^{2(\gamma-1)} z) x^{4\gamma-1} z^2 \\
&\quad + O(h^4), \ x \geq 0,
\end{aligned} \tag{3.4.18}$$

$$\mathbb{E}(\hat{S}_h^x - x)^4 = (3 + \gamma(26\gamma - 7)x^{2(\gamma-1)}z)x^{4\gamma}z^2 + O(h^4),\ x \geq 0, \quad (3.4.19)$$

$$\mathbb{E}(\hat{S}_h^x - x)^5 = 30\gamma x^{6\gamma-1}z^3 + O(h^4),\ x \geq 0, \quad (3.4.20)$$

$$\mathbb{E}(\hat{S}_h^x - x)^6 = 15x^{6\gamma}z^3 + O(h^4),\ x \geq 0, \quad (3.4.21)$$

$$|\mathbb{E}(\hat{S}_h^x - x)^7| = O(h^4),\ x \geq 0, \quad (3.4.22)$$

$$\mathbb{E}\Big[\max_{0 \leq s \leq \hat{S}_h^x} |f^{(8)}(s)|(\hat{S}_h^x - x)^8\Big] = \mathcal{O}(h^4). \quad (3.4.23)$$

We convert conditions (3.4.16)–(3.4.21) for the central moments of $\hat{S}_h^x$ into conditions for the noncentral moments:

$$\mathbb{E}(\hat{S}_h^x)^i = {}_3\hat{m}_i + O(h^4),\ i = 1, 2, \ldots, 6, \quad (3.4.24)$$

where the approximate moments ${}_3\hat{m}_i = {}_3\hat{m}_i(x, h),\ x \geq 0, h > 0,\ i = 1, 2, \ldots, 6$, are defined as

$$\begin{aligned}
{}_3\hat{m}_1 &= x,\\
{}_3\hat{m}_2 &= \tfrac{8}{3}\gamma(\gamma - \tfrac{1}{2})^2(\gamma - \tfrac{3}{4})x^{2(3\gamma-2)}z^3\\
&\quad + \gamma(\gamma - \tfrac{1}{2})x^{2(2\gamma-1)}z^2 + x^{2\gamma}z + x^2,\\
{}_3\hat{m}_3 &= \gamma(1 + 2\gamma)(\gamma - \tfrac{1}{2})(4\gamma - 1)x^{3(2\gamma-1)}z^3\\
&\quad + \tfrac{3}{2}\gamma(1 + 2\gamma)x^{4\gamma-1}z^2 + 3x^{1+2\gamma}z + x^3,\\
{}_3\hat{m}_4 &= 2\gamma(4\gamma - 1)(1 + 2\gamma)(1 + \gamma)x^{2(3\gamma-1)}z^3\\
&\quad + 3(1 + \gamma)(1 + 2\gamma)x^{4\gamma}z^2 + 6x^{2(1+\gamma)}z + x^4,\\
{}_3\hat{m}_5 &= \tfrac{10}{3}\gamma(3 + 2\gamma)(1 + 4\gamma)(1 + \gamma)x^{6\gamma-1}z^3\\
&\quad + 5(3 + 2\gamma)(1 + \gamma)x^{1+4\gamma}z^2 + 10x^{3+2\gamma}z + x^5,\\
{}_3\hat{m}_6 &= \tfrac{5}{2}(3 + 2\gamma)(1 + 2\gamma)(1 + 4\gamma)(2 + \gamma)x^{6\gamma}z^3\\
&\quad + \tfrac{15}{2}(2 + \gamma)(3 + 2\gamma)x^{2(1+2\gamma)}z^2 + 15x^{2(2+1\gamma)}z + x^6.
\end{aligned} \quad (3.4.25)$$

Remark 3.7. We noticed expressing $R_1^h f(x)$, $R_2^h f(x)$, and $R_3^h f(x)$ that (3.1.2) can be used finding $\mathbb{E}(S_h^x)^i$, $i \in \mathbb{N}_0$, with desired accuracy for more general b and $\tilde{\sigma}$. Indeed, let us find an expression of ${}_3\hat{m}_2$ such that

$$\mathbb{E}(S_h^x)^2 - {}_3\hat{m}_2 = O(h^4). \quad (3.4.26)$$

From $R_3^h f(x)$ we see that to eliminate the member with $f''(x)/2$ and to reach the desired accuracy, we need to force a coefficient at $f''(x)/2$ to be equal $O(h^4)$. Since we are looking for $\mathbb{E}f(S_h^x)$, it is obvious that the coefficient at $f''(x)/2$ equals $O(h^4)$ if

$$\mathbb{E}(\hat{S}_h^x - x)^2 - \mathbb{E}(S_h^x - x)^2 = O(h^4).$$

We use this remark to find ${}_3\hat{m}_2$. Since the first, second, and third powers of the generator of the stochastic part of (3.1.1) are (see Definition 3.2)(the subscript 0 indicates that $b = 0$)

$$\begin{aligned}
L_0 f &= \tfrac{1}{2}\,\tilde\sigma^2 f'',\\
L_0^2 f &= \tfrac{1}{2}\,\tilde\sigma^2\left(\tfrac{1}{2}\,\tilde\sigma^2 f''\right)'' = \tfrac{1}{2}\,\tilde\sigma^2\left(\tfrac{1}{2}\,\tilde\sigma^2 f''' + \tilde\sigma\tilde\sigma' f''\right)'\\
&= \tfrac{1}{4}\,\tilde\sigma^4 f^{(4)} + \tilde\sigma^3\tilde\sigma' f''' + \tfrac{1}{2}\,\tilde\sigma^2\left(\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) f'',\\
L_0^3 f &= L_0 L_0^2 f = \tfrac{1}{2}\,\tilde\sigma^2\left(\tfrac{1}{4}\,\tilde\sigma^4 f^{(4)} + \tilde\sigma^3\tilde\sigma' f''' + \tfrac{1}{2}\,\tilde\sigma^2\left(\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) f''\right)''\\
&= \tfrac{1}{8}\,\tilde\sigma^2\left(\tilde\sigma\left(\tilde\sigma^3 f^{(5)} + 8\,\tilde\sigma^2\tilde\sigma' f^{(4)} + 2\,\tilde\sigma\left(3\,\tilde\sigma\tilde\sigma'' + 7\,(\tilde\sigma')^2\right) f'''\right)\right)'\\
&\quad + \tfrac{1}{4}\,\tilde\sigma^2\left(\tilde\sigma\left(5\,\tilde\sigma\tilde\sigma'\tilde\sigma'' + \tilde\sigma^2\tilde\sigma''' + 2\,\left(\tilde\sigma'\right)^3\right) f''\right)'\\
&= \tfrac{1}{8}\,\tilde\sigma^6 f^{(6)} + \tfrac{3}{2}\,\tilde\sigma^5\tilde\sigma' f^{(5)} + \tfrac{1}{4}\,\tilde\sigma^4\left(7\,\tilde\sigma\tilde\sigma'' + 19\,\left(\tilde\sigma'\right)^2\right) f^{(4)}\\
&\quad + \tilde\sigma^3\left(\tilde\sigma^2\tilde\sigma''' + 7\,\tilde\sigma\tilde\sigma'\tilde\sigma'' + 4\,\left(\tilde\sigma'\right)^3\right) f'''\\
&\quad + \tfrac{1}{4}\,\tilde\sigma^4\left(\tilde\sigma\tilde\sigma^{(4)} + 5\left(\tilde\sigma''\right)^2 + 8\,\tilde\sigma'\tilde\sigma'''\right) f''\\
&\quad + \tfrac{1}{2}\,\tilde\sigma^2\left(\tilde\sigma'\right)^2\left(8\,\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) f'',
\end{aligned}$$

we collect the coefficients of f'' in these expressions and sum them:

$$\begin{aligned}
\mathbb{E}(S_h^x - x)^2 &= 2\sum_{k=1}^{3}\frac{(\text{coefficient at } f'' \text{ in } L_0^k f)}{k!}h^k + O(h^4)\\
&= \tilde\sigma^2 h + \tfrac{1}{2}\,\tilde\sigma^2\left(\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) h^2\\
&\quad + \tfrac{1}{12}\,\tilde\sigma^4\left(\tilde\sigma\tilde\sigma^{(4)} + 5\,\left(\tilde\sigma''\right)^2 + 8\,\tilde\sigma'\tilde\sigma'''\right) h^3\\
&\quad + \tfrac{1}{6}\,\tilde\sigma^2\left(\tilde\sigma'\right)^2\left(8\,\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) h^3 + O(h^4).
\end{aligned}$$

We know that

$$\mathbb{E}(S_h^x)^2 = \mathbb{E}(S_h^x - x)^2 + (\mathbb{E}(S_h^x))^2.$$

From this and the expression of $\mathbb{E}(S_h^x - x)^2$ we get

$${}_3\hat{m}_2 = x^2 + \tilde\sigma^2 h + \tfrac{1}{2}\,\tilde\sigma^2\left(\tilde\sigma\tilde\sigma'' + \left(\tilde\sigma'\right)^2\right) h^2$$

$$+\tfrac{1}{12}\,\tilde\sigma^4\left(\tilde\sigma\tilde\sigma^{(4)}+5\,\left(\tilde\sigma''\right)^2+8\,\tilde\sigma'\tilde\sigma'''\right)h^3$$
$$+\tfrac{1}{6}\,\tilde\sigma^2\left(\tilde\sigma'\right)^2\left(8\,\tilde\sigma\tilde\sigma''+\left(\tilde\sigma'\right)^2\right)h^3.$$

This means that we can calculate and use the moments of the stochastic part (3.3.1) from (3.4.25) with accuracy $O(z^4)$; in other words, using these expressions, we only lose the members having the multiplier h of degree 4 or higher.

Example 3.8. The approximate moments of $\mathrm{d}S_t^x=\sigma(S_t^x)^{1/2}\mathrm{d}B_t$ are:

$$\begin{aligned}
{}_3\hat m_1 &= x,\\
{}_3\hat m_2 &= xz+x^2,\\
{}_3\hat m_3 &= \tfrac{3}{2}xz^2+3x^2z+x^3,\\
{}_3\hat m_4 &= 3xz^3+9x^2z^2+6x^3z+x^4,\\
{}_3\hat m_5 &= 30x^2z^3+30x^3z^2+10x^4z+x^5,\\
{}_3\hat m_6 &= 150x^3z^3+75x^4z^2+15x^5z+x^6,
\end{aligned}$$

where $x\geq 0$, $z>0$.

Example 3.9. The approximate moments of $\mathrm{d}S_t^x=\sigma(S_t^x)^{2/3}\mathrm{d}B_t$ are:

$$\begin{aligned}
{}_3\hat m_1 &= x,\\
{}_3\hat m_2 &= -\tfrac{1}{243}z^3+\tfrac{1}{9}\,x^{2/3}z^2+x^{4/3}z+x^2,\\
{}_3\hat m_3 &= \tfrac{35}{81}\,xz^3+\tfrac{7}{3}\,x^{5/3}z^2+3\,x^{7/3}z+x^3,\\
{}_3\hat m_4 &= \tfrac{700}{81}\,x^2z^3+\tfrac{35}{3}\,x^{8/3}z^2+6\,x^{10/3}z+x^4,\\
{}_3\hat m_5 &= \tfrac{14300}{243}\,x^3z^3+\tfrac{325}{9}\,x^{11/3}z^2+10\,x^{13/3}z+x^5,\\
{}_3\hat m_6 &= \tfrac{20020}{81}\,x^4z^3+\tfrac{260}{3}\,x^{14/3}z^2+15\,x^{16/3}z+x^6,
\end{aligned}$$

where $x\geq 0$, $z>0$.

Chapter 4

First-order approximation

In this chapter, we construct a first-order discretization scheme for the solution of the CKLS model. In the first section of the chapter, we discuss all techniques used to find the first-order approximation. In Section 4.2, we find a general construction for a potential first-order discretization scheme of the stochastic part. Section 4.3 is dedicated to finding a discretization scheme for the CKLS equation. In Section 4.4, we give an algorithm for simulations, and finally, in Section 4.5, we provide numerical simulations illustrating the accuracy of CKLS approximations.

4.1 A first-order approximation

We construct our scheme applying methods described in Section 3.3. First, we split the CKLS model into the stochastic and deterministic parts. The solution of $\mathrm{d}D_t^x = (\theta - \beta D_t^x)\mathrm{d}t$, $D_0^x = x \geq 0$, is easy to find (see (3.3.2)), and then we focus on constructing a first-order discretization scheme for the solution of the stochastic part $\mathrm{d}S_t^x = \tilde{\sigma}(S_t^x)\,\mathrm{d}B_t$, $S_0^x = x \geq 0$. For this, we use approximate moments (see Remark 3.7) instead of using exact ones. We apply the found construction to the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma\,\mathrm{d}B_t$, $S_0^x = x \geq 0$, and we prove that constructed potential approximation is, in fact, a strongly potential approximation of the stochastic part of the CKLS equation. Theorem 3.3 allows us to merge split-step parts of equation (1.1.1) using composition (3.3.3), and in this way, we get a strongly potential first-order approximation of the CKLS equation.

4.2 A potential first-order approximation of the stochastic part

The generator of the stochastic part $\mathrm{d}S_t^x = \tilde{\sigma}(S_t^x)\mathrm{d}B_t$ (the stochastic part of (3.1.1)) for $f \in C^2(\overline{\mathbb{R}}_+)$ is

$$L_0 f(x) = \frac{1}{2}\tilde{\sigma}^2(x) f''(x),$$

(see Definition 3.2). If $f \in C^4(\overline{\mathbb{R}}_+)$ and $\mathbb{E}(S_h^x)^2$ is finite, then similarly to (3.4.1), we get that the discretization scheme $\hat{S}_h^x$ is a potential first-order approximation of the stochastic part (3.3.1) if

$$\mathbb{E}(\hat{S}_h^x - x) = 0,\ x \geq 0, \tag{4.2.1}$$

$$\mathbb{E}(\hat{S}_h^x - x)^2 = \tilde{\sigma}^2(x)h + O(h^2),\ x \geq 0, \tag{4.2.2}$$

$$|\mathbb{E}(\hat{S}_h^x - x)^3| = O(h^2),\ x \geq 0, \tag{4.2.3}$$

$$\mathbb{E}\Big[\max_{0 \leq s \leq \hat{S}_h^x} |f^{(4)}(s)|(\hat{S}_h^x - x)^4\Big] = \mathcal{O}(h^2). \tag{4.2.4}$$

It is obvious that

$$\mathbb{E}(\hat{S}_h^x - x) = \mathbb{E}(S_h^x - x) = 0.$$

From Remark 3.7 we know that condition (4.2.2) in fact means that

$$\mathbb{E}(\hat{S}_h^x - x)^2 - \mathbb{E}(S_h^x - x)^2 = O(h^2).$$

Using the equality

$$\mathbb{E}X^2 = \mathbb{E}(X - \mathbb{E}X)^2 + (\mathbb{E}X)^2$$

for any random variable X, we get the second approximate moment of S_h^x:

$$\mathbb{E}(S_h^x)^2 = \tilde{\sigma}^2(x)h + (\mathbb{E}(S_h^x))^2 + O(h^2).$$

Denote $\hat{m}_1 := {}_1\hat{m}_1 = \mathbb{E}(S_h^x) = x$ and $\hat{m}_2 := {}_1\hat{m}_2 = \tilde{\sigma}^2(x)h + \hat{m}_1^2$ and construct $\hat{S}_h^x$. Clearly, for $x = 0$, we can trivially take $\hat{S}_h^0 = 0$ for all

$h > 0$. For $x > 0$, let us first look for $\hat{S}_h^x$ satisfying (4.2.1) and (4.2.2) and taking two positive values x_1 and x_2 with probabilities p_1 and p_2, respectively, that is,

$$p_1 + p_2 = 1, \tag{4.2.5}$$

$$x_1 p_1 + x_2 p_2 = \hat{m}_1, \tag{4.2.6}$$

$$x_1^2 p_1 + x_2^2 p_2 = \hat{m}_2. \tag{4.2.7}$$

Denote, for short, $p = p_2$. Then from (4.2.5)–(4.2.7) we have

$$x_1 \frac{1-p}{\hat{m}_1} + x_2 \frac{p}{\hat{m}_1} = 1, \tag{4.2.8}$$

$$x_1^2 (1-p) + x_2^2 p = \hat{m}_2. \tag{4.2.9}$$

Consider $c = \frac{x_2 p}{\hat{m}_1} \in (0,1)$ as a parameter. Then from (4.2.8) we have $x_1 = \frac{\hat{m}_1(1-c)}{1-p}$ and $x_2 = \frac{\hat{m}_1 c}{p}$, and from (4.2.9) we get the following quadratic equation for p:

$$\hat{m}_2\, p^2 + \left(\hat{m}_1^2\,(1-2c) - \hat{m}_2\right) p + \hat{m}_1^2 c^2 = 0.$$

Since $\hat{m}_1^2 < \hat{m}_2 = \tilde{\sigma}^2(x)\, h + \hat{m}_1^2$, the discriminant of this equation

$$\left(\hat{m}_1^2\,(1-2c) - \hat{m}_2\right)^2 - 4\,\hat{m}_2\,\hat{m}_1^2 c^2 = \left(\hat{m}_2 - \hat{m}_1^2\right)\left(\hat{m}_2 - \hat{m}_1^2\,(1-2c)^2\right) > 0$$

as $(1-2c)^2 < 1$ for $c \in (0,1)$. So the quadratic equation has two real roots

$$\frac{\hat{m}_2 + (2c-1)\hat{m}_1^2 \mp \sqrt{(\hat{m}_2 - \hat{m}_1^2)\big(\hat{m}_2 - \hat{m}_1^2(2c-1)^2\big)}}{2\,\hat{m}_2},$$

which are both positive because their sum $\frac{\hat{m}_1^2}{\hat{m}_2}(2c-1) + 1$ and product $\frac{\hat{m}_1^2}{\hat{m}_2} c^2$ are clearly positive. Denote

$$\Delta := \sqrt{(\hat{m}_2 - \hat{m}_1^2)\big(\hat{m}_2 - \hat{m}_1^2(2c-1)^2\big)}.$$

Let us consider, say,

$$p_2 = p = \frac{\hat{m}_2 + (2c-1)\,\hat{m}_1^2 - \Delta}{2\,\hat{m}_2}. \tag{4.2.10}$$

Then

$$p_1 = 1 - p = \frac{\hat{m}_2 - (2\,c-1)\,\hat{m}_1^2 + \Delta}{2\,\hat{m}_2}, \tag{4.2.11}$$

$$x_1 = \frac{\hat{m}_1(1-c)}{p_1} = \frac{\hat{m}_2 - (2\,c-1)\,\hat{m}_1^2 - \Delta}{2\,\hat{m}_1\,(1-c)}, \tag{4.2.12}$$

$$x_2 = \frac{\hat{m}_1 c}{p_2} = \frac{\hat{m}_2 + (2\,c-1)\,\hat{m}_1^2 + \Delta}{2\,\hat{m}_1 c}. \tag{4.2.13}$$

Obviously, p_1, x_2, and x_1 are also positive. Let us now calculate the central moment $\mathbb{E}(\hat{S}_h^x - x)^3$, where $\hat{S}_h^x$ takes the values $x_{1,2}$ with probabilities $p_{1,2}$ defined in (4.2.10)–(4.2.13):

$$\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^3 &= \mathbb{E}(\hat{S}_h^x - \hat{m}_1)^3 \\
&= \mathbb{E}(\hat{S}_h^x)^3 - 3\,\mathbb{E}(\hat{S}_h^x)^2\hat{m}_1 + 3\,\mathbb{E}(\hat{S}_h^x)\,\hat{m}_1^2 - \hat{m}_1^3 \\
&= x_1^3\,p_1 + x_2^3\,p_2 - 3\,(x_1^2\,p_1 + x_2^2\,p_2)\,\hat{m}_1 + 2\,\hat{m}_1^3 \qquad (4.2.14)\\
&= \frac{\left(\hat{m}_2 - \hat{m}_1^2\right)^2 + \left(\hat{m}_2 - \hat{m}_1^2\right)(1-2\,c)\,\Delta}{2\,\hat{m}_1\,c\,(1-c)} \\
&= \frac{\tilde{\sigma}^4(x)\,h^2 + \tilde{\sigma}^3(x)\,(1-2\,c)\;h\sqrt{\left(\tilde{\sigma}^2(x)\,h + 4\,x^2 c\,(1-c)\,\right)h}}{2\,x\,c\,(1-c)}.
\end{aligned}$$

The second summand in the numerator of the last expression of (4.2.14) is of order $O(h^{3/2})$. To ensure (4.2.3), we have to eliminate it by taking $c = 1/2$. In such a case, from (4.2.10)–(4.2.13) we get the following solution of system (4.2.5)–(4.2.7):

$$x_{1,2} = \frac{\hat{m}_2}{\hat{m}_1} \mp \sqrt{\frac{\hat{m}_2(\hat{m}_2 - \hat{m}_1^2)}{\hat{m}_1^2}} \tag{4.2.15}$$

$$p_{1,2} = \frac{\hat{m}_1}{2\,x_{1,2}} = \frac{x}{2\,x_{1,2}}, \quad x > 0. \tag{4.2.16}$$

Note that the constructed potential first-order two-valued approximation of the stochastic part does not require knowing the exact finite moment $\mathbb{E}(S_h^x)^2$.

