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Discrete Universality of Absolutely Convergent Dirichlet Series

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Abstract. In the paper, an universality theorem of discrete type on the approximation of analytic functions by shifts of a special absolutely convergent Dirichlet series is obtained. These series is close in a certain sense to the periodic zeta-function and depends on a parameter.

Keywords: limit theorem, periodic zeta-function, universality, weak convergence.

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1 Introduction

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic zeta-function $\zeta(s;\mathfrak{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

In view of periodicity of the sequence $\mathfrak{a}$,

$$\zeta(s;\mathfrak{a}) = \frac{1}{q^s} \sum_{l=1}^{q} a_l \zeta\left(s, \frac{l}{q}\right), \tag{1.1}$$

where $\zeta(s,\alpha)$, $0 < \alpha \leqslant 1$, denotes the classical Hurwitz zeta-function, it follows that the function $\zeta(s;\mathfrak{a})$ is analytic in the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\hat{a} \stackrel{def}{=} \frac{1}{q} \sum_{l=1}^{q} a_l.$$

If $a_m \equiv 1$, then the function $\zeta(s;\mathfrak{a})$ reduces to the Riemann zeta-function $\zeta(s)$. If $a_m = \chi(m)$, where $\chi(m)$ is a Dirichlet character modulo q, then the function $\zeta(s;\mathfrak{a})$ becomes a Dirichlet L-function $L(s,\chi)$. Thus, the periodic zeta-function is a generalization of the classical Riemann zeta-function $\zeta(s)$ and Dirichlet L-function $L(s,\chi)$.

As the functions $\zeta(s)$ and $L(s,\chi)$, the function $\zeta(s;\mathfrak{a})$ for some sequences $\mathfrak{a}$ is universal in the sense that its shifts $\zeta(s+i\tau;\mathfrak{a})$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Universality of the function $\zeta(s;\mathfrak{a})$ was considered by various authors, among them B. Bagchi, J. Steuding, J. Kaczorowski, J. Sander, R. Macaitienė, R. Kačinskaitė, the second, third and fourth authors, and others. For a more precise account, we propose a survey paper [6]. The majority of the papers deal with the continuous universality of $\zeta(s;\mathfrak{a})$ when τ in shifts $\zeta(s+i\tau;\mathfrak{a})$ takes arbitrary real values. To our knowledge, the discrete universality of $\zeta(s;\mathfrak{a})$, when τ in $\zeta(s+i\tau;\mathfrak{a})$ takes real values from a certain discrete set, was discussed only in [3,4,5] with multiplicative sequence $\mathfrak{a}$, i. e., $a_{mn} = a_m a_n$ for all $(m,n) = 1$ and $a_1 = 1$. For example, Theorem 2 of [4] with $w(u) \equiv 1$ implies the following result. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let $\mathbb{P}$ be the set of all prime numbers, $\#A$ denotes the cardinality of the set A, and, for $h > 0$,

$$L(\mathbb{P},h,\pi) = \left\{(\log p : p \in \mathbb{P}), \frac{2\pi}{h}\right\}.$$

Under above notation, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0$$

provided that the sequence $\mathfrak{a}$ is multiplicative, and the set $L(\mathbb{P},h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$.

Latter, we will give a modified statement of the above theorem from [4], for it some notation is required.

In this paper, we will consider the discrete approximation of analytic functions by shifts of a certain absolutely convergent on D Dirichlet series with coefficients depending on a parameter. These series is closely connected to the function $\zeta(s;\mathfrak{a})$.

