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THE ASYMPTOTICS OF THE GEOMETRIC POLYNOMIALS

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ABSTRACT. The paper investigates the asymptotic behavior of the geometric polynomials, when the polynomial degree tends to infinity. Using the contour integration technique, we obtain an asymptotic formula, given explicitly in terms of the polynomial degree and variable. This type of asymptotics will be applied to derive limit theorems for combinatorial numbers.

1. Introduction

Geometric polynomials $\omega_n(x)$ are defined by the exponential generating function

$$\frac{1}{1-x(\mathrm{e}^t-1)}=\sum_{n=0}^{\infty}\omega_n(x)\frac{t^n}{n!}.$$

Alternatively, the geometric polynomial of degree n is given by

$$\omega_n(x)=\sum_{k=0}^{n}k!S(n,k)x^k,$$

where $S(n,k)$ stand for the Stirling numbers of the second kind. An overview of properties and applications of the geometric polynomials can be found in [8,11,19]. Here are the first five geometric polynomials,

$$\omega_0(x)=1,\qquad \omega_1(x)=x,\qquad \omega_2(x)=2x^2+x,$$
$$\omega_3(x)=6x^3+6x^2+x,\qquad \omega_4(x)=24x^4+36x^3+14x^2+x.$$

The geometric polynomials, as well as related exponential polynomials, are important instruments in obtaining limit theorems for combinatorial numbers satisfying a class of triangular arrays. In a series of works [4–7], we have received such central and local limit theorems for the combinatorial numbers associated with the Riemann zeta function.

The asymptotic expansion for the exponential polynomials recently has been established by Paris [21]. Similar asymptotic formulas for different polynomials also have attracted the attention of many researchers; see, for instance, the results of Alfaro et al. [1] for the generalized Freud polynomials, Barbero et al. [2] for the Appell polynomials, Corcino and Corcino [13] for the Genocchi polynomials, Lee and Wong [16] for the Tricomi–Carlitz polynomials, Li et al. [17] for the Wilson polynomials, Paris [20] for the generalized Hermite–Bell polynomials, Wang et al. [22, 23] for the Racah polynomials. This paper aims to extend these investigations and receive the geometric polynomials' asymptotics. The asymptotics will be applied to derive limit theorems for combinatorial numbers. Note that the asymptotic approximation for the special case of $x = 1$ has been addressed in [14, 15, 18].

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2. The asymptotics of $\omega_n(x)$ for large n

The exponential polynomials [3, 9, 10],

$$T_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

and the geometric polynomials are connected by the relation [8],

$$\omega_n(x) = \int_0^\infty T_n(xt)\mathrm{e}^{-t}\mathrm{d}t. \tag{2.1}$$

The exponential polynomials, for their part, have the integral representation (cf. formula (2.3) of [12: Section 2]),

$$T_n(x) = \frac{2n!}{\pi\tau^n}\int_0^\pi \sin(n\theta)\mathrm{e}^{x(\mathrm{e}^{\tau\cos\theta}\cos(\tau\sin\theta)-1)}\sin(x\mathrm{e}^{\tau\cos\theta}\sin(\tau\sin\theta))\mathrm{d}\theta. \tag{2.2}$$

Theorem 2.1. *Let $x > 0$ be fixed. Then*

$$\omega_n(x) = \frac{n!}{(1+x)\ln^{n+1}(1+1/x)}\left(1 + O\Big(1 + \frac{4\pi^2}{\ln^2(1+1/x)}\Big)^{-(n+1)/2}\right) \tag{2.3}$$

as $n \to \infty$.

P r o o f. First we reduce the expression of $\omega_n(x)$ to a simpler form. Combining (2.1) and (2.2) we obtain

$$\omega_n(x) = \frac{2n!}{\pi\tau^n}\int_0^\pi \sin(n\theta)\Big(\int_0^\infty \mathrm{e}^{-u(x,\tau,\theta)t}\sin(v(x,\tau,\theta)t)\mathrm{d}t\Big)\mathrm{d}\theta, \tag{2.4}$$

where

$$\begin{aligned} u(x,\tau,\theta) &= 1 + x - x\mathrm{e}^{\tau\cos\theta}\cos(\tau\sin\theta),\\ v(x,\tau,\theta) &= x\mathrm{e}^{\tau\cos\theta}\sin(\tau\sin\theta). \end{aligned} \tag{2.5}$$