4.3 A strongly potential first-order approximation of the CKLS equation

From Section 4.2 we get the first-order potential one-step approximation of the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma \,\mathrm{d}B_t$:

$$\begin{cases} x_1 = x + x^{2\gamma-1}\sigma^2 h - \sqrt{(x^{2\gamma} + x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h} > 0, & x > 0, \\ x_2 = x + x^{2\gamma-1}\sigma^2 h + \sqrt{(x^{2\gamma} + x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h} > 0, & x > 0, \end{cases} \tag{4.3.1}$$

$$\mathbb{P}\{\hat{S}_h^x = x_{1,2}\} = p_{1,2} = \frac{x}{2\,x_{1,2}}, \quad x > 0. \tag{4.3.2}$$

Remark 4.1. In the case of the CIR equation the two-valued discretization scheme of the stochastic part defined by (4.3.1)–(4.3.2) coincides with that of Mackevičius [33]. In turn, the latter is a particular case of the two-valued discretization scheme of Alfonsi [2, Sect. 2.2], who used it to approximate the CIR equation near zero in a second-order approximation.

Theorem 4.2 (Theorem 1.1). *Let $\hat{X}_t^x$ be the discretization scheme defined by composition* (3.3.3), *where $\hat{S}_h^x$ takes values x_1, x_2 with probabilities p_1, p_2 defined in* (4.3.1)–(4.3.2) ($\hat{S}_h^0 = 0$). *Then $\hat{X}_t^x$ is a strongly potential first-order discretization scheme for the CKLS equation* (1.1.1).

Proof. We have to check that scheme (4.3.1)–(4.3.2) satisfies conditions (3.4.4)–(3.4.5). For brevity, we further denote $z := \sigma^2 h$. For $x > 0$, by rather tedious calculations we have

$$\mathbb{E}(\hat{S}_h^x - x)^3 = (x_1 - x)^3 \frac{x}{2x_1} + (x_2 - x)^3 \frac{x}{2x_2} = 2x^{4\gamma-1} z^2, \tag{4.3.3}$$

$$\begin{aligned} \mathbb{E}(\hat{S}_h^x - x)^4 &= (x_1 - x)^4 \frac{x}{2x_1} + (x_2 - x)^4 \frac{x}{2x_2} \\ &= x^{4\gamma}(1 + 4x^{2(\gamma-1)} z) z^2. \end{aligned} \tag{4.3.4}$$

Now (3.4.4) follows from (4.3.3). If $f^{(4)}$ is bounded, then from (4.3.4) we immediately get (3.4.5). In fact, (3.4.5) is satisfied for every $f \in C^\infty_{\mathrm{pol}}(\mathbb{D})$. By the expression of the maximal value x_2 of $\hat{S}_h^x$ and the simple estimate $|P(x,y)| \le M(|x|^p + |y|^p)$ for all x, $y \in \mathbb{R}$ with some finite constant

M, where $P(x,y)$ is a pth-order homogeneous two-variable polynomial, condition (3.4.5) is satisfied for every $f \in C^{\infty}_{\text{pol}}(\mathbb{D})$ (suppose $|f^{(4)}(x)| \le C_4(1+x^{k_4})$):

$$\begin{aligned}
\mathbb{E}\Big[\max_{0\le s\le \hat{S}^x_h} |f^{(4)}(s)|(\hat{S}^x_h - x)^4\Big] &\le \max_{0\le s\le x_2} |f^{(4)}(s)|\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C_4(1+x_2^{k_4})\mathbb{E}(\hat{S}^x_h - x)^4 \\
&= C_4\left(1+\left(x+x^{2\gamma-1}z+\sqrt{(x^{2\gamma}+x^{2(2\gamma-1)}z)z}\right)^{k_4}\right)\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C_4\left(1+2^{k_4}\left(x+x^{2\gamma-1}z\right)^{k_4}\right)\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C_4\left(1+2^{k_4}\left(x+(1+x)\,z\right)^{k_4}\right)\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C_4\left(1+2^{k_4}\left(1+z\right)^{k_4}(1+x)^{k_4}\right)\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C_4\left(1+C_5\left(x^{k_4}+1\right)\right)\mathbb{E}(\hat{S}^x_h - x)^4 \\
&\le C(1+x^{k_4})\mathbb{E}(\hat{S}^x_h - x)^4 = \mathcal{O}(h^2).
\end{aligned}$$

We need to prove that the discretization scheme $\hat{S}^x_h$ has uniformly bounded moments. To this end, we need the following lemma.

Lemma 4.3. *For $p \in \mathbb{N}$,*

$$\begin{aligned}
&\mathbb{E}(\hat{S}^x_h)^p = x\hat{R}_p(x, x^{2\gamma-1}z), \ \textit{and} \\
&\mathbb{E}(\hat{S}^x_h - x)^{2p} = x(x^{2\gamma-1}z)^p\hat{Q}_p(x, x^{2\gamma-1}z),
\end{aligned}$$

where $\hat{R}_p = \hat{R}_p(x,y)$ and $\hat{Q}_p = \hat{Q}_p(x,y)$ are $(p-1)$th-order homogeneous two-variable polynomials with positive coefficients.

Proof. Substituting the expressions of $x_{1,2}$ and $p_{1,2}$ into $\mathbb{E}(\hat{S}^x_h)^p$, we get

$$\begin{aligned}
\mathbb{E}(\hat{S}^x_h)^p &= x_1^p p_1 + x_2^p p_2 = x_1^p\frac{x}{2x_1} + x_2^p\frac{x}{2x_2} = \frac{x}{2}\left(x_1^{p-1}+x_2^{p-1}\right) \\
&= \frac{x}{2}\Big\{\left(x+x^{2\gamma-1}z-\sqrt{(x^{2\gamma}+x^{2(2\gamma-1)}z)z}\right)^{p-1} \\
&\qquad + \left(x+x^{2\gamma-1}z+\sqrt{(x^{2\gamma}+x^{2(2\gamma-1)}z)z}\right)^{p-1}\Big\} \\
&= x\left\{\sum_{i=0}^{\lfloor (p-1)/2\rfloor}\binom{p-1}{2i}(x+x^{2\gamma-1}z)^{p-1-2i}((x+x^{2\gamma-1}z)x^{2\gamma-1}z)^i\right\}
\end{aligned}$$

$$
\begin{aligned}
&= x\left\{ \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \tbinom{p-1}{2i} (x + x^{2\gamma-1}z)^{p-1-i} (x^{2\gamma-1}z)^i \right\} \\
&= x\hat{R}_p(x, x^{2\gamma-1}z),
\end{aligned}
$$

where $\hat{R}_p(x,y) = \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p-1}{2i} (x+y)^{p-1-i} y^i$.

Similarly,

$$
\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^{2p} &= (x_1 - x)^{2p} p_1 + (x_2 - x)^{2p} p_2 \\
&= (x_1 - x)^{2p} \frac{x}{2x_1} + (x_2 - x)^{2p} \frac{x}{2x_2} \\
&= \frac{x}{2x_1 x_2} \Big\{ (x_1 - x)^{2p} x_2 + (x_2 - x)^{2p} x_1 \Big\} \\
&= \frac{1}{2(x + x^{2\gamma-1}z)} \Big\{ (x_1 - x)^{2p} (x_2 - x) + (x_2 - x)^{2p} (x_1 - x) \\
&\qquad\qquad + x((x_1 - x)^{2p} + (x_2 - x)^{2p}) \Big\} \\
&= \frac{-x^{2\gamma} z}{2(x + x^{2\gamma-1}z)} \Big\{ (x_1 - x)^{2p-1} + (x_2 - x)^{2p-1} \Big\} \\
&\quad + \frac{x}{2(x + x^{2\gamma-1}z)} \Big\{ (x_1 - x)^{2p} + (x_2 - x)^{2p} \Big\} \\
&= \frac{-x^{2\gamma} z}{(x + x^{2\gamma-1}z)} \Big\{ (x^{2\gamma-1}z)^{2p-1} \\
&\qquad\qquad + \sum_{i=1}^{p-1} \tbinom{2p-1}{2i} (x^{2\gamma-1}z)^{2p-1-2i} ((x^{2\gamma} + x^{2(2\gamma-1)}z)z)^i \Big\} \\
&\quad + \frac{x}{(x + x^{2\gamma-1}z)} \Big\{ (x^{2\gamma-1}z)^{2p} \\
&\qquad\qquad + \sum_{i=1}^{p} \tbinom{2p}{2i} (x^{2\gamma-1}z)^{2p-2i} ((x^{2\gamma} + x^{2(2\gamma-1)}z)z)^i \Big\} \\
&= x\Big\{ \sum_{i=1}^{p-1} \Big(\tbinom{2p}{2i} - \tbinom{2p-1}{2i} \Big) (x^{2\gamma-1}z)^{2p-i} (x + x^{2\gamma-1}z)^{i-1} \\
&\qquad + (x^{2\gamma-1}z)^p (x + x^{2\gamma-1}z)^{p-1} \Big\} \\
&= x(x^{2\gamma-1}z)^p \Big\{ \sum_{i=1}^{p-1} \Big(\tbinom{2p}{2i} - \tbinom{2p-1}{2i} \Big) (x^{2\gamma-1}z)^{p-i} (x + x^{2\gamma-1}z)^{i-1} \\
&\qquad\qquad + (x + x^{2\gamma-1}z)^{p-1} \Big\} \\
&= x(x^{2\gamma-1}z)^p \hat{Q}_p(x, x^{2\gamma-1}z),
\end{aligned}
$$

where $\hat{Q}_p(x,y) = \sum_{i=1}^{p-1} \left(\binom{2p}{2i} - \binom{2p-1}{2i} \right) y^{p-i} (x+y)^{i-1} + (x+y)^{p-1}$.

All coefficients of $\hat{Q}_p(x, x^{2\gamma-1}z)$ are positive integers because $\binom{2p}{2i} - \binom{2p-1}{2i} > 0$ for all $i = 1, \dots, p$. $\square$

From Lemma 4.3 we also easily obtain that, for all $z_0 > 0$ and $p \in \mathbb{N}$, there exists a constant C such that

$$\mathbb{E}(\hat{S}_h^x)^p \le x^p(1 + Cz) + Cz, \quad 0 < z \le z_0.$$

By Proposition 1.5 of [2], this leads to the following:

Lemma 4.4. *The discretization scheme $\hat{S}_h^x$ has uniformly bounded moments.*

In view of Theorem 3.3, our Theorem 4.2 now immediately follows from Lemma 4.4. $\square$

Remark 4.5. In addition to (3.4.2) and (3.4.3), we can set the requirement from the second-order calculations in Section (3.4)

$$\mathbb{E}(\hat{S}_h^x - x)^3 = 3\gamma x^{4\gamma-1} z^2 \tag{4.3.5}$$

although for our approximation, by Eq. (4.2.14), we actually have

$$\mathbb{E}(\hat{S}_h^x - x)^3 = 2x^{4\gamma-1} z^2.$$

However, we can achieve the fulfilment of Eqs. (3.4.2), (3.4.3), and (4.3.5) by modifying the discretization scheme (4.3.1)–(4.3.2) as follows:

$$\begin{cases} x_1 = x + x^{2\gamma-1}\sigma^2 h - \sqrt{(x^{2\gamma} + (3\,\gamma - 1)x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h}, & x > 0, \\ x_2 = x + x^{2\gamma-1}\sigma^2 h + \sqrt{(x^{2\gamma} + (3\,\gamma - 1)x^{2(2\gamma-1)}\sigma^2 h)\sigma^2 h}, & x > 0, \\ x_3 = 0, \quad x > 0, \end{cases} \tag{4.3.6}$$

$$\mathbb{P}\{\hat{S}_h^x = x_{1,2}\} = p_{1,2} = \frac{x}{2\,x_{1,2}}, \ \mathbb{P}\{\hat{S}_h^x = x_3\} = p_3 = 1 - p_1 - p_2, \ x > 0. \tag{4.3.7}$$

Note the appearance of the third value $x_3 = 0$ and the factor $3\gamma - 1$ inside the square root. Some improvement in accuracy of this modified scheme is further confirmed by simulation examples. Unfortunately, this scheme is well-defined only for $\gamma \in [1/2, 2/3]$ since we can easily check that for $\gamma > 2/3$, x_1 may take negative values.

4.4 Algorithm

We first give a short algorithm for calculating $\hat{X}_{(i+1)h}$ given $\hat{X}_{ih} = x$ at each simulation step i:

1. Draw a uniform random number U in the interval $[0, 1]$.

2. Given the value $x > 0$, generate a random variable $\hat{S}$ taking the values x_1 and x_2 with probabilities p_1 and p_2 defined in (4.3.1) and (4.3.2), respectively:
   if $U < p_1$, then $\hat{S} := x_1$, otherwise $\hat{S} := x_2$.
   If $x = 0$, then $\hat{S} := 0$.

3. Calculate (see (3.3.3))

$$\hat{X}_{(i+1)h} = D(\hat{S}, h).$$

4.5 Simulation examples

Using the discretization scheme defined in (4.3.1)–(4.3.2), we simulate the solution of CLKS equation (1.1.1) or its stochastic part (3.3.1) for $\gamma = 1/2$, $11/20$, $3/5$, $13/20$, $2/3$, $7/10$, $3/4$, $4/5$, $5/6$, $7/8$, and $9/10$ with test functions $f(x) = x^{19/10}$, $x^{9/5}$, $x^{17/10}$, $x^{8/5}$, x^2, x^3, and e^{-x}. Such a choice of γ and f is motivated by having explicit formulas for the expectations $\mathbb{E}f(S_t^x)$ (see Appendix) and

$$\mathbb{E}\mathrm{e}^{-X_t^x} = \left(\frac{\beta}{1/2\,\sigma^2\,(1-\mathrm{e}^{-\beta t}) + \beta} \right)^{2\,\frac{\theta}{\sigma^2}} \mathrm{e}^{-\frac{x\beta \mathrm{e}^{-\beta t}}{1/2\,\sigma^2\left(1-\mathrm{e}^{-\beta t}\right)+\beta}}, \qquad (4.5.1)$$

where $\gamma = 1/2$ (see, for example, [21, Prop. 6.2.5]). We also simulate the solution of CLKS equation (1.1.1) for $\gamma = 1/2$ (i.e., the CIR equation) with the discretization scheme defined in (4.3.6)–(4.3.7) and the test function $f(x) = \mathrm{e}^{-x}$.

In Tables 4.1, 4.2, 4.3, 4.4, 4.5, and 4.6, we give the values of the errors

$$|\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p|$$

for some p (for which we know explicit formulas for moments) with different approximation steps h. Tables 4.1, 4.3, and 4.5 represent calculations with "low" volatility ($\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$), and Tables 4.2, 4.4, and 4.6 represent those with "high" volatility ($\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$). In Tables 4.7 and 4.8, we give the values of $|\mathbb{E}e^{-X_1^x} - \mathbb{E}e^{-\hat{X}_1^x}|$, where $\hat{X}_1^x$ was calculated by generating $\hat{S}_1^x$ with schemes (4.3.1)–(4.3.2) (denoted A) and modified schemes (4.3.6)–(4.3.7) (denoted Am). Table 4.7 represents calculations with "low" volatility ($\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$), and Table 4.8 represents those with "high" volatility ($\sigma = 2$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$). Each table also contains the values of time steps h and the numbers $N \cdot 10^6$ of generated samples used calculating the corresponding error.

In Figures 4.1 and 4.2, we show $\mathbb{E}(S_1^x)^2$ (dashed lines) and $\mathbb{E}(\hat{S}_1^x)^2$ (solid lines); in Figures 4.3 and 4.4 we show $\mathbb{E}(S_1^x)^3$ (dashed lines) and $\mathbb{E}(\hat{S}_1^x)^3$ (solid lines); in Figures 4.5 and 4.6, we show $\mathbb{E}(S_1^x)^{\frac{19}{10}}$, $\mathbb{E}(S_1^x)^{\frac{9}{5}}$, $\mathbb{E}(S_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(S_1^x)^{\frac{8}{5}}$ (dashed lines) and $\mathbb{E}(\hat{S}_1^x)^{\frac{19}{10}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{9}{5}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(\hat{S}_1^x)^{\frac{8}{5}}$ (solid lines). All figures were generated by the same set of simulations that was used in the preparation of the tables. Figures 4.1, 4.3, and 4.5 represent values with "low" volatility ($\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$) and Figures 4.2, 4.4, and 4.6 with "high" volatility ($\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$). In Figures 4.7 and 4.8 we show $\mathbb{E}e^{-X_1^x}$ (exact) and $\mathbb{E}e^{-\hat{X}_1^x}$ generated using the discretization scheme (4.3.1)–(4.3.2) (denoted A) and $\mathbb{E}e^{-\hat{X}_1^x}$ generated using the modified discretization scheme (4.3.6)–(4.3.7) (denoted Am). Figure 4.7 represents the case of "low" volatility ($\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$) and Figure 4.8 represents the case of "high" volatility ($\sigma = 2$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$). In Figures 4.9 and 4.10, we show $\mathbb{E}e^{-X_1^x}$ for $\gamma = 1/2$ and $\mathbb{E}e^{-\hat{X}_1^x}$ for $\gamma = 1/2$, $2/3$, $3/4$, $5/6$, $7/8$, $9/10$, where $N = 10^6$ and $h = 0.25$, in the cases of "low" and "high" volatilities as before.

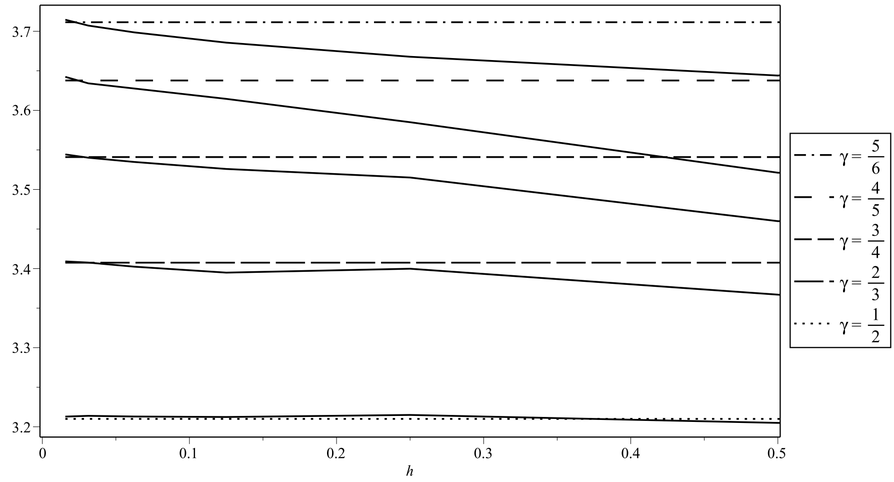

Figure 4.1: $\mathbb{E}(\hat{S}_1^x)^2$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^2$ are shown by horizontal dashed lines. $\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$.

