DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2022-0004
Math. Slovaca **72** (2022), No. 1, 51–66

JOINT APPROXIMATION BY DIRICHLET L-FUNCTIONS

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(*Communicated by István Gaál*)

ABSTRACT. In the paper, collections of analytic functions are simultaneously approximated by collections of shifts of Dirichlet L-functions $(L(s+\mathrm{i}\gamma_1(\tau),\chi_1),\dots,L(s+\mathrm{i}\gamma_r(\tau),\chi_r))$, with arbitrary Dirichlet characters $\chi_1,\dots,\chi_r$. The differentiable functions $\gamma_1(\tau),\dots,\gamma_r(\tau)$ and their derivatives satisfy certain growth conditions. The obtained results extend those of [PAŃKOWSKI, Ł.: *Joint universality for dependent L-functions*, Ramanujan J. **45** (2018), 181–195].

1. Introduction

Let χ be a Dirichlet character modulo $q\in\mathbb{N}$, i.e., χ is a function on $\mathbb{Z}$, $\chi(m)\not\equiv 0$, $\chi(m)=0$ for $(m,q)>1$, $\chi(m_1m_2)=\chi(m_1)\chi(m_2)$ for $m_1,m_2\in\mathbb{Z}$ and $\chi(m+q)=\chi(m)$ for all $m\in\mathbb{Z}$. The properties of Dirichlet characters can be found, for example, in [9]. The Dirichlet L-function $L(s,\chi)$, $s=\sigma+\mathrm{i}t$, is defined, for $\sigma>1$, by

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}=\prod_p\Big(1-\frac{\chi(p)}{p^{-s}}\Big)^{-1},$$

where the product is taken over all prime numbers. If $\chi(m)$ is the principal character ($\chi(m)=1$ if $(m,q)=1$), then

$$L(s,\chi)=\zeta(s)\prod_{p|q}\Big(1-\frac{1}{p^s}\Big),$$

where $\zeta(s)$ is the Riemann zeta-function,

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_p\Big(1-\frac{1}{p^s}\Big)^{-1},\qquad \sigma>1.$$

Therefore, in this case, the function $L(s,\chi)$ has analytic continuation to the whole complex plane, except for the point $s=1$ which is a simple pole with residue $\prod_{p|q}(1-1/p)$. If $\chi(m)$ is a non-principal character, then $L(s,\chi)$ has analytic continuation to an entire function.

Voronin discovered the universality property of the functions $L(s,\chi)$ concerning the approximation of analytic functions defined in the strip $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$. He proved [21] that if

2020 Mathematics Subject Classification: 11M41.
Keywords: Dirichlet L-functions, joint universality, functional independence, weak convergence.
The research of the first author is funded by the European Social Fund (project No. 09.3.3-LMT-K-712-01-0037) under grant agreement with the Research Council of Lithuania (LMT LT).
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$f(s)$ is a continuous non-vanishing function on the disc $|s| \le r$, $0 < r < 1/4$, and analytic in the interior of this disc, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s|\le r} |L(s + 3/4 + \mathrm{i}\tau, \chi) - f(s)| < \varepsilon.$$

Moreover, Voronin in [22] considered the functional independence of Dirichlet L-functions with pairwise non-equivalent Dirichlet characters $\chi_1, \dots, \chi_r$, and, for this, he in fact obtained in a non-explicit form a joint universality theorem for $L(s,\chi_1), \dots, L(s,\chi_r)$. Voronin's investigations were continued by Gonek [7], Bagchi [1,2] and the first author [11]. For the modern version of a joint universality theorem, we need some notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let meas$A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is true, see, for example, [19].

Theorem 1.1. *Let $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + \mathrm{i}\tau, \chi_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

We recall that if $\chi(m)$ for $(m,q) = 1$ has a period less than q, then the character χ is non-primitive. In opposite case, χ is primitive. Every non-primitive character χ is induced by a primitive character, i.e., there exists a primitive character χ_1 modulo q_1, $q_1 \mid q$, such that

$$\chi(m) = \begin{cases} \chi_1(m) & \text{if } (m, q_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Two Dirichlet characters are called non-equivalent if they are not induced by the same primitive character.

It is possible to consider the approximation of the collection $(f_1(s), \dots, f_r(s))$ by more general shifts $(L(s + \mathrm{i}\gamma_1(\tau), \chi_1), \dots, L(s + \mathrm{i}\gamma_r(\tau), \chi_r))$. Let $K_1 = \dots = K_r = K$. Then it follows from [8] that, under hypotheses of Theorem 1.1, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \sup_{1\le j\le r} \sup_{s\in K} |L(s + \mathrm{i}\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon\Big\} > 0,$$

where $\gamma_j(\tau) = \tau + \lambda_j$, with K satisfying $\hat{K}_k \cap \hat{K}_l = \emptyset$, $k \ne l$, where $\hat{K}_j = \{s + \mathrm{i}\lambda_j : s \in K\}$, $j = 1, \dots, r$. Nakamura [14] obtained the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \sup_{1\le j\le r} \sup_{s\in K} |L(s + \mathrm{i}\gamma_j(\tau), \chi) - f_j(s)| < \varepsilon\Big\} > 0,$$

where $\gamma_j(\tau) = a_j\tau$ with algebraic numbers $a_1, \dots, a_r \in \mathbb{R}$ linearly independent over the field of rational numbers $\mathbb{Q}$. The case $r = 2$ was studied in [15]–[17] with $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, $a_1 \ne \pm a_2$.

The most general result belongs to Pańkowski [18]. He proved the following theorem.