2 Definition of an absolutely convergent series

As usual, denote by $\Gamma(s)$ the Euler gamma-function, $\theta > 1/2$ is a fixed number, $u > 0$ and

$$l_u(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)u^s, \quad v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}, \quad m \in \mathbb{N}.$$

Define the series

$$\zeta_u(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m v_n(m)}{m^s}.$$

Since the sequence $\mathfrak{a}$ is bounded, the latter series is absolutely convergent, say, for $\sigma > 1/2$. In view of the Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s)b^{-s}\,\mathrm{d}s = \mathrm{e}^{-b}, \quad a, b > 0,$$

we have the equality

$$v_u(m) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right)\left(\frac{m}{u}\right)^{-s}\mathrm{d}s. \tag{2.1}$$

The latter formula also implies the bound $v_u(m) \ll_{\theta,u} m^{-\theta}$. Moreover, (2.1) shows that

$$\begin{aligned}\zeta_u(s;\mathfrak{a}) &= \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{a_m}{m^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{m}{u}\right)^{-z}\frac{\mathrm{d}z}{z}\\ &= \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{l_u(z)}{z}\sum_{m=1}^{\infty}\frac{a_m}{m^{s+z}}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z;\mathfrak{a})l_u(z)\frac{\mathrm{d}z}{z}.\end{aligned} \tag{2.2}$$

3 Statement of the main theorem

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. In this section, we define the $H(D)$-valued random element connected to the function $\zeta(s;\mathfrak{a})$. We start with traditional probability space. Let

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. The torus Ω is a compact topological Abelian group, therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\mathbb{X})$ is the Borel σ-field of the space $\mathbb{X}$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth component of an element $\omega \in \Omega$, $p \in \mathbb{P}$. Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element

$$\zeta(s,\omega;\mathfrak{a}) = \prod_{p\in\mathbb{P}}\left(1 + \sum_{\alpha=1}^{\infty}\frac{a_{p^\alpha}\omega^\alpha(p)}{p^{\alpha s}}\right).$$

The details of this definition can be found in [2]. We will prove the following statement.

Theorem 1. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P}, h, \pi)$ is linearly independent over $\mathbb{Q}$ and $u_N \to \infty$, $u_N \ll N^2 \log^A N$ with every $A > 0$ as $N \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_{u_N}(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon \right\}$$
$$= m_H \left\{ \omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

We also state a modified universality theorem for the function $\zeta(s; \mathfrak{a})$ mentioned in Introduction.

Theorem 2. *Suppose that the sequence $\mathfrak{a}$ is multiplicative, the set $L(\mathbb{P}, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - f(s)| < \varepsilon \right\}$$
$$= m_H \left\{ \omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Proof of Theorem 2 easily follows from a discrete limit theorem (Theorem 3 of [4]) by using the equivalent of weak convergence of probability measures in terms of continuity sets.

4 Approximation in the mean

Theorem 1 will be derived from Theorem 2. For this, we need a certain approximation result of $\zeta_{u_N}(s; \mathfrak{a})$ by $\zeta(s; \mathfrak{a})$.

Lemma 1. *Suppose that $u_N \to \infty$, $u_N \ll N^2 \log^A N$ with every $A > 0$ as $N \to \infty$. Then, for every compact set $K \subset D$ and $h > 0$,*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K} |\zeta(s+ikh; \mathfrak{a}) - \zeta_{u_N}(s+ikh; \mathfrak{a})| = 0.$$

Proof. Let $1/2 < \sigma < 1$ be fixed and $T \to \infty$. Then it is well known that

$$\int_{-T}^{T} |\zeta(\sigma+it, \alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} T, \quad \int_{-T}^{T} |\zeta'(\sigma+it, \alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} T.$$

From these estimates and (1.1), we have

$$\int_{-T}^{T} |\zeta(\sigma+it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T, \quad \int_{-T}^{T} |\zeta'(\sigma+it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T.$$

Hence, for $\tau \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma + i\tau + it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T(1 + |\tau|),$$
$$\int_0^T |\zeta'(\sigma + i\tau + it; \mathfrak{a})|^2 \, \mathrm{d}t \ll_{\sigma,\mathfrak{a}} T(1 + |\tau|).$$

