mathematics

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Central Limit Theorems for Combinatorial Numbers Associated with Laguerre Polynomials

Igoris Belovas

Faculty of Mathematics and Informatics, Institute of Data Science and Digital Technologies, Vilnius University, LT-04812 Vilnius, Lithuania; Igoris.Belovas@mif.vu.lt

Abstract: In this paper, we study limit theorems for numbers satisfying a class of triangular arrays, which are defined by a bivariate linear recurrence with bivariate linear coefficients. We obtain analytical expressions for the semi-exponential generating function of several classes of the numbers, including combinatorial numbers associated with Laguerre polynomials. We apply these results to prove the numbers' asymptotic normality and specify the convergence rate to the limiting distribution.

Keywords: limit theorems; combinatorial numbers; generating functions; asymptotic enumeration; asymptotic normality; Laguerre polynomials

MSC: 05A15; 05A16; 33C45; 39A14; 60F05

Citation: Belovas, I. Central Limit Theorems for Combinatorial Numbers Associated with Laguerre Polynomials. *Mathematics* **2022**, *10*, 865. https://doi.org/10.3390/math10060865

Academic Editor: Patrick Solé

Received: 15 February 2022
Accepted: 7 March 2022
Published: 9 March 2022

1. Introduction

In this research, we establish limit theorems for combinatorial numbers satisfying a class of triangular arrays, extending, particularly, the investigations of Canfield [1], Kyriakoussis [2], Kyriakoussis and Vamvakari [3–6], and Belovas [7]. We consider numbers, which are defined by a bivariate linear recurrence with bivariate linear coefficients.

Definition 1. *Let Ψ be a real non-zero matrix (generating matrix),*

$$\Psi = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}, \tag{1}$$

then

$$a_{n,k} = \begin{cases} 1, & \text{for } n = 0 \text{ and } k = 0, \\ 0, & \text{for } \min(n,k,n-k) < 0, \\ (\psi_{1,1}n + \psi_{1,2}k + \psi_{1,3})a_{n-1,k-1} + & \\ (\psi_{2,1}n + \psi_{2,2}k + \psi_{2,3})a_{n-1,k}\,, & \text{otherwise.} \end{cases} \tag{2}$$

The numbers defined above involve binomial coefficients, k-permutations of n without repetition, Morgan numbers, Stirling numbers of the first kind and the second kind, non-central Stirling numbers, Eulerian numbers, Lah numbers, as well as some generalizations of the numbers mentioned above (see [8,9] and the references therein).

The paper is organized as follows. The first part is the introduction. In Section 2, we receive generating functions and analytic expressions for particular numbers, satisfying a class of triangular arrays, using general partial differential equations. Section 3 establishes a connection between numbers satisfying a class of triangular arrays and generalized Lah numbers. The result is used to obtain generating functions and analytic expressions for other numbers, satisfying a class of triangular arrays. In Section 4, we prove asymptotic normality for the said numbers and specify the rates of convergence. In Section 5, we establish central limit theorems for numbers satisfying a class of triangular arrays associated with Laguerre polynomials and determine convergence rates to the limiting distribution. Section 6 of the study contains concluding remarks.

Throughout this paper, we denote by C_n^k the binomial coefficients, by $\Gamma(x)$ the gamma function, $E_1(x)$ stands for the exponential integral,

$$E_1(x) = \int_x^\infty e^{-t} t^{-1} dt = \int_1^\infty e^{-tx} t^{-1} dt, \qquad x > 0,$$

and $\Phi(x)$ stands for the cumulative distribution function of the standard normal distribution,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt, \qquad x \in \mathbb{R}.$$

Let $L_{\alpha;n}(x)$ be the generalized Laguerre polynomials,

$$L_{\alpha;n}(x) = \sum_{k=0}^n (-1)^k \frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)} \frac{x^k}{k!}.$$

The generating function of the generalized Laguerre polynomials is [10]

$$\frac{1}{(1-x)^{1+\alpha}} \exp\left(t - \frac{t}{1-x}\right) = \sum_{n=0}^\infty x^n L_{\alpha;n}(t). \tag{3}$$

All limits, unless specified, are taken as $n \to \infty$.

2. Generating Functions and Analytic Expressions of the Combinatorial Numbers $a_{n,k}$

We may view the recurrent expression for the numbers $a_{n,k}$ (2) as a partial difference equation with linear coefficients. First, let us introduce the semi-exponential generating function of the numbers,

$$F(x,y) = \sum_{n=0}^\infty \sum_{k=0}^\infty a_{n,k} \frac{x^n}{n!} y^k = \sum_{n=0}^\infty \sum_{k=0}^n a_{n,k} \frac{x^n}{n!} y^k. \tag{4}$$

This expression, contrary to ordinary or exponential ones, leads us to a first-order characteristic differential equation (see Equation (6) in Theorem 1). In contrast, ordinary and exponential generating functions satisfy second-order partial differential equations.

Definition 2. *For the numbers satisfying a class of triangular arrays* (2)*, we define their dual counterparts* $\tilde{a}_{n,k} := a_{n,n-k}$.

Remark 1. *In view of Definintion 2, the generating matrix of the dual numbers is*

$$\begin{pmatrix} \psi_{2,1} + \psi_{2,2} & -\psi_{2,2} & \psi_{2,3} \\ \psi_{1,1} + \psi_{1,2} & -\psi_{1,2} & \psi_{1,3} \end{pmatrix}. \tag{5}$$

Lemma 1. *The double semi-exponential generating function* $\tilde{F}(x,y)$ *of the dual numbers* (5) *equals* $\tilde{F}(x,y) = F(xy, y^{-1})$.

Proof. By Definition 2, we have

$$\tilde{F}(x,y) = \sum_{n=0}^\infty \sum_{k=0}^n \underbrace{a_{n,n-k}}_{=\tilde{a}_{nk}} \frac{x^n}{n!} y^k = \sum_{n=0}^\infty \sum_{j=0}^n a_{n,j} \frac{(xy)^n}{n!} y^{-j},$$

yielding us the statement of the lemma. □

In [7], we have received subsequent theorems (see Theorems 1 and 2) for the generating functions of the numbers satisfying a class of triangular arrays (2).

Theorem 1 (Belovas)**.** *The generating function $F(x,y)$ satisfies the linear first-order partial differential equation*

$$(1-\psi_{1,1}xy-\psi_{2,1}x)F_x-(\psi_{1,2}y^2+\psi_{2,2}y)F_y=(\xi_1 y+\xi_2)F,\quad \xi_1=\psi_{1,1}+\psi_{1,2}+\psi_{1,3},\quad \xi_2=\psi_{2,1}+\psi_{2,3}, \tag{6}$$

with the initial condition $F|_{x=0}=1$.