Theorem 1.2. *Suppose that $\chi_1, \dots, \chi_r$ are Dirichlet characters, $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, $a_1, \dots, a_r \in \mathbb{R}^+$, and $b_1, \dots, b_r$ are such that*

$$b_j \in \begin{cases} \mathbb{R} & \textit{if} \quad a_j \notin \mathbb{N}, \\ (-\infty, 0] \cup (1 + \infty) & \textit{if} \quad a_j \in \mathbb{N}, \end{cases}$$

and $a_j \ne a_k$ or $b_j \ne b_k$ if $k \ne j$. Moreover, let $K \in \mathcal{K}$, $f_1, \dots, f_r \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [2,T] : \sup_{1\le j\le r} \sup_{s\in K} |L(s + \mathrm{i}\alpha_j \tau^{a_j} \log^{b_j} \tau, \chi_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

It is very important to stress that, in Theorem 1.2, $\chi_1, \dots, \chi_r$ are arbitrary, not necessarily pairwise non-equivalent, Dirichlet characters. The proof is based on the uniform distribution modulo 1.

Our aim is to obtain the joint universality for Dirichlet L-functions with other functions $\gamma_j(\tau)$ without using the uniform distribution theory. Moreover, we approximate in different sets $K_1, \dots, K_r \in \mathcal{K}$.

Suppose that, for $j = 1, \dots, r$, $\gamma_j(\tau)$ is an increasing to infinity real continuously differentiable functions on $[T_0, \infty)$, $T_0 > 0$, with derivative

(i)
$$\gamma_j'(\tau) = \hat{\gamma}_j(\tau)(1 + o(1)),$$
where $\hat{\gamma}_j(\tau)$ is monotonic such that

(ii)
$$\hat{\gamma}_1(\tau) = o(\hat{\gamma}_2(\tau)), \dots, \hat{\gamma}_{r-1}(\tau) = o(\hat{\gamma}_r(\tau))$$
and

(iii)
$$\gamma_j(2\tau) \max_{\tau \le u \le 2\tau} \frac{1}{\gamma_j'(u)} \ll \tau$$
as $\tau \to \infty$.

Denote the class of r-tuples $(\gamma_1, \dots, \gamma_r)$ satisfying the above hypotheses by U_r. Then the following joint universality theorem for Dirichlet L-functions is valid.

Theorem 1.3. *Suppose that $\chi_1, \dots, \chi_r$ are arbitrary Dirichlet characters, and $(\gamma_1, \dots, \gamma_r) \in U_r$. Let, for $j = 1, \dots, r$, $K_j \in \mathcal{K}$ and $f_j \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*
$$\liminf_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\Big\{\tau \in [T_0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + \mathrm{i}\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon\Big\} > 0.$$
Moreover, the limit
$$\lim_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\Big\{\tau \in [T_0, T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + \mathrm{i}\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon\Big\} > 0$$
exists for all but at most countably many $\varepsilon > 0$.

For example, the system of polynomials $\gamma_1(\tau) = \tau + 1$, $\gamma_2(\tau) = \tau^2 + \tau + 1, \dots, \gamma_r(\tau) = \tau^r + \tau^{r-1} + \dots + 1$ is a member of the class U_r. Also $(\tau \log \tau, \dots, \tau^r \log \tau) \in U_r$ and $(\tau(\Gamma'(\tau)/\Gamma(\tau)), \dots, \tau^r(\Gamma'(\tau)/\Gamma(\tau))) \in U_r$, where $\Gamma(\cdot)$ is the Euler gamma-function. We note that $(\tau \log \tau, \dots, \tau^r \log \tau)$ does not satisfy hypotheses of Theorem 1.2.

Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Theorem 1.3 can be generalized for some compositions. We will give only one theoretical example. Denote by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Thus, $H_0(K) \subset H(K)$. Let $S = \{g \in H(D) : g(s) \ne 0 \text{ or } g(s) \equiv 0\}$.

Theorem 1.4. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$ and $F\colon H^r(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*
$$\liminf_{T\to\infty} \frac{1}{T - T_0} \operatorname{meas}\Big\{\tau \in [T_0, T] : \sup_{s \in K} \big|F\big(L(s + \mathrm{i}\gamma_1(\tau), \chi_1), \dots, L(s + \mathrm{i}\gamma_r(\tau), \chi_r)\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty}\frac{1}{T-T_0}\text{meas}\Big\{\tau\in[T_0,T]:\\ \sup_{s\in K}\big|F\big(L(s+\mathrm{i}\gamma_1(\tau),\chi_1),\dots,L(s+\mathrm{i}\gamma_r(\tau),\chi_r)\big)-f(s)\big|<\varepsilon\Big\}>0$$

exists for all but at most countably many $\varepsilon>0$.

It is known that the sets of values taking by zeta or L-functions are in some sense dense. First, Bohr [4] obtained that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip $\{s\in\mathbb{C}:1<\sigma<1+\delta\}$ for any $\delta>0$. Bohr and Courant [5] obtained that, for any fixed σ, $1/2<\sigma<1$, the set

$$\{\zeta(\sigma+\mathrm{i}t):t\in\mathbb{R}\}$$

is dense in $\mathbb{C}$. Voronin extended and generalized the above results. He proved [20] that the set

$$\{(\zeta(s_1+\mathrm{i}\tau),\dots,\zeta(s_n+\mathrm{i}\tau)):\tau\in\mathbb{R}\}$$

with any fixed different $s_1,\dots,s_n$, $1/2<\mathrm{Re}s_k<1$, $1\le k\le n$, and the set

$$\Big\{\Big(\zeta(\sigma+\mathrm{i}t),\zeta'(\sigma+\mathrm{i}t),\dots,\zeta^{(n-1)}(\sigma+\mathrm{i}t)\Big):t\in\mathbb{R}\Big\}$$