Now the latter estimates and the application of the Gallagher lemma that connects discrete and continuous mean squares for some differentiable functions (Lemma 1.4 of [8]), lead to the bound

$$\begin{aligned}\sum_{k=0}^{N} |\zeta(\sigma + ikh + i\tau; \mathfrak{a})|^2 &\ll_h \int_0^{Nh} |\zeta(\sigma + i\tau + it; \mathfrak{a})|^2 \, \mathrm{d}t \\ &\quad + \left(\int_0^{Nh} |\zeta(\sigma + i\tau + it; \mathfrak{a})|^2 \, \mathrm{d}t \int_0^{Nh} |\zeta'(\sigma + i\tau + it; \mathfrak{a})|^2 \, \mathrm{d}t \right)^{1/2} \\ &\ll_{\sigma,\mathfrak{a},h} N(1 + |\tau|). \end{aligned} \tag{4.1}$$

Let $K \subset D$ be an arbitrary compact set. Then there exists $\varepsilon > 0$ such that K lies in the strip $\{s \in \mathbb{C} : 1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon\}$. For s from that strip, we have $\theta_1 = \sigma - \frac{1}{2} - \varepsilon > 0$. Therefore, the representation (2.2) and the residue theorem give

$$\zeta_{u_N}(s; \mathfrak{a}) - \zeta(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s + z; \mathfrak{a}) l_{u_N}(z) \frac{\mathrm{d}z}{z} + \frac{\hat{a} l_{u_N}(1 - s)}{1 - s}.$$

Therefore, for $s \in K$,

$$\begin{aligned} &|\zeta(s + ikh; \mathfrak{a}) - \zeta_{u_N}(s + ikh; \mathfrak{a})| \\ &\ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \varepsilon + ikh + it + i\tau; \mathfrak{a}\right) \right| \frac{|l_{u_N}(1/2 + \varepsilon - \sigma + i\tau)|}{|1/2 + \varepsilon - \sigma + i\tau|} \, \mathrm{d}\tau \\ &\quad + \frac{|\hat{a}||l_{u_N}(1 - s - ikh)|}{|1 - s - ikh|} \\ &\ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \varepsilon + ikh + i\tau; \mathfrak{a}\right) \right| \sup_{s \in K} \frac{|l_{u_N}(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} \, \mathrm{d}\tau \\ &\quad + \frac{|\hat{a}||l_{u_N}(1 - s - ikh)|}{|1 - s - ikh|}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{N+1} \sum_{k=0}^{N} |\zeta(s + ikh; \mathfrak{a}) - \zeta_{u_N}(s + ikh; \mathfrak{a})| \\ &\ll \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=0}^{N} \left| \zeta\left(\frac{1}{2} + \varepsilon + ikh + i\tau; \mathfrak{a}\right) \right| \right) \sup_{s \in K} \frac{|l_{u_N}(1/2 + \varepsilon - s + i\tau)|}{|1/2 + \varepsilon - s + i\tau|} \, \mathrm{d}\tau \\ &\quad + |\hat{a}| \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} \frac{|l_{u_N}(1 - s - ikh)|}{|1 - s - ikh|} \overset{def}{=} I + Z. \end{aligned} \tag{4.2}$$

The Cauchy inequality and (4.1) show that

$$\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau;\mathfrak{a}\right)\right|$$
$$\leqslant\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau;\mathfrak{a}\right)\right|^2\right)^{1/2}\ll_{\varepsilon,\mathfrak{a},h}(1+|\tau|)^{1/2}. \qquad (4.3)$$

It is well known that there exists a constant $c>0$ such that

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\} \qquad (4.4)$$

uniformly in $\sigma_1<\sigma<\sigma_2$ for arbitrary $\sigma_1<\sigma_2$. Thus, by the definition of $l_u(s)$, for all $s\in K$,

$$\frac{l_{u_N}(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\ll_\theta u_N^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\varepsilon-it+i\tau\right)\right)\right|$$
$$\ll_\theta u_N^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|\tau-t|\right\}\ll_{\theta,K}u_N^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|\tau|\right\}.$$

This together with (4.3) implies the bound

$$I\ll_{\varepsilon,\mathfrak{a},h,\theta,K}u_N^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\left\{-\frac{c}{\theta}|\tau|\right\}\,\mathrm{d}\tau\ll_{\varepsilon,\mathfrak{a},h,\theta,K}u_N^{-\varepsilon}. \qquad (4.5)$$