Remark 2. *Solving the linear first-order partial differential Equation (6), we obtain the generating function $F(x,y)$. The formal Taylor series in two variables for the generating function equals*

$$F(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\partial^{n+k}}{\partial x^n\partial y^k}F(x,y)\bigg|_{(0,0)}\right)\frac{x^n y^k}{n!k!}.$$

Hence, the partial differentiation of the double semi-exponential generating function $F(x,y)$ at $(0,0)$ yields us the analytic expressions of the numbers

$$a_{nk}=\frac{1}{k!}\frac{\partial^{n+k}}{\partial x^n\partial y^k}F(x,y)\bigg|_{(0,0)}. \tag{7}$$

Theorem 2 (Belovas)**.** *For $\psi_{1,2},\psi_{2,1},\psi_{2,2}\neq 0$, numbers generated by the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}$$

(i) *have the generating function*

$$F(x,y)=(1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}}\left(1+\frac{\psi_{1,2}}{\psi_{2,2}}y\left(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}}\right)\right)^{-\frac{\xi_1}{\psi_{1,2}}}, \tag{8}$$

(ii) *and the analytic expression*

$$a_{n,k}=\frac{\prod_{j=1}^{k}(\psi_{1,2}j+\psi_{1,3})}{k!(\psi_{2,2})^k}\sum_{m=0}^{k}(-1)^m C_k^m\prod_{l=1}^{n}(\psi_{2,2}(k-m)+\psi_{2,1}l+\psi_{2,3}). \tag{9}$$

Using a substitution $F(x,y)=\Lambda(x,y)A(y)$, we can reduce the linear partial differential Equation (6) into its homogeneous form. First, we formulate an auxiliary lemma [11].

Lemma 2.

(i) *Let $v=ax+by$; then, the principal integral of the first-order partial differential equation*

$$[f(v)+bxg(v)]w_x+[h(v)-axg(v)]w_y=0 \tag{10}$$

is

$$\Xi=xE-\int\frac{f(v)Edv}{af(v)+bh(v)}, \tag{11}$$

where

$$E=\exp\left(-b\int\frac{g(v)dv}{af(v)+bh(v)}\right). \tag{12}$$

(ii) *Let $f_x = g_y$; then, the principal integral of the first order partial differential equation*

$$f(x,y)w_x - g(x,y)w_y = 0 \tag{13}$$

is

$$\Xi = \int_{y_0}^{y} f(x_0,t)dt + \int_{x_0}^{x} g(t,y)dt, \tag{14}$$

where x_0 and y_0 are arbitrary constants.

Proof.

(i) See 2.9.3.4 in Polyanin et al. [11];
(ii) See 2.9.3.10 in Polyanin et al. [11].

□

Theorem 3. *Under conditions of Theorem 1, the function $\Lambda(x,y) = F(x,y)A_m(y)$ satisfies a linear first-order homogeneous partial differential equation*

(i) *for $\psi_{1,2}, \psi_{2,2} \neq 0$,*

$$(1 - \psi_{1,1}xy - \psi_{2,1}x)\Lambda_x - (\psi_{1,2}y^2 + \psi_{2,2}y)\Lambda_y = 0, \tag{15}$$

with the initial condition

$$\Lambda|_{x=0} = \underbrace{(\psi_{1,2}y + \psi_{2,2})^{\frac{\xi_1}{\psi_{1,2}} - \frac{\xi_2}{\psi_{2,2}}} y^{\frac{\xi_2}{\psi_{2,2}}}}_{:=A_1(y)}. \tag{16}$$

The principal integral of Equation (15) is

$$L_1(x,y) = x(\psi_{1,2}y + \psi_{2,2})^{\alpha} y^{\beta} + \int (\psi_{1,2}y + \psi_{2,2})^{\alpha-1} y^{\beta-1} dy, \tag{17}$$

where

$$\alpha = \frac{\psi_{2,1}}{\psi_{2,2}} - \frac{\psi_{1,1}}{\psi_{1,2}}, \qquad \beta = -\frac{\psi_{2,1}}{\psi_{2,2}}.$$

(ii) *for $\psi_{1,2} = 0$, $\psi_{2,2} \neq 0$,*

$$(1 - \psi_{1,1}xy - \psi_{2,1}x)\Lambda_x - \psi_{2,2}y\Lambda_y = 0, \qquad \Lambda|_{x=0} = \underbrace{e^{\frac{\xi_1}{\psi_{2,2}}y} y^{\frac{\xi_2}{\psi_{2,2}}}}_{:=A_2(y)}. \tag{18}$$

The principal integral of Equation (18) is

$$L_2(x,y) = xy^{-\frac{\psi_{2,1}}{\psi_{2,2}}} e^{-\frac{\psi_{1,1}}{\psi_{2,2}}y} + \psi_{2,2}^{-1} \int y^{-\frac{\psi_{2,1}}{\psi_{2,2}}-1} e^{-\frac{\psi_{1,1}}{\psi_{2,2}}y} dy. \tag{19}$$

(iii) *for $\psi_{1,2} \neq 0$, $\psi_{2,2} = 0$,*

$$(1 - \psi_{1,1}xy - \psi_{2,1}x)\Lambda_x - \psi_{1,2}y^2\Lambda_y = 0, \qquad \Lambda|_{x=0} = \underbrace{y^{\frac{\xi_1}{\psi_{1,2}}} e^{-\frac{\xi_2}{\psi_{1,2}y}}}_{:=A_3(y)} \tag{20}$$

The principal integral of Equation (20) is

$$L_3(x,y) = xy^{-\frac{\psi_{1,1}}{\psi_{1,2}}} e^{\frac{\psi_{2,1}}{\psi_{1,2}y}} + \psi_{1,2}^{-1} \int y^{-\frac{\psi_{1,1}}{\psi_{1,2}}-2} e^{\frac{\psi_{2,1}}{\psi_{1,2}y}} dy. \tag{21}$$

Proof. First, substituting $F(x,y) = \Lambda(x,y)/A_1(y)$ into (6), we obtain

$$(1-\psi_{1,1}xy-\psi_{2,1}x)\frac{\Lambda_x}{A_1} - (\psi_{1,2}y^2+\psi_{2,2}y)\left(\frac{\Lambda_y}{A_1}+(1/A_1)'\Lambda\right) = (\xi_1 y+\xi_2)\frac{\Lambda}{A_1},$$

$$(1-\psi_{1,1}xy-\psi_{2,1}x)\Lambda_x - (\psi_{1,2}y^2+\psi_{2,2}y)(\Lambda_y-(\ln A_1(y))'\Lambda,$$

$$(1-\psi_{1,1}xy-\psi_{2,1}x)\Lambda_x - (\psi_{1,2}y^2+\psi_{2,2}y)\left(\Lambda_y-\frac{\xi_1 y+\xi_2}{\psi_{1,2}y^2+\psi_{2,2}y}\Lambda\right) = (\xi_1 y+\xi_2)\Lambda.$$

Thus, since $F|_{x=1}$, we have

$$(1-\psi_{1,1}xy-\psi_{2,1}x)\Lambda_x - (\psi_{1,2}y^2+\psi_{2,2}y)\Lambda_y = 0, \qquad \Lambda|_{x=0} = A_1(y),$$

yielding us the first part of the first statement of the lemma.

