mathematics

Article

Diversity of Bivariate Concordance Measures

Martynas Manstavičius

Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania; martynas.manstavicius@mif.vu.lt

Abstract: We revisit the axioms of Scarsini, defining bivariate concordance measures for a pair of continuous random variables (X, Y); such measures can be understood as functions of the bivariate copula C associated with (X, Y). Two constructions, investigated in the works of Edwards, Mikusiński, Taylor, and Fuchs, are generalized, yielding, in particular, examples of higher than degree-two polynomial-type concordance measures, along with examples of non-polynomial-type concordance measures, and providing an incentive to investigate possible further characterizations of such concordance measures, as was achieved by Edwards and Taylor for the degree-one case.

Keywords: scarsini axioms; bivariate copula; transformation; polynomial-type concordance measure; multiplicative function

MSC: 62H20; 62H05; 58D19

check for updates

Citation: Manstavičius, M. Diversity of Bivariate Concordance Measures. *Mathematics* **2022**, *10*, 1103. https://doi.org/10.3390/math10071103

Academic Editors: Darius Plikynas, Leonidas Sakalauskas and Saulius Minkevičius

Received: 24 February 2022
Accepted: 28 March 2022
Published: 29 March 2022

1. Introduction

When modeling the dependence between two or more random variables, one often tries to gauge the strength of their relationship by statistically estimating some measure(s) of association. This typically involves computing one or several functionals that depend on the copula of those variables (Spearman ρ, Kendall's τ, Gini's γ, etc.; see Section 2) and, possibly, on the one-dimensional marginals (e.g., Pearson's correlation coefficient).

The interest in dependence measures stems from two complementary directions: on the one hand, it comes from a desire to measure the strength of dependence in collected data samples, i.e., to empirically estimate some association or dependence measure among variables of interest using parametric or nonparametric approaches (the latter typically involves using ranks and their functions); on the other hand, the interest arises from the need to compute dependence measures in a given theoretical (population) model, i.e., to find the theoretical dependence level among the variables. Such results often end up being used in statistical goodness-of-fit tests when choosing competing models for the data at hand.

There are many different ways to measure dependence, and one can naturally wonder about key properties that such measures should have. In the literature, one encounters several sets of axioms listing the most important properties of such measures as functionals on the set of copulas. For the measures of concordance, there are the axioms of Scarsini [1] in the bivariate case, and their generalizations to the multivariate setting; see the works of Taylor [2,3], Joe [4], Nelsen [5], and Dolati and Úbeda-Flores [6], just to name a few. Alternatively, we also have the axioms of Rényi [7] for the measures of dependence of a pair of random variables. In the copula setting (when the considered variables have continuous distributions), these axioms are provided, e.g., in [8], Definition 5.3.1. For more recent results in a similar direction, we recommend a paper by Borroni [9] and the references cited therein.

Many authors have considered various properties of dependence measures, how they compare with each other, what inequalities are satisfied in general and for particular families of copulas, what dependence measures say about symmetry/asymmetry of distributions, etc. Among many such works, as examples, we could mention recent works by Kokol Bukovšek et al. [10] and Mroz and Trutschnig [11]. A significant amount of research has been conducted to extend bivariate measures of concordance to the multivariate setting. The interested reader is referred to the works of Joe [4], Nelsen [5,12], Úbeda-Flores [13], Fuchs [14,15], and Mesfioui and Quessy [16], just to name a few. Many classical facts about concordance measures can also be found in the books by Nelsen [8], Joe [17], and Durante and Sempi [18].

Regarding computational aspects and practical applications of concordance measures, we can mention recent works by Dalessandro and Peters [19], Derumigny and Fermanian [20], and Denuit et al. [21].

In this paper, we focus on the theoretical aspects of bivariate measures of concordance, which preserve the so-called concordance order, and primarily try to investigate the diversity of the set of such measures. This interest has been stimulated by the works of Edwards and Taylor [22] and Taylor [2,3]. The authors, among other questions, suggested characterizing degree-$k \geq 2$ polynomial-type measures of concordance (see Section 2 for definitions). Degree-one measures of concordance were characterized in [22] (see also the earlier works with Mikusiński [23,24]), and three construction methods were provided. Yet another construction of degree-one concordance measures is provided by Borroni [9]. In higher dimensions, similar constructions were considered by Fuchs [14,15,25], extending earlier work by Fuchs and Schmidt [26]. To the best of our knowledge, there are well-known degree-one (Spearman's ρ, Gini's γ, etc.) and degree-two (e.g., Kendall's τ, or its generalization τ_ϕ in [9]) polynomial-type concordance measures, but no higher-degree examples are present in the literature. Thus, before trying to solve the suggested characterization problem, one has to have several examples to work with. In this paper, we take the first step in this direction and provide two constructions of concordance measures of odd-degree polynomial-type, along with those of non-polynomial-type. This is achieved in Theorems 1 and 2, which generalize ([23] Theorem 0.6) and ([22] Theorem 2). In addition, we give also give an easy method to get some even-degree polynomial-type measures; see Remark 1. Our search for more nontrivial extensions for even-degree polynomial-type concordance measures that would generalize Kendall's τ encountered several technical hurdles that were overcome only after a preliminary version of this paper was submitted for a review. We expect to provide the new findings regarding generalized Kendall's τ in a companion paper [27] in the near future.