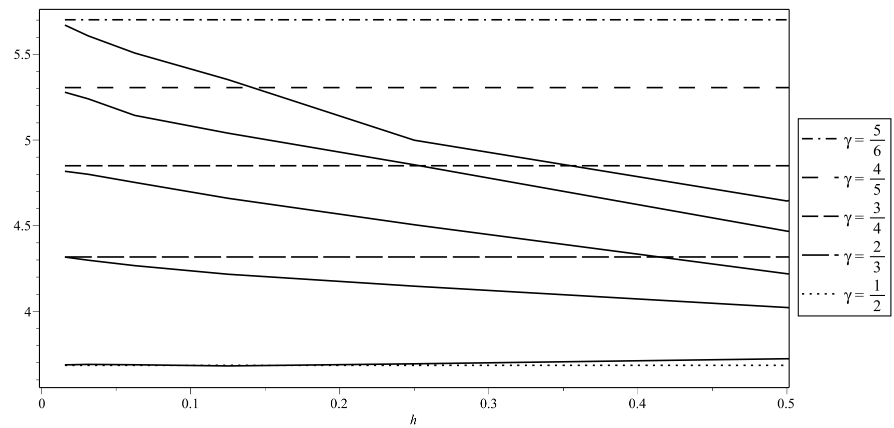

Figure 4.2: $\mathbb{E}(\hat{S}_1^x)^2$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^2$ are shown by horizontal dashed lines. $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$.

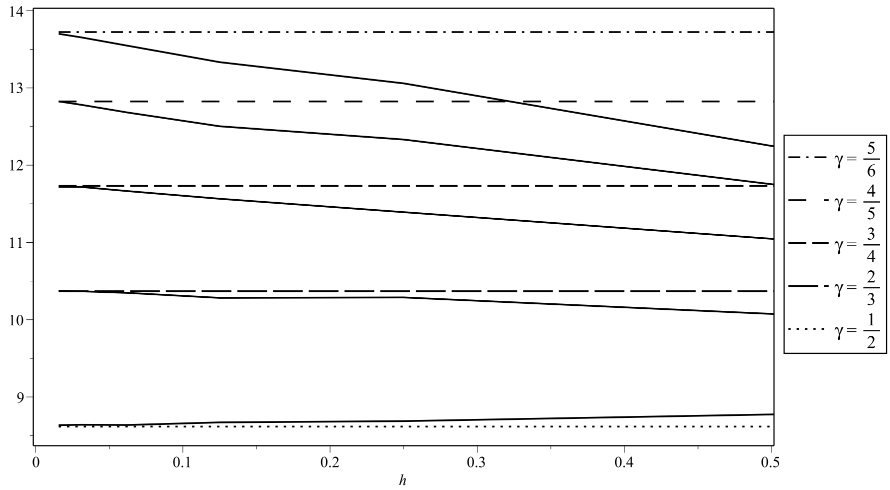

Figure 4.3: $\mathbb{E}(\hat{S}_1^x)^3$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^3$ are shown by horizontal dashed lines. $\theta = 0$, $\beta = 0$, $x_0 = 1.5$, $\sigma = 0.8$.

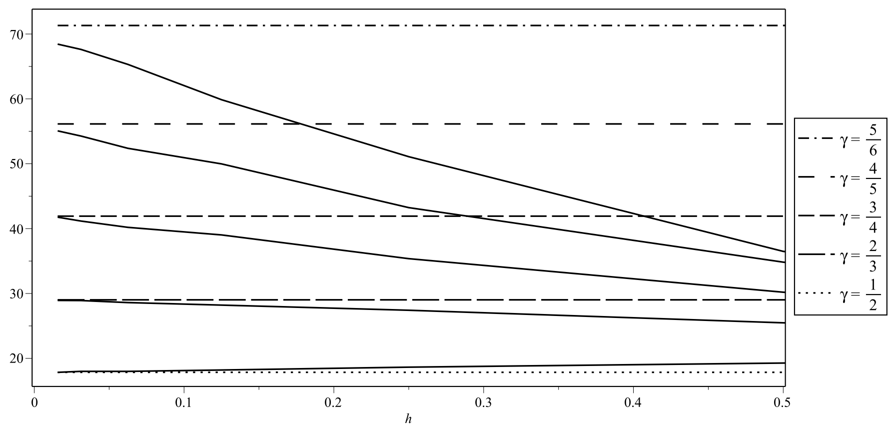

Figure 4.4: $\mathbb{E}(\hat{S}_1^x)^3$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^3$ are shown by horizontal dashed lines. $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$.

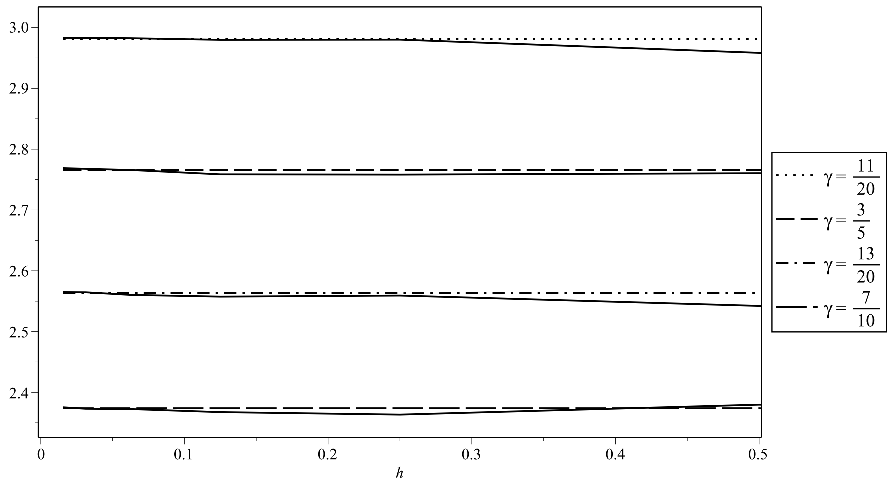

Figure 4.5: $\mathbb{E}(\hat{S}_1^x)^{\frac{19}{10}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{9}{5}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(\hat{S}_1^x)^{\frac{8}{5}}$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^{\frac{19}{10}}$, $\mathbb{E}(S_1^x)^{\frac{9}{5}}$, $\mathbb{E}(S_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(S_1^x)^{\frac{8}{5}}$ are shown by horizontal dashed lines. $\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$.

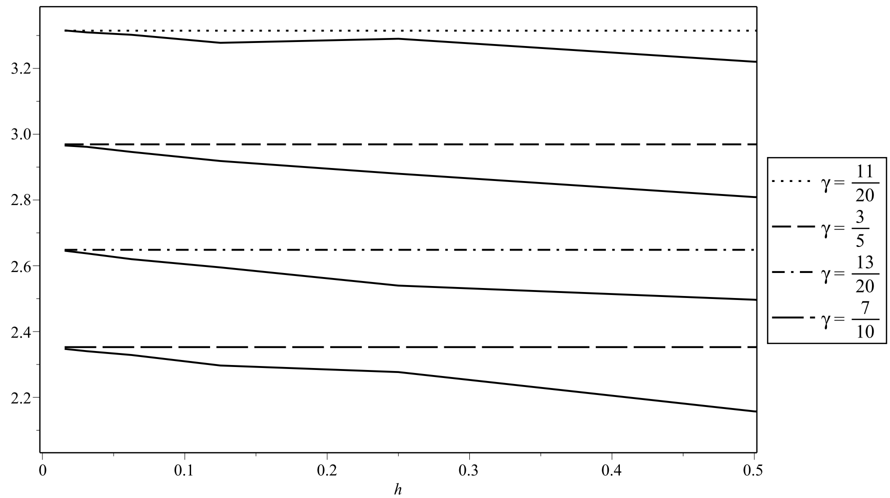

Figure 4.6: $\mathbb{E}(\hat{S}_1^x)^{\frac{19}{10}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{9}{5}}$, $\mathbb{E}(\hat{S}_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(\hat{S}_1^x)^{\frac{8}{5}}$ (solid lines) as functions of h for some values of γ. The exact values of $\mathbb{E}(S_1^x)^{\frac{19}{10}}$, $\mathbb{E}(S_1^x)^{\frac{9}{5}}$, $\mathbb{E}(S_1^x)^{\frac{17}{10}}$, and $\mathbb{E}(S_1^x)^{\frac{8}{5}}$ are shown by horizontal dashed lines. $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$.

Table 4.1: Values of $|\mathbb{E}(S_1^x)^2 - \mathbb{E}(\hat{S}_1^x)^2|$ for $\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$, $h = 1/2^n$.

γ	$n=6$ $N=102.4$	$n=5$ $N=25.6$	$n=4$ $N=6.4$	$n=3$ $N=1.6$	$n=2$ $N=0.4$	$n=1$ $N=0.1$
			$|\mathbb{E}(S_1^x)^2 - \mathbb{E}(\hat{S}_1^x)^2|$			
1/2	0.002915	0.003857	0.003016	0.002397	0.004984	0.005049
2/3	0.001559	0.000121	0.005077	0.012592	0.007669	0.040422
3/4	0.003340	0.000786	0.006177	0.015025	0.025857	0.081000
4/5	0.004578	0.003604	0.010149	0.023229	0.052740	0.116562
5/6	0.003147	0.004245	0.012843	0.025792	0.043640	0.067414

Table 4.2: Values of $|\mathbb{E}(S_1^x)^2 - \mathbb{E}(\hat{S}_1^x)^2|$ for $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$, $h = 1/2^n$.

γ	$n=6$ $N=102.4$	$n=5$ $N=25.6$	$n=4$ $N=6.4$	$n=3$ $N=1.6$	$n=2$ $N=0.4$	$n=1$ $N=0.1$
			$|\mathbb{E}(S_1^x)^2 - \mathbb{E}(\hat{S}_1^x)^2|$			
1/2	0.002717	0.005503	0.003268	0.003579	0.008855	0.038444
2/3	0.000509	0.018897	0.050594	0.100763	0.170371	0.295007
3/4	0.032290	0.049884	0.097300	0.189880	0.344428	0.630473
4/5	0.027158	0.065583	0.162247	0.265843	0.450578	0.837778
5/6	0.031016	0.093902	0.194255	0.349305	0.703000	1.057874

Table 4.3: Values of $|\mathbb{E}(S_1^x)^3 - \mathbb{E}(\hat{S}_1^x)^3|$ for $\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$, $h = 1/2^n$.

γ	$n=6$ $N=102.4$	$n=5$ $N=25.6$	$n=4$ $N=6.4$	$n=3$ $N=1.6$	$n=2$ $N=0.4$	$n=1$ $N=0.1$
			$|\mathbb{E}(S_1^x)^3 - \mathbb{E}(\hat{S}_1^x)^3|$			
1/2	0.018900	0.023367	0.020630	0.054256	0.070708	0.157102
2/3	0.006975	0.000945	0.021329	0.085945	0.080353	0.293248
3/4	0.011071	0.013691	0.066994	0.166196	0.339563	0.684186
4/5	0.000602	0.044261	0.143507	0.321432	0.493142	1.071639
5/6	0.022714	0.069883	0.177082	0.389716	0.664207	1.474358

Table 4.4: Values of $|\mathbb{E}(S_1^x)^3 - \mathbb{E}(\hat{S}_1^x)^3|$ for $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$, $h = 1/2^n$.

γ	$n=6$ $N=102.4$	$n=5$ $N=25.6$	$n=4$ $N=6.4$	$n=3$ $N=1.6$	$n=2$ $N=0.4$	$n=1$ $N=0.1$
	$|\mathbb{E}(S_1^x)^3 - \mathbb{E}(\hat{S}_1^x)^3|$					
1/2	0.004807	0.147367	0.144550	0.353186	0.792581	1.425949
2/3	0.091082	0.109581	0.423729	0.812364	1.605744	3.542449
3/4	0.159271	0.759191	1.726704	2.894023	6.554272	11.724430
4/5	1.068226	1.869902	3.757507	6.158224	12.900163	21.305779
5/6	2.872209	3.692346	5.996176	11.439493	20.224748	34.820765

Table 4.5: Values of $|\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p|$ for some p and γ. $\sigma = 0.8$, $\theta = 0$, $\beta = 0$, $x_0 = 1.5$, $h = 1/2^n$.

p	γ	$n=6$ $N=102.4$	$n=5$ $N=25.6$	$n=4$ $N=6.4$
		$|\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p|$		
19/10	11/20	0.001803	0.001705	0.001113
9/5	3/5	0.002859	0.001819	0.000148
17/10	13/20	0.001340	0.001132	0.003159
8/5	7/10	0.001526	0.000849	0.001378

p	γ	$n=3$ $N=1.6$	$n=2$ $N=0.4$	$n=1$ $N=0.1$
		$|\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p|$		
19/10	11/20	0.001510	0.001197	0.022932
9/5	3/5	0.007329	0.007740	0.005487
17/10	13/20	0.006003	0.004246	0.021242
8/5	7/10	0.006399	0.010495	0.005900

Table 4.6: Values of $|\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p|$ for some p and γ. $\sigma = 1.5$, $\theta = 0$, $\beta = 0$, $x_0 = 1.1$, $h = 1/2^n$.

p	γ	$n = 6$ $N = 102.4$	$n = 5$ $N = 25.6$	$n = 4$ $N = 6.4$
			$\lvert\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p\rvert$	
19/10	11/20	0.000755	0.005139	0.012418
9/5	3/5	0.003443	0.007615	0.023056
17/10	13/20	0.002730	0.010925	0.028374
8/5	7/10	0.005346	0.012419	0.023916

p	γ	$n = 3$ $N = 1.6$	$n = 2$ $N = 0.4$	$n = 1$ $N = 0.1$
			$\lvert\mathbb{E}(S_1^x)^p - \mathbb{E}(\hat{S}_1^x)^p\rvert$	
19/10	11/20	0.036881	0.024518	0.094347
9/5	3/5	0.050630	0.089301	0.160324
17/10	13/20	0.053472	0.108758	0.151686
8/5	7/10	0.055905	0.075857	0.195324

Table 4.7: Values of $|\mathbb{E}e^{-X_1^x} - \mathbb{E}e^{-\hat{X}_1^x}|$ for $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$, $\gamma = 1/2$, $h = 1/2^n$. "A" denotes scheme (4.3.1)–(4.3.2), "Am" denotes scheme (4.3.6)–(4.3.7).

	$n = 6$ $N = 102.4$	$n = 5$ $N = 25.6$	$n = 4$ $N = 6.4$	$n = 3$ $N = 1.6$	$n = 2$ $N = 0.4$	$n = 1$ $N = 0.1$
			$\lvert\mathbb{E}e^{-X_1^x} - \mathbb{E}e^{-\hat{X}_1^x}\rvert$			
A	0.000290	0.000430	0.000769	0.001231	0.002891	0.005404
Am	0.000273	0.000258	0.000306	0.000449	0.000327	0.002275

Table 4.8: Values of $|\mathbb{E}e^{-X_1^x} - \mathbb{E}e^{-\hat{X}_1^x}|$ for $\sigma = 2$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$, $\gamma = 1/2$, $h = 1/2^n$. "A" denotes scheme (4.3.1)–(4.3.2), "Am" denotes scheme (4.3.6)–(4.3.7).

	$n = 6$ $N = 102.4$	$n = 5$ $N = 25.6$	$n = 4$ $N = 6.4$	$n = 3$ $N = 1.6$	$n = 2$ $N = 0.4$	$n = 1$ $N = 0.1$
			$\lvert\mathbb{E}e^{-X_1^x} - \mathbb{E}e^{-\hat{X}_1^x}\rvert$			
A	0.000491	0.001331	0.002349	0.005248	0.012123	0.027630
Am	0.000418	0.000557	0.001046	0.001707	0.003434	0.007223

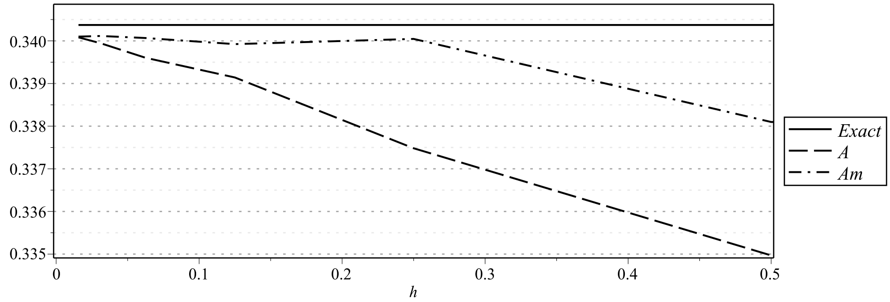

Figure 4.7: $\mathbb{E}e^{-X_1^x}$(exact) and $\mathbb{E}e^{-\hat{X}_1^x}$(“$A$” denotes scheme (4.3.1)–(4.3.2) and “Am” denotes scheme (4.3.6)–(4.3.7)) as functions of h for $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$, $\gamma = 1/2$.

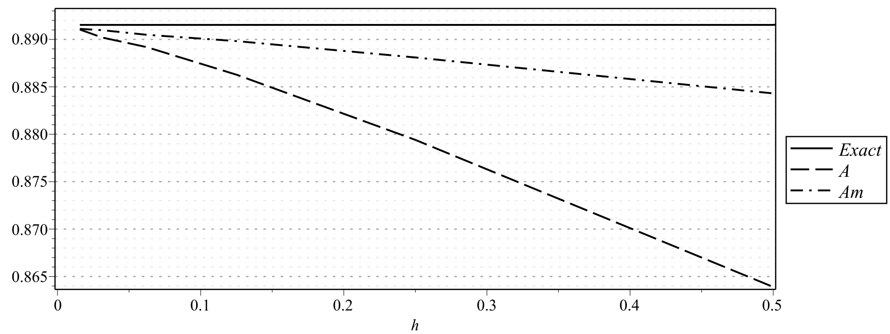

Figure 4.8: $\mathbb{E}e^{-X_1^x}$(exact) and $\mathbb{E}e^{-\hat{X}_1^x}$ (“A” denotes scheme (4.3.1)–(4.3.2) and “Am” denotes scheme (4.3.6)–(4.3.7)) for $\sigma = 2$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$, $\gamma = 1/2$.

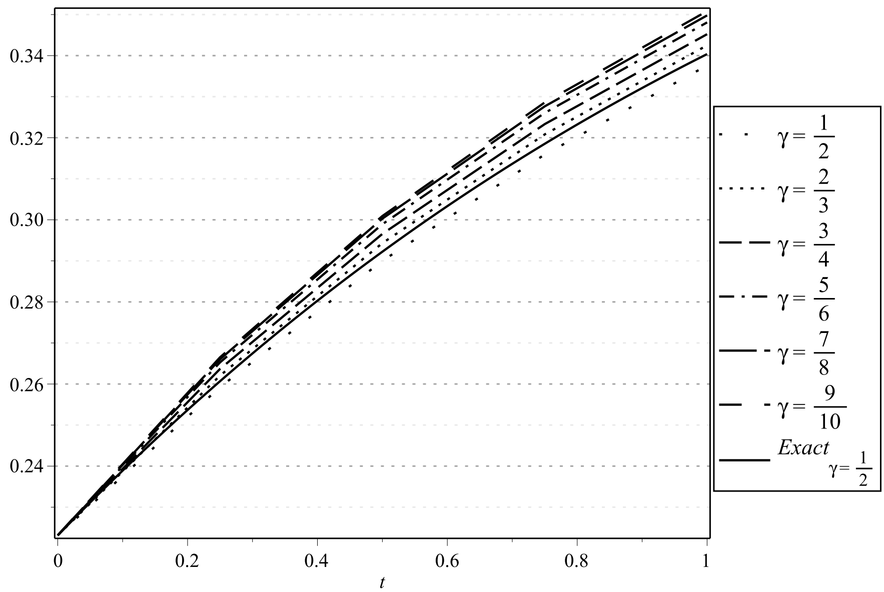

Figure 4.9: $\mathbb{E}e^{-X_1^x}$(exact) and $\mathbb{E}e^{-\hat{X}_1^x}$ for $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$, $h = 0.25$.