Next, substituting

$$a=0,\ b=1,\ f(y)=1,\ g(y)=-\psi_{1,1}y-\psi_{2,1},\ h(y)=-\psi_{1,2}y^2-\psi_{2,2}y$$

into (11) and (12) of Lemma 2, we receive the second part of the first statement of the lemma.

The second and the third statements are proved analogically. □

Corollary 1.

(i) *Let* $\psi_{1,2}, \psi_{2,2} \neq 0$, $\psi_{2,1}+\psi_{2,2}=0$ *and* $\psi_{1,1}+2\psi_{1,2}=0$; *then, the numbers generated by the matrix* Ψ (1) *have the generating function*

$$F(x,y) = (1+\psi_{1,2}xy)^{\frac{\psi_{1,3}}{\psi_{1,2}}-\frac{\psi_{2,3}}{\psi_{2,2}}}(1+\psi_{1,2}xy+\psi_{2,2}x)^{\frac{\psi_{2,3}}{\psi_{2,2}}-1}. \tag{22}$$

(ii) *For* $\psi_{1,1}, \psi_{2,2} \neq 0$, *the numbers generated by the matrix*

$$\begin{pmatrix} \psi_{1,1} & 0 & \psi_{1,3} \\ 0 & \psi_{2,2} & \psi_{2,3} \end{pmatrix}$$

have the generating function

$$F(x,y) = y^{-\frac{\psi_{2,3}}{\psi_{2,2}}}\left(\frac{\psi_{2,2}}{\psi_{1,1}y}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right)-\psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)\right)^{\frac{\psi_{2,3}}{\psi_{2,2}}}$$
$$\times\exp\left(\frac{\xi_1}{\psi_{1,1}}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right)-\psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)-\frac{\xi_1}{\psi_{2,2}}y\right).$$

Proof. First, by (ii) of Lemma 2, we receive the principal integral

$$L(x,y) = y + x(\psi_{1,2}y^2+\psi_{2,2}y).$$

Next, by (16), we have the norming function

$$A(y) = \underbrace{(\psi_{1,2}y+\psi_{2,2})^{\frac{\psi_{1,3}}{\psi_{1,2}}-\frac{\psi_{2,3}}{\psi_{2,2}}}y^{\frac{\psi_{2,3}}{\psi_{2,2}}-1}}_{=\Lambda(0,y)}.$$

Hence, the solution to the corresponding Cauchy problem (15) and (16) is

$$\Lambda(x,y) = (\psi_{1,2}L(x,y)+\psi_{2,2})^{\frac{\psi_{1,3}}{\psi_{1,2}}-\frac{\psi_{2,3}}{\psi_{2,2}}}(L(x,y))^{\frac{\psi_{2,3}}{\psi_{2,2}}-1}.$$

Recollecting that $F(x,y) = \Lambda(x,y)/A(y)$ and simplifying the expression, we obtain the first statement of the lemma.

Second, by (19) of Theorem 3, the general solution of the corresponding differential equation is

$$E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right) - \psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y} = C.$$

Using the general solution and the condition

$$\Lambda|_{x=0} = A_2(y) = e^{\frac{\xi_1}{\psi_{2,2}}y}y^{\frac{\psi_{2,3}}{\psi_{2,2}}}$$

we obtain the solution to the Cauchy problem,

$$\Lambda(x,y) = \left(\frac{\psi_{2,2}}{\psi_{1,1}}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right) - \psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)\right)^{\frac{\psi_{2,3}}{\psi_{2,2}}} \times e^{\frac{\xi_1}{\psi_{1,1}}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right)-\psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)},$$

yielding us the statement of the lemma,

$$F(x,y) = \frac{\Lambda(x,y)}{A_2(y)} = \left(\frac{\psi_{2,2}}{\psi_{1,1}y}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right) - \psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)\right)^{\frac{\psi_{2,3}}{\psi_{2,2}}} \times e^{\frac{\xi_1}{\psi_{1,1}}E_1^{-1}\left(E_1\left(\frac{\psi_{1,1}}{\psi_{2,2}}y\right)-\psi_{2,2}xe^{-\frac{\psi_{1,1}}{\psi_{2,2}}y}\right)-\frac{\xi_1}{\psi_{2,2}}y} y^{-\frac{\psi_{2,3}}{\psi_{2,2}}}.$$

□

In the next section, we will use the following auxiliary result [7].

Theorem 4 (Belovas)**.** *Numbers generated by the matrix*

$$\begin{pmatrix} \psi_{1,1} & 0 & \psi_{1,3} \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}$$

(i) *have the generating function*

$$F(x,y) = (1 - (\psi_{1,1}y + \psi_{2,1})x)^{-\frac{\xi_1 y+\xi_2}{\psi_{1,1}y+\psi_{2,1}}}, \tag{23}$$

(ii) *and the analytic expression*

$$\begin{aligned} a_{n,k} &= \sum_{k_1+\cdots+k_n=k,\ k_j\in\{0,1\}} \prod_{j=1}^{n}(\psi_{2-k_j,1}j + \psi_{2-k_j,3}) \\ &= \sum_{k_1+\ldots+k_n=k,\ k_j\in\{0,1\}} \prod_{j=1}^{n}(b(k_j) + (j-1)c(k_j)), \end{aligned} \tag{24}$$

where $b(0) = \psi_{2,1} + \psi_{2,3}$, $b(1) = \psi_{1,1} + \psi_{1,3}$, $c(0) = \psi_{2,1}$, $c(1) = \psi_{1,1}$.

3. Numbers Satisfying a Class of Triangular Arrays and Generalized Lah Numbers

Definition 3. *Generalized Lah numbers [12] are defined by the recurrent expression*

$$L_{n,k} = L_{n-1,k-1} + m_{n,k}L_{n-1,k}, \tag{25}$$

here $L_{0,0} = 1$ *and* $L_{n,k} = 0$ *for* $\min(n,k,n-k) < 0$.