Let $c_0 = \min(\ln(1+x^{-1}), \pi/2)$. Note that for $x > 0$, $\theta \in (0,\pi)$ and $\tau \in (0, c_0)$, we have $u(x,\tau,\theta) > 0$. Thus,

$$\omega_n(x) = \frac{2n!}{\pi\tau^n}\int_0^\pi \frac{v(x,\tau,\theta)\sin(n\theta)}{u^2(x,\tau,\theta) + v^2(x,\tau,\theta)}\mathrm{d}\theta. \tag{2.6}$$

Next, we notice that the integral expression (2.6) does not yield to the techniques of elementary calculus. We will evaluate it using contour integration. Let us denote the denominator in (2.6) by $D(x,\tau,\theta)$,

$$\begin{aligned} D(x,\tau,\theta) &= u^2(x,\tau,\theta) + v^2(x,\tau,\theta)\\ &= (1+x)^2 - 2x(1+x)\mathrm{e}^{\tau\cos\theta}\cos(\tau\sin\theta) + x^2\mathrm{e}^{2\tau\cos\theta}\\ &= \Big((\mathrm{e}^{\tau\mathrm{e}^{\mathrm{i}\theta}} - 1)x - 1\Big)\Big((\mathrm{e}^{\tau\mathrm{e}^{-\mathrm{i}\theta}} - 1)x - 1\Big). \end{aligned} \tag{2.7}$$

Zeros of the function are

$$\theta_{n,k} = \arctan\left(\frac{2\pi k}{\ln(1+x^{-1})}\right) + 2\pi n \pm \frac{\mathrm{i}}{2}\ln\left(\ln^2(1+x^{-1}) + (2\pi k)^2\right) \mp \mathrm{i}\ln\tau, \tag{2.8}$$

where $n, k \in \mathbb{Z}$.

Suppose $x_0 = 1/(\mathrm{e}-1)$. Let us consider two cases.

Case 1. First we consider the integral (2.6) for $x \in (0, x_0)$. Noticing that $u(x,\tau,\theta)$ is even in θ and $v(x,\tau,\theta)$ is odd in θ, we get

$$\omega_n(x) = \frac{n!}{\pi\tau^n} \int_{-\pi}^{\pi} \frac{v(x,\tau,\theta)}{D(x,\tau,\theta)} \sin n\theta \mathrm{d}\theta = \frac{n!}{\pi\tau^n} \Im \underbrace{\int_{-\pi}^{\pi} \frac{v(x,\tau,\theta)}{D(x,\tau,\theta)} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta}_{:=J_n(x,\tau)} . \tag{2.9}$$

To evaluate the integral (2.9), we apply the contour integral of the corresponding complex-valued function over the rectangular contour,

$$\oint_\gamma \underbrace{\frac{v(z,\tau,z)}{D(x,\tau,z)} \mathrm{e}^{\mathrm{i}nz}}_{:=f(z)} \mathrm{d}z. \tag{2.10}$$

Let us denote the vertices of the rectangular contour γ as $A(-\pi,0)$, $B(\pi,0)$, $C(\pi,R)$ and $D(-\pi,R)$. By the residue theorem, we have that, while R goes to infinity,

$$\oint_\gamma = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 2\pi\mathrm{i} \sum_{k=-\infty}^{+\infty} \operatorname*{Res}_{z=\theta_{0,k}^+} f(z), \tag{2.11}$$

here

$$\theta_{0,k}^+ = \arctan\left(\frac{2\pi k}{\ln(1+x^{-1})}\right) + \frac{\mathrm{i}}{2} \ln\left(\ln^2(1+x^{-1}) + (2\pi k)^2\right) - \mathrm{i}\ln\tau, \tag{2.12}$$

and $f(z)$ stands for an integrand. Note that

$$\mathrm{e}^{\mathrm{i}\theta_{0,k}^+} = \underbrace{\tau(\ln(1+x^{-1}) - \mathrm{i}2\pi k)^{-1}}_{:=\xi_k}. \tag{2.13}$$

Next we evaluate integrals over the sides of the rectangular contour γ (cf. (2.11)).