To summarize our contributions, we have (i) generalized popular degree-one polynomial-type concordance measures (Spearman's ρ, Gini's γ, Blomqvist's β, etc.) to higher-degree (odd and also some even) polynomial and non-polynomial type measures and (ii) present various examples illustrating how the new measures could be computed. In the process we have found that the set of concordance measures is much bigger and deserves further study, both from the theoretical and practical points of view.

The rest of the paper is structured as follows. In Section 2, we give the preliminaries by introducing the needed concepts, including Scarsini's axioms of concordance measures for bivariate copulas, emphasizing the role of the symmetries of the unit square $[0,1]^2$. Sections 3 and 4 provide our main results, Section 5 gives various examples of their application, and Section 6 includes a short discussion and directions for future research.

2. Preliminaries

In this section, we succinctly provide several basic notions and facts about copulas, their transformations obtained from the symmetries of the unit square, and the axioms of Scarsini for a functional on the set of bivariate copulas to be called a concordance measure. For more on the copula theory, see the books by Nelsen [8], Durante and Sempi [18], and Joe [17].

2.1. Basic Notions of Copula Theory

We begin with the notion of a bivariate copula. Let $\mathbb{I} := [0,1]$.

Definition 1 ([8] Definition 2.2.2)**.** *By a bivariate copula (a copula, for short) C (though one can also consider n-variate copulas for any $n \geq 2$ (see, e.g., [8,17,18]), but we will only be concerned with bivariate copulas in this paper), we mean a function defined on $\mathbb{I}^2$ with values in $\mathbb{I}$ such that*

(i) $C(x, 0) = C(0, x) = 0$ *for any* $x \in \mathbb{I}$,
(ii) $C(x, 1) = C(1, x) = x$ *for any* $x \in \mathbb{I}$, *and*
(iii) *(2-increasingness) for all* $x, x', y, y' \in \mathbb{I}$ *with* $x \leq x'$ *and* $y \leq y'$,

$$V_C([x, x'] \times [y, y']) := C(x', y') - C(x, y') - C(x', y) + C(x, y) \geq 0.$$

The first two conditions for C are also called the boundary conditions, and the set of all bivariate copulas will be denoted by $\mathcal{C}$.

Among many examples of copulas in the literature (see [8,17,18] and the references therein), one often discusses the comonotonicity copula $M(x, y) = \min\{x, y\}$, independence copula $\Pi(x, y) = xy$, and countermonotonicity copula $W(x, y) = \max\{x + y - 1, 0\}$ for $(x, y) \in \mathbb{I}^2$.

With each (bivariate) copula $C \in \mathcal{C}$, one can associate a Borel measure μ_C which is doubly stochastic, i.e., $\mu_C(A \times \mathbb{I}) = \mu(\mathbb{I} \times A) = \lambda(A)$ for any Borel set $A \subset \mathcal{B}(\mathbb{I}^2)$, and where λ denotes the Lebesgue measure, and such that $\mu_C((0, x] \times (0, y]) = C(x, y)$, for any $x, y \in \mathbb{I}$, and vice versa (see, e.g., ([18] Theorem 3.1.2), where the result is stated for a general dimension $d \geq 2$). In what follows, integrals with respect to a copula $C \in \mathcal{C}$, e.g., $\int_{\mathbb{I}^2} f dC$, will mean $\int_{\mathbb{I}^2} f d\mu_C$.