with every fixed σ, $1/2<\sigma<1$, are dense in $\mathbb{C}^n$. Moreover, Voronin obtained a joint generalization of the later result for Dirichlet L-functions. Namely, he proved [22] that if $\chi_1,\dots,\chi_r$ are pairwise non-equivalent Dirichlet characters, then the set

$$\Big\{\Big(L(\sigma+\mathrm{i}t,\chi_1),L'(\sigma+\mathrm{i}t,\chi_1),\dots,L^{(n-1)}(\sigma+\mathrm{i}t,\chi_1),\dots,\\ L(\sigma+\mathrm{i}t,\chi_r),L'(\sigma+\mathrm{i}t,\chi_r),\dots,L^{(n-1)}(\sigma+\mathrm{i}t,\chi_r)\Big)\ t\in\mathbb{R}\Big\}$$

is everywhere dense in $\mathbb{C}^{r\times n}$ for every fixed σ, $1/2<\sigma<1$.

Theorem 1.3 has the following corollary.

Corollary 1.4.1. *Suppose that* $\chi_1,\dots,\chi_r$ *are arbitrary Dirichlet characters, and* $(\gamma_1,\dots,\gamma_r)\in U_r$. *Then, for every fixed* σ, $1/2<\sigma<1$, *the set*

$$\Big\{\Big(L(\sigma+\mathrm{i}\gamma_1(t),\chi_1),L'(\sigma+\mathrm{i}\gamma_1(t),\chi_1),\dots,L^{(n-1)}(\sigma+\mathrm{i}\gamma_1(t),\chi_1),\dots,\\ L(\sigma+\mathrm{i}\gamma_r(t),\chi_r),L'(\sigma+\mathrm{i}\gamma_r(t),\chi_r),\dots,L^{(n-1)}(\sigma+\mathrm{i}\gamma_r(t),\chi_r)\Big):t\ge T_0\Big\}$$

is everywhere dense in $\mathbb{C}^{r\times n}$.

The proof of the corollary uses Theorem 1.3 and repeats Voronin's arguments.

Corollary 1.4.1 implies the following functional independence property of Dirichlet L-functions.

Corollary 1.4.2. *Suppose that* $\chi_1,\dots,\chi_r$ *are arbitrary Dirichlet characters,* $\Phi\colon\mathbb{C}^{r\times n}\to\mathbb{C}$ *is a continuous function, and*

$$\Phi\Big(L(s,\chi_1),L'(s,\chi_1),\dots,L^{(n-1)}(s,\chi_1),\dots,L(s,\chi_r),L'(s,\chi_r),\dots,L^{(n-1)}(s,\chi_r)\Big)=0$$

identically for s. *Then* $\Phi\equiv0$.

For the proof of universality theorems, we apply a method different from that of [18]. This method is probabilistic, and is based on weak convergence of probability measures in the space of analytic functions, see [1, 10, 12] and [19].

2. Lemmas

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by γ the unit circle on the complex plane. Define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\mathbb{P}$ is the set of all prime numbers, and $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. Then, again, Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m^r_H exists. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m^r_H)$. For $j = 1, \ldots, r$, denote by $\omega_j(p)$ the pth component of an element $\omega_j \in \Omega_j$, $p \in \mathbb{P}$. Let $\omega = (\omega_1, \ldots, \omega_r)$, $\omega_j \in \Omega_j$, be the elements of Ω^r.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q^r_T(A) = \frac{1}{T-T_0}\operatorname{meas}\Big\{\tau \in [T_0, T] : \Big((p^{-\mathrm{i}\gamma_1(\tau)} : p \in \mathbb{P}), \ldots, (p^{-\mathrm{i}\gamma_r(\tau)} : p \in \mathbb{P})\Big) \in A\Big\}.$$

Lemma 2.1 (Main lemma)**.** *Suppose that $(\gamma_1, \ldots, \gamma_r) \in U_r$. Then Q^r_T converges weakly to the Haar measure m^r_H as $T \to \infty$.*

Proof. Let $g_{Q^r_T}(\underline{k}_1, \ldots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P})$, $j = 1, \ldots, r$, be the Fourier transform of Q^r_T, i.e.,

$$g_{Q^r_T}(\underline{k}_1, \ldots, \underline{k}_r) = \int\limits_{\Omega^r} \bigg(\prod_{j=1}^{r} {\prod_{p\in\mathbb{P}}}' \omega^{k_{jp}}(p)\bigg)\,\mathrm{d}Q^r_T,$$

where the sign "$'$" means that only a finite number of integers k_{jp}, $j = 1, \ldots, r$, are distinct from zero. Thus, by the definition of Q^r_T,

$$g_{Q^r_T}(\underline{k}_1, \ldots, \underline{k}_r) = \frac{1}{T-T_0}\int\limits_{T_0}^{T} \exp\bigg\{-\mathrm{i}\sum_{j=1}^{r}\gamma_j(\tau){\sum_{p\in\mathbb{P}}}' k_{jp}\log p\bigg\}\,\mathrm{d}\tau. \tag{2.1}$$