Side BC. We parametrize the side BC by $z = \pi + \mathrm{i}t$, $\mathrm{d}z = \mathrm{i}\mathrm{d}t$ and evaluate the integral using Watson's lemma. We have

$$\begin{aligned}\int_{BC} &= \mathrm{i}\int_0^R \frac{v(x,\tau,\pi+\mathrm{i}t)}{D(x,\tau,\pi+\mathrm{i}t)} \mathrm{e}^{\mathrm{i}n(\pi+\mathrm{i}t)} \mathrm{d}t = -\mathrm{i}^2 \mathrm{e}^{\mathrm{i}\pi n} x \int_0^R \frac{\mathrm{e}^{-\tau\cosh t}\sinh(\tau\sinh t)}{\left((\mathrm{e}^{-\tau\mathrm{e}^{-t}}-1)x-1\right)\left((\mathrm{e}^{-\tau\mathrm{e}^{t}}-1)x-1\right)} \mathrm{e}^{-nt}\mathrm{d}t \\ &= \frac{(-1)^n x}{n^2} \frac{\tau\mathrm{e}^{-\tau}}{\left((\mathrm{e}^{-\tau}-1)x-1\right)^2} \Big(1 + O\Big(\frac{1}{n}\Big)\Big). \end{aligned} \tag{2.14}$$

Side DA. Similarly, we parametrize the side DA by $z = -\pi + \mathrm{i}t$, $\mathrm{d}z = \mathrm{i}\mathrm{d}t$ and evaluate the integral using Watson's lemma. We receive

$$\begin{aligned}\int_{DA} &= -\mathrm{i}\int_0^R \frac{v(x,\tau,-\pi+\mathrm{i}t)}{D(x,\tau,-\pi+\mathrm{i}t)} \mathrm{e}^{\mathrm{i}n(-\pi+\mathrm{i}t)} \mathrm{d}t \\ &= \mathrm{i}^2 \mathrm{e}^{-\mathrm{i}\pi n} x \int_0^R \frac{\mathrm{e}^{-\tau\cosh t}\sinh(\tau\sinh t)}{\left((\mathrm{e}^{-\tau\mathrm{e}^{-t}}-1)x-1\right)\left((\mathrm{e}^{-\tau\mathrm{e}^{t}}-1)x-1\right)} \mathrm{e}^{-nt}\mathrm{d}t \\ &= \frac{(-1)^{n+1} x}{n^2} \frac{\tau\mathrm{e}^{-\tau}}{\left((\mathrm{e}^{-\tau}-1)x-1\right)^2} \Big(1 + O\Big(\frac{1}{n}\Big)\Big). \end{aligned} \tag{2.15}$$

Side CD. We parametrize the side CD by $z = t + \mathrm{i}R$, $\mathrm{d}z = \mathrm{d}t$. We get

$$\int\limits_{CD} = -\int\limits_{-\pi}^{\pi} \frac{v(x,\tau,t+\mathrm{i}R)}{D(x,\tau,t+\mathrm{i}R)} \mathrm{e}^{\mathrm{i}n(t+\mathrm{i}R)}\mathrm{d}t = -\mathrm{e}^{-nR}\int\limits_{-\pi}^{\pi} \frac{v(x,\tau,t+\mathrm{i}R)}{D(x,\tau,t+\mathrm{i}R)} \mathrm{e}^{\mathrm{i}nt}\mathrm{d}t \underset{R\to\infty}{\to} 0. \tag{2.16}$$

Next, by applying the residue theorem to the integral (2.10), we obtain

$$\oint\limits_{\gamma} f(z)\mathrm{d}z = \oint\limits_{\gamma} \frac{v(x,\tau,z)}{\big((\mathrm{e}^{\tau\mathrm{e}^{\mathrm{i}z}}-1)x-1\big)\big((\mathrm{e}^{\tau\mathrm{e}^{-\mathrm{i}z}}-1)x-1\big)}\mathrm{e}^{\mathrm{i}nz}\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=-\infty}^{+\infty} \operatorname*{Res}_{z=\theta^+_{0,k}} f(z). \tag{2.17}$$

Note that (cf. (2.12))

$$(\exp(\tau\exp(-\mathrm{i}\theta^+_{0,k}))-1)x-1=0 \quad\Longrightarrow\quad \exp(\tau\xi_k^{-1}) = 1+\frac{1}{x}, \tag{2.18}$$