Similarly, using (4.4), we obtain that, for all $s\in K$,

$$\frac{l_{u_N}(1-\sigma-it+ikh)}{1-\sigma-it+ikh}\ll_{\theta,K}u_N^{1-\sigma}\exp\left\{-\frac{ch}{\theta}k\right\}.$$

Therefore,

$$Z\ll_{\theta,K,\mathfrak{a}}\frac{u_N^{1/2-2\varepsilon}}{N}\sum_{k=0}^{N}\exp\left\{-\frac{ch}{\theta}k\right\}\ll_{\theta,K,\mathfrak{a},h}u_N^{1/2-2\varepsilon}\frac{\log N}{N}.$$

Since $u_N\ll N^2\log^A N$, this together with (4.5) and (4.2) proves the lemma. □

5 Proof of universality

For the proof of Theorem 1, we will apply the probabilistic approach based on weak convergence of probability measures and distribution functions. First we will recall the proof of Theorem 2.

For $A\in\mathcal{B}(H(D))$, define

$$P_{N,\mathfrak{a},h}(A)=\frac{1}{N+1}\#\{0\leqslant k\leqslant N:\zeta(s+ikh;\mathfrak{a})\in A\},$$

and let $P_{\zeta,\mathfrak{a}}$ denote the distribution of the random element $\zeta(s,\omega;\mathfrak{a})$, i. e.,

$$P_{\zeta,\mathfrak{a}}(A)=m_H\{\omega\in\Omega:\zeta(s,\omega;\mathfrak{a})\in A\}.$$

Lemma 2. *Suppose that the sequence* $\mathfrak{a}$ *is multiplicative and the set* $L(\mathbb{P}, h, \pi)$ *is linearly independent over* $\mathbb{Q}$*. Then* $P_{N,\mathfrak{a},h}$ *converges weakly to* $P_{\zeta,\mathfrak{a}}$ *as* $N \to \infty$*. Moreover, the support of the measure* $P_{\zeta,\mathfrak{a}}$ *is the set* $\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\} \overset{def}{=} S$*.*

Proof. The first assertion of the lemma is contained in Theorem 3 of [4] with weight function $w(u) \equiv 1$. Observe that in [4] the restriction $h \geqslant 1$ in the definition of the set $L(\mathbb{P}, h, \pi)$ is involved, however, in the case $w(u) \equiv 1$, the latter restriction is not needed.

The second assertion of the lemma can be found in [2]. □

Proof. (Proof of Theorem 2). Since $f(s) \neq 0$ on K, by the Mergelyan theorem on the approximation of analytic functions by polynomials [7], there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\left|f(s) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{5.1}$$

Obviously, $\mathrm{e}^{p(s)} \in S$. Therefore, the set

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K}\left|g(s) - \mathrm{e}^{p(s)}\right| < \frac{\varepsilon}{2}\right\}$$

is open and contains an element of the support of the measure $P_{\zeta,\mathfrak{a}}$. Hence,

$$P_{\zeta,\mathfrak{a}}(G_\varepsilon) > 0. \tag{5.2}$$

Define the open neighbourhood of the function $f(s)$

$$\mathcal{G}_\varepsilon = \left\{g \in H(D) : \sup_{s\in K}|g(s) - f(s)| < \varepsilon\right\}.$$

The boundary $\partial\mathcal{G}_\varepsilon$ of the set $\mathcal{G}_\varepsilon$ is the set

$$\left\{g \in H(D) : \sup_{s\in K}|g(s) - f(s)| = \varepsilon\right\}.$$

Therefore, the boundaries $\partial\mathcal{G}_{\varepsilon_1}$ and $\partial\mathcal{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. This implies that $P_{\zeta,\mathfrak{a}}(\partial\mathcal{G}_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$, i. e., in other words, the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$ for all but at most countably many $\varepsilon > 0$. Thus, an application of the first assertion of Lemma 2 together with the equivalent of weak convergence of probability measures in terms of continuity sets, see, for example, [1], gives

