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Weak approximations of Wright–Fisher equation

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Abstract. We construct weak approximations of the Wright-Fisher model and illustrate their accuracy by simulation examples.
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Introduction

We consider Wright–Fisher process defined by the stochastic differential equation

$$\mathrm{d}X_t^x = \left(a - bX_t^x\right)\mathrm{d}t + \sigma\sqrt{X_t^x\left(1 - X_t^x\right)}\,\mathrm{d}B_t, \quad X_0^x = x, \tag{1}$$

where B is a standard Brownian motion, $0 \leqslant a \leqslant b$, $\sigma > 0$, and $x \in [0,1]$.

The Wright–Fisher model (Fisher 1930; Wright 1931) takes the values in the interval $[0,1]$ and explicitly accounts for the effects of various evolutionary forces – random genetic drift, mutation, selection – on allele frequencies over time. This model can also accommodate the effect of demographic forces such as variation in population size through time and/or migration connecting populations [5].

In this note, we present a simple first-order weak approximation of the solution of Eq. (1) by discrete random variables that take two values at each approximation step. Recall the definition of such an approximation. By a discretization scheme with time step $h > 0$ we mean any time-homogeneous Markov chain $\widehat{X}^h = \{\widehat{X}^h_{kh}, k = 0, 1, \ldots\}$. We say that a family of discretization schemes $\widehat{X}^h$, $h > 0$, is a first-order weak approximation of the solution X^x of (1) in the interval $[0,T]$ if

$$\left|\mathbb{E}f\left(\widehat{X}_T^h\right) - \mathbb{E}f\left(X_T^x\right)\right| \leqslant Ch, \quad h = \frac{T}{N} \leqslant h_0, \tag{2}$$

for a "sufficiently wide" class of functions $f : [0,1] \to \mathbb{R}$ and some constants C and $h_0 > 0$ (depending on the function f), where $N \in \mathbb{N}$. Note that because of the Markovity, the one-step approximation $\widehat{X}^h_h$ completely defines (in distribution) a weak approximation $\widehat{X}^h_{kh}$, $k = 0, 1, \ldots$. Thus, with some ambiguity, we also call it an approximation and denote it by $\widehat{X}^x_h$, with x indicating its starting point.

In our context, we introduce the following "sufficiently wide" function class of infinitely differentiable functions with "not too fast" growing derivatives:

$$C^\infty_*[0,1] := \left\{ f \in C^\infty[0,1] : \limsup_{k\to\infty} \frac{1}{k!} \sup_{x\in[0,1]} \left| f^{(k)}(x) \right| < \infty \right\}.$$

We easily see that all functions from this class can be expanded by the Taylor series in the interval $[0,1]$ around arbitrary $x_0 \in [0,1]$ (which, in fact, converges on the whole real line $\mathbb{R}$) and contain, for example, all polynomials and exponential functions.

Approximation

Let us first construct an approximation for the "stochastic" part of Wright–Fisher equation, that is, the solution S^x_t of Eq. (1) with $a = b = 0$. Similarly to [4] (see also [3]), we look for an approximation $\hat{S}^x_h$ as a two-valued discrete random variable taking values $x_{1,2} \in [0,1]$ with probabilities $p_{1,2}$ such that

$$\mathbb{E}\big(\hat{S}^x_h - x\big) = 0, \quad x \in [0,1], \tag{3}$$

$$\mathbb{E}\big(\hat{S}^x_h - x\big)^2 = \sigma^2 x(1-x)h + O\big(h^2\big), \quad x \in [0,1], \tag{4}$$

$$\big|\mathbb{E}\big(\hat{S}^x_h - x\big)^3\big| = O(h^2), \quad x \in [0,1], \tag{5}$$

$$\mathbb{E}\big[\big(\hat{S}^x_h - x\big)^4\big] = O(h^2), \quad x \in [0,1]. \tag{6}$$