On the set of bivariate copulas, one can consider a pointwise partial order relation defined as follows:

Definition 2 ([8] Definition 2.8.1)**.** *For any $C_1, C_2 \in \mathcal{C}$, we say that C_1 is* smaller *(resp.* larger*) than C_2 and denote it by $C_1 \prec C_2$ (resp. $C_1 \succ C_2$) if $C_1(x, y) \leq C_2(x, y)$ (resp. $C_1(x, y) \geq C_2(x, y)$) for any $(x, y) \in \mathbb{I}^2$.*

Concordance order, in the general d-dimensional setting (when $d \geq 2$), is defined as

$$\begin{aligned} C_1 \prec C_2 \quad \iff \quad & C_1(x_1, \ldots, x_d) \leq C_2(x_1, \ldots, x_d) \\ & \text{and } \overline{C}_1(x_1, \ldots, x_d) \leq \overline{C}_2(x_1, \ldots, x_d) \quad \forall (x_1, \ldots, x_d) \in \mathbb{I}^d, \end{aligned}$$

where $\overline{C}(x_1, \ldots, x_d) = \mathbb{P}(U_1 > x_1, \ldots, U_d > x_d)$, $U_1, \ldots, U_d \sim U(\mathbb{I})$ are uniformly distributed random variables on $\mathbb{I}$ whose copula is C. In other words, $\overline{C}$ is the survival function associated with copula C. For $d = 2$, $\overline{C}(x, y) = 1 - x - y + C(x, y)$, so concordance order for bivariate copulas is equivalent to pointwise order.

Then, the famous Fréchet–Hoeffding bounds (see, e.g., ([8] Equation (2.2.5)) can be written succinctly as $W \prec C \prec M$ for any $C \in \mathcal{C}$. For any reasonable concordance measure $\kappa_{X,Y}$ in the sense of Scarsini (Kendall's τ, Spearman's ρ, and Gini's γ are examples; see ([8] Definition 5.1.7)), when measuring the dependence between continuous random variables X and Y whose copula is C, an increase in C in concordance order means an increase in $\kappa_{X,Y}$, which justifies the name of the order.

2.2. Transformations of Copulas Generated by Symmetries of Their Domain

In relation to the axioms of concordance measures, of particular importance are the transformations of bivariate (or, more generally, multivariate) copulas that are induced by the symmetries of their domain $\mathbb{I}^2$ (or $\mathbb{I}^d$ for $d > 2$ in higher dimensions). The group of symmetries of the unit square $\mathbb{I}^2$ can be generated by involutions $\pi : \mathbb{I}^2 \to \mathbb{I}^2$ (permutation, i.e., reflection with respect to the main diagonal) and $\sigma_1 : \mathbb{I}^2 \to \mathbb{I}^2$ (partial reflection, i.e., symmetry with respect to the vertical line $x = 1/2$), given by

$$\pi(x, y) = (y, x) \qquad \text{and} \qquad \sigma_1(x, y) = (1 - x, y).$$

Involution means that $\pi^2 = \sigma_1^2 = e$, the identity transformation. Additionally, one can get the partial reflection $\sigma_2(x,y) = (x, 1-y)$ with respect to the axis $y = 1/2$ as $\sigma_2(x,y) = (\pi \circ \sigma_1 \circ \pi)(x,y)$. Combining the two reflections, we get the so-called total reflection

$$\tau(x,y) = (\sigma_1 \circ \sigma_2)(x,y) = (\sigma_2 \circ \sigma_1)(x,y) = (1-x, 1-y).$$

Altogether, the group of symmetries of the unit square, also called the dihedral group D_4, has $8 = 2!\,2^2$ elements:

$$D_4 = \{e, \pi, \sigma_1, \sigma_2, \tau, \pi \circ \sigma_1, \pi \circ \sigma_2, \pi \circ \tau\}.$$

Given a symmetry $\xi \in D_4$, there is a corresponding transformation $\xi^* : \mathcal{C} \to \mathcal{C}$ given by

$$\xi^*(C)(x,y) := \mu_C(\xi([0,x] \times [0,y])), \qquad (x,y) \in \mathbb{I}^2. \tag{1}$$

For the partial reflections σ_1, σ_2 and total reflection τ, one easily gets

$$\begin{aligned}
\sigma_1^*(C)(u,v) &= \mu_C([1-u,1] \times [0,v]) \\
&= \mu_C([0,1] \times [0,v]) - \mu_C([0,1-u] \times [0,v]) \\
&= C(1,v) - C(1-u,v) = v - C(1-u,v), \\
\sigma_2^*(C)(u,v) &= C(u,1) - C(u,1-v) = u - C(u,1-v), \\
\tau^*(C)(u,v) &= u + v - 1 + C(1-u,1-v), \qquad u,v \in \mathbb{I},
\end{aligned}$$

and the transpose of C is given by $C^T(u,v) := \pi^*(C)(v,u)$.