Remark 3. *For* $m_{n,k}$ *linear in* n *and* k, *we have* $L_{n,k} = a_{n,k}$, *generated by the matrix*

$$\begin{pmatrix} 0 & 0 & 1 \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}.$$

Lemma 3. *Let* $a_{n,k}$ *be a class of triangular arrays* (2) *and* $L_{n,k}$ *be generalized Lah numbers* (25) *with nonlinear coefficient*

$$\begin{aligned} m_{n,k} &= (\psi_{2,1}n + \psi_{2,2}k + \psi_{2,3})\frac{\xi_{n-1,k}}{\xi_{n,k}}, \\ \xi_{n,k} &= \prod_{j=1}^{k}[\psi_{1,1}(n-j+1) + \psi_{1,2}(k-j+1) + \psi_{1,3}], \end{aligned} \tag{26}$$

where $\xi_{0,0} = 1$ *and* $\xi_{n,k} = 0$ *for* $\min(n,k,n-k) < 0$. *Then*

$$a_{n,k} = \xi_{n,k} L_{n,k}. \tag{27}$$

Proof. We induct on n. By Definition 1, Definition 3, and (26), we have (see Table 1) that the statement (27) holds for $n = 0$ and $n = 1$.

Table 1. Numbers $a_{n,k}$, $\xi_{n,k}$ and $L_{n,k}$ for $n,k = 0,1$.

$a_{n,k}$	0	1	$\xi_{n,k}$	0	1	$L_{n,k}$	0	1
0	1	0	0	1	0	0	1	0
1	ξ_2	ξ_1	1	1	ξ_1	1	ξ_2	1

Let (27) is true for $n = r-1$, then, by (26),

$$\begin{aligned} \xi_{r,k}L_{r,k} &= \xi_{r,k}L_{r-1,k-1} + m_{r,k}\xi_{r,k}L_{r-1,k} \\ &= (\psi_{1,1}r + \psi_{1,2}k + \psi_{1,3})\underbrace{\xi_{r-1,k-1}L_{r-1,k-1}}_{=a_{r-1,k-1}} \\ &\quad + (\psi_{2,1}r + \psi_{2,2}k + \psi_{2,3})\underbrace{\xi_{r-1,k}L_{r-1,k}}_{=a_{r-1,k}}, \end{aligned}$$

yielding us the statement of the lemma. □

Remark 4. *Note that the coefficient* m_{nk} (26) *is linear if*

(i) $\psi_{1,1} = 0$, then
$$m_{n,k} = \psi_{2,1}n + \psi_{2,2}k + \psi_{2,3};$$

(ii) $\psi_{1,2} = \psi_{2,2} = 0$ and $\psi_{1,1}/\psi_{2,1} = \psi_{1,3}/\psi_{2,3}$, then
$$m_{n,k} = \psi_{2,1}n - \psi_{2,1}k + \psi_{2,3}.$$

Corollary 2. *For* $\psi_{13}, \psi_{21}, \psi_{22} \neq 0$, *numbers generated by the matrix*

$$\begin{pmatrix} 0 & 0 & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}$$

(i) *have the generating function*

$$F(x,y) = (1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}} \exp\left(\frac{\psi_{1,3}}{\psi_{2,2}} y((1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}} - 1)\right), \tag{28}$$

(ii) *and the analytic expression*

$$a_{n,k} = \frac{(\psi_{1,3})^k}{k!(\psi_{2,2})^k} \sum_{m=0}^{k} (-1)^m C_k^m \prod_{l=1}^{n} (\psi_{2,2}(k-m) + \psi_{2,1}l + \psi_{2,3}). \tag{29}$$

Proof. The first statement of the corollary can be obtained by solving the general differential Equation (6) by the method of characteristics. However, calculating the limit of the generating function (8), while $\psi_{1,2} \to 0$,

$$\lim_{\psi_{1,2}\to 0} (1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}} \left(1 + \frac{\psi_{1,2}}{\psi_{2,2}} y(1-(1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}})\right)^{-\frac{\psi_{1,2}+\psi_{1,3}}{\psi_{1,2}}}$$

$$= (1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}} \lim_{\psi_{1,2}\to 0} \left(1 - \frac{\psi_{1,2}}{\psi_{2,2}} y((1-\psi_{2,1}x)^{-\frac{\psi_{2,2}}{\psi_{2,1}}} - 1)\right)^{-\frac{\psi_{1,3}}{\psi_{1,2}}},$$

and applying the formula

$$\lim_{t\to 0} (1-at)^{-\frac{b}{t}} = e^{ab},$$

we receive the first statement of the corollary immediately.

We obtain the analytic expression (29) by differentiating the generating function (cf. (7)), or substituting $\psi_{1,2} = 0$ into (9) directly. □

Remark 5. *Note that Corollary 2 yields us the analytic expression for the ordinary Lah numbers, which are generated by the matrix*

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

$$a_{n,k} = \frac{1}{k!} \sum_{m=0}^{k} C_k^m \prod_{l=1}^{n} (k-m+l-1).$$

Next, we establish the following result.

Corollary 3. *For $\psi_{1,2}, \psi_{2,1} \neq 0$, numbers generated by the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}$$

(i) *have the generating function*

$$F(x,y) = (1-\psi_{2,1}x)^{-\frac{\xi_2}{\psi_{2,1}}} \left(1 + \frac{\psi_{1,2}}{\psi_{2,1}} y \ln(1-\psi_{2,1}x)\right)^{-\frac{\xi_1}{\psi_{1,2}}}, \tag{30}$$

(ii) *and the analytic expression*

$$a_{n,k} = \left(\prod_{j=1}^{k} (\psi_{1,2}j + \psi_{1,3})\right) \left(\sum_{\substack{k_1+\ldots+k_n=k,\\ k_j\in\{0,1\}}} \prod_{j=1}^{n} (b(k_j) + (j-1)c(k_j))\right), \tag{31}$$

where $b(k_j) = (1 - (\psi_{2,1} + \psi_{2,3}))k_j + (\psi_{2,1} + \psi_{2,3})$ and $c(k_j) = \psi_{2,1}(1 - k_j)$.