By solving the equation system (3)–(4) with respect to x_1, x_2, p_1, p_2, we get the solution

$$x_1 = x + (1-x)\sigma^2 h - \sqrt{\big(x + (1-x)\sigma^2 h\big)(1-x)\sigma^2 h}, \quad x \in [0,1], \tag{7}$$

$$x_2 = x + (1-x)\sigma^2 h + \sqrt{\big(x + (1-x)\sigma^2 h\big)(1-x)\sigma^2 h}, \quad x \in [0,1] \tag{8}$$

with $p_{1,2} = \frac{x}{2x_{1,2}}$. It also satisfies conditions (5)–(6). However, for the values of x near 1, the values of x_2 a slightly greater than 1, which is unacceptable. We overcome this problem by using the symmetry of the solution of the stochastic part with respect to the point $\frac{1}{2}$; to be precise, $S^x_t \overset{d}{=} 1 - S^{1-x}_t$. Therefore, in the interval $[0, 1/2]$, we can use the values $x_{1,2}$ defined by (7)–(8), whereas in the interval $(1/2, 1]$, we use the values corresponding to the process $1 - \hat{S}^{1-x}_t$, that is,

$$\hat{x}_{1,2} = \hat{x}_{1,2}(x,h) := 1 - x_{1,2}(1-x,h) = x - x\sigma^2 h \pm \sqrt{\big(1 - x + x\sigma^2 h\big)x\sigma^2 h} \tag{9}$$

with probabilities $\hat{p}_{1,2} = \frac{1-x}{2x_{1,2}(1-x,h)}$. Thus we obtain a correct (i.e., with values in $[0,1]$) approximation $\hat{S}^x_h$ taking the values

$$\tilde{x}_{1,2} := \begin{cases} x_{1,2}(x,h) \text{ with probabilities } p_{1,2} = \frac{x}{2x_{1,2}(x,h)}, & x \in [0,1/2], \\ 1 - x_{1,2}(1-x,h) \text{ with probabilities } p_{1,2} = \frac{1-x}{2x_{1,2}(1-x,h)}, & x \in (1/2,1]. \end{cases}$$

Now for the initial equation (1), we obtain an approximation $\widehat{X}_h^x$ by a simple "split-step" procedure (again, see, e.g., [4] or [3]):

$$\widehat{X}_h^x := \hat{S}_h^x e^{-bh} + \frac{a}{b}\bigl(1 - e^{-bh}\bigr). \tag{10}$$

Now we can state the following:

Theorem 1. *Let $\hat{X}_t^x$ be the discretization scheme defined by one-step approximation* (10)*. Then $\hat{X}_t^x$ is a first-order weak approximation of equation* (1) *for functions $f \in C_*^\infty[0,1]$.*

Backward Kolmogorov equation

The constructed approximation is in fact a so-called *potential* first-order weak approximation of Eq. (1) (for a definition, see, e.g., Alfonsi [1], Section 2.3.1). The proof that, indeed, it is a first-order weak approximation, is based on the following:

Theorem 2. *Let $f \in C_*^\infty[0,1]$. The $u(t,x) := \mathbb{E} f(X_t^x)$ is a C^∞ function on $[0,1] \times \mathbb{R}$ that solves the backward Kolmogorov equation*

$$\partial_t u(t,x) = Au(t,x), \quad x \in [0,1],\ t \geqslant 0.$$

In particular,

$$\forall\, T > 0,\ \forall\, l,m \in \mathbb{N},\ \exists\, C_{l,m} : \bigl|\partial_l \partial_m u(t,x)\bigr| \leqslant C_{l,m},\ t \in [0,T],\ x \in [0,1].$$

Such theorem is stated for $f \in C^\infty[0,1]$ in [1, Thm. 6.1.12], based on the results of [2]. Our class of functions f is slightly narrower, but our proof of the theorem is significantly simpler and is based on the estimates of the moments of X_t^x, which show that they grow slower than factorials. The recurrent relations of the moments $\mathbb{E}[(X_t^x)^k]$ show that they are infinitely differentiable with respect to t and x, which allows us to infinitely differentiate the series

$$u(t,x) = \mathbb{E} f\bigl(X_t^x\bigr) = \sum_{k=0}^{\infty} c_k \mathbb{E}\bigl[\bigl(X_t^x\bigr)^k\bigr]$$

termwise with respect to t and x, where $f(x) = \sum_{k=0}^\infty c_k x^k$ is the Taylor expansion of f.