hence, by (2.13), we have

$$\begin{aligned} v(x,\tau,\theta^+_{0,k}) &= x\exp\Big(\tau\frac{\mathrm{e}^{\mathrm{i}\theta^+_{0,k}}+\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}}}{2}\Big)\frac{\mathrm{e}^{\mathrm{i}\tau\sin\theta^+_{0,k}}-\mathrm{e}^{-\mathrm{i}\tau\sin\theta^+_{0,k}}}{2\mathrm{i}} \\ &= \frac{x}{2\mathrm{i}}\Big(\exp\Big(\mathrm{e}^{\mathrm{i}\theta^+_{0,k}}\Big)\Big)^{\tau/2}\Big(\exp\Big(\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}}\Big)\Big)^{\tau/2} \\ &\quad\times\Big(\exp\Big(\mathrm{i}\tau\frac{\mathrm{e}^{\mathrm{i}\theta^+_{0,k}}-\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}}}{2\mathrm{i}}\Big)-\exp\Big(-\mathrm{i}\tau\frac{\mathrm{e}^{\mathrm{i}\theta^+_{0,k}}-\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}}}{2\mathrm{i}}\Big)\Big) \\ &= \frac{x}{2\mathrm{i}}(\exp\xi_k)^{\tau/2}\big(\exp\xi_k^{-1}\big)^{\tau/2} \\ &\quad\times\Big(\exp\Big(\tau\frac{\xi_k-\xi_k^{-1}}{2}\Big)-\exp\Big(-\tau\frac{\xi_k-\xi_k^{-1}}{2}\Big)\Big) \\ &= x\frac{\mathrm{e}^{\tau\xi_k}-\mathrm{e}^{\tau\xi_k^{-1}}}{2\mathrm{i}} = x\frac{\exp(\tau\mathrm{e}^{\mathrm{i}\theta^+_{0,k}})-(1+x^{-1})}{2\mathrm{i}}, \end{aligned} \tag{2.19}$$

and, by (2.13), (2.18) and (2.19), we calculate the residues

$$\begin{aligned} \operatorname*{Res}_{z=\theta^+_{0,k}} f(z) &= \frac{v(x,\tau,\theta^+_{0,k})\mathrm{e}^{\mathrm{i}n\theta^+_{0,k}}}{(\exp(\tau\mathrm{e}^{\mathrm{i}\theta^+_{0,k}})-1)x-1}\lim_{z\to\theta^+_{0,k}}\frac{z-\theta^+_{0,k}}{(\mathrm{e}^{\tau\mathrm{e}^{-\mathrm{i}z}}-1)x-1} \\ &= \frac{x(\exp(\tau\mathrm{e}^{\mathrm{i}\theta^+_{0,k}})-(1+x^{-1}))\mathrm{e}^{\mathrm{i}n\theta^+_{0,k}}}{2\mathrm{i}((\exp(\tau\mathrm{e}^{\mathrm{i}\theta^+_{0,k}})-1)x-1)(x\exp(\tau\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}})\tau\mathrm{e}^{-\mathrm{i}\theta^+_{0,k}}(-i))} \\ &= \frac{(\mathrm{e}^{\tau\xi_k}-(1+x^{-1}))\xi_k^{n+1}}{2((\mathrm{e}^{\tau\xi_k}-1)x-1)(1+x^{-1})\tau} = \frac{\xi_k^{n+1}}{2(x+1)\tau}. \end{aligned} \tag{2.20}$$

Combining (2.11), (2.14), (2.15) and (2.16), we obtain that

$$\Im J_n(x,\tau) = \frac{\pi}{(1+x)\tau}\Im\Big(\mathrm{i}\sum_{k=-\infty}^{+\infty}\xi_k^{n+1}\Big) = \frac{\pi}{(1+x)\tau}\Re\sum_{k=-\infty}^{+\infty}\xi_k^{n+1}. \tag{2.21}$$

Next, combining (2.9) and (2.21), we get

$$\begin{aligned}\omega_n(x) &= \frac{n!}{\pi\tau^n}\Im J_n(x,\tau) = \frac{n!}{(1+x)\tau^{n+1}}\Re\sum_{k=-\infty}^{+\infty}\xi_k^{n+1}\\ &= \frac{n!}{(1+x)\tau^{n+1}}\Re\sum_{k=-\infty}^{+\infty}\frac{\tau^{n+1}}{\left(\ln\left(1+x^{-1}\right)-\mathrm{i}2\pi k\right)^{n+1}}\\ &= \frac{n!}{1+x}\sum_{k=-\infty}^{+\infty}\frac{\cos\left((n+1)\eta_k\right)}{\left(\ln^2\left(1+x^{-1}\right)+(2\pi k)^2\right)^{\frac{n+1}{2}}}\\ &= \frac{n!}{1+x}\left(\frac{1}{\ln^{n+1}\left(1+x^{-1}\right)}+\sum_{k=1}^{\infty}\frac{2\cos\left((n+1)\eta_k\right)}{\left(\ln^2\left(1+x^{-1}\right)+(2\pi k)^2\right)^{\frac{n+1}{2}}}\right),\end{aligned}\tag{2.22}$$

where $\eta_k = \arctan\left(\frac{2\pi k}{\ln(1+x^{-1})}\right)$. Note that $\ln\left(1+x^{-1}\right) > 1$. Thus, we receive the statement of the theorem

$$\omega_n(x) = \frac{n!}{(1+x)\ln^{n+1}\left(1+x^{-1}\right)}\left(1+O\left(\left(1+\frac{4\pi^2}{\ln^2\left(1+x^{-1}\right)}\right)^{-\frac{n+1}{2}}\right)\right)\tag{2.23}$$

for the first interval.