Partial reflections σ_1 and σ_2 can also be written using the star-product, introduced and developed by Darsow, Nguyen, and Olsen [28]:

$$(A * B)(x,y) := \int_0^1 \partial_2 A(x,y) \partial_1 B(t,y) dt, \qquad A, B \in \mathcal{C},\ (x,y) \in \mathbb{I}^2,$$

where $\partial_i f$ denotes the partial derivative of f with respect to the ith variable. Then, straightforward computations show:

$$\begin{aligned}
(W * C)(x,y) &= y - C(1-x,y) = \sigma_1^*(C)(x,y), \\
(C * W)(x,y) &= x - C(x,1-y) = \sigma_2^*(C)(x,y), \\
(M * C)(x,y) &= (C * M)(x,y) = C(x,y), \\
(\Pi * C)(x,y) &= (C * \Pi)(x,y) = \Pi(x,y), \qquad \forall C \in \mathcal{C}.
\end{aligned} \tag{2}$$

2.3. Scarsini's Axioms of Concordance Measures

In this paper, we are concerned with the family of functionals on the set of copulas $\mathcal{C}$, which measure the "degree of association" of continuous random variables having a given copula and preserve concordance order. This family was axiomatized by Scarsini in 1984 (see [1,29]; for extensions to the multidimensional case, see [2,6]).

Definition 3 ([18] Definition 2.4.7)**.** *A measure of concordance is a mapping* $\kappa : \mathcal{C} \to \mathbb{R}$ *such that*

(κ1) *κ is defined for every copula $C \in \mathcal{C}$,*
(κ2) *for every $C \in \mathcal{C}$, $\kappa(C) = \kappa(C^T)$,*
(κ3) *$\kappa(C_1) \le \kappa(C_2)$ whenever $C_1 \prec C_2$,*
(κ4) *$\kappa(C) \in [-1,1]$,*
(κ5) *$\kappa(\Pi) = 0$,*
(κ6) *$\kappa(\sigma_1^*(C)) = \kappa(\sigma_2^*(C)) = -\kappa(C)$ for the partial reflections σ_1 and σ_2, and any $C \in \mathcal{C}$,*
(κ7) *(continuity) if $C_n \to C$ uniformly (for copulas, pointwise convergence is enough) as $n \to \infty$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.*

One can observe that some authors, e.g., Nelsen ([8] Definition 5.1.17, Property 2) and Fuchs ([15] Section 30) also required

(κ_5') $\kappa(M) = 1$,

which can be achieved by a simple normalization if the original concordance measure does not satisfy this condition.

The list of the best known concordance measures includes Spearman's ρ, Kendall's τ, Gini's γ, Blomqvist's β, see ([8] Chapter 5), ([18] Section 2.4). On the other hand, Spearman's foot-rule is not a concordance measure, see ([8] Exercise 5.21).

These measures, including Spearman's foot-rule, are typically defined in terms of the so-called biconvex form (a more precise name, in our opinion, would be convex-combinations-restricted bilinear form, which is much longer, albeit clearer), given by

$$[C, D] := \int_{\mathbb{I}^2} C dD, \qquad C, D \in \mathcal{C}, \tag{3}$$

which is linear in each place with respect to convex combinations of copulas, hence the terminology. Indeed (see [8]), for a copula $C \in \mathcal{C}$,

- Spearman's ρ is given by $\rho(C) = 12[C, \Pi] - 3 = 12[C - \Pi, \Pi]$,
- Kendall's τ is defined as $\tau(C) = 4[C, C] - 1 = 4([C, C] - [\Pi, \Pi])$,
- Gini's γ is $\gamma(C) = 4([C, M] + [C, W]) - 2$, and
- Spearman's foot-rule is $\phi(C) = 6[M, C] - 2$.

One can alternatively use Kruskal's [30] concordance function

$$Q(C, D) := 4[D, C] - 1$$

(which is, in fact, symmetric in the bivariate case) to define the above measures. An exception is Blomqvist's β defined by $\beta(C) = 4C(1/2, 1/2) - 1$. It can be written in a similar form as

$$\beta(C) = \int_{\mathbb{I}^2} (C - \Pi) d\mu,$$

with the Borel measure $\mu = 4\delta_{(1/2,1/2)}$, where δ_z denotes Dirac's delta at the point z. Note that such a μ is not doubly-stochastic and so cannot come from a copula.

Many concordance measures κ in the literature have the property that the map

$$t \mapsto \kappa(tC_1 + (1 - t)C_2), \qquad t \in \mathbb{I}, \tag{4}$$

is a polynomial in t for any fixed $C_1, C_2 \in \mathcal{C}$. Thus, it makes sense to introduce

Definition 4 ([22] Definition 2)**.** *A measure of concordance, κ, is of polynomial type if for every choice of $C_1, C_2 \in \mathcal{C}$ the mapping in Equation (4) is a polynomial in t.*

Definition 5 ([22] Definition 3)**.** *The degree of a measure of concordance of polynomial type, κ, is defined as*

$$\deg \kappa = \sup\{\deg \kappa(tA + (1 - t)B) : A, B \in \mathcal{C}\}.$$

It is known that the popular Spearman's ρ, Gini's γ, and Blomqvist's β are all degree-one concordance measures, and Kendall's τ is a degree-two concordance measure.