Proof. We can prove the first statement of the corollary by applying the limit

$$\lim_{\psi_{2,2}\to 0} \frac{1-(1-ax)^{-\frac{\psi_{2,2}}{b}}}{\psi_{2,2}} = \frac{\ln(1-ax)}{b}$$

to the general generating function (8). The second statement we get using Lemma 3. By the relation (27), we have (note that $\psi_{1,1} = 0$)

$$\xi_{n,k} = \prod_{j=1}^{k}[\psi_{1,2}(k-j+1)+\psi_{1,3}] = \prod_{j=1}^{k}[\psi_{1,2}j+\psi_{1,3}], \qquad \xi_{n-1,k}/\xi_{n,k} = 1. \tag{32}$$

Thus, by (26), we get $m_{n,k} = (\psi_{2,1}n + \psi_{2,2}k + \psi_{2,3})$ linear and (see Remark 3) $L_{n,k}$ are generated by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}.$$

Combining (32) with the result of the Theorem 4 for $L_{n,k}$ (see (24)), and substituting into (27), we receive the second statement of the corollary. □

4. Limit Theorems for Numbers Satisfying a Class of Triangular Arrays

Limit theorems for numbers satisfying a class of triangular arrays can be established using properties of ordinary or semi-exponential generating functions (cf. [13,14]). Let Ω_n be an integral random variable with the probability mass function

$$\underbrace{P(\Omega_n = k)}_{:=p_{n,k}} = \frac{a_{n,k}}{\sum_{k=0}^{n} a_{n,k}}, \quad k = 0, \ldots, n. \tag{33}$$

Definition 4. *Numbers $a_{n,k}$ are asymptotically normal with mean μ_n and variance σ_n^2 if*

$$\lim_{n\to\infty}\sup_x \left| \sum_{k \leqslant \vartheta_n x + \eta_n} p_{n,k} - \Phi(x) \right| = 0. \tag{34}$$

We use a general central limit theorem by Bender [15], based on the nature of the generating function $\sum a_{n,k} z^n w^k$, to prove the asymptotic normality of the numbers.

Lemma 4 (Bender). *Let $f(z,w)$ have a power series expansion*

$$f(z,w) = \sum_{n,k\geqslant 0} a_{n,k} z^n w^k \tag{35}$$

with non-negative coefficients. Suppose there exists

(i) *An $A(s)$ continuous and non-zero near 0,*
(ii) *An $r(s)$ with bounded third derivative near 0,*
(iii) *A non-negative integer m, and*
(iv) *$\varepsilon, \delta > 0$ such that*

$$\left(1 - \frac{z}{r(s)}\right)^m f(z,e^s) - \frac{A(s)}{1 - z/r(s)} \tag{36}$$

is analytic and bounded for

$$|s| < \varepsilon, \qquad |z| < |r(0)| + \delta. \tag{37}$$

Define

$$\eta = -\frac{r'(0)}{r(0)}, \qquad \theta^2 = \eta^2 - \frac{r''(0)}{r(0)}. \tag{38}$$

If $\theta \neq 0$*, then* (34) *holds with* $\eta_n = n\eta$ *and* $\vartheta_n^2 = n\vartheta^2$*.*

Let us formulate the central limit theorem.

Theorem 5. *Let the coefficients* $\psi_{1,2}, \psi_{2,1}$ *be positive and* $\psi_{2,3}$ *be non-negative, then the numbers generated by the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & 0 \\ \psi_{2,1} & 0 & \psi_{2,3} \end{pmatrix}$$

are asymptotically normal with mean $\mu_n = n\mu$ *and variance* $\sigma_n^2 = n\sigma^2$*, where*

$$\mu = \frac{\beta_1}{e^{\beta_1} - 1}, \qquad \sigma^2 = \frac{\beta_1(1 - (1-\beta_1)e^{\beta_1})}{(e^{\beta_1} - 1)^2}, \\ \beta_1 = \psi_{2,1}/\psi_{1,2}, \quad \beta_2 = \psi_{2,3}/\psi_{2,1}. \tag{39}$$

Proof. Let us transform the numbers $a_{n,k}$ into $\alpha_{n,k}$, $\alpha_{n,k} := (\psi_{1,2})^{-n} a_{n,k}$. For the numbers $\alpha_{n,k}$ we have

$$p_{n,k} = \frac{a_{n,k}}{\sum_{k=0}^n a_{n,k}} = \frac{\alpha_{n,k}}{\sum_{k=0}^n \alpha_{n,k}}.$$

By (30) of Corollary 3, the generating function of the numbers is

$$f(z, e^s) = (1 - \beta_1 z)^{-1-\beta_2} \Big(1 + \beta_1^{-1} e^s \ln(1 - \beta_1 z)\Big)^{-1}. \tag{40}$$

The crucial part of the proof is the selection of functions $r(s)$ and $A(s)$. Let $r(s)$ (cf. Lemma 3) be the root of the function

$$h(z, e^s) = 1 + \beta_1^{-1} e^s \ln(1 - \beta_1 z),$$

i.e.,

$$r(s) = \beta_1^{-1}(1 - \exp(-\beta_1 e^{-s})).$$

Calculating the derivatives, we receive

$$r'(s) = -\beta_1^{-1} \exp(-s - \beta_1 e^{-s})), \\ r''(s) = \beta_1^{-1}(1 - \beta_1 e^{-s}) \exp(-s - \beta_1 e^{-s})).$$

Hence, by Lemma 4,

$$\mu = -\frac{r'(0)}{r(0)} = \frac{\beta_1^{-1} e^{-\beta_1}}{\beta_1^{-1}(1 - e^{-\beta_1})} = \frac{\beta_1}{e^{\beta_1} - 1}, \\ \sigma^2 = \mu^2 - \frac{r''(0)}{r(0)} = \frac{\beta_1^2}{(e^{\beta_1} - 1)^2} - \frac{\beta_1^{-1}(1 - \beta_1)e^{-\beta_1}}{\beta_1^{-1}(1 - e^{-\beta_1})} = \frac{\beta_1(1 - (1-\beta_1)e^{\beta_1})}{(e^{\beta_1} - 1)^2}.$$

Note that $\sigma \neq 0$, since for $\beta_1 \neq 0$, we have $1 - (1 - \beta_1)e^{\beta_1} > 0$.

As Bender indicates (see Section 3 in [15]), the easiest way for verifying the (36) and (37) conditions of Lemma 4 is to show that $f(z, e^s)$ is continuous for $s \leqslant \varepsilon$ and z in the set

$$\{|z| \leqslant |r(0) + \delta|\} \cap \{|z - r(s)| \geqslant \eta\}$$

for some η. Since this is a compact set, f and hence (36) is bounded here. For $|z - r(s)| \leqslant \eta$, we can expand $f(z, e^s)$ in a Laurent series about $r(s)$ and show that the coefficient of the error term is bounded.