<u>*Case 2*</u>. Now let us consider the integral (2.6) for $x \in (x_0, +\infty)$. We compute $\omega_n(x)$ (cf. (2.9)),

$$\omega_n(x) = \frac{n!}{\pi\tau^n}\int_{-\pi}^{\pi}\frac{v(x,\tau,\theta)}{D(x,\tau,\theta)}\sin n\theta\mathrm{d}\theta = -\frac{n!}{\pi\tau^n}\Im\underbrace{\int_{-\pi}^{\pi}\frac{v(x,\tau,\theta)}{D(x,\tau,\theta)}\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta}_{:=Y_n(x,\tau)},\tag{2.24}$$

integrating

$$\oint_{\delta}\underbrace{\frac{v(x,\tau,z)}{D(x,\tau,z)}\mathrm{e}^{-\mathrm{i}nz}}_{:=g(z)}\mathrm{d}z\tag{2.25}$$

over the rectangular contour δ with vertices $A(-\pi,0)$, $B(\pi,0)$, $C'(\pi,-R)$ and $D'(-\pi,-R)$. By the residue theorem, we have that, while R goes to infinity,

$$\oint_{\delta} = \int_{AB}+\int_{BC'}+\int_{C'D'}+\int_{D'A} = -2\pi\mathrm{i}\sum_{k=-\infty}^{+\infty}\operatorname*{Res}_{z=\theta_{0,k}^-}g(z),\tag{2.26}$$

here

$$\theta_{0,k}^- = \arctan\left(\frac{2\pi k}{\ln\left(1+x^{-1}\right)}\right)-\frac{\mathrm{i}}{2}\ln\left(\ln^2(1+x^{-1})+(2\pi k)^2\right)+\mathrm{i}\ln\tau,\tag{2.27}$$

and $g(z)$ stands for an integrand. Note that

$$\mathrm{e}^{\mathrm{i}\theta_{0,k}^-} = \underbrace{\tau^{-1}(\ln(1+x^{-1})+\mathrm{i}2\pi k)}_{:=\zeta_k}.\tag{2.28}$$

Next we evaluate integrals over the sides of the rectangular contour δ (cf. (2.26)).

<u>*Side BC'.*</u> We parametrize the side BC' by $z=\pi-\mathrm{i}t$, $\mathrm{d}z=-\mathrm{i}\mathrm{d}t$ and evaluate the integral using Watson's lemma. We get

$$\begin{aligned}\int\limits_{BC'} = -\mathrm{i}\int\limits_0^R \frac{v(x,\tau,\pi-\mathrm{i}t)}{D(x,\tau,\pi-\mathrm{i}t)}\mathrm{e}^{-\mathrm{i}n(\pi-\mathrm{i}t)}\mathrm{d}t &= \mathrm{i}^2\mathrm{e}^{-\mathrm{i}\pi n}x\int\limits_0^R \frac{\mathrm{e}^{-\tau\cosh t}\sinh(\tau\sinh t)}{\big((\mathrm{e}^{-\tau\mathrm{e}^{t}}-1)x-1\big)\big((\mathrm{e}^{-\tau\mathrm{e}^{-t}}-1)x-1\big)}\mathrm{e}^{-nt}\mathrm{d}t\\ &= \frac{(-1)^{n+1}x}{n^2}\frac{\tau\mathrm{e}^{-\tau}}{((\mathrm{e}^{-\tau}-1)x-1)^2}\Big(1+O\Big(\frac{1}{n}\Big)\Big).\end{aligned} \tag{2.29}$$