3. Main Results

This section contains our main results and their proofs. We begin with our generalization of ([23] Theorem 0.6).

Theorem 1. *Let μ be a Borel measure on $(0,1)^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ a multiplicative function, i.e., $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in \mathbb{R}$. Then, the mapping*

$$\kappa_{\phi,\mu}(C) = \gamma \int_{(0,1)^2} \phi(C - \Pi) d\mu, \qquad C \in \mathcal{C}_2, \tag{5}$$

defines a measure of concordance for some $\gamma > 0$ if and only if

(a) μ *is regular and* D_4*-invariant;*
(b) ϕ *is continuous, odd, nondecreasing,* $\phi(0) = 0$, $\phi(1) = 1$, *and*
(c) $0 < \int_{(0,1)^2} \phi(M - \Pi) d\mu < +\infty$, $\gamma = \gamma_{\phi,\mu}$, *where*

$$\gamma_{\phi,\mu} := \left(\int_{(0,1)^2} \phi(M - \Pi) d\mu \right)^{-1}. \tag{6}$$

Remark 1. *Theorem 0.6 of [23] considers only $\phi(x) \equiv x$. On the other hand, our theorem allows taking*

(i) $\phi(x) = x^{2\ell+1}$ *for* $\ell \in \mathbb{N} \cup \{0\}$;
(ii) $\phi(x) = \operatorname{sgn}(x)|x|^\alpha$, *where* $\alpha \in (0, \infty)$ *and* sgn *denotes the sign function, i.e.,*

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

The proof of Theorem 1 requires appropriate generalizations of Lemmas 0.4 and 0.5 [23]. First, we have

Lemma 1. *Let μ and ν be nonnegative, regular Borel measures on $(0,1)^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ be an odd, continuous, multiplicative function such that $\phi(0) = 0$ and $\phi(1) = 1$. If*

$$\int_{(0,1)^2} \phi(C - \Pi) d\mu = \int_{(0,1)^2} \phi(C - \Pi) d\nu \qquad \forall C \in \mathcal{C}_2,$$

then $\mu = \nu$.

Proof. The proof mimics that of ([23] Lemma 0.4). The key differences are places where the original $\phi(x) = x$ is replaced by a more general odd, continuous, and multiplicative function ϕ.

First, observe that it is enough to establish that $\mu(R) = \nu(R)$ for any rectangle $R = [x_1, x_2] \times [y_1, y_2] \subset (0,1)^2$. Following Edwards, Mikusiński, and Taylor, for any $\delta > 0$ sufficiently small, so that $[x_1 - \delta, y_1 - \delta] \times [x_2 + \delta, y_2 + \delta] \subset (0,1)^2$, we consider

$$Q_{R,\delta} := \frac{1}{\delta^2}(C_{R,\delta} - \Pi), \tag{7}$$

where $C_{R,\delta}$ is an absolutely continuous copula with density

$$c_{R,\delta}(x,y) = \frac{\partial^2 C_{R,\delta}}{\partial x \partial y}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [x_1 - \delta, x_1] \times [y_2, y_2 + \delta] \\ & \text{or } (x,y) \in [x_2, x_2 + \delta] \times [y_1 - \delta, y_1]; \\ 2, & \text{if } (x,y) \in [x_1 - \delta, x_1] \times [y_1 - \delta, y_1] \\ & \text{or } (x,y) \in [x_2, x_2 + \delta] \times [y_2, y_2 + \delta]; \\ 1, & \text{elsewhere on } [0,1]^2. \end{cases}$$

This $Q_{R,\delta}$ is an approximation of the indicator function $\mathbf{1}_R$: for any $(x,y) \notin R$, there is a sufficiently small $\delta_0 > 0$ such that $Q_{R,\delta_0}(x,y) = 0$; on the other hand, for any $(x,y) \in R$, $Q_{R,\delta}(x,y) = 1$. Moreover, as $\delta \to 0+$, $Q_{R,\delta} \to \mathbf{1}_R$.