Let us consider the function $A(s)$ from (36) of Lemma 4 as the limit

$$A(s) = \lim_{z \to r(s)} f(z, e^s)\left(1 - \frac{z}{r(s)}\right).$$

Calculating $A(s)$, we obtain

$$\begin{aligned} A(s) &= \lim_{z \to r(s)} \frac{r(s) - z}{\underbrace{r(s)(1 - \beta_1 z)^{1+\beta_2}}_{\neq 0}\left(1 + \beta_1^{-1} e^s \ln(1 - \beta_1 z)\right)} \\ &= \frac{-1}{r(s)(1 - \beta_1 r(s))^{1+\beta_2} \frac{-\beta_1^{-1}\beta_1 e^s}{1 - \beta_1 r(s)}} = \frac{e^{-s}}{r(s)(1 - \beta_1 r(s))^{\beta_2}} \\ &= \frac{\beta_1 e^{-s}}{(1 - \exp(-\beta_1 e^{-s}))\exp(-\beta_1 \beta_2 e^{-s})}. \end{aligned}$$

The function

$$f(z, e^s) - \frac{A(s)}{1 - z/r(s)} = \frac{(1 - \beta_1 z)^{-1-\beta_2}}{1 + \beta_1^{-1} e^s \ln(1 - \beta_1 z)} - \frac{\beta_1 \exp(\beta_1 \beta_2 e^{-s})}{1 - \beta_1 z - \exp(-\beta_1 e^{-s})}$$

is analytic and bounded for

$$|s| < \varepsilon, \qquad |z| < |r(0)| + \delta = \beta_1^{-1}(1 - e^{-\beta_1}) + \delta.$$

Thus, conditions (i)–(iv) of Lemma 4 are satisfied. This concludes the proof of the theorem. □

Remark 6. *Note that the expression for the mean μ in Theorem 5 (see (39)) is the generating function $g(t)$ of the Bernoulli numbers B_n,*

$$g(t) = \frac{t}{e^t - 1} = \frac{t}{2}\left(\coth\frac{t}{2} - 1\right).$$

Remark 7. *For the numbers generated by the matrix*

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

we have

$$\begin{aligned} \mu_n &= \frac{n}{e - 1}\left(1 + \frac{c_1}{n} + O\left(\frac{1}{n^2}\right)\right), \\ \sigma_n^2 &= \frac{n}{(e - 1)^2}\left(1 + \frac{d_1}{n} + O\left(\frac{1}{n^2}\right)\right), \end{aligned}$$

where $c_1 = 2 - e$ and $d_1 = 1$.

Theorem 5 allows us to receive a symmetric result for the dual numbers (5). We can formulate the subsequent corollary.

Corollary 4. *Let the coefficients $\psi_{1,2}, \psi_{2,1}$ be positive and $\psi_{2,3}$ be non-negative; then, the numbers generated by the matrix*

$$\begin{pmatrix} \psi_{2,1} & 0 & \psi_{2,3} \\ \psi_{1,2} & -\psi_{1,2} & 0 \end{pmatrix}$$

are asymptotically normal with mean $\mu_n = n\tilde{\mu}$ and variance $\sigma_n^2 = n\sigma^2$, where

$$\tilde{\mu} = 1 - \mu = 1 - \frac{\beta_1}{e^{\beta_1} - 1}, \quad \sigma^2 = \frac{\beta_1(1 - (1 - \beta_1)e^{\beta_1})}{(e^{\beta_1} - 1)^2}, \quad \beta_1 = \psi_{2,1}/\psi_{1,2}.$$

Further, we use Hwang's result on the convergence rate in the central limit theorem for combinatorial structures (see Corollary 2 from Section 4 in [16]) to establish central limit theorems and specify the rate of convergence to the limiting distribution.

The moment generating function of the random variable Ω_n (33) equals

$$M_n(s) = E(e^{\Omega_n s}) = \sum_{k=0}^{n} P(\Omega_n = k)e^{ks} = \left(\sum_{k=0}^{n} a_{n,k}\right)^{-1} \sum_{k=0}^{n} a_{n,k}e^{ks}. \tag{41}$$

Combining the definition of the semi-exponent generating function (4) and (41), we obtain

$$F(x, e^s) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} a_{n,k}e^{sk} = \sum_{n=0}^{\infty} \frac{x^n}{n!} S_n M_n(s),$$

where

$$S_n = \sum_{k=0}^{n} a_{n,k}.$$

Thus, the partial differentiation of the double semi-exponential generating function $F(x, y)$ at $x = 0$ yields us the moment generating function

$$M_n(s) = S_n^{-1} \frac{\partial^n}{\partial x^n} F(x, e^s)\bigg|_{x=0}. \tag{42}$$

Since $M_n(0) = 1$, the formula for the sum S_n follows,

$$S_n = \frac{\partial^n}{\partial x^n} F(x, e^s)\bigg|_{(0,0)}. \tag{43}$$

Lemma 5 (Hwang). *Let $P_n(z)$ be a probability generating function of the random variable Ω_n, taking only non-negative integral values, with expectation μ_n and variance σ_n^2. Suppose that, for each fixed $n \geqslant 1$, $P_n(z)$ is a Hurwitz polynomial. If $\sigma_n \to \infty$, then, Ω_n satisfies*

$$P\left(\frac{\Omega_n - \mu_n}{\sigma_n} < x\right) = \Lambda(x) + O\left(\frac{1}{\sigma_n}\right), \quad x \in \mathbb{R}. \tag{44}$$

Theorem 6. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with the probability mass function (33) of the numbers generated by the matrix*

$$\begin{pmatrix} 0 & \psi_{1,2} & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix}. \tag{45}$$

Let the coefficients $\psi_{1,2}, \psi_{2,1}, \psi_{2,2}$ be positive, and

$$\text{(i)} \quad \frac{\psi_{2,3}}{\psi_{2,1}} = \frac{\psi_{1,3}}{\psi_{1,2}}, \qquad \text{(ii)} \quad \psi_{2,2} = \psi_{2,1},$$

then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O(n^{-1/2}), \qquad x \in \mathbb{R}. \tag{46}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ *and the variance* $\mathrm{Var}(\Omega_n) = \sigma_n^2$ *are equal to*

$$\mu_n = \frac{n\rho}{1+\rho}, \quad \sigma_n^2 = \frac{n\rho}{(1+\rho)^2}, \quad \rho = \psi_{1,2}/\psi_{2,1}, \tag{47}$$

respectively.

Proof. Let

$$\nu = 1 + \frac{\psi_{2,3}}{\psi_{2,1}} = 1 + \frac{\psi_{1,3}}{\psi_{1,2}}.$$

By (8) of Theorem 3, the generating function of numbers (45) is

$$\begin{aligned} F(x, e^s) &= (1-\psi_{2,1}x)^{-1-\nu}\Big(1+\rho e^s(1-(1-\psi_{2,1}x)^{-1})\Big)^{-1-\nu} \\ &= (1-\psi_{2,1}x(1+\rho e^s))^{-1-\nu}. \end{aligned} \tag{48}$$