<u>*Side $D'A$.*</u> Similarly, we parametrize the side $D'A$ by $z=-\pi-\mathrm{i}t$, $\mathrm{d}z=-\mathrm{i}\mathrm{d}t$ and evaluate the integral using Watson's lemma. We have

$$\begin{aligned}\int\limits_{D'A} = \mathrm{i}\int\limits_0^R \frac{v(x,\tau,-\pi-\mathrm{i}t)}{D(x,\tau,-\pi-\mathrm{i}t)}\mathrm{e}^{-\mathrm{i}n(-\pi-\mathrm{i}t)}\mathrm{d}t &= \mathrm{i}^2\mathrm{e}^{\mathrm{i}\pi n}x\int\limits_0^R \frac{\mathrm{e}^{-\tau\cosh t}\sinh(\tau\sinh t)}{\big((\mathrm{e}^{-\tau\mathrm{e}^{t}}-1)x-1\big)\big((\mathrm{e}^{-\tau\mathrm{e}^{-t}}-1)x-1\big)}\mathrm{e}^{-nt}\mathrm{d}t\\ &= \frac{(-1)^{n+1}}{n^2}\frac{\tau\mathrm{e}^{-\tau}}{((\mathrm{e}^{-\tau}-1)x-1)^2}\Big(1+O\Big(\frac{1}{n}\Big)\Big).\end{aligned} \tag{2.30}$$

<u>*Side $C'D'$.*</u> We parametrize the side $C'D'$ by $z=t-\mathrm{i}R$, $\mathrm{d}z=\mathrm{d}t$. We get

$$\int\limits_{C'D'} = -\int\limits_{-\pi}^{\pi} \frac{v(x,\tau,t-\mathrm{i}R)}{D(x,\tau,t-\mathrm{i}R)}\mathrm{e}^{-\mathrm{i}n(t-\mathrm{i}R)}\mathrm{d}t = -\mathrm{e}^{-nR}\int\limits_{-\pi}^{\pi} \frac{v(x,\tau,t-\mathrm{i}R)}{D(x,\tau,t-\mathrm{i}R)}\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t \underset{R\to\infty}{\to} 0. \tag{2.31}$$

Next, by applying the residue theorem to the integral (2.25), we receive

$$\oint\limits_{\delta} g(z)\mathrm{d}z = \oint\limits_{\delta} \frac{v(x,\tau,z)}{\big((\mathrm{e}^{\mathrm{e}^{\mathrm{i}z}}-1)x-1\big)\big((\mathrm{e}^{\mathrm{e}^{-\mathrm{i}z}}-1)x-1\big)}\mathrm{e}^{-\mathrm{i}nz}\mathrm{d}z = -2\pi\mathrm{i}\sum_{k=-\infty}^{+\infty}\operatorname*{Res}_{z=\theta^-_{0,k}} g(z). \tag{2.32}$$

Note that (cf. (2.27) and (2.28))

$$(\exp(\tau\exp(\mathrm{i}\theta^-_{0,k}))-1)x-1=0 \quad\Longrightarrow\quad \exp(\tau\zeta_k)=1+\frac{1}{x}, \tag{2.33}$$

hence,

$$\begin{aligned} v(x,\tau,\theta^-_{0,k}) &= x\exp\Big(\tau\frac{\mathrm{e}^{\mathrm{i}\theta^-_{0,k}}+\mathrm{e}^{-\mathrm{i}\theta^-_{0,k}}}{2}\Big)\frac{\mathrm{e}^{\mathrm{i}\tau\sin\theta^-_{0,k}}-\mathrm{e}^{-\mathrm{i}\tau\sin\theta^-_{0,k}}}{2\mathrm{i}}\\ &= \frac{x}{2\mathrm{i}}\Big(\exp\Big(\mathrm{e}^{\mathrm{i}\theta^-_{0,k}}\Big)\Big)^{\tau/2}\Big(\exp\Big(\mathrm{e}^{-\mathrm{i}\theta^-_{0,k}}\Big)\Big)^{\tau/2}\\ &\quad\times\Big(\exp\Big(\mathrm{i}\tau\frac{\mathrm{e}^{\mathrm{i}\theta^-_{0,k}}-\mathrm{e}^{-\mathrm{i}\theta^-_{0,k}}}{2\mathrm{i}}\Big)-\exp\Big(-\mathrm{i}\tau\frac{\mathrm{e}^{\mathrm{i}\theta^-_{0,k}}-\mathrm{e}^{-\mathrm{i}\theta^-_{0,k}}}{2\mathrm{i}}\Big)\Big)\\ &= \frac{x}{2i}\left(\exp\zeta_k\right)^{\tau/2}\left(\exp\zeta_k^{-1}\right)^{\tau/2}\\ &\quad\times\Big(\exp\Big(\tau\frac{\zeta_k-\zeta_k^{-1}}{2}\Big)-\exp\Big(-\tau\frac{\zeta_k-\zeta_k^{-1}}{2}\Big)\Big)\\ &= x\frac{\mathrm{e}^{\tau\zeta_k}-\mathrm{e}^{\tau\zeta_k^{-1}}}{2\mathrm{i}} = x\frac{(1+x^{-1})-\exp(\tau\mathrm{e}^{-\mathrm{i}\theta^-_{0,k}})}{2\mathrm{i}},\end{aligned} \tag{2.34}$$