$$\lim_{N\to\infty} P_{N,\mathfrak{a},h}(\mathcal{G}_\varepsilon) = P_{\zeta,\mathfrak{a}}(\mathcal{G}_\varepsilon) \tag{5.3}$$

for all but at most countably many $\varepsilon > 0$. Moreover, in view of (5.1),

$$\sup_{s\in K}|g(s) - f(s)| \leqslant \sup_{s\in K}\left|g(s) - \mathrm{e}^{p(s)}\right| + \sup_{s\in K}\left|f(s) - \mathrm{e}^{p(s)}\right| < \varepsilon$$

for $g \in \mathcal{G}_\varepsilon$. This shows that $\mathcal{G}_\varepsilon \supset G_\varepsilon$. Therefore, by (5.2), we have the inequality $P_{\zeta,\mathfrak{a}}(\mathcal{G}_\varepsilon) > 0$. This inequality, the definitions of $P_{N,\mathfrak{a},h}$ and $\mathcal{G}_\varepsilon$, and (5.3) prove the theorem. $\square$

We observe that the limit measure $P_{\zeta,\mathfrak{a}}$ is independent of the number h, and this is conditioned by the linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P}, h, \pi)$. The case of the linear dependence of the set $L(\mathbb{P}, h, \pi)$ is more complicated and requires a new probability space in place of $(\Omega, \mathcal{B}(\Omega), m_H)$. This case will be considered elsewhere.

Proof. (Proof of Theorem 1). We will deduce Theorem 1 from Theorem 2 by using Lemma 1 and the weak convergence of distribution functions.

Define the functions

$$\begin{aligned} F_{N,\mathfrak{a},h}(\varepsilon) &= P_{N,\mathfrak{a},h}(\mathcal{G}_\varepsilon) \\ &= \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\Big\}, \\ \hat{F}_{N,\mathfrak{a},h}(\varepsilon) &= \frac{1}{N+1}\#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)| < \varepsilon\Big\}, \\ F_{\zeta,\mathfrak{a}}(\varepsilon) &= P_{\zeta,\mathfrak{a}}(\mathcal{G}_\varepsilon) = m_H\Big\{\omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega;\mathfrak{a}) - f(s)| < \varepsilon\Big\}. \end{aligned}$$

The functions $F_{N,\mathfrak{a},h}(\varepsilon)$, $\hat{F}_{N,\mathfrak{a},h}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as functions of the variable ε are distribution functions. Actually, they are non-decreasing, left continuous, at $+\infty$ take values 1 and at $-\infty$ take values 0. Moreover, since

$$\begin{aligned} &\partial\mathcal{G}_\varepsilon \\ &= \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| \leqslant \varepsilon\Big\} \setminus \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}, \end{aligned}$$

we have

$$\begin{aligned} P_{\zeta,\mathfrak{a}}(\partial\mathcal{G}_\varepsilon) &= m_H\Big\{\omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega;\mathfrak{a}) - f(s)| \leqslant \varepsilon\Big\} \\ &- m_H\Big\{\omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega;\mathfrak{a}) - f(s)| < \varepsilon\Big\} = F_{\zeta,\mathfrak{a}}(\varepsilon+0) - F_{\zeta,\mathfrak{a}}(\varepsilon). \end{aligned}$$

Hence, $F_{\zeta,\mathfrak{a}}(\varepsilon+0) = F_{\zeta,\mathfrak{a}}(\varepsilon)$ if and only if $P_{\zeta,\mathfrak{a}}(\partial\mathcal{G}_\varepsilon) = 0$. Therefore, the point ε is a continuity point of the distribution function $F_{\zeta,\mathfrak{a}}$ if and only if the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\mathfrak{a}}$. By (5.3), $P_{N,\mathfrak{a},h}$ converges weakly to $P_{\zeta,\mathfrak{a}}$ as $N \to \infty$ for all continuity sets $\mathcal{G}_\varepsilon$ of $P_{\zeta,\mathfrak{a}}$. Thus, by the above remarks, the distribution function $F_{N,\mathfrak{a},h}(\varepsilon)$ as $N \to \infty$ converges to the distribution function $F_{\zeta,\mathfrak{a}}(\varepsilon)$ at all its continuity points ε ($F_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $N \to \infty$).