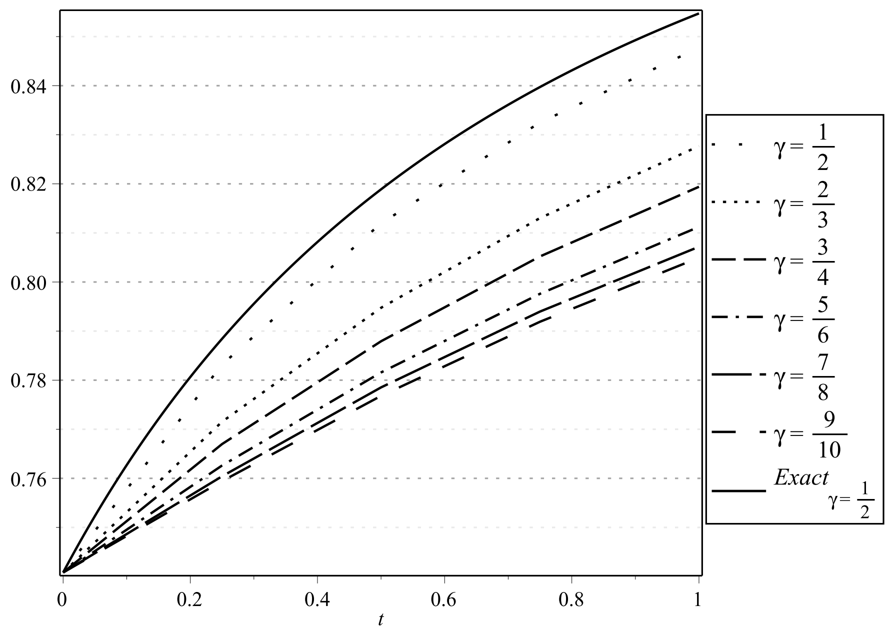

Figure 4.10: $\mathbb{E}e^{-X_1^x}$(exact) and $\mathbb{E}e^{-\hat{X}_1^x}$ for $\sigma = 1.5$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$, $h = 0.25$.

Chapter 5

Second-order approximation

In this chapter, we construct a second-order discretization scheme for the solution of the CKLS model. In the first section of the chapter, we discuss all techniques used to find the second-order approximations. In Section 5.2, we find a second-order discretization scheme for the CIR equation. Section 5.3 is dedicated to finding a third-order approximation for the stochastic part of the CIR equation. In Section 5.4, we find a second-order discretization scheme for the CKLS equation. In the next section, we give algorithm for simulations, and finally, in the last section of the chapter, we provide numerical simulations illustrating the accuracy of CKLS approximations.

5.1 A second-order approximation

We construct our schemes applying methods described in Section 3.3. First, we split CKLS model into the stochastic and deterministic parts. The solution of $\mathrm{d}D_t^x = (\theta - \beta D_t^x)\mathrm{d}t$, $D_0^x = x \geq 0$, is easy to find (see (3.3.2)) and we focus on constructing a second-order discretization scheme for the solution of the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma \,\mathrm{d}B_t$, $S_0^x = x \geq 0$. Here we find probability p_1, p_2, and p_3 expressions through values of a three-valued (x_1, x_2, and x_3) random variable and the first three moments of the corresponding stochastic part (with particular γ). Then we find suitable values expressions of the three-valued random variable. For the CIR equation we manage to find probability p_1, p_2, p_3, and p_4 expressions through values of a four-valued (x_1, x_2, x_3, and x_4) random variable and the first four moments of the corresponding stochastic part. Then we find suitable values expressions of the four-valued random variable.

Theorem 3.4 allows us to merge split-step parts of equation (1.1.1) using the composition (3.3.4) and we get a strongly potential second-order approximation of the CKLS equation.

5.2 A strongly potential second-order approximation of the CIR equation

In this section, we construct a strongly potential second-order approximation for the CIR equation ($\gamma = 1/2$) using a three-valued random variable at each generation step, without switching to another scheme in a neighborhood of zero. The approximate moments (3.4.15) in conditions (3.4.14) for the noncentral moments $\mathbb{E}(\hat{S}_h^x)^i$ in this case become as follows (recall that $z := ah = \sigma^2 h$; let us denote $\hat{m}_i := {}_2\hat{m}_i$, $i = 1, 2, \ldots, 6$):

$$\begin{aligned}
\hat{m}_1 &= x, \\
\hat{m}_2 &= x^2 + xz, \\
\hat{m}_3 &= x^3 + 3x^2 z + \tfrac{3}{2} x z^2, \\
\hat{m}_4 &= x^4 + 6x^3 z + 9x^2 z^2, \\
\hat{m}_5 &= x^5 + 10x^4 z + 30x^3 z^2, \\
\hat{m}_6 &= x^6 + 15x^5 z + 75x^4 z^2.
\end{aligned} \tag{5.2.1}$$

We look for approximations $\hat{S}_h^x$ taking three positive values x_1, x_2, and x_3 with probabilities p_1, p_2, and p_3 such that

$$\mathbb{E}(\hat{S}_h^x)^i = \hat{m}_i + O(h^3), \ i = 1, 2, \ldots, 6, \tag{5.2.2}$$

where $x \geq 0$, $h > 0$, together with obvious requirement

$$p_1 + p_2 + p_3 = 1. \tag{5.2.3}$$

Denote $m_p = m_p(h,x) := \mathbb{E}(S_h^x)^p$, $p \in \mathbb{N}$. We have (see Section 7 [Appendix])

$$m_1 = \hat{m}_1 = x,$$
$$m_2 = \hat{m}_2 = x^2 + xz,$$
$$m_3 = \hat{m}_3 = x^3 + 3\,x^2 z + \tfrac{3}{2}\,xz^2.$$

Solving the system

$$x_1^i p_1 + x_2^i p_2 + x_3^i p_3 = m_i,\ \ i = 1,2,3,$$

with respect to unknowns x_1, x_2, and x_3, we get:

$$\begin{aligned} p_1 &= \frac{m_1 x_2 x_3 - m_2(x_2 + x_3) + m_3}{x_1(x_3 - x_1)(x_2 - x_1)}, \\ p_2 &= \frac{m_2(x_1 + x_3) - m_1 x_1 x_3 - m_3}{x_2(x_2 - x_1)(x_3 - x_2)}, \\ p_3 &= \frac{m_1 x_1 x_2 - m_2(x_1 + x_2) + m_3}{x_3(x_3 - x_2)(x_3 - x_1)}. \end{aligned} \tag{5.2.4}$$

We can get analogous expressions from the last three equations of system (5.2.2) (with m_4, m_5, m_6 instead of m_1, m_2, m_3). However, trying to directly solve the obtained six equations with respect to all unknowns x_1, x_2, x_3, p_1, p_2, p_3 gave no satisfactory results. In view of the form of approximations presented by Alfonsi [2] and Mackevičius [33] for the CIR equation and of our first-order approximations for the CKLS equations (4.3.1), after a number of experiments, we arrived at the following conclusions:

- the values of discretization scheme $\hat{S}_h^x$ may be chosen of the following form:

$$x_{1,3} = x + A_1 z \mp \sqrt{(Bx + Cz)z}, \quad x_2 = x + A_2 z, \tag{5.2.5}$$

  with parameters $A_1, A_2, B, C > 0$;

- Instead of the exact matching of moments $\mathbb{E}(\hat{S}_h^x)^i = m_i$ for $i = 4,5,6$, it is more convenient to require $\mathbb{E}(\hat{S}_h^x)^i = \hat{m}_i$, $i = 4,5,6$.

Solving system (5.2.2)–(5.2.4) with x_1, x_2, x_3 of the form (5.2.5), together with ensuring the nonnegativity of the solution $\{x_1, x_2, x_3, p_1, p_2, p_3\}$, still is a rather technical and long task, even with the help of MAPLE. Note that the right-hand sides $O(h^3)$ in conditions (5.2.2) give us certain flexibility in finding relatively simple expressions of solutions. We postpone the details to Appendix.

This way we get a family of second-order discretization schemes $\hat{S}_h^x$ depending on the parameter $A \in [3/4, 3/2]$:

$$\begin{aligned} x_{1,3} &= x + (A + \tfrac{3}{4})\sigma^2 h \mp \sqrt{\big(3x + (A + \tfrac{3}{4})^2\sigma^2 h\big)\sigma^2 h}, \\ x_2 &= x + A\sigma^2 h, \end{aligned} \tag{5.2.6}$$

with probabilities p_1, p_2, and p_3 given by (5.2.4). The interval of possible values of the parameter A is conditioned by the necessary nonnegativity of the solution $\{x_1, x_2, x_3, p_1, p_2, p_3\}$. In particular, the value $A = (3 + \sqrt{3})/4 \approx 1.183$ ensures the exact matching of the fourth moment, $\mathbb{E}(\hat{S}_h^x)^4 = m_4$, in addition to the exact matching of the first three moments.

Theorem 5.1 (Theorem 1.2). *Let $\hat{X}_t^x$ be the discretization scheme defined by composition* (3.3.4)*, where $\hat{S}_h^x$ takes the values x_1, x_2, and x_3 defined in* (5.2.6) *with probabilities p_1, p_2, and p_3 defined in* (5.2.4) ($\hat{S}_h^0 = 0$)*. Then $\hat{X}_t^x$ is a strongly potential second-order discretization scheme for the CIR equation.*

Proof. Let us first check that

$$x_1 = x + (A + \tfrac{3}{4})z - \sqrt{(3x + (A + \tfrac{3}{4})^2 z)z} \geq 0$$

for all $x \geq 0$ and $z > 0$. This is equivalent to

$$(A + \tfrac{3}{4})^2 z^2 + 2(A + \tfrac{3}{4})xz + x^2 \geq (A + \tfrac{3}{4})^2 z^2 + 3xz,$$

which in turn is equivalent to

$$(4A - 3)xz + 2x^2 \geq 0.$$

This implies that $x_1 \geq 0$ for all $x, z \geq 0$, provided that $A \geq 3/4$. Obviously, $x_2, x_3 \geq x_1 \geq 0$.

Now let us check the nonnegativity of p_1, p_2, and p_3. For p_1, we have

$$\begin{aligned} p_1 &= \frac{m_1x_2x_3 - m_2(x_2+x_3) + m_3}{x_1(x_3-x_1)(x_2-x_1)} \\ &= \frac{8xz((4A^2-5A+3)z + 4x - (1-A)\sqrt{((4A+3)^2z+48x)z})}{((4A+3)z+4x-\sqrt{((4A+3)^2z+48x)z})\sqrt{((4A+3)^2z+48x)z}} \\ &\quad \times \frac{1}{(\sqrt{((4A+3)^2z+48x)z} - 3z)}, \end{aligned}$$

where $x \geq 0, z > 0$. We have already checked the nonnegativity of

$$4x + (4A+3)z - \sqrt{(48x+(4A+3)^2z)z} = 4x_1.$$

The positivity of $\sqrt{(4A+3)^2z+48x)z} - 3z$ is obvious, and $4A^2 - 5A + 3 > 0$ for all $A \in \mathbb{R}$. Thus, clearly, $p_1 \geq 0$ if $A \geq 1$. Now let $A < 1$. Then $p_1 \geq 0$ if and only if

$$((4A^2-5A+3)z+4x)^2 \geq (1-A)^2((4A+3)^2z+48x)z$$

or, equivalently,

$$-A(4A-3)(2A-3)z^2 - 2(2A-1)(A-3)xz + 4x^2 \geq 0,$$

which clearly holds for all $x \geq 0$ and $z > 0$ if $A \in [3/4, 3/2]$. Thus $p_1 \geq 0$ for $x \geq 0$ and $z > 0$ if $A \in [3/4, 3/2]$. For p_2, we obviously have

$$\begin{aligned} p_2 &= \frac{m_2(x_1+x_3) - m_1x_1x_3 - m_3}{x_2(x_2-x_1)(x_3-x_2)} \\ &= \frac{32xz}{(-3z+\sqrt{((4A+3)^2z+48x)z})(3z+\sqrt{((4A+3)^2z+48x)z})} \\ &= \frac{32xz}{16A^2z^2+24Az^2+48xz} = \frac{4x}{2A^2z+3Az+6x} \geq 0 \end{aligned}$$

for $x \geq 0, z > 0$. Finally, for p_3, we have

$$\begin{aligned} p_3 &= \frac{m_1x_1x_2 - m_2(x_1+x_2) + m_3}{x_3(x_3-x_2)(x_3-x_1)} \\ &= \frac{8xz((4A^2-5A+3)z + 4x - (A-1)\sqrt{((4A+3)^2z+48x)z})}{((4A+3)z+4x+\sqrt{((4A+3)^2z+48x)z})\sqrt{((4A+3)^2z+48x)z}} \end{aligned}$$

$$\times \frac{1}{(\sqrt{((4A+3)^2z+48x)z}+3z)}$$

for $x \geq 0$ and $z > 0$. The nominator is obviously positive, and the nonnegativity of the denominator follows similarly to that of p_1.

Let us check that, indeed, the central moments of $\hat{S}^x_h$ satisfy conditions (3.4.8)–(3.4.13) (with $\gamma = 1/2$). The first three are obvious, since the moments of the random variable $\hat{S}^x_h$ exactly match the three first moments of S^x_h, so they also match the first three central moments:

$$\mathbb{E}(\hat{S}^x_h - x) = \mathbb{E}(S^x_h - x) = 0, \quad \mathbb{E}(\hat{S}^x_h - x)^2 = \mathbb{E}(S^x_h - x)^2 = xz,$$

$$\mathbb{E}(\hat{S}^x_h - x)^3 = \mathbb{E}(S^x_h - x)^3 = 3xz^2/2.$$

Conditions (3.4.11), (3.4.12), and (3.4.13a) are satisfied, since, respectively,

$$\mathbb{E}(\hat{S}^x_h - x)^4 = (-2A^2 + 3A + 9/4)xz^3 + 3x^2z^2 = 3x^2z^2 + O(h^3),$$

$$\begin{aligned}|\mathbb{E}(\hat{S}^x_h - x)^5| &= |(-6A^3 + 3A^2 + 9A + 27/8)xz^4 + (6A+9)x^2z^3| \\ &= O(h^3),\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\hat{S}^x_h - x)^6 = (-14A^4 &- 3A^3 + (45/2)A^2 + (81/4)A + 81/16)xz^5 \\ &+ (6A^2 + 36A + 81/4)x^2z^4 + 9x^3z^3 = O(h^3)\end{aligned}$$

for $A \in [3/4, 3/2]$.

For a pth-order homogeneous two-variable polynomial $P(x,y)$, we have the simple estimate $|P(x,y)| \leq M(|x|^p + |y|^p)$ for all x, $y \in \mathbb{R}$ with some finite constant M.

By the the relation $\mathbb{E}(\hat{S}^x_h - x)^6 = O(h^3)$ and the expression of the maximal value x_3 of $\hat{S}^x_h$, condition (3.4.13) is satisfied for every $f \in C^\infty_{\mathrm{pol}}(\mathbb{D})$ (suppose $|f^{(6)}(x)| \leq C_6(1 + x^{k_6})$):

$$\mathbb{E}\Big[\max_{0\leq s\leq \hat{S}^x_h} |f^{(6)}(s)|(\hat{S}^x_h - x)^6\Big] \leq \max_{0\leq s\leq x_3} |f^{(6)}(s)|\mathbb{E}(\hat{S}^x_h - x)^6$$

$$
\begin{aligned}
&\le C_6(1+x_3^{k_6})\mathbb{E}(\hat{S}_h^x - x)^6 \\
&= C_6\left(1+\left(x+(A+\tfrac{3}{4})z+\sqrt{\left(3x+(A+\tfrac{3}{4})^2z\right)z}\right)^{k_6}\right)\mathbb{E}(\hat{S}_h^x - x)^6 \\
&\le C_6\left(1+2^{k_6}\left(x+\tfrac{9}{4}\,z\right)^{k_6}\right)\mathbb{E}(\hat{S}_h^x - x)^6 \\
&\le C_6\left(1+C_7\left(x^{k_6}+z^{k_6}\right)\right)\mathbb{E}(\hat{S}_h^x - x)^6 \\
&\le C(1+x^{k_6})\mathbb{E}(\hat{S}_h^x - x)^6 = \mathcal{O}(h^3).
\end{aligned}
$$

It remains to check that the discretization scheme $\hat{S}_h^x$ has uniformly bounded moments, that is, there exists $h_0 > 0$ such that

$$
\sup_{0<h\le h_0}\ \sup_{t\in\Delta^h} \mathbb{E}(|\hat{S}(x,t)|^p) < +\infty,\ \ p\in\mathbb{N},\ \ x\ge 0.
$$

By elementary but tedious calculations that we postpone to Appendix, we arrive at the following expression for the moments:

$$
\begin{aligned}
\mathbb{E}(\hat{S}_h^x)^p &= x^p + \tfrac{p(p-1)}{2}\,x^{p-1}\,z + \tfrac{p(p-1)^2(p-2)}{8}\,x^{p-2}\,z^2 + \cdots \\
&\le x^p + C(1+x^p)h = x^p(1+Ch) + Ch,\ \ x\ge 0,\ \ h\le h_0 = \tfrac{1}{\sigma^2},
\end{aligned}
$$

where the constant $C > 0$ depends on p and σ, from which the boundedness of the moments of the approximation follows in a standard way (see [2, Prop. 1.5]). □

5.3 A potential third-order approximation for the stochastic part of the CIR equation

By a similar procedure we can obtain a potential third-order weak approximation of the stochastic part (3.3.1) of the CIR equation (1.1.1) ($\gamma = 1/2$). Although then composition (3.3.4) theoretically gives only second-order approximation, numerical simulations show that, practically, it gives a slightly better accuracy of approximation than with second-order approximation of the stochastic part.

Let $m_i = m_i(x,h) = \mathbb{E}(S_h^x)^p$, $i = 1,2,3,4$, and $\hat{m}_i := {}_3\hat{m}_i$, $i = 1,2,\ldots,8$. We look at a discretization scheme $\hat{S}_h^x$ taking four values x_1, x_2, x_3, x_4 with probabilities $p_1.p_2, p_3, p_4$ such that

$$
x_1^i p_1 + x_2^i p_2 + x_3^i p_3 + x_4^i p_4 = m_i, i = 1,2,3,4, \tag{5.3.1}
$$

and

$$x_1^i p_1 + x_2^i p_2 + x_3^i p_3 + x_4^i p_4 = \hat{m}_i + O(h^3),\ i = 5, 6, 7, 8. \tag{5.3.2}$$

Its solution with respect to x_1, x_2, x_3, and x_4 is as follows:

$$\begin{aligned} p_1 &= -\frac{m_1\, x_2\, x_3\, x_4 - m_2(x_2\, x_3 + x_2\, x_4 + x_3\, x_4) + m_3(x_2 + x_3 + x_4) - m_4}{x_1\,(x_1 - x_4)(x_1 - x_3)(x_1 - x_2)}, \\ p_2 &= \frac{m_1\, x_1\, x_3\, x_4 - m_2(x_1\, x_3 + x_1\, x_4 + x_3\, x_4) + m_3(x_1 + x_3 + x_4) - m_4}{(x_1 - x_2)x_2\,(x_2 - x_4)(x_2 - x_3)}, \\ p_3 &= -\frac{m_1\, x_1\, x_2\, x_4 - m_2(x_1\, x_2 + x_1\, x_4 + x_2\, x_4) + m_3(x_1 + x_2 + x_4) - m_4}{(x_2 - x_3)(x_1 - x_3)x_3\,(x_3 - x_4)}, \\ p_4 &= \frac{m_1\, x_1\, x_2\, x_3 - m_2(x_1\, x_2 + x_1\, x_3 + x_2\, x_3) + m_3(x_1 + x_2 + x_3) - m_4}{x_4\,(x_3 - x_4)(x_2 - x_4)(x_1 - x_4)}. \end{aligned} \tag{5.3.3}$$

Again, after a number of experiments, we have chosen to look for a solution of (5.3.1)–(5.3.2), together with $\sum_i p_i = 1$ and $p_i \geq 0$, in the form

$$\begin{aligned} x_{1,3} &= x + A_1 z \mp \sqrt{(B_1 x + C_1 z)z} \geq 0, \\ x_{2,4} &= x + A_2 z \mp \sqrt{(B_2 x + C_2 z)z} \geq 0, \end{aligned}$$

with parameters $A_1, A_2, B_1, B_2, C_1, C_2 > 0$ and probabilities p_1, p_2, p_3, p_4 defined in (5.3.3). The main difficulty was obtaining a nonnegative solution $\{x_1, x_2, x_3, x_4, p_1,\ p_2,\ p_3,\ p_4\}$.

The final result of current section is bellow.