and, by (2.28), (2.34) and (2.35), we obtain the residues

$$\begin{aligned}\operatorname*{Res}_{z=\theta_{0,k}^-} g(z) &= \frac{v(x,\tau,\theta_{0,k}^-)\mathrm{e}^{-\mathrm{i}n\theta_{0,k}^-}}{(\exp(\tau\mathrm{e}^{-\mathrm{i}\theta_{0,k}^-})-1)x-1}\lim_{z\to\theta_{0,k}^-}\frac{z-\theta_{0,k}^-}{(\mathrm{e}^{\tau\mathrm{e}^{\mathrm{i}z}}-1)x-1}\\ &= \frac{x(1+x^{-1}-\exp(\tau\mathrm{e}^{-\mathrm{i}\theta_{0,k}^-}))\mathrm{e}^{-\mathrm{i}n\theta_{0,k}^-}}{2\mathrm{i}((\exp(\tau\mathrm{e}^{-\mathrm{i}\theta_{0,k}^-})-1)x-1)(x\exp(\tau\mathrm{e}^{\mathrm{i}\theta_{0,k}^-})\tau\mathrm{e}^{\mathrm{i}\theta_{0,k}^-}\mathrm{i})} \qquad (2.35)\\ &= -\frac{(1+x^{-1}-\mathrm{e}^{\tau\zeta_k^{-1}})\zeta_k^{-(n+1)}}{2((\mathrm{e}^{\tau\zeta_k^{-1}}-1)x-1)(1+x^{-1})\tau} = \frac{\zeta_k^{-(n+1)}}{2(x+1)\tau}.\end{aligned}$$

Combining (2.26), (2.29), (2.30) and (2.31), we receive that

$$\Im Y_n(x,\tau) = -\frac{\pi}{\tau(1+x)}\Im\Big(\mathrm{i}\sum_{k=-\infty}^{+\infty}\zeta_k^{-(n+1)}\Big) = -\frac{\pi}{\tau(1+x)}\Re\sum_{k=-\infty}^{+\infty}\zeta_k^{-(n+1)}. \qquad (2.36)$$

Next, combining (2.24) and (2.36), we get

$$\begin{aligned}\omega_n(x) &= -\frac{n!}{\pi\tau^n}\Im Y_n(x,\tau) = \frac{n!}{(1+x)\tau^{n+1}}\Re\sum_{k=-\infty}^{+\infty}\zeta_k^{-(n+1)}\\ &= \frac{n!}{(1+x)\tau^{n+1}}\Re\sum_{k=-\infty}^{+\infty}\frac{\tau^{n+1}}{(\ln(1+x^{-1})+\mathrm{i}2\pi k)^{n+1}} \qquad (2.37)\\ &= \frac{n!}{1+x}\Big(\frac{1}{\ln^{n+1}(1+x^{-1})}+\sum_{k=1}^{\infty}\frac{2\cos((n+1)\eta_k)}{\big(\ln^2(1+x^{-1})+(2\pi k)^2\big)^{\frac{n+1}{2}}}\Big),\end{aligned}$$

where $\eta_k = \arctan\Big(\frac{2\pi k}{\ln(1+x^{-1})}\Big)$. Note that $\ln\big(1+x^{-1}\big)\in(0,1)$. Thus, we receive the statement of the theorem

$$\omega_n(x) = \frac{n!}{(1+x)\ln^{n+1}(1+x^{-1})}\Big(1+O\Big(\Big(1+\frac{4\pi^2}{\ln^2(1+x^{-1})}\Big)^{-\frac{n+1}{2}}\Big)\Big) \qquad (2.38)$$

for the second interval.