Denote by $\varphi_{N,\mathfrak{a},h}(v)$, $\hat{\varphi}_{N,\mathfrak{a},h}(v)$ and $\varphi_{\zeta,\mathfrak{a}}(v)$, $v \in \mathbb{R}$, the characteristic functions of the distribution functions $F_{N,\mathfrak{a},h}(\varepsilon)$, $\hat{F}_{N,\mathfrak{a},h}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$, respectively. Since $F_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $N \to \infty$, the continuity theorem for characteristic functions implies the equality

$$\lim_{N\to\infty} \varphi_{N,\mathfrak{a},h}(v) = \varphi_{\zeta,\mathfrak{a}}(v) \tag{5.4}$$

for all $v \in \mathbb{R}$. We have to show that $\hat{\varphi}_{N,\mathfrak{a},h}(v)$ also converges to $\varphi_{\zeta,\mathfrak{a}}(v)$ as $N \to \infty$.

By the definition of characteristic functions,

$$\begin{aligned}\hat{\varphi}_{N,\mathfrak{a},h}(v) - \varphi_{N,\mathfrak{a},h}(v) &= \int_{-\infty}^{\infty} \mathrm{e}^{iv\varepsilon}\,\mathrm{d}\left(\hat{F}_{N,\mathfrak{a},h}(\varepsilon) - F_{N,\mathfrak{a},h}(\varepsilon)\right)\\ &= \frac{1}{N+1}\sum_{k=0}^{\infty}\left(\exp\left\{iv\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)|\right\}\right.\\ &\quad\left. - \exp\left\{iv\sup_{s\in K}|\zeta(s+ikh;\mathfrak{a}) - f(s)|\right\}\right).\end{aligned}$$

Hence,

$$\begin{aligned}|\hat{\varphi}_{N,\mathfrak{a},h}(v) - \varphi_{N,\mathfrak{a},h}(v)| &\leqslant \frac{1}{N+1}\sum_{k=0}^{N}\left|\exp\left\{iv\left(\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a}) - f(s)|\right.\right.\right.\\ &\left.\left.\left. - \sup_{s\in K}|\zeta(s+ikh;\mathfrak{a})-f(s)|\right)\right\} - 1\right| \leqslant \frac{|v|}{N+1}\sum_{k=0}^{N}\left|\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-f(s)|\right.\\ &\left. - \sup_{s\in K}|\zeta(s+ikh;\mathfrak{a})-f(s)|\right| \leqslant \frac{|v|}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta_{u_N}(s+ikh;\mathfrak{a})-\zeta(s+ikh;\mathfrak{a})|.\end{aligned}$$

Therefore, in view of Lemma 1,

$$\hat{\varphi}_{N,\mathfrak{a},h}(v) - \varphi_{N,\mathfrak{a},h}(v) = o(1)$$

as $N \to \infty$ uniformly in $|v| \leqslant C$ with every $\infty > C > 0$. This and (5.4) show that

$$\lim_{N\to\infty}\hat{\varphi}_{N,\mathfrak{a},h}(v) = \varphi_{\zeta,\mathfrak{a}}(v)$$

uniformly in $|v| \leqslant C$. Therefore, by the continuity theorem, we obtain that the distribution function $\hat{F}_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ as $N \to \infty$, or $\hat{F}_{N,\mathfrak{a},h}(\varepsilon)$ converges weakly to $F_{\zeta,\mathfrak{a}}(\varepsilon)$ in all continuity points ε of $F_{\zeta,\mathfrak{a}}(\varepsilon)$. However, the distribution function has at most countably many of discontinuity points. This and the definitions of $\hat{F}_{N,\mathfrak{a},h}(\varepsilon)$ and $F_{\zeta,\mathfrak{a}}(\varepsilon)$ prove the theorem. □

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