Theorem 5.2. *Approximation $\hat{S}_h^x$ taking the four values*

$$\begin{aligned} x_{1,2} =& x + \tfrac{3}{2}\,\sigma^2 h \mp \sqrt{\big((3-\sqrt{6})x + \tfrac{3}{4}\,\sigma^2 h\big)\sigma^2 h}, \\ x_{3,4} =& x + \big(\tfrac{3}{2} + \tfrac{1}{2}\sqrt{6}\big)\sigma^2 h \mp \sqrt{\big((3+\sqrt{6})x + \big(\tfrac{15}{4} + \tfrac{3}{2}\sqrt{6}\big)\sigma^2 h\big)\sigma^2 h}, \end{aligned} \tag{5.3.4}$$

with the corresponding probabilities p_i, $i = 1, 2, 3, 4$, given by (5.3.3), *is a potential third-order weak approximation of the stochastic part* (3.3.1) *of the CIR equation.*

Proof. Let us first check that

$$x_1 = x + \tfrac{3}{2}\,z - \sqrt{\big((3-\sqrt{6})x + \tfrac{3}{4}\,z\big)z} > 0$$

for all $x \geq 0$ and $z > 0$. This is equivalent to

$$x^2 + 3\,xz + \tfrac{9}{4}\,z^2 > (3-\sqrt{6})xz + \tfrac{3}{4}\,z^2,$$

which in turn is equivalent to

$$\tfrac{1}{4}\,(\sqrt{6}z + 2\,x)^2 > 0.$$

Let us check then that

$$x_3 = x + \big(\tfrac{3}{2} + \tfrac{1}{2}\sqrt{6}\big)z - \sqrt{\big((3+\sqrt{6})x + \big(\tfrac{15}{4} + \tfrac{3}{2}\sqrt{6}\big)z\big)z} \geq 0$$

for all $x \geq 0$ and $z > 0$. This is equivalent to

$$x^2 + \big(3+\sqrt{6}\big)xz + \big(\tfrac{15}{4} + \tfrac{3}{2}\sqrt{6}\big)z^2 \geq (3+\sqrt{6})xz + \big(\tfrac{15}{4} + \tfrac{3}{2}\sqrt{6}\big)z^2,$$

which in turn is equivalent to

$$x^2 \geq 0.$$

This implies that x_1, $x_3 \geq 0$ for all $x \geq 0$, $z > 0$. Obviously, $x_2 \geq x_1 \geq 0$ and $x_4 \geq x_3 \geq 0$.

Now let us check the nonnegativity of p_1, p_2, p_3, and p_4. For p_1, we have

$$\begin{aligned} p_1 &= \frac{m_2(x_2\,x_3 + x_2\,x_4 + x_3\,x_4) - m_1\,x_2\,x_3\,x_4 - m_3(x_2+x_3+x_4) + m_4}{x_1\,\left(x_1-x_4\right)\left(x_1-x_3\right)\left(x_1-x_2\right)} \\ &= \frac{{}_np_1}{{}_dp_1}, \end{aligned}$$

where

$$\begin{aligned} {}_np_1 =& 2\left(3-\sqrt{6}\right)\left(z\sqrt{6} + 2\,x + 3\,z\right)xz \\ &+ \sqrt{4\,\left(3-\sqrt{6}\right)xz + 3\,z^2}\left(8\,x + 6\,z + 4\,\sqrt{6}\,(x+z)\right)x, \\ {}_dp_1 =& 3\,\left(2-\sqrt{6}\right)\left(z\sqrt{6} + 4\,x + 3\,z\right)\left(z\sqrt{6} + 12\,x + 12\,z\right)z \\ &+ \left(12\,z^2\left(3 + 4\,\sqrt{6}\right) + 32\,\sqrt{6}x^2 + 24\,\left(5\,\sqrt{6} - 3\right)xz\right) \\ &\times 1/2\,\sqrt{4\,\left(3-\sqrt{6}\right)xz + 3\,z^2}, \quad x \geq 0, \quad z > 0. \end{aligned}$$

The nonnegativity of ${}_np_1$ is obvious. Let us check the positivity of ${}_dp_1$ for all $x \geq 0$ and $z > 0$. This is equivalent to

$$\begin{aligned}
0 \leq & 6\left(3-\sqrt{6}\right)\left(6\,xz\sqrt{6}-3\,z^2\sqrt{6}-16\,x^2-60\,xz-24\,z^2\right)^2 \\
& -18\left(z\sqrt{6}+4\,x+3\,z\right)z\left(5-2\sqrt{6}\right)\left(z\sqrt{6}+12\,x+12\,z\right)^2 \\
= & 12\left(3-\sqrt{6}\right)\left(16\,xz\sqrt{6}+6\,z^2\sqrt{6}+32\,x^2+24\,xz+15\,z^2\right) \\
& \times\left(z\sqrt{6}+2\,x\right)^2
\end{aligned}$$

for $x \geq 0$ and $z > 0$.

For p_2, we have

$$\begin{aligned}
p_2 &= \frac{m_1\,x_1\,x_3\,x_4 - m_2(x_1\,x_3+x_1\,x_4+x_3\,x_4)+m_3(x_1+x_3+x_4)-m_4}{(x_1-x_2)\,x_2\,(x_2-x_4)\,(x_2-x_3)} \\
&= \frac{{}_np_2}{{}_dp_2}
\end{aligned}$$

where

$$\begin{aligned}
{}_np_2 =& \sqrt{\left(3-\sqrt{6}\right)\left(z\sqrt{6}+4\,x+3\,z\right)z}\left(2\sqrt{6}\,(x+z)+4\,x+3\,z\right)x \\
& -\left(3-\sqrt{6}\right)\left(z\sqrt{6}+2\,x+3\,z\right)xz, \\
{}_dp_2 =& \left(8\sqrt{6}\,x^2+6\left(5\sqrt{6}-3\right)xz+3\left(3+4\sqrt{6}\right)z^2\right) \\
& \times\sqrt{\left(3-\sqrt{6}\right)\left(z\sqrt{6}+4\,x+3\,z\right)z} \\
& +72\left(\sqrt{6}-2\right)x^2z+6\left(13\sqrt{6}-18\right)xz^2 \\
& +9\left(2\sqrt{6}+1\right)z^3, \quad x \geq 0, \quad z > 0.
\end{aligned}$$

The positivity of ${}_dp_2$ is obvious. Let us check the nonnegativity of ${}_np_2$ for all $x \geq 0$ and $z > 0$. This is equivalent to

$$\begin{aligned}
0 \leq & 1/2\left(3+\sqrt{6}\right)\left(z\sqrt{6}+4\,x+3\,z\right)\left(z\sqrt{6}-4\,x-6\,z\right)^2 \\
& -3\,z\left(5-2\sqrt{6}\right)\left(z\sqrt{6}+2\,x+3\,z\right)^2 \\
= & \left(3+\sqrt{6}\right)\left(z\sqrt{6}+8\,x+3\,z\right)\left(z\sqrt{6}+2\,x\right)^2
\end{aligned}$$

for $x \geq 0$ and $z > 0$.

For p_3, we have

$$p_3 = \frac{m_2(x_1\,x_2+x_1\,x_4+x_2\,x_4)-m_1\,x_1\,x_2\,x_4-m_3(x_1+x_2+x_4)+m_4}{(x_2-x_3)\,(x_1-x_3)\,x_3\,(x_3-x_4)}$$

$$= \frac{{}_{n}p_3}{{}_{d}p_3}$$

where

$$\begin{aligned}
{}_{n}p_3 =& 2\left(3-\sqrt{6}\right)x^2 z + 4\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\left(\sqrt{6}-2\right)x^2,\\
{}_{d}p_3 =& \sqrt{6}\left(3\,z\sqrt{6}+8\,x+6\,z-4\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\right)\\
&\times\left(z\sqrt{6}+2\,x+3\,z-2\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\right)\\
&\times\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}, \quad x \geq 0, \quad z > 0.
\end{aligned}$$

The nonnegativity of ${}_{n}p_3$ is obvious. Let us check the positivity of ${}_{d}p_3$ for all $x \geq 0$ and $z > 0$. This is equivalent checking that

$$\begin{aligned}
0 &< \left(3\,z\sqrt{6}+8\,x+6\,z\right)^2 - 16\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z\\
&= 6\,(2\,\sqrt{6}+5)z^2 + 16\,(2\,\sqrt{6}+3)xz + 64\,x^2
\end{aligned}$$

and

$$0 \leq \left(z\sqrt{6}+2\,x+3\,z\right)^2 - 4\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z = 4\,x^2$$

for $x \geq 0$ and $z > 0$.

Finally, for p_4, we have

$$\begin{aligned}
p_4 &= \frac{m_1\,x_1\,x_2\,x_3 - m_2(x_1\,x_2 + x_1\,x_3 + x_2\,x_3) + m_3(x_1+x_2+x_3) - m_4}{x_4\,(x_3-x_4)\,(x_2-x_4)\,(x_1-x_4)}\\
&= \frac{{}_{n}p_4}{{}_{d}p_4}
\end{aligned}$$

where

$$\begin{aligned}
{}_{n}p_4 =& 2\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\left(\sqrt{6}-2\right)x^2 - \left(3-\sqrt{6}\right)x^2 z,\\
{}_{d}p_4 =& \left(3\left(3+\sqrt{6}\right)z + 2\,\sqrt{6}\left(2\,x+\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\right)\right)\\
&\times\left(\left(3+\sqrt{6}\right)z+2\,x+2\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}\right)\\
&\times\sqrt{\left((3+\sqrt{6})x + \left(\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6}\right)z\right)z}, \quad x \geq 0, \quad z > 0.
\end{aligned}$$

The positivity of ${}_dp_4$ is obvious. Let us check the nonnegativity of ${}_np_4$ for all $x \geq 0$ and $z > 0$. This is equivalent checking that

$$
\begin{aligned}
0 \leq 4 &\left(\sqrt{6}-2\right)^2 \sqrt{((3+\sqrt{6})x + (\tfrac{15}{4}+\tfrac{3}{2}\sqrt{6})z)z} - \left(3-\sqrt{6}\right)^2 z^2 \\
&= \left(3-\sqrt{6}\right)\left(3\,z\sqrt{6}+8\,x+3\,z\right) z
\end{aligned}
$$

for $x \geq 0$ and $z > 0$.

Let us check that, indeed, the central moments of $\hat{S}_h^x$ satisfy conditions (3.4.16)–(3.4.19) (with $\gamma = 1/2$). The first four are obvious, since the moments of the random variable $\hat{S}_h^x$ exactly match the four first moments of S_h^x, so they also match the first four central moments:

$$
\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x) &= \mathbb{E}(S_h^x - x) = 0, \\
\mathbb{E}(\hat{S}_h^x - x)^2 &= \mathbb{E}(S_h^x - x)^2 = xz, \\
\mathbb{E}(\hat{S}_h^x - x)^3 &= \mathbb{E}(S_h^x - x)^3 = \tfrac{3}{2}\,xz^2, \\
\mathbb{E}(\hat{S}_h^x - x)^4 &= \mathbb{E}(S_h^x - x)^4 = 3\,(x+z)xz^2.
\end{aligned}
$$

Conditions (3.4.20)–(3.4.22) are satisfied, since, respectively,

$$
\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^5 &= \tfrac{27}{4} xz^4 + 15\,x^2z^3 = 15\,x^2z^3 + O(h^4), \\
\mathbb{E}(\hat{S}_h^x - x)^6 &= \tfrac{63}{4}\,xz^5 + 3\,(\sqrt{6}+21)x^2z^4 + 15\,x^3z^3 = 15\,x^3z^3 + O(h^4), \\
|\mathbb{E}(\hat{S}_h^x - x)^7| &= |\tfrac{297}{8}\,xz^6 + \tfrac{1}{4}\,(1053+135\,\sqrt{6})x^2z^5 + (6\,\sqrt{6}+\tfrac{279}{2})x^3z^4| \\
&= O(h^4).
\end{aligned}
$$

Then we estimate that

$$
\begin{aligned}
\mathbb{E}(\hat{S}_h^x - x)^8 = &\tfrac{351}{4}\,xz^7 + (252\,\sqrt{6}+1161)x^2z^6 \\
&+ (108\,\sqrt{6}+918)x^3z^5 + 81\,x^4z^4 = O(h^4).
\end{aligned}
$$

Finally, by the last relation and the expression of the maximal value x_4 of $\hat{S}_h^x$, condition (3.4.23) is satisfied for every $f \in C^{\infty}_{\text{pol}}(\mathbb{D})$ (suppose $|f^{(8)}(x)| \leq C_8(1+x^{k_8})$):

$$
\begin{aligned}
\mathbb{E}\Big[\max_{0\leq s\leq \hat{S}_h^x} |f^{(8)}(s)|(\hat{S}_h^x - x)^8\Big] &\leq \max_{0\leq s\leq x_4} |f^{(8)}(s)|\mathbb{E}(\hat{S}_h^x - x)^8 \\
&\leq C_8(1+x_4^{k_8})\mathbb{E}(\hat{S}_h^x - x)^8 \leq C(1+x^{k_8+1})\mathbb{E}(\hat{S}_h^x - x)^8 = \mathcal{O}(h^4).
\end{aligned}
$$

□

5.4 A strongly potential second-order approximation of the CKLS equation

In this section, we apply to the CKLS equations the method of constructing second-order approximations used in the previous section in the CIR case. As an example, we present strongly potential second-order approximations in the cases $\gamma = 3/4$ and $\gamma = 5/6$, where the results look relatively simple.

Let $\gamma = 3/4$ in the CKLS equation (1.1.1). Then for the stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^{3/4}\,\mathrm{d}B_t$, $S_0^x = x \ge 0$, we have (see Section 7 [Appendix])

$$\begin{aligned}
m_1 &= x,\\
m_2 &= x^2 + x^{3/2}z + \tfrac{3}{16}\,xz^2,\\
m_3 &= x^3 + 3\,x^{5/2}z + \tfrac{45}{16}\,x^2z^2 + \tfrac{15}{16}\,x^{3/2}z^3 + \tfrac{45}{512}\,xz^4.
\end{aligned}$$

Let us use (4.2.15) with $\hat{m}_i = m_i$, $i = 1, 2$, for obtaining values of a two-valued approximation of the stochastic part. In particular, for $\gamma = 3/4$,

$$x_{1,2} = x + x^{1/2}z + \tfrac{3}{16}z^2 \mp \sqrt{\big(x^{3/2} + \tfrac{19}{16}\,xz + \tfrac{3}{8}\,x^{1/2}z^2 + \tfrac{9}{256}\,z^3\big)z}.$$

This motivated us to look for the second-order approximations with values of the following form:

$$\begin{aligned}
x_{1,3} &= x + A_1x^{1/2}z + A_2z^2 \mp \sqrt{\big(B_1x^{3/2} + B_2xz + B_3x^{1/2}z^2 + B_4z^3\big)z},\\
x_2 &= x + C_1x^{1/2}z + C_2z^2, \quad A_1, A_2, B_1, B_2, B_3, B_4, C_1, C_2 > 0,
\end{aligned}$$

with probabilities (5.2.4). Using the same method as in the CIR case, after tedious and rather complex calculations, we arrive at the scheme with values

$$\begin{aligned}
x_{1,3} &= x + \tfrac{5}{2}\,x^{1/2}\sigma^2h + \tfrac{15}{64}\,(\sigma^2h)^2\\
&\quad \mp \sqrt{\big(3\,x^{3/2} + \tfrac{103}{16}\,x\sigma^2h + \tfrac{75}{64}\,x^{1/2}(\sigma^2h)^2 + \tfrac{225}{4096}\,(\sigma^2h)^3\big)\sigma^2h},\\
x_2 &= x + \tfrac{11}{8}\,x^{1/2}\sigma^2h + \tfrac{15}{64}\,(\sigma^2h)^2,
\end{aligned} \tag{5.4.1}$$

and probabilities p_1, p_2, and p_3 defined in (5.2.4).

Similarly, in the case $\gamma = 5/6$, we have

$$
\begin{aligned}
m_1 &= x, \\
m_2 &= x^2 + x^{5/3} z + \tfrac{5}{18}\, x^{4/3} z^2 + \tfrac{5}{243}\, x z^3, \\
m_3 &= x^3 + 3\, x^{8/3} z + \tfrac{10}{3}\, x^{7/3} z^2 + \tfrac{140}{81}\, x^2 z^3 + \tfrac{35}{81}\, x^{5/3} z^4 \\
&\qquad + \tfrac{35}{729}\, x^{4/3} z^5 + \tfrac{35}{19683}\, x z^6.
\end{aligned}
$$

The corresponding approximation takes the values

$$
\begin{aligned}
x_{1,3} &= x + \tfrac{3}{2}\, x^{2/3}\sigma^2 h + \tfrac{485}{816}\, x^{1/3}(\sigma^2 h)^2 + \tfrac{1681}{22032}\, (\sigma^2 h)^3 \\
&\quad \mp \big((3\, x^{5/3} + \tfrac{2077}{612}\, x^{4/3}\sigma^2 h + \tfrac{125695}{66096}\, x(\sigma^2 h)^2 + \tfrac{1162907}{1997568}\, x^{2/3}(\sigma^2 h)^3 \\
&\quad + \tfrac{815285}{8989056}\, x^{1/3}(\sigma^2 h)^4 + \tfrac{2825761}{485409024}(\sigma^2 h)^5)\sigma^2 h\big)^{1/2}, \qquad (5.4.2) \\
x_2 &= x + \tfrac{1}{4}\, x^{2/3}\sigma^2 h + \tfrac{5}{72}\, x^{1/3}(\sigma^2 h)^2 + \tfrac{1}{72}(\sigma^2 h)^3,
\end{aligned}
$$

with probabilities p_1, p_2, and p_3 defined in (5.2.4).

In summary, we have the following:

Theorem 5.3 (Theorem 1.3)**.** *Let $\hat{X}_t^x$ be the discretization scheme defined by composition* (3.3.4)*, where $\hat{S}_h^x$ takes the values x_1, x_2, and x_3 defined in* (5.4.1) *in the case $\gamma = 3/4$ or in* (5.4.2) *in the case $\gamma = 5/6$ with probabilities p_1, p_2, and p_3 defined in* (5.2.4) ($\hat{S}_h^0 = 0$)*. Then $\hat{X}_t^x$ is a strongly potential second-order discretization scheme for the CKLS equation with $\gamma = 3/4$ or $\gamma = 5/6$, respectively.*

5.5 Algorithm

We give a short algorithm for calculating $\hat{X}_{(i+1)h}$ given $\hat{X}_{ih} = x$ at each simulation step i:

1. Substitute $x := D(x, h/2)$.

2. Draw a uniform random number U in the interval $[0, 1]$.

3. Generate a random variable $\hat{S}$ taking the values x_1, x_2, and x_3 defined by (5.2.6), (5.4.1), or (5.4.2) (for ${}_{1/2}\hat{S}_h^x$, ${}_{3/4}\hat{S}_h^x$, or ${}_{5/6}\hat{S}_h^x$, respectively) with probabilities p_1, p_2 and p_3 defined in (5.2.4):

if $U < p_1$, then $\hat{S} := x_1$; otherwise, if $U < p_1 + p_2$, then $\hat{S} := x_2$; otherwise, $\hat{S} := x_3$. If $x = 0$, then $\hat{S} := 0$.

4. Calculate (see (3.3.4))

$$\hat{X}_{(i+1)h} = D\Big(\hat{S}, h/2\Big).$$

In the case of a strongly potential *third-order* approximation of ${}_{1/2}\hat{S}^x_h$, step (3) should be replaced by

(3') Generate a random variable $\hat{S}$ taking the values x_1, x_2, x_3, and x_4 defined by (5.3.4) with probabilities p_1, p_2, p_3, and p_4 defined in (5.3.3):

if $U < p_1$, then $\hat{S} := x_1$; otherwise, if $U < p_1 + p_2$, then $\hat{S} := x_2$; otherwise, if $U < p_1 + p_2 + p_3$, then $\hat{S} := x_3$; otherwise, $\hat{S} := x_4$. If $x = 0$, then $\hat{S} := 0$.