Simulation examples

We illustrate our approximation for $f(x) = x^4$ and $f(x) = \exp\{-x\}$. Since we do not explicitly know the moments $\mathbb{E}\exp\{-X_t^x\}$, we use the approximate equality $\exp\{-x\} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$. In Figs. 1 and 2, we compare the moments $\mathbb{E} f(\widehat{X}_t^x)$ and $\mathbb{E} f(X_t^x)$ as functions of t (left plots, $h = 0.001$) and as functions of discretization step h (right plots, $t = 1$). As expected, the approximations agree with exact values pretty well.

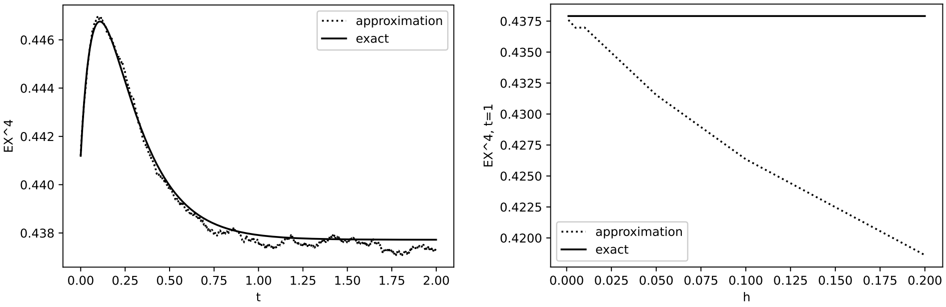

Fig. 1. Comparison of $\mathbb{E}f(\widehat{X}_t^x)$ and $\mathbb{E}f(X_t^x)$ as functions of t and h for $f(x) = x^4$: $x = 0.815$, $\sigma^2 = 0.5$, $a = 4$, $b = 5$, the number of iterations $N = 500.000$.

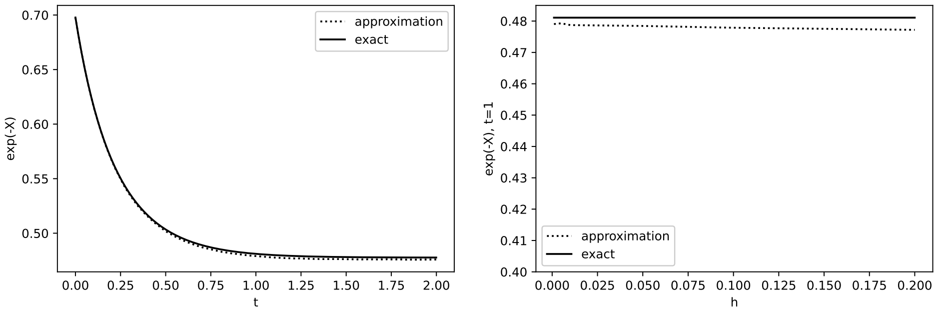

Fig. 2. Comparison of $\mathbb{E}f(\widehat{X}_t^x)$ and $\mathbb{E}f(X_t^x)$ as functions of t and h for $f(x) = \exp\{-x\}$: $x = 0.36$, $\sigma^2 = 0.6$, $a = 3$, $b = 4$, $N = 100.000$.

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REZIUMĖ

Wright–Fisher lygties silpnosios aproksimacijos

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Sukonstruota silpnoji pirmos eilės aproksimacija stochastinei Wright–Fisher lygčiai. Pavyzdžiais iliustruojamas jos tikslumas.

Raktiniai žodžiai: Wright–Fisher modelis; modeliavimas; silpnoji aproksimacija