Let $a_{n,k}$=$(\psi_{2,1})^n\alpha_{n,k}$. For the numbers $\alpha_{n,k}$, we have the generating function $\tilde{F}(u,v) = F(u/\psi_{2,1}, v)$. Note that

$$P(\Omega_n = k) = \frac{a_{n,k}}{\sum_{k=0}^n a_{n,k}} = \frac{\alpha_{n,k}}{\sum_{k=0}^n \alpha_{n,k}},$$

thus, $M_n(s) = \tilde{M}_n(s)$ (cf.(41)). Taking into account the formula for the nth derivative,

$$\left(\frac{1}{(1-kx)^\gamma}\right)^{(n)} = \frac{\Gamma(\gamma+n)k^n}{\Gamma(\gamma)(1-kx)^{\gamma+n}}, \tag{49}$$

we calculate the partial derivative of the double semi-exponential generating function,

$$\begin{aligned} \frac{\partial^n}{\partial x^n}\tilde{F}(x,e^s)\Big|_{x=0} &= \frac{\Gamma(\nu+n)}{\Gamma(\nu)}\frac{(1+\rho e^s)^n}{(1-(1+\rho e^s)x)^{\nu+n}}\Big|_{x=0} \\ &= \frac{\Gamma(\nu+n)}{\Gamma(\nu)}(1+\rho e^s)^n. \end{aligned} \tag{50}$$

Hence, the moment generating function (cf. (42)) and the probability generating function are

$$M_n(s) = \frac{(1+\rho e^s)^n}{(1+\rho)^n}, \qquad P_n(z) = M_n(\ln z) = \frac{(1+\rho z)^n}{(1+\rho)^n}, \tag{51}$$

respectively. The Hurwitz polynomial is a polynomial whose zeros are located in the left halfplane of the complex plane or on the imaginary axis. Since $\rho > 0$, $P_n(z)$ is a Hurwitz polynomial.

Note that the moment generating function $M_n(s)$ is the moment generating function of the binomial distribution $M_{p,q;n}(s) = (q + pe^s)^n$ with parameters

$$p = \frac{\rho}{1+\rho}, \qquad q = \frac{1}{1+\rho}.$$

Thus,

$$\mu_n = np = \frac{n\rho}{1+\rho}, \qquad \sigma_n^2 = npq = \frac{n\rho}{(1+\rho)^2}, \tag{52}$$

with $\sigma_n \to \infty$, yielding us, by Lemma 5, the statement of the theorem. □

Theorem 6 allows us to receive the symmetric result for the dual numbers. We can formulate the subsequent corollary.

Corollary 5. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with the probability mass function (33) of the numbers generated by the matrix*

$$\begin{pmatrix} 2\psi_{2,1} & -\psi_{2,1} & \psi_{2,3} \\ \psi_{1,2} & -\psi_{1,2} & \psi_{1,3} \end{pmatrix} \tag{53}$$

Let the coefficients $\psi_{1,2}, \psi_{2,1}, \psi_{2,2}$ be positive, and $\psi_{2,3}/\psi_{2,1} = \psi_{1,3}/\psi_{1,2}$, then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O(n^{-1/2}), \qquad x \in \mathbb{R}. \tag{54}$$

The expectation $E(\Omega_n) = \mu_n$ and the variance $\mathrm{Var}(\Omega_n) = \sigma_n^2$ are equal to

$$\mu_n = \frac{n}{1+\rho}, \quad \sigma_n^2 = \frac{n\rho}{(1+\rho)^2}, \quad \rho = \psi_{1,2}/\psi_{2,1}, \tag{55}$$

respectively.

5. Limit Theorems for Numbers Satisfying a Class of Triangular Arrays Associated with Laguerre Polynomials

We will use the following result on asymptotics of ratios of Laguerre polynomials [10].

Lemma 6 (Deaño et al.)**.** *Let $u, v > -1$ and $z \in \mathbb{C} \setminus [0, \infty)$; then, the ratio of arbitrary Laguerre polynomials has an asymptotic expansion*

$$\frac{L_{u;n+j}(z)}{L_{v;n}(z)} \sim \left(-\frac{z}{n}\right)^{\frac{v-u}{2}} \sum_{m=0}^{\infty} \frac{U_m(u,v,j,z)}{n^{m/2}}, \tag{56}$$

where the first coefficients are

$$\begin{aligned} U_0(u,v,j,z) &= 1, \\ U_1(u,v,j,z) &= \frac{v^2 - u^2 + 2z(v-u-2j)}{4\sqrt{-z}}. \end{aligned} \tag{57}$$

Theorem 7. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with the probability mass function (33) of the numbers generated by the matrix*

$$\begin{pmatrix} 0 & 0 & \psi_{1,3} \\ \psi_{2,1} & \psi_{2,2} & \psi_{2,3} \end{pmatrix} \tag{58}$$

Let the coefficients $\psi_{2,1}, \psi_{1,3}$ be positive, and $\psi_{2,2} = \psi_{2,1}$, then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O(n^{-1/4}), \qquad x \in \mathbb{R}. \tag{59}$$

The expectation $\mathrm{E}(\Omega_n) = \mu_n$ and the variance $\mathrm{Var}(\Omega_n) = \sigma_n^2$ are equal to

$$\begin{aligned} \mu_n &= \theta_1 \left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} - 1 \right), \\ \sigma_n^2 &= -\theta_1^2 \left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} \right)^2 + (\theta_1^2 - \theta_1\theta_2) \frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} + (\theta_1\theta_2 + n\theta_1), \end{aligned} \tag{60}$$

respectively. Here, $\theta_1 = \psi_{1,3}/\psi_{2,1}$ and $\theta_2 = \psi_{2,3}/\psi_{2,1}$.

Proof. First, we derive the moment generating function. Let $a_{n,k}$=$(\psi_{2,1})^n\alpha_{n,k}$. For the numbers $\alpha_{n,k}$, we have the generating function $\tilde{F}(u,v) = F(u/\psi_{2,1}, v)$. Note that $M_n(s) = \tilde{M}_n(s)$ (cf. (41)). By Corollary 2 (see (28)), the semi-exponential generating function equals

$$\tilde{F}(x,y) = \frac{e^{-\theta_1 y}}{(1-x)^{1+\theta_2}} \exp\left(\frac{\theta_1 y}{1-x}\right). \tag{61}$$

Thus, by (3), we receive

$$\tilde{F}(x,y) = \sum_{n=0}^{\infty} x^n L_{\theta_2;n}(-\theta_1 y), \tag{62}$$

and

$$\frac{\partial^n}{\partial x^n}\tilde{F}(x,y)\bigg|_{x=0} = n! L_{\theta_2;n}(-\theta_1 y), \tag{63}$$

Combining (42), (43) and (63), we have that the moment generating function equals

$$M_n(s) = \frac{L_{\theta_2;n}(-\theta_1 e^s)}{L_{\theta_2;n}(-\theta_1)}. \tag{64}$$

Hence, the probability-generating functon is

$$P_n(z) = M_n(\ln z) = \frac{L_{\theta_2;n}(-\theta_1 z)}{L_{\theta_2;n}(-\theta_1)}. \tag{65}$$

The Hurwitz polynomial is a polynomial whose zeros are located in the left halfplane of the complex plane or on the imaginary axis. If α is non-negative, then all roots of the generalized Laguerre polynomial $L_{\alpha;n}(x)$ are real and positive. Since $\theta_1 > 0$, the polynomial (65) is a Hurwitz polynomial.