5.6 Simulation examples

We indicate a particular γ of the stochastic part (3.3.1) by the left subscript γ as in ${}_\gamma S^x_t$. Using our discretization schemes, we simulate the solutions of the CLKS equation (1.1.1) or its stochastic part (3.3.1) for $\gamma = 1/2$, $3/4$, and $5/6$ with test functions $f(x) = x^3$, x^4, x^5, and e^{-x}. Such a choice of f is motivated by having explicit formulas for the expectations $\mathbb{E}f(S^x_t)$ (see Section 7 [Appendix]) and, in the case $\gamma = 1/2$, (4.5.1) (see, e.g., [21, Prop. 6.2.5]). We also simulate the solution of the CLKS equation (1.1.1) for $\gamma = 1/2$ (i.e., the CIR equation) with discretization scheme defined in (5.3.3)–(5.3.4) and test function $f(x) = \mathrm{e}^{-x}$.

Below we present the results by a number of figures, were the exact and approximate expectations are given as functions of the approximation step size h. For the reader's convenience, we give a list of graphs in the figures:

- Figs. 5.1 and 5.2: $\mathbb{E}\mathrm{e}^{-({}_{1/2}X^x_1)}$ and $\mathbb{E}\mathrm{e}^{-({}_{1/2}\hat{X}^x_1)}$ with the same parameters as in Alfonsi [2];

- Figs. 5.3 and 5.4: $\mathbb{E}({}_{3/4}S_1^x)^3$ and $\mathbb{E}({}_{3/4}\hat{S}_1^x)^3$;
- Figs. 5.5 and 5.6: $\mathbb{E}({}_{3/4}S_1^x)^4$ and $\mathbb{E}({}_{3/4}\hat{S}_1^x)^4$;
- Figs. 5.7 and 5.8: $\mathbb{E}({}_{3/4}S_1^x)^5$ and $\mathbb{E}({}_{3/4}\hat{S}_1^x)^5$;
- Figs. 5.9 and 5.10: $\mathbb{E}({}_{5/6}S_1^x)^3$ and $\mathbb{E}({}_{5/6}\hat{S}_1^x)^3$;
- Figs. 5.11 and 5.12: $\mathbb{E}({}_{5/6}S_1^x)^4$ and $\mathbb{E}({}_{5/6}\hat{S}_1^x)^4$;
- Figs. 5.13 and 5.14: $\mathbb{E}({}_{5/6}S_1^x)^5$ and $\mathbb{E}({}_{5/6}\hat{S}_1^x)^5$;
- Figs. 5.15 and 5.16: $\mathbb{E}\mathrm{e}^{-({}_{3/4}\hat{X}_1^x)}$;
- Figs. 5.17 and 5.18: $\mathbb{E}\mathrm{e}^{-({}_{5/6}\hat{X}_1^x)}$.

Figures 5.1, 5.15, and 5.17 represent the values of $\mathbb{E}\mathrm{e}^{-X_1^x}$ with "low" volatility ($\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$). Figures 5.2, 5.16, and 5.18 represent the values of $\mathbb{E}\mathrm{e}^{-X_1^x}$ with "high" volatility ($\sigma = 2.0$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$). Figures 5.3, 5.5, 5.7, 5.9, 5.11, and 5.13 represent values of $\mathbb{E}f(S_1^x)$ with "low" volatility ($\sigma = 0.8$, $x_0 = 1.5$). Figures 5.4, 5.6, 5.8, 5.10, 5.12, 5.14 and represent the values of $\mathbb{E}f(S_1^x)$ with "high" volatility ($\sigma = 1.5$, $x_0 = 0.3$). In all the graphs, the error bars show 95% confidence intervals. To shorten the bars, for approximation time-step sizes $h = 1/2^i$, $i = 0, 1, 2, 3, 4, 5$, we have generated $N = 90{,}000 \cdot 4^i$ samples of approximations.

In the legends of figures, we use the following notation.

1. "First ord. GLVM": the modified first-order scheme for the CIR (Rem. 4.5, [27, Rem. 4]) (for comparison with higher-order schemes);
2. "Second ord. GLVM": our second-order scheme for the CIR (Thm. 5.1);
3. "Third ord. GLVM": the second-order composition (3.3.4) with our third-order scheme $\hat{S}_h^x$ taking the values x_1, x_2, x_3, and x_4 defined in (5.3.4) with probabilities p_1, p_2, p_3, and p_4 defined in (5.3.3);

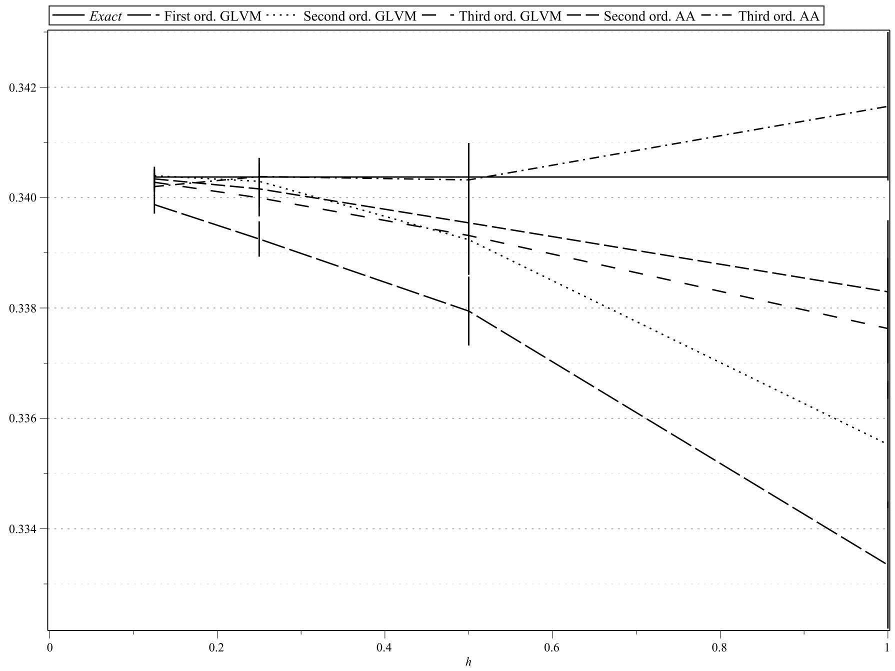

Figure 5.1: $\mathbb{E}e^{-(1/2\hat{X}_1^x)}$ as functions of h: $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$.

4. "First ord.": our first-order scheme for the CKLS (Thm. 4.2, [27, Thm. 2]);

5. "Second ord.": our second-order schemes for the CKLS (Thm. 5.3);

6. "Second ord. AA": the second-order scheme of Alfonsi for the CIR [2, Thm. 2.8];

7. "Third ord. AA": the third-order scheme of Alfonsi for the CIR [2, Thm. 3.7].

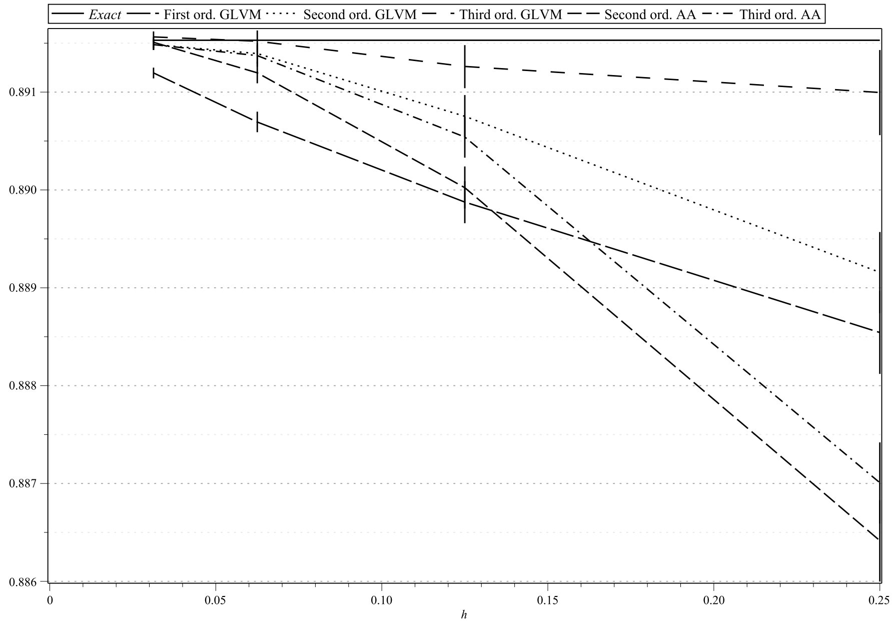

Figure 5.2: $\mathbb{E}e^{-({}_{1/2}\hat{X}_1^x)}$ as functions of h: $\sigma = 2.0$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$.

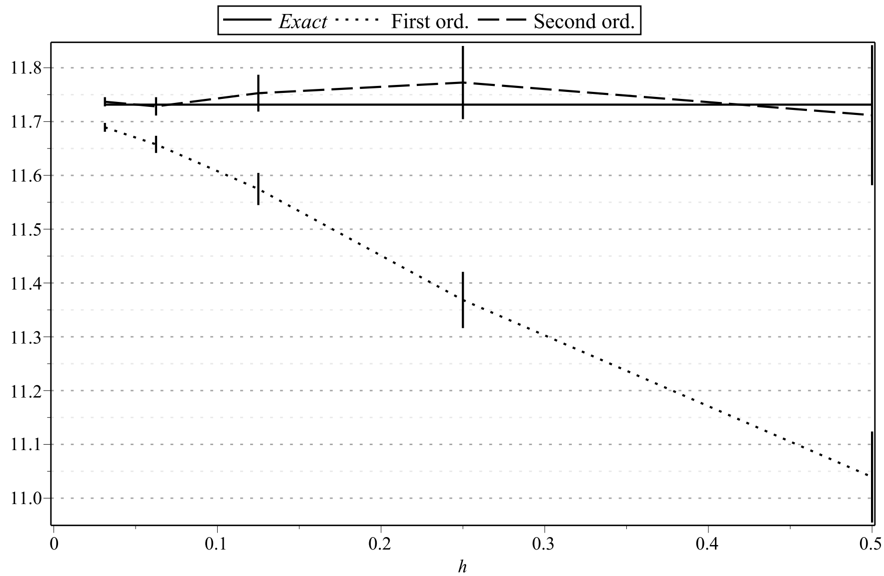

Figure 5.3: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^3$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

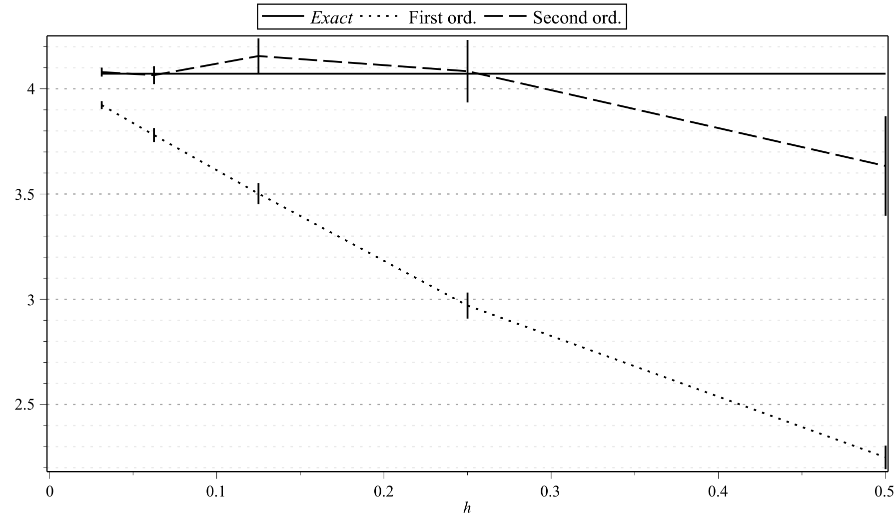

Figure 5.4: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^3$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

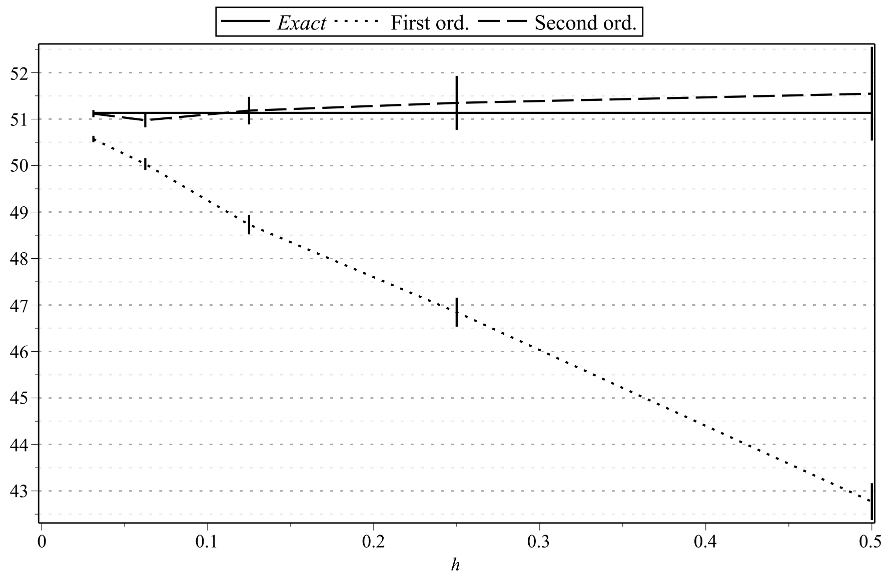

Figure 5.5: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^4$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

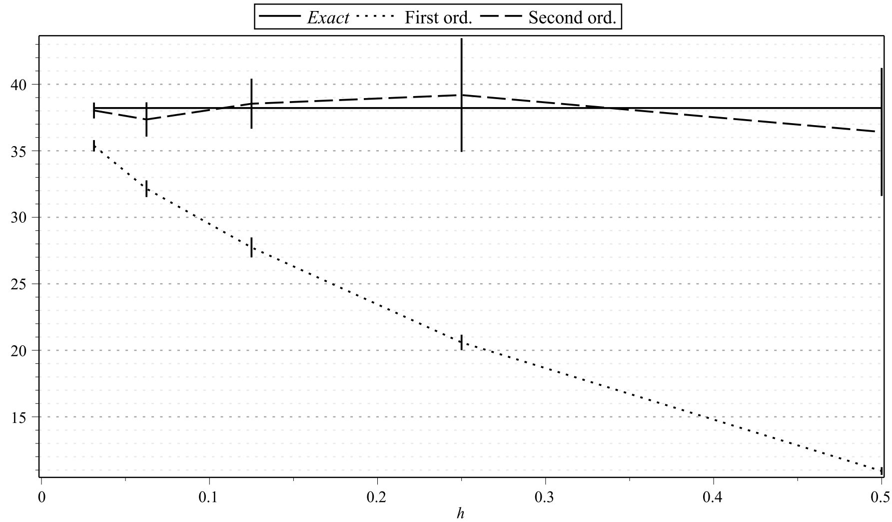

Figure 5.6: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^4$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

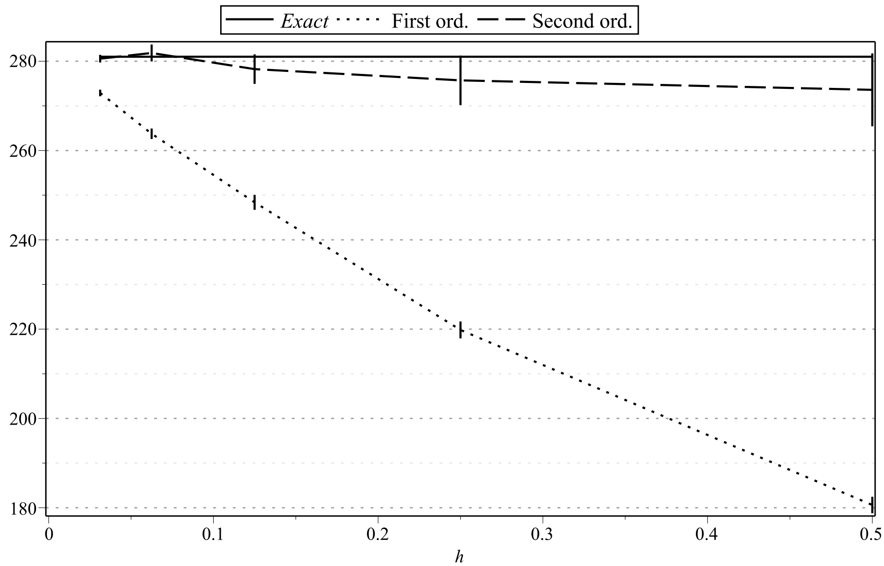

Figure 5.7: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^5$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

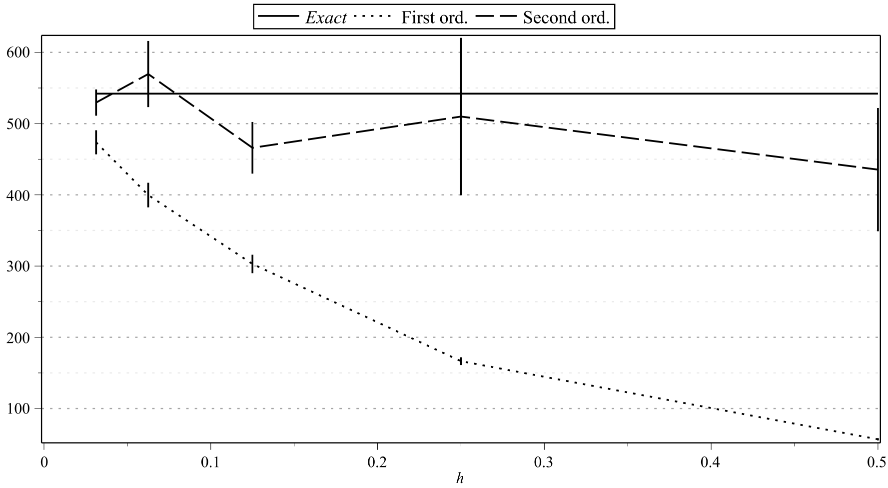

Figure 5.8: $\mathbb{E}({}_{3/4}\hat{S}_1^x)^5$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

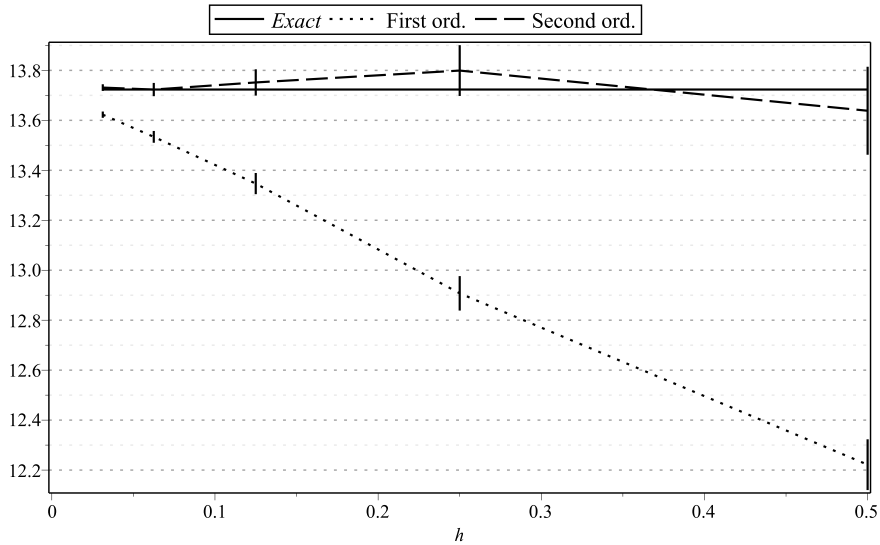

Figure 5.9: $\mathbb{E}({}_{5/6}\hat{S}_1^x)^3$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