Observing that $\phi(\mathbf{1}_R) = \mathbf{1}_R$ and using the continuity and multiplicativity assumptions on ϕ, together with the dominated convergence theorem ($|Q_{R,\delta}| \leq 1$ for any rectangle $R \subset (0,1)^2$ and any $\delta > 0$ sufficiently small), we get

$$\begin{aligned}\mu(R) &= \int_{(0,1)^2} \mathbf{1}_R d\mu = \int_{(0,1)^2} \phi(\mathbf{1}_R) d\mu = \lim_{\delta\to 0} \int_{(0,1)^2} \phi(Q_{R,\delta}) d\mu \\ &= \lim_{\delta\to 0} \phi(\delta^{-2}) \int_{(0,1)^2} \phi(C_{R,\delta} - \Pi) d\mu \\ &= \lim_{\delta\to 0} \phi(\delta^{-2}) \int_{(0,1)^2} \phi(C_{R,\delta} - \Pi) d\nu \\ &= \int_{(0,1)^2} \phi(\mathbf{1}_R) d\nu = \int_{(0,1)^2} \mathbf{1}_R d\nu = \nu(R),\end{aligned}$$

completing the proof of this lemma. □

Second, we need

Lemma 2. *If μ is a Borel measure on $(0,1)^2$ and $\phi : \mathbb{R} \to \mathbb{R}$ is an odd, Borel measurable function, then μ is D_4-invariant if and only if for any $C \in \mathcal{C}_2$,*

$$\begin{aligned}&(i) \quad \int_{(0,1)^2} \phi(C^T - \Pi) d\mu = \int_{(0,1)^2} \phi(C - \Pi) d\mu, \\ &(ii) \quad \int_{(0,1)^2} \phi((W * C) - \Pi) d\mu = -\int_{(0,1)^2} \phi(C - \Pi) d\mu,\end{aligned}$$

provided the integrals on the left-hand sides or on the right-hand sides are defined (possibly infinite).

Proof. The proof is almost identical to that of ([23] Lemma 0.5), and hence is omitted. It essentially employs the definition of D_4-invariance and a change of variable formula for integrals (where only measurability of integrands is needed), and the oddness property of ϕ is needed to be able to get the minus sign outside the integral. □

Proof of Theorem 1. (Necessity) Suppose $\kappa := \kappa_{\phi,\mu}$ in (5) defines a concordance measure. Consider a subfamily of Fréchet–Mardia copulas

$$C_t := tM + (1-t)\Pi, \qquad t \in \mathbb{I}.$$

Then, it is clear that $C_{t_1} \prec C_{t_2}$ if $0 \leq t_1 \leq t_2 \leq 1$—that is, the family of copulas $\{C_t\}_{t\in\mathbb{I}}$ is positively ordered with respect to concordance order. Using the fact that ϕ is multiplicative, we get, for any $t \in \mathbb{I}$,

$$\kappa(C_t) = \gamma \int_{(0,1)^2} \phi(t(M - \Pi)) d\mu = \phi(t) \left(\gamma \int_{(0,1)^2} \phi(M - \Pi) d\mu \right).$$

Since κ is assumed to be a concordance measure,

$$0 = \kappa(\Pi) = \kappa(C_0) = \phi(0) \left(\gamma \int_{(0,1)^2} \phi(M - \Pi) d\mu \right), \tag{8}$$

$$1 = \kappa(M) = \kappa(C_1) = \phi(1) \left(\gamma \int_{(0,1)^2} \phi(M - \Pi) d\mu \right). \tag{9}$$

Moreover, since ϕ is assumed multiplicative,

$$\phi(1) = (\phi(1))^2 \quad \iff \quad \phi(1) = 0 \text{ or } \phi(1) = 1.$$

The choice $\phi(1) = 0$ contradicts (9), so we are left with $\phi(1) = 1$, which gives

$$\phi(t) = \kappa(C_t) \geq \kappa(\Pi) = \phi(0) = 0, \qquad \forall t \in \mathbb{I}.$$

However, $1 = \phi(t \cdot (1/t)) = \phi(t)\phi(1/t)$ due to multiplicativity, and so $\phi(t) > 0$ for any $t \in (0,1]$, and hence also for $t > 1$. Furthermore,

$$1 = \gamma \int_{(0,1)^2} \phi(M - \Pi) d\mu \quad \Longrightarrow 0 < \int_{(0,1)^2} \phi(M - \Pi) d\mu < +\infty,$$

implying $\gamma = \gamma_{\phi,\mu}$ as in (6). As $\phi(t) = \kappa(C_t)$ for any $t \in \mathbb{I}$, by multiplicativity,

$$\phi(t) = \big(\phi(1/t)\big)^{-1} = \big(\kappa(C_{1/t})\big)^{-1}, \qquad \forall t > 1. \tag{10}$$

This shows that ϕ must be continuous on $\mathbb{I}$ and hence also on $(1, \infty)$. Indeed, for any $t_0 \in \mathbb{I}$ we have $C_t \to C_{t_0}$ pointwise (and also uniformly) if $t \to t_0$ and so, using Axiom $(\kappa 7)$ of concordance measures, $\phi(t) = \kappa(C_t) \to \kappa(C_{t_0}) = \phi(t_0)$.