Let, for brevity,

$$a_j = {\sum_{p\in\mathbb{P}}}' k_{jp}\log p, \qquad j = 1, \ldots, r.$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, $a_j = 0$ if and only if $\underline{k}_j = \underline{0}$, $j = 1, \ldots, r$. Clearly, in view of (2.1),

$$g_{Q^r_T}(\underline{0}, \ldots, \underline{0}) = 1. \tag{2.2}$$

Now, suppose that $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$. Since $(\gamma_1, \ldots, \gamma_r) \in U_r$, we have

$$\bigg(\sum_{j=1}^{r} a_j\gamma_j(\tau)\bigg)' = \sum_{j=1}^{r} a_j\gamma'_j(\tau) = \sum_{j=1}^{r} a_j\hat{\gamma}_j(\tau)(1+o(1)) = a_{j_0}\hat{\gamma}_{j_0}(\tau)(1+o(1))$$

as $\tau \to \infty$, where $j_0 = \max(j : a_j \neq 0)$. Hence,

$$\bigg(\sum_{j=1}^{r} a_j\gamma'_j(\tau)\bigg)^{-1} = \frac{1}{a_{j_0}\hat{\gamma}_{j_0}(\tau)(1+o(1))} = \frac{1}{a_{j_0}\hat{\gamma}_{j_0}(\tau)}(1+o(1)) \tag{2.3}$$

as $\tau \to \infty$. Moreover, since $\gamma_j(\tau) \to \infty$ as $\tau \to \infty$, we have, in view of (iii) of the class U_r, that

$$\frac{1}{\hat{\gamma}_j(\tau)} = o(\tau) \tag{2.4}$$

as $\tau \to \infty$, $j = 1, \ldots, r$. Let $A(\tau) = \sum_{j=1}^{r} a_j \gamma_j(\tau)$. Then (2.3), (2.4), the monotonicity of $\hat{\gamma}_j(\tau)$, and the second mean value theorem show that

$$\begin{aligned}
\int_{T_0}^{T} \cos A(\tau)\,\mathrm{d}\tau &= \int_{\log T}^{T} \cos A(\tau)\,\mathrm{d}\tau + O(\log T) = \int_{\log T}^{T} \frac{1}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= \int_{\log T}^{T} \frac{1}{a_{j_0}\hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + \int_{\log T}^{T} \frac{o(1)}{a_{j_0}\hat{\gamma}_{j_0}(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \qquad (2.5) \\
&= \int_{\log T}^{T} \frac{1}{a_{j_0}\hat{\gamma}_{j_0}(\tau)}\,\mathrm{d}(\sin A(\tau)) + \int_{\log T}^{T} \frac{o(1)(1+o(1))}{A'(\tau)} \cos A(\tau)\,\mathrm{d}A(\tau) + O(\log T) \\
&= o(T) + \int_{\log T}^{T} o(1) \cos A(\tau)\,\mathrm{d}\tau + O(\log T) = o(T)
\end{aligned}$$

as $T \to \infty$. Similarly, we find that

$$\int_{T_0}^{T} \sin A(\tau)\,\mathrm{d}\tau = o(T)$$

as $T \to \infty$. This, (2.5) and (2.1) show that, in the case $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$,

$$g_{Q_T^r}(\underline{k}_1, \ldots, \underline{k}_r) = o(1), \qquad T \to \infty.$$

Thus, in view of (2.2),

$$\lim_{T\to\infty} g_{Q_T^r}(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r, the lemma follows by a continuity theorem for probability measures on compact groups. □

Lemma 2.1, by a standard way, implies a joint limit theorem in the space $H^r(D)$ for absolutely convergent Dirichlet series. Let $\sigma_0 > 1/2$ be a fixed number, $\underline{\chi} = (\chi_1, \ldots, \chi_r)$, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\},$$

and

$$\underline{L}_n(s, \underline{\chi}) = \left(L_n(s, \chi_1), \ldots, L_n(s, \chi_r)\right),$$

where

$$L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s}, \quad j = 1, \ldots, r,$$

and

$$\underline{L}_n(s,\omega,\underline{\chi}) = (L_n(s,\omega_1,\chi_1),\dots,L_n(s,\omega_r,\chi_r)),$$

where

$$L_n(s,\omega_j,\chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega_j(m)v_n(m)}{m^s}, \quad j=1,\dots,r,$$

and, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j=1,\dots,r.$$

Then the series for $L_n(s,\chi_j)$ and $L(s,\omega_j,\chi_j)$ are absolutely convergent for $\sigma > 1/2$, $j=1,\dots,r$. Define the function $u_n\colon \Omega^r \to H^r(D)$ by the formula

$$u_n(\omega) = \underline{L}_n(s,\omega,\underline{\chi}), \qquad \omega \in \Omega^r.$$

The absolute convergence of the series for $L_n(s,\omega_j,\chi_j)$, $j=1,\dots,r$, implies the continuity of the function u_n. Let $V_n = m_H^r u_n^{-1}$, where

$$V_n(A) = m_H^r u_n^{-1}(A) = m_H^r(u_n^{-1}A), \qquad A \in \mathcal{B}(H^r(D)).$$

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,n}(A) = \frac{1}{T-T_0}\operatorname{meas}\left\{\tau \in [T_0,T] : \underline{L}_n(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi}) \in A\right\},$$

where $\underline{\gamma}(\tau) = (\gamma_1(\tau),\dots,\gamma_r(\tau))$ and

$$\underline{L}_n(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi}) = (L_n(s+\mathrm{i}\gamma_1(\tau),\chi_1),\dots,L_n(s+\mathrm{i}\gamma_r(\tau),\chi_r)).$$

Then Lemma 2.1, the continuity of u_n and [3: Theorem 5.1] lead to the following statement.

Lemma 2.2. *Suppose that $(\gamma_1,\dots,\gamma_r) \in U_r$. Then $P_{T,n}$ converges weakly to the measure V_n as $T \to \infty$.*

The family of probability measures $\{V_n : n \in \mathbb{N}\}$ is very important for the investigation of the collection

$$\underline{L}(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi}) = (L(s+\mathrm{i}\gamma_1(\tau),\chi_1),\dots,L(s+\mathrm{i}\gamma_r(\tau),\chi_r)).$$

We recall that the family of probability measures $\{P\}$ on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$ is tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1-\varepsilon$$

for all $P \in \{P\}$.