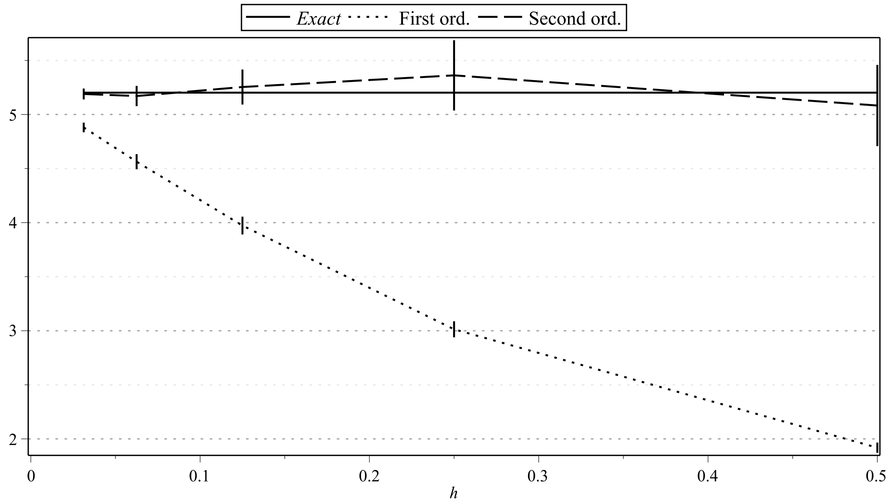

Figure 5.10: $\mathbb{E}({}_{5/6}\hat{S}_1^x)^3$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

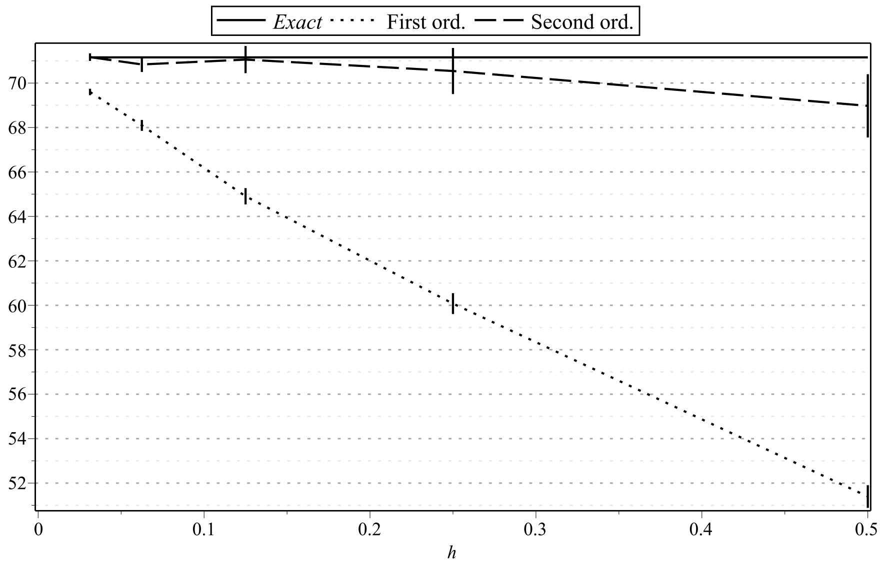

Figure 5.11: $\mathbb{E}({}_{5/6}\hat{S}_1^x)^4$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

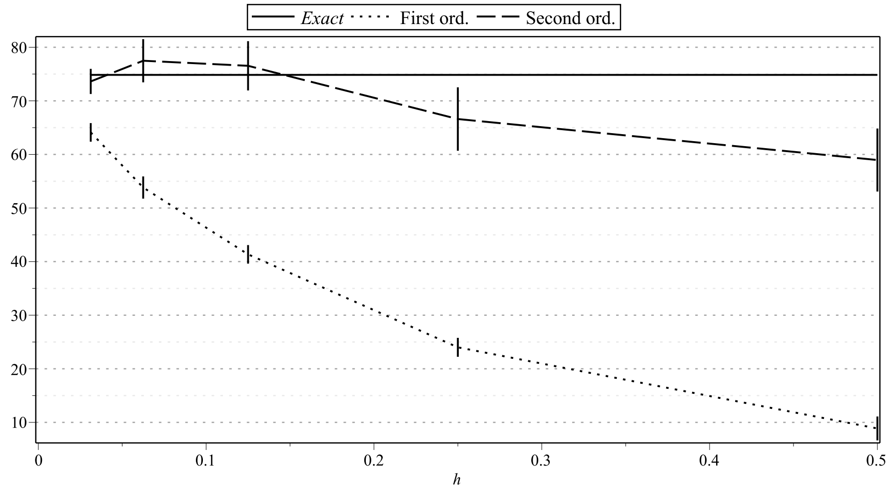

Figure 5.12: $\mathbb{E}({}_{5/6}\hat{S}_1^x)^4$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

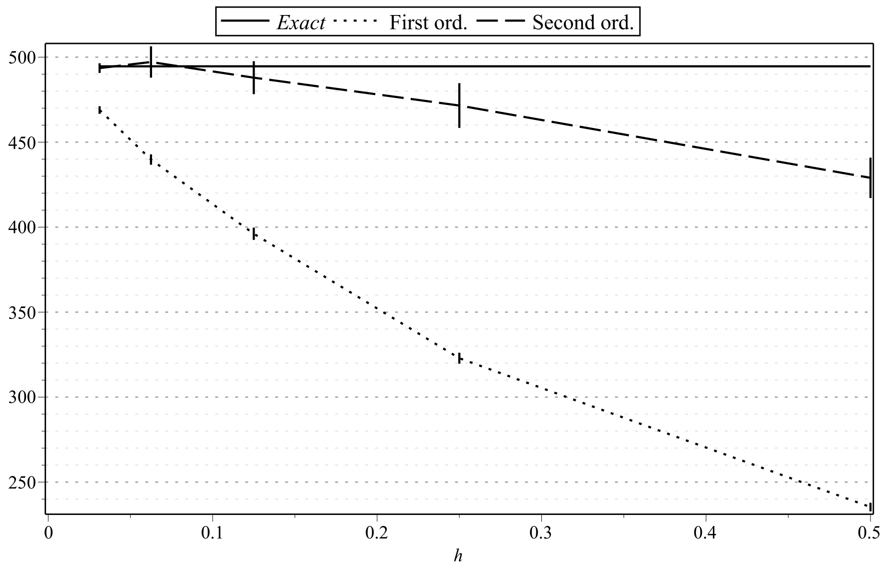

Figure 5.13: $\mathbb{E}({}_{5/6}\hat{S}_1^x)^5$ as functions of h: $\sigma = 0.8$, $x_0 = 1.5$.

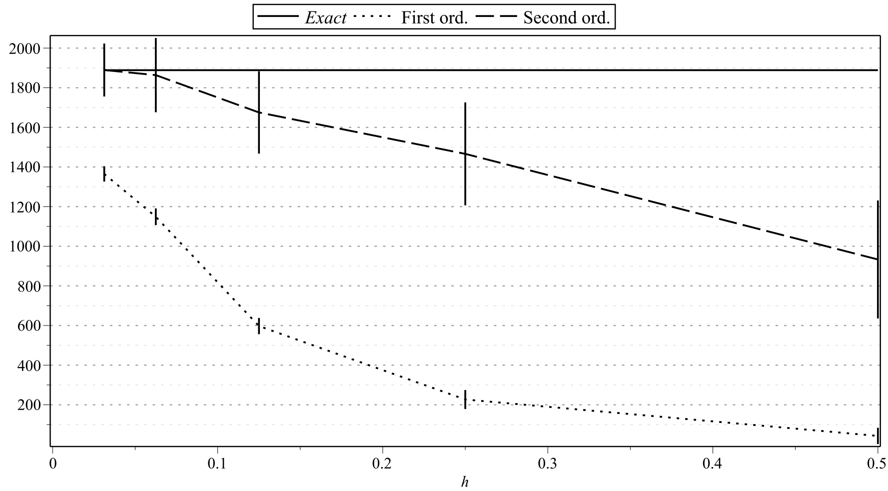

Figure 5.14: $\mathbb{E}(_{5/6}\hat{S}_1^x)^5$ as functions of h: $\sigma = 1.5$, $x_0 = 0.3$.

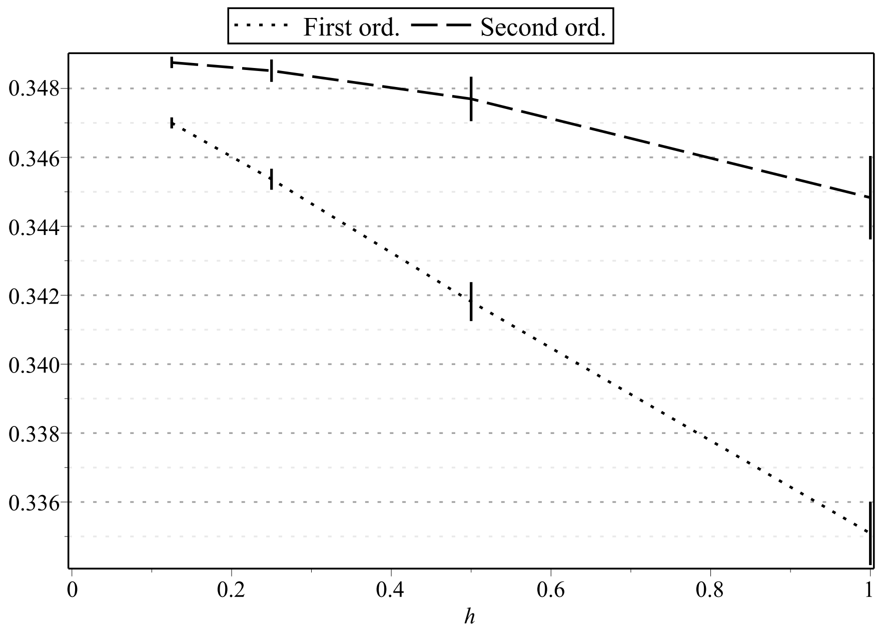

Figure 5.15: $\mathbb{E}e^{-(_{3/4}\hat{X}_1^x)}$ as functions of h: $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$.

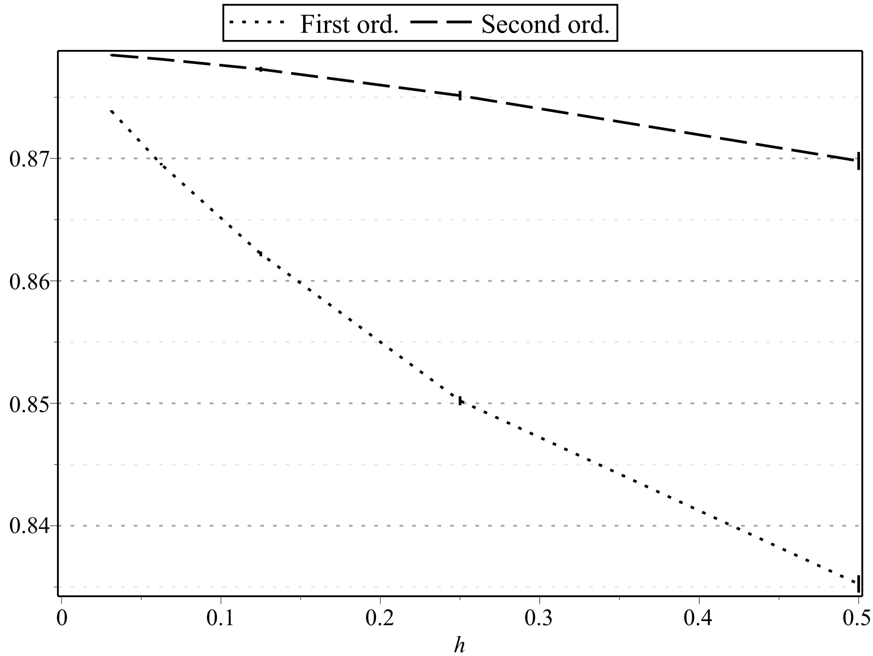

Figure 5.16: $\mathbb{E}\mathrm{e}^{-(3/4\hat{X}_1^x)}$ as functions of h: $\sigma = 2.0$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$.

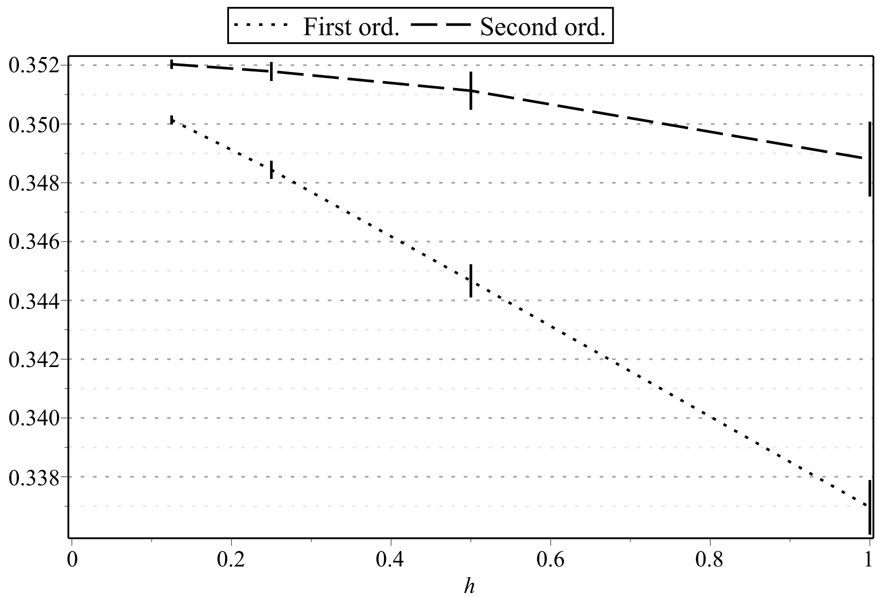

Figure 5.17: $\mathbb{E}\mathrm{e}^{-(5/6\hat{X}_1^x)}$ as functions of h: $\sigma = 0.8$, $\theta = 0.5$, $\beta = 0.5$, $x_0 = 1.5$.

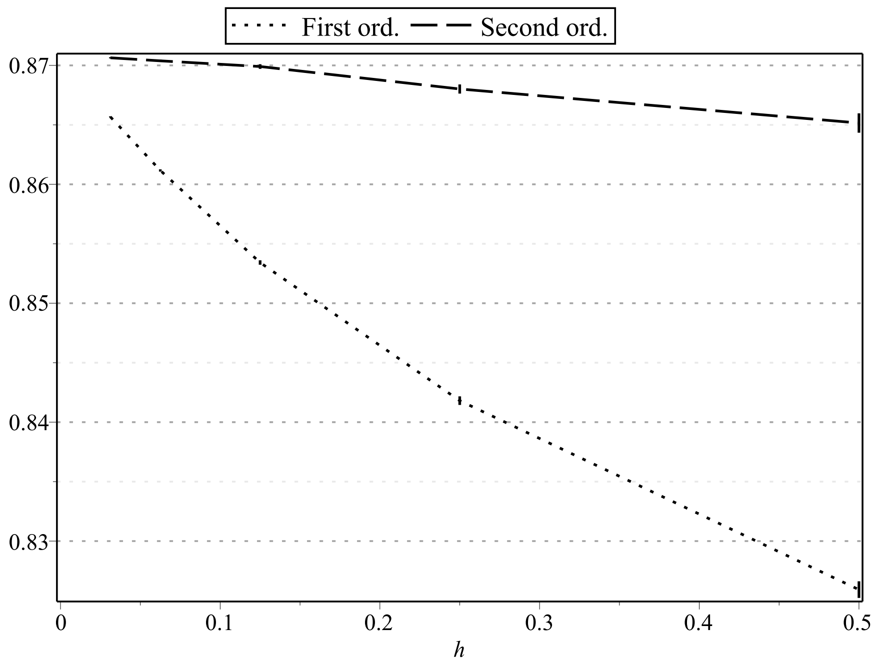

Figure 5.18: $\mathbb{E}e^{-({}_{5/6}\hat{X}_1^x)}$ as functions of h: $\sigma = 2.0$, $\theta = 0.04$, $\beta = 0.1$, $x_0 = 0.3$.

Chapter 6

Conclusions

In the doctoral thesis, we construct first- and second-order weak approximations for the CKLS model using split-step, moments matching, and approximate moment matching techniques. Split-step technique allows us to divide the model into deterministic and stochastic parts, so that we need to construct a discretization scheme for the stochastic part only, as the deterministic part is easily solvable in explicit way. Moment matching and approximate moments techniques help to construct discrete random variables so that we get weak approximations of the desired order.

The following contributions are the main results of the thesis:

- A construction of potential first-order two-valued approximations of a stochastic part that does not require knowing the exact finite moments of the stochastic part;

- A strongly potential first-order approximation of the CKLS equation;

- A new construction method for second-order weak approximations based on a particular form of approximating random variable and using approximate moments in moment matching;

- A strongly potential second-order approximation for the CIR equation without switching to another scheme near zero;

- A strongly potential second-order approximation for the CKLS equation;
- A potential third-order approximation for the stochastic part of the CIR equation without switching to another scheme near zero.

Some ideas for the research in the future:

- Investigate the restrictions and possible extensions of the construction of potential first-order two-valued approximations of a stochastic part that does not require knowing the exact finite moments of the stochastic part;
- Investigate the restrictions and possible extensions of the new construction method for second-order weak approximations based on a particular form of approximating random variable and using approximate moments in moment matching;
- Apply in Heston model for option pricing the strongly potential second-order approximation for the CIR equation without switching to another scheme near zero;
- Apply to other classes of equations the idea of construction of the strongly potential second-order approximation for the CKLS equation.

Chapter 7

Appendix

In this chapter, we provide additional calculations which we think would only distract the reader if placed elsewhere in the text.