Next, we calculate the expectation μ_n and the variance σ_n^2. The derivatives of the generalized Laguerre polynomials satisfy the following expression,

$$L^{(k)}_{\alpha;n}(x) = (-1)^k L_{\alpha+k;n-k}(x), \qquad k \leqslant n. \tag{66}$$

Hence,

$$\begin{aligned} M_n'(s) &= \frac{-\theta_1 e^s L'_{\theta_2;n}(-\theta_1 e^s)}{L_{\theta_2;n}(-\theta_1)} = \frac{\theta_1 e^s L_{\theta_2+1;n-1}(-\theta_1 e^s)}{L_{\theta_2;n}(-\theta_1)}, \\ M_n''(s) &= \frac{\theta_1^2 e^{2s} L''_{\theta_2;n}(-\theta_1 e^s) - \theta_1 e^s L'_{\theta_2;n}(-\theta_1 e^s)}{L_{\theta_2;n}(-\theta_1)} \\ &= \frac{\theta_1^2 e^{2s} L_{\theta_2+2;n-2}(-\theta_1 e^s) + \theta_1 e^s L_{\theta_2+1;n-1}(-\theta_1 e^s)}{L_{\theta_2;n}(-\theta_1)}. \end{aligned} \tag{67}$$

Using the properties of the generalized Laguerre polynomials,

$$\begin{aligned} L_{\alpha+1;n-1}(x) &= L_{\alpha+1;n}(x) - L_{\alpha;n}(x), \\ L_{\alpha+2;n-2}(x) &= \frac{\alpha+1-x}{x} L_{\alpha+1;n}(x) - \frac{\alpha+1-x+n}{x} L_{\alpha;n}(x). \end{aligned} \tag{68}$$

We obtain the expectation

$$\mu_n = M_n'(0) = \theta_1\left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} - 1\right) \tag{69}$$

and the variance

$$\begin{aligned}\sigma_n^2 &= M_n''(0) - M_n'^2(0) \\ &= \frac{(-\theta_1\theta_2 - \theta_1^2)L_{\theta_2+1;n}(-\theta_1) + (\theta_1\theta_2 + \theta_1^2 + n\theta_1)L_{\theta_2;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} \\ &- \theta_1^2\left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} - 1\right)^2 \\ &= -\theta_1^2\left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)}\right)^2 + (\theta_1^2 - \theta_1\theta_2)\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} + (\theta_1\theta_2 + n\theta_1).\end{aligned} \tag{70}$$

Using the asymptotic formula for the ratio of Laguerre polynomials (56) and (57), we obtain

$$\begin{aligned}\frac{L_{\alpha+1;n}(-x)}{L_{\alpha;n}(-x)} &= \sqrt{\frac{n}{x}}\left(1 + \frac{U_1(\alpha+1,\alpha,0,-x)}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right) \\ &= \sqrt{\frac{n}{x}}\left(1 + \frac{x-\alpha-1/2}{2\sqrt{xn}} + O\left(\frac{1}{n}\right)\right).\end{aligned} \tag{71}$$

Substituting (71) into (70), we get

$$\begin{aligned}\sigma_n^2 &= -\theta_1 n\left(1 + \frac{\theta_1-\theta_2-1/2}{2\sqrt{\theta_1 n}} + O\left(\frac{1}{n}\right)\right)^2 \\ &= -\theta_1 n\left(1 + \frac{\theta_1-\theta_2-1/2}{\sqrt{\theta_1 n}} + O\left(\frac{1}{n}\right)\right) \\ &+ (\theta_1-\theta_2)\sqrt{n}\theta_1\left(1 + \frac{\theta_1-\theta_2-1/2}{2\sqrt{\theta_1 n}} + O\left(\frac{1}{n}\right)\right) + \theta_1\theta_2 + n\theta_1 \\ &+ (\theta_1-\theta_2)\sqrt{n}\theta_1\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) + \theta_1\theta_2 + n\theta_1 = \frac{\sqrt{n\theta_1}}{2} + O(1).\end{aligned} \tag{72}$$

Thus, $\sigma_n \to \infty$, yielding us, by Lemma 5, the statement of the theorem. □

Theorem 7 allows us to receive the symmetric result for the dual numbers. We can establish the following corollary.

Corollary 6. *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable Ω_n with the probability mass function (33) of the numbers generated by the matrix*

$$\begin{pmatrix} 2\psi_{2,1} & -\psi_{2,1} & \psi_{2,3} \\ 0 & 0 & \psi_{1,3} \end{pmatrix} \tag{73}$$

Let the coefficients $\psi_{2,1}, \psi_{2,3}, \psi_{1,3}$ be positive, then

$$F_n(\sigma_n x + \mu_n) = \Phi(x) + O(n^{-1/4}), \qquad x \in \mathbb{R}. \tag{74}$$

The expectation $E(\Omega_n) = \mu_n$ and the variance $\mathrm{Var}(\Omega_n) = \sigma_n^2$ are equal to

$$\begin{aligned}\mu_n &= n - \theta_1\left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} - 1\right), \\ \sigma_n^2 &= -\theta_1^2\left(\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)}\right)^2 + (\theta_1^2 - \theta_1\theta_2)\frac{L_{\theta_2+1;n}(-\theta_1)}{L_{\theta_2;n}(-\theta_1)} + (\theta_1\theta_2 + n\theta_1),\end{aligned} \tag{75}$$

where $\theta_1 = \psi_{1,3}/\psi_{2,1}$ and $\theta_2 = \psi_{2,3}/\psi_{2,1}$.

6. Conclusions

We have proved limit theorems for three categories of numbers satisfying a class of triangular arrays (see Theorems 5 and 6), which are defined by a bivariate linear recurrence with bivariate linear coefficients, including combinatorial numbers associated with Laguerre polynomials (see Theorem 7). We have established the asymptotic normality of these combinatorial numbers and have specified convergence rates to the limiting distribution. Apart from the theoretical value (generating functions are a very important tool to derive the identities, connections, and interpolation functions for polynomials, or limit theorems for corresponding combinatorial numbers), these results can be applied to the construction of efficient algorithms for the calculation of the values of special functions. We have used similar limit theorems for the combinatorial numbers in calculations of the Riemann zeta function (see Theorem 3 in [14] and Algorithm 3 in [17]). Moreover, the presented asymptotic normality results may have also an important utilization in choosing a suitable cumulative distribution function or a cumulative intensity function for models in insurance [18].

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the anonymous reviewer for the careful reading of the manuscript and providing constructive comments and suggestions, which have helped to improve the quality of the paper.

Conflicts of Interest: The author declares no conflict of interest.

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