Lemma 2.3. *The family $\{V_n : n \in \mathbb{N}\}$ is tight.*

Proof. For $j=1,\dots,r$, let $m_{H,j}$ be the probability Haar measure on $(\Omega_j,\mathcal{B}(\Omega_j))$, and $u_{n,j} : \Omega_j \to H(D)$ be given by the formula

$$u_{n,j}(\omega_j) = L_n(s,\omega_j,\chi_j).$$

Then $V_{n,j} = m_{H,j} u_{n,j}^{-1}$, $j = 1, \dots, r$, are the marginal measures of V_n. Actually, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} V_n\Big(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D)\Big) \\ = m_H^r u_n^{-1}\Big(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D)\Big) \\ = m_H^r \Big(u_n^{-1}\Big(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D)\Big)\Big) \\ = m_H^r\big(u_{n,j}^{-1} A\big) = m_{H,j} u_{n,j}^{-1}(A). \end{aligned}$$

It is easy to see using the absolute convergence of the series for $L_n(s, \chi_j)$, see, for example, the proof of Lemma 4.11 from [19] for more general functions from the Selberg class, that the families $\{V_{n,j} : n \in \mathbb{N}\}$, $j = 1, \dots, r$, are tight. Therefore, for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that

$$V_{n,j}(K_j) > 1 - \frac{\varepsilon}{r}, \qquad j = 1, \dots, r \tag{2.6}$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \dots \times K_r$ is compact in the space $H^r(D)$, and, in view of (2.6),

$$\begin{aligned} V_n(H^r(D) \setminus K) = V_n\Big(\bigcup_{j=1}^{r}\Big(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \dots \times H(D)\Big)\Big) \\ \le \sum_{j=1}^{r} V_{n,j}\left(H(D) \setminus K_j\right) \le \varepsilon \end{aligned}$$

for all $n \in \mathbb{N}$. The lemma is proved. □

3. Mean square estimates

Mean square estimates play an important role in the universality theory of zeta- and L-functions. In this section, we present estimates for generalized mean squares of Dirichlet L-functions.

Lemma 3.1. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$. Then, for fixed σ, $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^{T} |L(\sigma + \mathrm{i}t + \mathrm{i}\gamma_j(\tau), \chi_j)|^2 \, \mathrm{d}\tau \ll_\sigma T(1 + |t|), \qquad j = 1, \dots, r.$$

P r o o f. It is well known that, for fixed σ, $1/2 < \sigma < 1$,

$$\int_{T_0}^{T} |L(\sigma + \mathrm{i}t, \chi_j)|^2 \, \mathrm{d}t \ll_\sigma T.$$

Therefore, for all $t \in \mathbb{R}$,

$$\int_{0}^{|t|+\gamma_j(\tau)} |L(\sigma + \mathrm{i}u, \chi_j)|^2 \, \mathrm{d}u \ll_\sigma (|t| + \gamma_j(\tau)).$$

Thus, for $X \geq T_0$,

$$\begin{aligned}\int_X^{2X} |L(\sigma+\mathrm{i}t+\mathrm{i}\gamma_j(\tau),\chi_j)|^2\,\mathrm{d}\tau &= \int_X^{2X} \frac{1}{\gamma_j'(\tau)} |L(\sigma+\mathrm{i}t+\mathrm{i}\gamma_j(\tau),\chi_j)|^2\,\mathrm{d}\gamma_j(\tau)\\ &\ll \max_{X\leq\tau\leq 2X} \frac{1}{\gamma_j'(\tau)} \int_X^{2X} \mathrm{d}\Bigg(\int_0^{t+\gamma_j(\tau)} |L(\sigma+\mathrm{i}u,\chi_j)|^2\,\mathrm{d}u\Bigg)\\ &\ll_\sigma (|t|+\gamma_j(2X)) \max_{X\leq\tau\leq 2X} \frac{1}{\gamma_j'(\tau)} \ll_\sigma X(1+|t|)\end{aligned}$$

in virtue of properties of U_r. Now, taking $X = T2^{-k-1}$ and summing over $k = 0, 1, \ldots$, we get the estimate of the lemma. □

Lemma 3.1 allows to obtain the approximation in the mean for $\underline{L}(s,\underline{\chi})$ by $\underline{L}_n(s,\underline{\chi})$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s)-g_2(s)|}{1+\sup_{s\in K_l} |g_1(s)-g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. Then ρ is a metric in $H(D)$ inducing the topology of uniform convergence on compacta. For $\underline{g}_1 = (g_{11}, \ldots, g_{1r})$, $\underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D)$, define

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leq j\leq r} \rho(g_{1j}, g_{2j}).$$

Then $\underline{\rho}$ is a metric in $H^r(D)$ inducing the product topology.

Lemma 3.2. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \underline{\rho}\left(\underline{L}(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi}), \underline{L}_n(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi})\right)\,\mathrm{d}\tau = 0$$

holds.