Moments of the CKLS model

Using Itô's formula for the solution of stochastic part $\mathrm{d}S_t^x = \sigma(S_t^x)^\gamma \mathrm{d}B_t$, $S_0^x = x$, with $\gamma = [1/2, 1)$, we have

$$(S_t^x)^p = x^p + p\sigma \int\limits_0^t (S_s^x)^{\gamma+p-1} \mathrm{d}B_s + \tfrac{p(p-1)\sigma^2}{2} \int\limits_0^t (S_s^x)^{2\gamma+p-2} \mathrm{d}s,$$

and thus

$$\mathbb{E}(S_t^x)^p = x^p + \tfrac{p(p-1)\sigma^2}{2} \int\limits_0^t \mathbb{E}(S_s^x)^{2\gamma+p-2} \mathrm{d}s. \tag{7.0.1}$$

In particular,

$$\mathbb{E}(S_t^x) = x, \tag{7.0.2}$$

$$\mathbb{E}(S_t^x)^2 = x^2 + \sigma^2 \int\limits_0^t \mathbb{E}(S_s^x)^{2\gamma} \mathrm{d}s, \tag{7.0.3}$$

$$\mathbb{E}(S_t^x)^3 = x^3 + 3\sigma^2 \int\limits_0^t \mathbb{E}(S_s^x)^{2\gamma+1} \mathrm{d}s. \tag{7.0.4}$$

From the recurrence relation (7.0.1) we can get $\mathbb{E}(S_t^x)^2$ and $\mathbb{E}(S_t^x)^3$ if $\gamma = k/(k+1)$, where $k = 1, 2, \dots$ (we further indicate a particular γ of the stochastic part (3.3.1) by the left subscript γ as in ${}_{\gamma}S_t^x$). In these cases the recurrence calculation "stops" at $\mathbb{E}({}_{\gamma}S_t^x)^p$, where $p \in \mathbb{N}_0$, and we already know them. For example, if $\gamma = 3/4$, then using (7.0.3), we get

$$\mathbb{E}({}_{3/4}S_t^x)^2 = x^2 + \sigma^2 \int_0^t \mathbb{E}({}_{3/4}S_s^x)^{3/2}\mathrm{d}s.$$

Using (7.0.1) and (7.0.2), we get

$$\begin{aligned}\mathbb{E}({}_{3/4}S_t^x)^{3/2} =& x^{3/2} + \tfrac{3}{8}\sigma^2 \int_0^t \mathbb{E}({}_{3/4}S_s^x)\mathrm{d}s = x^{3/2} + \tfrac{3}{8}\sigma^2 \int_0^t x\mathrm{d}s \\ =& x^{3/2} + \tfrac{3}{8}x\sigma^2 t.\end{aligned}$$

For brevity, we use the notation $w := \sigma^2 t$. We have

$$\begin{aligned}\mathbb{E}({}_{3/4}S_t^x)^2 =& x^2 + \sigma^2 \int_0^t (x^{3/2} + \tfrac{3}{8}\sigma^2 xs)\mathrm{d}s \\ =& x^2 + x^{3/2}\sigma^2 t + \tfrac{3}{16}x\sigma^4 t^2 \end{aligned} \tag{7.0.5}$$

or

$$\mathbb{E}({}_{3/4}S_t^x)^2 = x^2 + x^{3/2}w + \tfrac{3}{16}xw^2.$$

Using (7.0.4), (7.0.1), and (7.0.5), we have

$$\mathbb{E}({}_{3/4}S_t^x)^3 = x^3 + 3\sigma^2 \int_0^t \mathbb{E}({}_{3/4}S_s^x)^{5/2}\mathrm{d}s,$$

$$\begin{aligned}\mathbb{E}({}_{3/4}S_t^x)^{5/2} &= x^{5/2} + \tfrac{15}{8}\sigma^2 \int_0^t \mathbb{E}({}_{3/4}S_s^x)^2\mathrm{d}s \\ &= x^{5/2} + \tfrac{15}{8}\sigma^2 \int_0^t (x^2 + \sigma^2 x^{3/2}s + \tfrac{3}{16}\sigma^4 xs^2)\mathrm{d}s \\ &= x^{5/2} + \tfrac{15}{8}\sigma^2 x^2 t + \tfrac{15}{16}\sigma^4 x^{3/2}t^2 + \tfrac{15}{128}\sigma^6 xt^3,\end{aligned}$$

and finally,

$$\mathbb{E}({}_{3/4}S_t^x)^3 = x^3 + 3\sigma^2 \int_0^t (x^{5/2} + \tfrac{15}{8}\sigma^2 x^2 s + \tfrac{15}{16}\sigma^4 x^{3/2}s^2 + \tfrac{15}{128}\sigma^6 xs^3)\mathrm{d}s$$

$$= x^3 + 3\sigma^2 x^{5/2} t + \tfrac{45}{16}\sigma^4 x^2 t^2 + \tfrac{15}{16}\sigma^6 x^{3/2} t^3 + \tfrac{45}{512}\sigma^8 x t^4$$
$$= x^3 + 3x^{5/2} w + \tfrac{45}{16} x^2 w^2 + \tfrac{15}{16} x^{3/2} w^3 + \tfrac{45}{512} x w^4.$$

Then we have:

$$\begin{aligned}
\mathbb{E}({}_{1/2}S_t^x)^2 &= x^2 + xw,\\
\mathbb{E}({}_{2/3}S_t^x)^2 &= x^2 + x^{4/3}w + \tfrac{1}{9}\, x^{2/3}w^2 - \tfrac{1}{243}\, w^3,\\
\mathbb{E}({}_{3/4}S_t^x)^2 &= x^2 + x^{3/2}w + \tfrac{3}{16}\, xw^2,\\
\mathbb{E}({}_{4/5}S_t^x)^2 &= x^2 + x^{8/5}w + \tfrac{6}{25}\, x^{6/5}w^2 + \tfrac{6}{625}\, x^{4/5}w^3 - \tfrac{3}{15625}\, x^{2/5}w^4\\
&\quad + \tfrac{9}{1953125}\, w^5,\\
\mathbb{E}({}_{5/6}S_t^x)^2 &= x^2 + x^{5/3}w + \tfrac{5}{18}\, x^{4/3}w^2 + \tfrac{5}{243}\, xw^3,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}({}_{1/2}S_t^x)^3 &= x^3 + 3\, x^2w + \tfrac{3}{2}\, xw^2,\\
\mathbb{E}({}_{2/3}S_t^x)^3 &= x^3 + 3\, x^{7/3}w + \tfrac{7}{3}\, x^{5/3}w^2 + \tfrac{35}{81}\, xw^3,\\
\mathbb{E}({}_{3/4}S_t^x)^3 &= x^3 + 3\, x^{5/2}w + \tfrac{45}{16}\, x^2w^2 + \tfrac{15}{16}\, x^{3/2}w^3 + \tfrac{45}{512}\, xw^4,\\
\mathbb{E}({}_{4/5}S_t^x)^3 &= x^3 + 3\, x^{13/5}w + \tfrac{78}{25}\, x^{11/5}w^2 + \tfrac{858}{625}\, x^{9/5}w^3 + \tfrac{3861}{15625}\, x^{7/5}w^4\\
&\quad + \tfrac{27027}{1953125}\, xw^5,\\
\mathbb{E}({}_{5/6}S_t^x)^3 &= x^3 + 3\, x^{8/3}w + \tfrac{10}{3}\, x^{7/3}w^2 + \tfrac{140}{81}\, x^2w^3 + \tfrac{35}{81}\, x^{5/3}w^4\\
&\quad + \tfrac{35}{729}\, x^{4/3}w^5 + \tfrac{35}{19683}\, xw^6,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}({}_{1/2}S_t^x)^4 &= x^4 + 6\, x^3w + 9\, x^2w^2 + 3\, xw^3,\\
\mathbb{E}({}_{3/4}S_t^x)^4 &= x^4 + 6\, x^{7/2}w + \tfrac{105}{8}\, x^3w^2 + \tfrac{105}{8}\, x^{5/2}w^3 + \tfrac{1575}{256}\, x^2w^4\\
&\quad + \tfrac{315}{256}\, x^{3/2}w^5 + \tfrac{315}{4096}\, xw^6,\\
\mathbb{E}({}_{5/6}S_t^x)^4 &= x^4 + 6\, x^{11/3}w + \tfrac{44}{3}\, x^{10/3}w^2 + \tfrac{1540}{81}\, x^3w^3 + \tfrac{385}{27}\, x^{8/3}w^4\\
&\quad + \tfrac{1540}{243}\, x^{7/3}w^5 + \tfrac{10780}{6561}\, x^2w^6 + \tfrac{1540}{6561}\, x^{5/3}w^7 + \tfrac{1925}{118098}\, x^{4/3}w^8\\
&\quad + \tfrac{1925}{4782969}\, xw^9,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}({}_{1/2}S_t^x)^5 &= x^5 + 10\, x^4w + 30\, x^3w^2 + 30\, x^2w^3 + \tfrac{15}{2}\, xw^4,\\
\mathbb{E}({}_{3/4}S_t^x)^5 &= x^5 + 10\, x^{9/2}w + \tfrac{315}{8}\, x^4w^2 + \tfrac{315}{4}\, x^{7/2}w^3 + \tfrac{11025}{128}\, x^3w^4\\
&\quad + \tfrac{6615}{128}\, x^{5/2}w^5 + \tfrac{33075}{2048}\, x^2w^6 + \tfrac{4725}{2048}\, x^{3/2}w^7 + \tfrac{14175}{131072}\, xw^8,
\end{aligned}$$

$$\begin{aligned}\mathbb{E}({}_{5/6}S_t^x)^5 &= x^5 + 10\,x^{14/3}w + \tfrac{385}{9}\,x^{13/3}w^2 + \tfrac{25025}{243}\,x^4w^3 + \tfrac{25025}{162}\,x^{11/3}w^4\\ &+ \tfrac{110110}{729}\,x^{10/3}w^5 + \tfrac{1926925}{19683}\,x^3w^6 + \tfrac{275275}{6561}\,x^{8/3}w^7\\ &+ \tfrac{1376375}{118098}\,x^{7/3}w^8 + \tfrac{9634625}{4782969}\,x^2w^9 + \tfrac{1926925}{9565938}\,x^{5/3}w^{10}\\ &+ \tfrac{875875}{86093442}\,x^{4/3}w^{11} + \tfrac{875875}{4649045868}\,xw^{12},\end{aligned}$$

$$\begin{aligned}\mathbb{E}({}_{1/2}S_t^x)^6 &= x^6 + 15\,x^5w + 75\,x^4w^2 + 150\,x^3w^3 + \tfrac{225}{2}\,x^2w^4 + \tfrac{45}{2}\,xw^5,\\ \mathbb{E}({}_{3/4}S_t^x)^6 &= x^6 + 15\,x^{11/2}w + \tfrac{1485}{16}\,x^5w^2 + \tfrac{2475}{8}\,x^{9/2}w^3 + \tfrac{155925}{256}\,x^4w^4\\ &+ \tfrac{93555}{128}\,x^{7/2}w^5 + \tfrac{1091475}{2048}\,x^3w^6 + \tfrac{467775}{2048}\,x^{5/2}w^7\\ &+ \tfrac{7016625}{131072}\,x^2w^8 + \tfrac{779625}{131072}\,x^{3/2}w^9 + \tfrac{467775}{2097152}\,xw^{10},\\ \mathbb{E}({}_{5/6}S_t^x)^6 &= x^6 + 15\,x^{17/3}w + \tfrac{595}{6}\,x^{16/3}w^2 + \tfrac{30940}{81}\,x^5w^3 + \tfrac{77350}{81}\,x^{14/3}w^4\\ &+ \tfrac{1191190}{729}\,x^{13/3}w^5 + \tfrac{38713675}{19683}\,x^4w^6 + \tfrac{11061050}{6561}\,x^{11/3}w^7\\ &+ \tfrac{60835775}{59049}\,x^{10/3}w^8 + \tfrac{2129252125}{4782969}\,x^3w^9 + \tfrac{425850425}{3188646}\,x^{8/3}w^{10}\\ &+ \tfrac{387136750}{14348907}\,x^{7/3}w^{11} + \tfrac{1354978625}{387420489}\,x^2w^{12} + \tfrac{104229125}{387420489}\,x^{5/3}w^{13}\\ &+ \tfrac{74449375}{6973568802}\,x^{4/3}w^{14} + \tfrac{14889875}{94143178827}\,xw^{15}.\end{aligned}$$

For γ other than $k/(k+1)$, $k \in \mathbb{N}$, we cannot calculate $\mathbb{E}({}_\gamma S_t^x)^2$ or $\mathbb{E}({}_\gamma S_t^x)^3$ because the recurrence calculation requires, for example, to use unknown $\mathbb{E}({}_\gamma S_t^x)^p$, where $p < 0$. However, we want to test our approximation with γ from the interval $(1/2, 3/4)$. From the recurrence relation (7.0.1) we notice that the recurrence calculation immediately "stops" at $\mathbb{E}({}_\gamma S_t^x) = x$ if

$$2\gamma + p - 2 = 1, \quad 1/2 \le \gamma < 1, \quad p \ge 0.$$

Using this simple equation, we get that for $\gamma = 11/20$, we can calculate $\mathbb{E}({}_{11/20}S_t^x)^{19/10}$. Indeed,

$$\begin{aligned}\mathbb{E}({}_{11/20}S_t^x)^{19/10} &= x^{19/10} + \tfrac{171}{200}\,\sigma^2\int_0^t \mathbb{E}(S_s^x)\mathrm{d}s = x^{19/10} + \tfrac{171}{200}\,\sigma^2 xt\\ &= x^{19/10} + \tfrac{171}{200}\,wx. \qquad (7.0.6)\end{aligned}$$

In the same way, we calculate the following moments for $\gamma = 3/5$, $13/20$, and $7/10$:

$$\begin{aligned}
\mathbb{E}({}_{11/20}S^x_t)^{19/10} &= y^{19/10} + \tfrac{171}{200}\, yw,\\
\mathbb{E}({}_{3/5}S^x_t)^{9/5} &= y^{9/5} + \tfrac{18}{25}\, yw,\\
\mathbb{E}({}_{13/20}S^x_t)^{17/10} &= y^{17/10} + \tfrac{119}{200}\, yw,\\
\mathbb{E}({}_{7/10}S^x_t)^{8/5} &= y^{8/5} + \tfrac{12}{25}\, yw.
\end{aligned}$$

Finding values of a three-valued random variable

We find the coefficients A_1, B, and C in a three-valued random variable (5.2.5):

$$x_{1,3} = x + A_1 z \mp \sqrt{(Bx + Cz)z}, \quad x_2 = x + A_2 z,$$

where the corresponding probabilities p_1, p_2, and p_3 are given in (5.2.4). For this purpose, we will calculate the differences

$$d_i := x_1^i p_1 + x_2^i p_2 + x_3^i p_3 - \hat{m}_i,\ i = 1, \dots, 6,$$

where $\hat{m}_i$ are of $\gamma = 1/2$ (3.4.15). Since $d_i = 0$, $i = 1, 2, 3$, because $\hat{m}_i - m_i = 0$, $i = 1, 2, 3$, and m_i, $i = 1, 2, 3$, are used in (5.2.4), we proceed with $i = 4, 5, 6$:

$$\begin{aligned}
d_4 &= (B\,(1 - A_2) + 2\,A_1 + A_2 - 9/2)\, x^2 z^2 + O(h^3),\\
d_5 &= (B\,(5 - 4\,A_2) + 4\,(2\,A_1 + A_2) - 21)\, x^3 z^2 + O(h^3),\\
d_6 &= (5\,B\,(3 - 2\,A_2) + 10\,(2\,A_1 + A_2) - 60)\, x^4 z^2 + O(h^3).
\end{aligned}$$

From these expressions we get the system of equations

$$\begin{cases}
B\,(1 - A_2) + 2\,A_1 + A_2 - 9/2 = 0,\\
B\,(5 - 4\,A_2) + 4\,(2\,A_1 + A_2) - 21 = 0,\\
5\,B\,(3 - 2\,A_2) + 20\,A_1 + 10\,A_2 - 60 = 0,
\end{cases} \tag{7.0.7}$$

which has the solution $\{B = 3, A_1 = A_2 + 3/4\}$. We substitute B and A_2 into (5.2.5) with expressions found and get

$$x_{1,3} = x + (A_2 + 3/4)z \mp \sqrt{(3x + Cz)z}, \quad x_2 = x + A_2 z.$$

Then we solve the equation

$$\Sigma\, p_i = 1$$

or

$$\begin{aligned}&(\text{Denominator of } \Sigma) - (\text{Numerator of } \Sigma)\\ &\qquad = A_2\,(A_2\,(16\,A_2 + 24) - 16\,C + 9)\,z^3 = 0,\end{aligned}$$

where $\Sigma = p_1 + p_2 + p_3$. The equation has two solutions:

$$\left\{C = A_2^2 + 3/2\,A_2 + \frac{9}{16}\right\} \text{ and } \{A_2 = 0\}.$$

$\{A_2 = 0\}$ does not suit because we cannot ensure the nonnegativity of x_1. We substitute C into (5.2.5) with found expression and get

$$x_{1,3} = x + (A_2 + \tfrac{3}{4})z \mp \sqrt{(3x + \left(A_2 + \tfrac{3}{4}\right)^2 z)z}, \quad x_2 = x + A_2 z,$$

which matches (5.2.6).

Uniformly bounded moments of a second-order discretization scheme of the CIR equation

$$\begin{aligned}\mathbb{E}(\hat{S}_h^x)^q =& \frac{1}{(x_1 - x_3)\,(x_1 - x_2)\,(x_2 - x_3)}\\ &\times \Big((m_1\,x_2\,x_3 - m_2\,(x_2 + x_3) + m_3)\,{x_1}^{q-1}\,(x_2 - x_3)\\ &- (m_1\,x_1\,x_3 - m_2\,(x_1 + x_3) + m_3)\,{x_2}^{q-1}\,(x_1 - x_3)\\ &+ (m_1\,x_1\,x_2 - m_2\,(x_1 + x_2) + m_3)\,{x_3}^{q-1}\,(x_1 - x_2)\Big)\\ =& \frac{1}{2\,(((4\,A + 3)^2\,z + 48\,x)z)^{1/2}\,(A\,(2\,A + 3)\,z + 6\,x)}\\ &\times \Big(8\,x(((4\,A + 3)^2\,z + 48\,x)z)^{1/2}(x + Az)^q\\ &+ (((4\,A + 3)^2\,z + 48\,x)z)^{1/2}\,(A\,(2\,A + 3)\,z + 2\,x)\,({x_1}^q + {x_3}^q)\\ &+ \Big(A\,(2\,A + 3)\,(4\,A + 3)\,z^2 + 6\,(4\,A + 1)\,xz\Big)\,({x_1}^q - {x_3}^q)\Big)\\ =& \Bigg[\, x_{\mp} := {x_1}^q \mp {x_3}^q\\ &\qquad = \Big(x + (A + 3/4)\,z - \sqrt{(3\,x + (A + 3/4)^2 z)z}\Big)^q\end{aligned}$$

$$
\begin{aligned}
&\qquad \mp \left(x + (A+3/4)\, z + \sqrt{(3\,x + (A+3/4)^2 z) z}\right)^q \\
&\quad = \sum_{k=0}^{q} \binom{q}{k} (x + (A+3/4)\, z)^{q-k} \\
&\qquad \times \left(-\sqrt{(3\,x + (A+3/4)^2 z) z}\right)^k \\
&\qquad \mp \sum_{k=0}^{q} \binom{q}{k} (x + (A+3/4)\, z)^{q-k} \\
&\qquad \times \left(\sqrt{(3\,x + (A+3/4)^2 z) z}\right)^k, \\
&x_- = -2 \sum_{k=0}^{\lceil q/2 \rceil} \binom{q}{2\,k+1} (x + (A+3/4)\, z)^{q-2\,k-1} \\
&\quad \times \left(\sqrt{(3\,x + (A+3/4)^2 z) z}\right)^{2\,k+1} \\
&\quad = -1/2\, (((4\,A+3)^2\, z + 48\,x) z)^{1/2} \\
&\quad \times \sum_{k=0}^{\lceil q/2 \rceil} \binom{q}{2\,k+1} (x + (A+3/4)\, z)^{q-2\,k-1} \\
&\quad \times (3\,x + (A+3/4)^2 z)^k z^k, \\
&x_-^* := -1/2 \sum_{k=0}^{\lceil q/2 \rceil} \binom{q}{2\,k+1} (x + (A+3/4)\, z)^{q-2\,k-1} \\
&\quad \times (3\,x + (A+3/4)^2 z)^k z^k, \\
&x_+ = 2 \sum_{k=0}^{\lfloor q/2 \rfloor} \binom{q}{2\,k} (x + (A+3/4)\, z)^{q-2\,k} \\
&\quad \times (3\,x + (A+3/4)^2 z)^k z^k. \Bigg] \\
&= \frac{1}{2\,\left(A\,(2\,A+3)\, z + 6\,x\right)} \Bigg[8\,x\,(x + Az)^q \\
&\quad + (A\,(2\,A+3)\, z + 2\,x)\, x_+ \\
&\quad + \left(A\,(2\,A+3)\,(4\,A+3)\, z^2 + 6\,\,(4\,A+1)\, xz\right) x_-^* \Bigg] \\
&= \frac{1}{2\,\left(A\,(2\,A+3)\, z + 6\,x\right)} \Bigg[8\,x \Big[x^q + q x^{q-1} Az + \tbinom{q}{2} x^{q-2} (Az)^2 \\
&\quad + \tbinom{q}{3} x^{q-3} (Az)^3 + \ldots \Big] + 2\,(A\,(2\,A+3)\, z + 2\,x)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[(x+(A+3/4)z)^q + \tbinom{q}{2}\,(x+(A+3/4)z)^{q-2}\\
&\times (3\,x+(A+3/4)^2z)z\\
&+ \tbinom{q}{4}\,(x+(A+3/4)z)^{q-4}(3\,x+(A+3/4)^2z)^2z^2\\
&+ \tbinom{q}{6}\,(x+(A+3/4)z)^{q-6}(3\,x+(A+3/4)^2z)^3z^3+...\Big]\\
&+(-1/2)\,\big(A\,(2\,A+3)\,(4\,A+3)\,z^2+6\,(4\,A+1)\,xz\big)\\
&\times \Big[q\,(x+(A+3/4)z)^{q-1} + \tbinom{q}{3}\,(x+(A+3/4)z)^{q-3}\\
&\times (3\,x+(A+3/4)^2z)z + \tbinom{q}{5}\,(x+(A+3/4)z)^{q-5}\\
&\times (3\,x+(A+3/4)^2z)^2z^2\\
&+ \tbinom{q}{7}\,(x+(A+3/4)z)^{q-7}(3\,x+(A+3/4)^2z)^3z^3+...\Big]\Bigg]\\
\le{}& x^q+\tfrac{q(q-1)}{2}\,x^{q-1}\,z+\tfrac{q(q-1)^2(q-2)}{8}\,x^{q-2}\,z^2+...\\
\le{}& x^q+C\,(1+x^q)\,z\\
\le{}& x^q(1+C\,z)+C\,z,\ 0<z\le z_0.
\end{aligned}
$$

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