In addition, since $C_{t_1} \prec C_{t_2}$ if $0 \le t_1 \le t_2 \le 1$, we also get that $\phi(t)$ must be nondecreasing on $[0,1]$, and hence also on $(1, \infty)$ by Equation (10).

To prove that ϕ is odd, we first need the fact that μ is necessarily nonnegative on any rectangle $R = [x_1, x_2] \times [y_1, y_2] \subset (0,1)^2$. As in the proof of Theorem 0.6 [23], for any $\delta > 0$ small enough, we consider $Q_{R,\delta}$ given by (7). Then, $\Pi \prec C_{R,\delta}$, and since ϕ is multiplicative, Axiom $(\kappa 3)$ gives

$$\int_{(0,1)^2} \phi(Q_{R,\delta}) d\mu = \phi(\delta^{-2}) \int_{(0,1)^2} \phi(C_{R,\delta} - \Pi) d\mu \ge \phi(\delta^{-2}) \int_{(0,1)^2} \phi(\Pi - \Pi) d\mu = 0. \tag{11}$$

Letting $\delta \to 0+$, we get

$$Q_{R,\delta} \to \mathbf{1}_R \quad \Longrightarrow \quad \phi(Q_{R,\delta}) \to \phi(\mathbf{1}_R) = \mathbf{1}_R,$$

so that the left-hand side of (11) tends to $\mu(R)$ by the dominated convergence theorem, remaining nonnegative in the limit.

As ϕ is multiplicative, for any $x \in \mathbb{R}$,

$$\phi(-x) = \phi(-1)\phi(x), \quad 1 = \phi(1) = (\phi(-1))^2.$$

Thus, $\phi(-1) = \pm 1$, but the value $\phi(-1) = 1$ cannot be taken, since

$$-1 = \kappa(\sigma_1^*(M)) = \kappa(W) = \gamma \int_{(0,1)^2} \phi(W - \Pi) d\mu = \phi(-1) \underbrace{\gamma \int_{(0,1)^2} \phi(\Pi - W) d\mu}_{\ge 0}.$$

This means that only $\phi(-1) = -1$ can be taken, and so ϕ must be an odd function on $\mathbb{R}$.

To show that μ is regular, we argue by contradiction and slightly modify the corresponding lines in ([23] p. 1510). Suppose there is a compact set $K \subset (0,1)^2$ such that $\mu(K) = +\infty$. Then, setting

$$m := \min_{(x,y) \in K} (M - \Pi)(x,y) > 0,$$

we have

$$\infty > \int_{(0,1)^2} \phi(M - \Pi) d\mu \ge \int_K \phi(M - \Pi) d\mu \ge \phi(m) \int_K d\mu = \phi(m)\mu(K) = +\infty,$$

since $\phi(m) > 0$, a contradiction. Therefore, μ must indeed be a regular measure.

Finally, since κ is a measure of concordance,

$$\int_{(0,1)^2} \phi(C^T - \Pi)d\mu = \frac{\kappa(C^T)}{\gamma} = \frac{\kappa(C)}{\gamma} = \int_{(0,1)^2} \phi(C - \Pi)d\mu,$$
$$\int_{(0,1)^2} \phi((W * C) - \Pi)d\mu = \frac{\kappa(W * C)}{\gamma} = -\frac{\kappa(C)}{\gamma} = -\int_{(0,1)^2} \phi(C - \Pi)d\mu,$$

so that μ is D_4-invariant by Lemma 2.

(Sufficiency) Suppose μ, ϕ, and γ satisfy conditions (a)–(c) of the theorem. Then, using Lemma 2 and Equation (2), we have

$$\int_{(0,1)^2} \phi(W - \Pi)d\mu = \int_{(0,1)^2} \phi(W * M - \Pi)d\mu = -\int_{(0,1)^2} \phi(M - \Pi)d\mu,$$

and using Fréchet–Hoeffding bounds (i.e., $W \prec C \prec M$) and nondecreasingness of ϕ, we get $-1 = \kappa(W) \leq \kappa(C) \leq \kappa(M) = 1$, for any copula $C \in \mathcal{C}$, verifying Axiom ($\kappa 4$). Additionally, it is clear that $\kappa(\Pi) = 0$ as $\phi(0) = 0$, showing that Axiom ($\kappa 5$) holds.

Since μ is D_4-invariant, by Lemma 2, we have $\kappa(W * C) = -\kappa(C)$ and $\kappa(C^T) = \kappa(C)$. Additionally,

$$\kappa(C * W) = \kappa\big((W * C^T)^T\big) = \kappa(W * C^T) = -\kappa(C^T) = -\kappa(C),$$

so that Axiom ($\kappa 6$) is also verified.