Proof. From the definitions of the metrics ρ and $\underline{\rho}$, it follows that it suffices to prove that, for every compact set $K \subset D$ and all $j = 1, \ldots, r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \sup_{s\in K} |L(s+\mathrm{i}\gamma_j(\tau),\chi_j) - L_n(s+\mathrm{i}\gamma_j(\tau),\chi_j)|\,\mathrm{d}\tau = 0. \tag{3.1}$$

Let

$$l_n(s) = \frac{s}{\sigma_0}\Gamma\left(\frac{s}{\sigma_0}\right) n^{-s},$$

where the number σ_0 is from the definition of $v_n(m)$. Then an application of the Mellin formula leads to the representation

$$L_n(s,\chi) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty} L(s+z,\chi) l_n(z) \frac{\mathrm{d}z}{z},$$

where χ is an arbitrary Dirichlet character modulo q. Let K be an arbitrary fixed compact set of the strip D. We fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \le \sigma \le 1-\varepsilon$ for points $s \in K$. The residue theorem for $\hat{\sigma}_0 > 0$ implies

$$L_n(s,\chi) - L(s,\chi) = \frac{1}{2\pi \mathrm{i}} \int\limits_{-\hat{\sigma}_0-\mathrm{i}\infty}^{-\hat{\sigma}_0+\mathrm{i}\infty} L(s+z,\chi) l_n(z) \frac{\mathrm{d}z}{z} + R_n(s), \tag{3.2}$$

where

$$R_n(s) = \begin{cases} 0 & \text{if} \quad \chi \text{ is a non-principal character,} \\ \prod\limits_{p|q} \left(1 - \frac{1}{p}\right) \frac{l_n(1-s)}{1-s} & \text{otherwise.} \end{cases}$$

Denote by $s = \sigma + \mathrm{i}v$ the points of K, and take

$$\hat{\sigma}_0 = \sigma - \varepsilon - \frac{1}{2}, \qquad \sigma_0 = \frac{1}{2} + \varepsilon.$$

Let $\gamma(\tau)$ be one of the functions $\gamma_j(\tau)$, $j = 1, \dots, r$. Then, by (3.2),

$$\begin{aligned} &|L_n(s+\mathrm{i}\gamma(\tau),\chi) - L(s+\mathrm{i}\gamma(\tau),\chi)| \\ &\quad \le \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |L(s+\mathrm{i}\gamma(\tau) - \hat{\sigma}_0 + \mathrm{i}t,\chi)| \frac{|l_n(-\hat{\sigma}_0 + \mathrm{i}t)|}{|-\hat{\sigma}_0 + \mathrm{i}t|} \,\mathrm{d}t + |R_n(s+\mathrm{i}\gamma(\tau)| \\ &\quad = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |L(1/2+\varepsilon+\mathrm{i}(t+\gamma(\tau)),\chi)| \frac{|l_n(1/2+\varepsilon-s+\mathrm{i}t)|}{|1/2+\varepsilon-s+\mathrm{i}t|} \,\mathrm{d}t + |R_n(s+\mathrm{i}\gamma(\tau)| \end{aligned}$$

after a shift $t + v \to t$. Thus,

$$\begin{aligned} &\frac{1}{T-T_0} \int\limits_{T_0}^{T} \sup_{s\in K} |L(s+\mathrm{i}\gamma(\tau),\chi) - L_n(s+\mathrm{i}\gamma(\tau),\chi)| \,\mathrm{d}\tau \\ &\quad \le \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left(\frac{1}{T-T_0} \int\limits_{T_0}^{T} |L(1/2+\varepsilon+\mathrm{i}(t+\gamma(\tau)),\chi)| \,\mathrm{d}\tau \right) \sup_{s\in K} \frac{|l_n(1/2+\varepsilon-s+\mathrm{i}t)|}{|1/2+\varepsilon-s+\mathrm{i}t|} \,\mathrm{d}t \\ &\qquad + \frac{1}{T-T_0} \int\limits_{T_0}^{T} \sup_{s\in K} |R_n(s+\mathrm{i}\gamma(\tau)| \,\mathrm{d}\tau \\ &\quad \stackrel{\text{def}}{=} I_1 + I_2. \end{aligned} \tag{3.3}$$

For the function $\Gamma(s)$, the well-known estimate

$$\Gamma(\sigma + \mathrm{i}t) \ll \exp\{-c|t|\}, \qquad c > 0,$$

uniform in $\sigma_1 \le \sigma \le \sigma_2$ is valid. Therefore, the definition of $l_n(s)$ implies the bound

$$\frac{l_n(1/2+\varepsilon-s+\mathrm{i}t)}{1/2+\varepsilon-s+\mathrm{i}t} \ll \frac{n^{1/2+\varepsilon-\sigma}}{\sigma_0}\exp\left\{-\frac{c}{\sigma_0}|t-v|\right\} \ll_K n^{-\varepsilon}\exp\{-c|t|\}.$$

Thus, by Lemma 3.1,

$$I_1 \ll_{K,\varepsilon} n^{-\varepsilon}\int_{-\infty}^{\infty}\left(1+|t|^{1/2}\right)\exp\{-c|t|\}\,\mathrm{d}t \ll_{K,\varepsilon} n^{-\varepsilon}. \tag{3.4}$$

Similarly, we find that

$$I_2 \ll_{K,q} n^{1/2-2\varepsilon}\frac{1}{T-T_0}\int_{T_0}^{T}\exp\{-c\gamma(\tau)\}\,\mathrm{d}\tau \ll_{K,q} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\int_{\log T}^{T}\exp\{-c\gamma(\tau)\}\,\mathrm{d}\tau\right)$$

$$\ll_{K,q} n^{1/2-2\varepsilon}\left(\frac{\log T}{T}+\frac{1}{T}\exp\left\{-\frac{c}{2}\gamma(\log T)\right\}\int_{\log T}^{T}\exp\left\{-\frac{c}{2}\gamma(\tau)\right\}\mathrm{d}\tau\right)=o(T)$$

as $T\to\infty$ because $\gamma(\tau)\to\infty$ as $\tau\to\infty$. This, (3.4) and (3.3) prove (3.1). The lemma is proved. □

4. Limit theorem

In this section, we consider the weak convergence for

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T-T_0}\,\text{meas}\left\{\tau\in[T_0,T] : \underline{L}(s+\mathrm{i}\underline{\gamma}(\tau),\underline{\chi})\in A\right\}, \qquad A\in\mathcal{B}(H^r(D)),$$

as $T\to\infty$. For this, we recall the useful property of convergence in distribution ($\xrightarrow{\mathcal{D}}$).