Finally, considering two one-sided limits, $\lim_{x\nearrow x_0}\omega_n(x)$ (see (2.23), *Case 1*) and $\lim_{x\searrow x_0}\omega_n(x)$ (see (2.38), *Case 2*), we receive

$$\omega_n(x_0) = (1-\mathrm{e}^{-1})n!\Big(1+O\Big(\Big(\frac{1}{1+4\pi^2}\Big)^{\frac{n+1}{2}}\Big)\Big),$$

thus concluding the proof of the theorem. □

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REFERENCES

[1] ALFARO, M.—MORENO-BALCÁZAR, J. J.—PEÑA, A.—REZOLA, M. L.: *Asymptotic formulae for generalized Freud polynomials*, J. Math. Anal. Appl. **421**(1) (2015), 474–488.

[2] FERNANDO BARBERO, J.—SALAS, J.—VILLASEÑOR, E. J. S.: *On the asymptotics of the rescaled Appell polynomials*, Adv. Appl. Math. **113** (2020), Art. ID 101962.

[3] BELL, E. T.: *Exponential polynomials*, Ann. Math. **35**(2) (1934), 258–277.

[4] BELOVAS, I.: *A central limit theorem for coefficients of the modified Borwein method for the calculation of the Riemann zeta-function*, Lith. Math. J. **59**(1) (2019), 17–23.
[5] BELOVAS, I.: *A local limit theorem for coefficients of modified Borwein's method*, Glas. Mat. Ser. III. **54**(1) (2019), 1–9.
[6] BELOVAS, I.—SABALIAUSKAS, M.: *Series with binomial-like coefficients for the evaluation and 3D visualization of zeta functions*, Informatica **31**(4) (2020), 659–680.
[7] BELOVAS, I.—SAKALAUSKAS, L.: *Limit theorems for the coefficients of the modified Borwein method for the calculation of the Riemann zeta-function values*, Colloq. Math. **151**(2) (2018), 217–227.
[8] BOYADZHIEV, K. N.: *A series transformation formula and related polynomials*, Int. J. Math. Math. **2005** (2005), Art. ID 792107.
[9] BOYADZHIEV, K. N.: *Exponential polynomials, Stirling numbers, and evaluation of some gamma integrals*, Abstr. Appl. Anal. **2009** (2009), Art. ID 168672.
[10] BOYADZHIEV, K. N.: *Close encounters with the Stirling numbers of the second kind*, Math. Mag. **85**(4) (2012), 252–266.
[11] BOYADZHIEV, K. N.—DIL, A.: *Geometric polynomials: properties and applications to series with zeta values*, Anal. Math. **42** (2016), 203–224.
[12] BRYCHKOV, Yu. A.: *Some properties of the exponential polynomials*, Integral Transforms Spec. Funct. **29**(7) (2018), 571–579.
[13] CORCINO, C. B.—CORCINO, R. B.: *Asymptotics of Genocchi polynomials and higher order Genocchi polynomials using residues*, Afr. Mat. **31** (2020), 781–792.
[14] GROSS, O. A.: *Preferential arrangements*, Amer. Math. Monthly **69**(1) (1962), 4–8.
[15] KNUTH, D. E.: *Minimum-Comparison Sorting (Section 5.3.1, Exercise 4)*. In: The Art of Computer Programming, Vol. 3, Addison Wesley, 1998, p. 194 and p. 653.
[16] LEE, K. F.—WONG, R.: *Asymptotic expansion of the Tricomi–Carlitz polynomials and their zeros*, J. Comput. Appl. Math. **265** (2014), 220–242.
[17] LI, Y.-T.—WANG, X.-S.—WONG, R.: *Asymptotics of the Wilson polynomials*, Anal. Appl. **18**(2) (2020), 237–270.
[18] MAHADEV, N. V. R.—PELED, U. N.: *Enumeration (Chapter 17)*. In: Threshold Graphs and Related Topics, Annals of Discrete Mathematics 56, Elsevier, 1995, p. 474.
[19] MIHOUBI, M.—TAHARBOUCHET, S.: *Identities and congruences involving the geometric polynomials*, Miskolc Math. Notes **20**(1) (2020), 395–408.
[20] PARIS, R. B.: *The asymptotics of the generalised Hermite–Bell polynomials*, J. Comput. Appl. Math. **232**(2) (2009), 216–226.
[21] PARIS, R. B.: *The asymptotics of the Touchard polynomials*, Math. Aeterna **6**(5) (2016), 765–779.
[22] WANG, X.-S.—WONG, R.: *Asymptotics of Racah polynomials with varying parameters*, J. Math. Anal. Appl. **436**(2) (2016), 1149–1164.
[23] WANG, X.-S.—WONG, R.: *Asymptotics of Racah polynomials with fixed parameters*, Proc. Amer. Math. Soc. **146** (2018), 1083–1096.

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