Proposition 4.1. *Suppose that the space $(\mathbb{X},d)$ is separable, and X_{kn} and X_n, $k\in\mathbb{N}$, $n\in\mathbb{N}$, are $\mathbb{X}$-valued random elements defined on the same probability space with measure ν. If $X_{kn}\xrightarrow[n\to\infty]{\mathcal{D}} Z_k \xrightarrow[k\to\infty]{\mathcal{D}} X$ and, for every $\varepsilon>0$,*

$$\lim_{k\to\infty}\limsup_{n\to\infty}\nu\left\{d(X_{kn},X_n)\ge\varepsilon\right\}=0,$$

then $X_n\xrightarrow[n\to\infty]{\mathcal{D}} X$.

The proof of the proposition is given in [3].

Define

$$\underline{L}(s,\omega,\underline{\chi})=\left(L(s,\omega_1,\chi_1),\dots,L(s,\omega_r,\chi_r)\right),$$

where

$$L(s,\omega_j,\chi_j)=\sum_{m=1}^{\infty}\frac{\chi_j(m)\omega_j(m)}{m^s}, \qquad j=1,\dots,r,$$

and denote by $P_{\underline{L}}$ the distribution of the $H^r(D)$-valued random element $\underline{L}(s,\omega,\underline{\chi})$, i.e.,

$$P_{\underline{L}}(A)=m_H^r\left\{\omega\in\Omega^r : \underline{L}(s,\omega,\underline{\chi})\in A\right\}, \qquad A\in\mathcal{B}(H^r(D)).$$

Theorem 4.2. *Suppose that $(\gamma_1,\dots,\gamma_r)\in U_r$. Then P_T converges weakly to $P_{\underline{L}}$ as $T\to\infty$.*

P r o o f. On a certain probability space with measure μ, define a random variable ξ_T and assume that ξ_T is uniformly distributed on $[T_0, T]$. On the above probability space, define the $H^r(D)$-valued random element

$$\underline{X}_{T,n} = \underline{X}_{T,n}(s) = L_n(s + \mathrm{i}\underline{\gamma}(\xi_T), \underline{\chi}),$$

and denote by $\underline{\hat{X}}_n = \underline{\hat{X}}_n(s)$ the $H^r(D)$-valued random element with distribution V_n, where V_n is the limit measure in Lemma 2.2. Then Lemma 2.2 implies the relation

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{\hat{X}}_n. \tag{4.1}$$

By Lemma 2.3, the family $\{V_n : n \in \mathbb{N}\}$ is tight, therefore, in view of the Prokhorov theorem [3: Theorem 6.1], it is relatively compact. Thus, there exists a subsequence $\{V_{n_k}\} \subset \{V_n\}$ weakly convergent to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. This is equivalent to the relation

$$\underline{\hat{X}}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{4.2}$$

Define one more $H^r(D)$-valued random element

$$\underline{X}_T = \underline{X}_T(s) = \underline{L}(s + \mathrm{i}\underline{\gamma}(\xi_T), \underline{\chi}).$$

Then, using Lemma 3.2, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}(\underline{X}_T, \underline{X}_{T,n}) \ge \varepsilon\right\} \\
&\le \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{(T-T_0)\varepsilon} \int_{T_0}^{T} \underline{\rho}\left(\underline{L}(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\chi}), \underline{L}_n(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\chi})\right) \mathrm{d}\tau = 0.
\end{aligned}$$

This, (4.1), (4.2) and Proposition 4.1 show that

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P, \tag{4.3}$$

i.e., P_T converges weakly to P as $T \to \infty$.

It remains to prove that $P = P_{\underline{L}}$. The relation (4.3) shows that the measure P is independent of the choice of the sequence $\{V_{n_k}\}$. Hence, we have that

$$\underline{\hat{X}}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or V_n converges weakly to P as $n \to \infty$. In [6], a discrete limit theorem for Dirichlet L-functions was discussed, and it was obtained that the limit measure P of V_n, as $n \to \infty$, is $P_{\underline{L}}$. This remark and (4.3) complete the proof of the theorem. □

Theorem 4.2 implies a limit theorem for the compositions $F(\underline{L}(s, \underline{\chi}))$.

Theorem 4.3. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$, and $F\colon H^r(D) \to H(D)$ is a continuous operator. Then*

$$P_{T,F}(A) \overset{\text{def}}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : F\left(\underline{L}(s+\mathrm{i}\underline{\gamma}(\tau), \underline{\chi})\right) \in A\right\} \qquad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{L}}F^{-1}$ as $T \to \infty$.

P r o o f. The theorem follows from Theorem 4.2, continuity of F and [3: Theorem 5.1]. □

5. Support

For proving of universality, we need the explicit form of the support of the measure $P_{\underline{L}}$. Since the space $H^r(D)$ is separable, the support $S_{P_{\underline{L}}}$ of $P_{\underline{L}}$ is a minimal closed set of $H^r(D)$ such that $P_{\underline{L}}(S_{P_{\underline{L}}}) = 1$. The set $S_{P_{\underline{L}}}$ consists of all $\underline{g} \in H^r(D)$ such that, for every open neighbourhood $\underline{G}$ of $\underline{g}$, the inequality $P_{\underline{L}}(\underline{G}) > 0$ is satisfied.

We recall that

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Proposition 5.1. *The support of $P_{\underline{L}}$ is the set S^r.*

Proof. Let, for $\omega \in \Omega$,

$$L(s, \omega, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega(m)}{m^s} = \prod_p \Big(1 - \frac{\chi(p)\omega(p)}{p^s}\Big)^{-1},$$

and P_L be the distribution of the $H(D)$-valued random element $L(s, \omega, \chi)$. Then it is well known, see, for example, [1], [11], that the support of P_L is the set S. We will apply this remark for the support of $P_{\underline{L}}$.

Since the space $H^r(D)$ is separable, it is known that [3]

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_{r}.$$

Therefore, it suffices to consider the measure $P_{\underline{L}}$ on the sets $A \in H^r(D)$ of the form

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in H(D).$$

The Haar measure m_H^r is the product of the Haar measures $m_{H,j}$ on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Therefore,

$$\begin{aligned} P_{\underline{L}}(A) &= m_H^r \big\{\omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A\big\} \\ &= \prod_{j=1}^{r} m_{H,j} \{\omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j\} = \prod_{j=1}^{r} P_{L_j}(A_j), \end{aligned}$$

where P_{L_j} is the distribution of the random element $L(s, \omega_j, \chi_j)$. Since, for all $j = 1, \dots, r$, the support of P_{L_j} is the set S, the minimality of the support proves the proposition. □

Proposition 5.2. *Let $F \colon H^r(D) \to H(D)$ be a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S^r$ is non-empty. Then the support of the measure $P_{\underline{L}}F^{-1}$ is the whole of $H(D)$.*

Proof. Let $g \in H(D)$ be an arbitrary element, and G its any open neighbourhood. Then the set $F^{-1}G$ is open as well, and contains an element of the set S^r. Thus, in view of Proposition 5.1, $F^{-1}G$ is an open neighbourhood of an element of the support of the measure $P_{\underline{L}}$. Hence,

$$P_{\underline{L}}F^{-1}(G) = P_{\underline{L}}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the proposition. □

6. Proof of universality

We recall the Mergelyan theorem on the approximation of analytic functions by polynomials [13]. Let $K \subset \mathbb{C}$ be a compact set with connected complements, and $f(s)$ be a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

P r o o f o f T h e o r e m 1.3. By the Mergelyan theorem, there exist polynomials $p_j(s)$ such that

$$\sup_{1 \le j \le r} \sup_{s \in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{6.1}$$

Define the set

$$G_\varepsilon^r = \Big\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \le j \le r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \Big\}.$$

Then G_ε^r, in view of Proposition 5.1, is an open neighbourhood of the element $\left(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)}\right)$ of the support of the measure $P_{\underline{L}}$. Therefore

$$P_{\underline{L}}(G_\varepsilon^r) > 0. \tag{6.2}$$

Thus, Theorem 4.2 and the equivalent of weak convergence of probability measures in terms of open sets ([3: Theorem 2.1]) show that

$$\liminf_{T \to \infty} P_T(G_\varepsilon^r) > 0.$$

This, (6.1) and the definitions of P_T and G_ε^r prove the first part of the theorem.

To prove the second part of the theorem, define one more set

$$\hat{G}_\varepsilon^r = \Big\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \le j \le r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \Big\}.$$

Then the boundary $\partial \hat{G}_\varepsilon^r$ of $\hat{G}_\varepsilon^r$ lies in the set

$$\Big\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \le j \le r} \sup_{s \in K_j} |g_j(s) - f_j(s)| = \varepsilon \Big\},$$

therefore, $\partial \hat{G}_{\varepsilon_1}^r \cap \partial \hat{G}_{\varepsilon_2}^r = \emptyset$ for different positive ε_1 and ε_2. From this remark, it follows that the set $\hat{G}_\varepsilon^r$ is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. Thus, Theorem 4.2 and the equivalent of weak convergence of probability measures in terms of continuity sets ([3: Theorem 2.1]) imply the equality

$$\lim_{T \to \infty} P_T(\hat{G}_\varepsilon^r) = P_{\underline{L}}(\hat{G}_\varepsilon^r) \tag{6.3}$$

for all but at most countably many $\varepsilon > 0$. In view of (6.1), $G_\varepsilon^r \subset \hat{G}_\varepsilon^r$. Therefore, $P_{\underline{L}}(\hat{G}_\varepsilon^r) > 0$ by (6.2). This, (6.3) and the definitions of P_T and $\hat{G}_\varepsilon^r$ prove the second part of the theorem. □

P r o o f o f T h e o r e m 1.4. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{6.4}$$

Define the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \Big\}.$$

Then, by Proposition 5.2, G_ε is an open neighbourhood of the element $p(s)$ of the support of the measure $P_{\underline{L}}F^{-1}$. Therefore,

$$P_{\underline{L}}F^{-1}(G_\varepsilon) > 0. \tag{6.5}$$

From this, Theorem 4.3 and the equivalent of weak convergence of probability measures in terms of open sets, we obtain that

$$\liminf_{T\to\infty} P_{T,F}(G_\varepsilon) \geq P_{\underline{L}}F^{-1}(G_\varepsilon) > 0,$$

and the definitions of $P_{T,F}$ and G_ε, and (6.4) prove the first part of the theorem.

Define one more set

$$\hat{G}_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Then we have that this set is a continuity set of the measure $P_L F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 4.3 and the equivalent of weak convergence of probability measures in terms of continuity sets show that

$$\lim_{T\to\infty} P_{T,F}(\hat{G}_\varepsilon) = P_{\underline{L}}F^{-1}(\hat{G}_\varepsilon) \tag{6.6}$$

for all but at most countably many $\varepsilon > 0$. Moreover, in view of (6.4), we have that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, by (6.5), the inequality $P_{\underline{L},F^{-1}}(\hat{G}_\varepsilon) > 0$ holds. This together with (6.6) proves the second part of the theorem. □

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Received 14. 11. 2020
Accepted 3. 2. 2021

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