As μ is a nonnegative measure and ϕ is nondecreasing, $C_1 \prec C_2$ implies $\kappa(C_1) \leq \kappa(C_2)$; i.e., Axiom ($\kappa 2$) holds.

Finally, if $C_n \to C$ pointwise as $n \to \infty$, then due to the continuity of ϕ and by dominated convergence theorem, $\kappa(C_n) \to \kappa(C)$, yielding Axiom ($\kappa 7$). □

4. Inner and Outer Averages

Edwards and Taylor ([22] Theorems 2 and 3) also presented two more characterizations of degree-one bivariate concordance measures. The first is obtained by averaging (5) (with $\phi(x) \equiv x$) over the symmetries $\xi \in D_4$ of $\mathbb{I}^2$ for an arbitrary copula $C \in \mathcal{C}$ and using the fact that $\kappa(\xi^*(C)) = (-1)^{|\xi|}\kappa(C)$ (see Equation (1) and Axioms ($\kappa 2$) and ($\kappa 6$)), with

$$|\xi| = \begin{cases} 0, & \text{for } \xi = e, \pi; \\ 1, & \text{for } \xi = \sigma_1, \sigma_2, \pi\sigma_1, \pi\sigma_2; \\ 2, & \text{for } \xi = \sigma_1\sigma_2, \pi\sigma_1\sigma_2. \end{cases}$$

In other words, $|\xi|$, called the order of ξ, counts the minimal number of partial reflections (i.e., σ_1 and σ_2) needed to describe $\xi \in D_4$.

By ([22] Theorem 2), κ is degree one measure of concordance if and only if there is a Borel measure ν on $[0,1]^2$, putting zero mass on the boundary of $[0,1]^2$ and normalizing it such that $\int_{\mathbb{I}^2}(M - \Pi)d\nu = 1$, so that

$$\kappa(C) = \int_{(0,1)^2} \frac{1}{8} \sum_{\xi \in D_4} (-1)^{|\xi|}\xi^*(C)d\nu, \qquad C \in \mathcal{C}.$$

Here $\nu = \gamma\mu$, where μ is the measure from ([23] Theorem 0.6) (the same ν also appears in ([22] Theorem 1)). Observe that here the measure ν need not be unique: several Borel measures on $\mathbb{I}^2$ can give the same measure of concordance; ν is unique only among the Borel measures with the above-specified properties!

Note also that in [22], the authors construct ν from a given degree-one concordance measure κ, and in [23], concordance measure κ is constructed from a given measure ν.

With a more general function ϕ, we can consider two kinds of average:

- (outer)

$$\kappa(C) = \int_{(0,1)^2} \frac{1}{8} \sum_{\xi \in D_4} (-1)^{|\xi|} \phi\Big(\xi^*(C) - \Pi\Big) d\nu, \qquad C \in \mathcal{C},$$

which clearly gives another (symmetrized) form of the same concordance measure κ as in Theorem 1, provided γ, $\mu = \nu/\gamma$, and ϕ satisfy the conditions of that theorem;

- (inner)

$$\hat{\kappa}(C) := \int_{(0,1)^2} \phi\left(\frac{1}{8} \sum_{\xi \in D_4} (-1)^{|\xi|} \xi^*(C)\right) d\nu, \qquad C \in \mathcal{C}, \tag{12}$$

where after averaging Π disappears.

Following similar ideas and generalizing the definitions of concordance measures to the multidimensional setting, by using the biconvex form (3), Fuchs [15] considers the mappings

$$C \mapsto \frac{[\psi_\Lambda(C), A] - [\Pi, A]}{[M, A] - [\Pi, A]} =: \kappa^\bullet_{A,\Lambda}(C), \tag{13}$$

$$C \mapsto \frac{[\psi_\Lambda(C), A] - [\Pi, \Pi]}{[M, A] - [\Pi, \Pi]} =: \kappa^\circ_{A,\Lambda}(C), \tag{14}$$

where $A \in \mathcal{C}$ is a fixed copula, and $\psi_\Lambda : \mathcal{C} \to \mathcal{C}$ is a transformation of copulas related to a certain subgroup Λ of the group Γ of copula transformations, which is generated by all partial reflections and permutations. The author gives conditions for the maps in (13) and (14) to be concordance measures when ψ_Λ is the arithmetic average

$$\psi_\Lambda(C) := \frac{1}{|\Lambda|} \sum_{\xi \in \Lambda} \xi^*(C),$$

where $|\Lambda|$ denotes the number of elements in Λ. This $\psi_\Lambda(C)$ is again a copula (as a convex combination of copulas) and Λ-invariant, i.e., $\xi^*(\psi_\Lambda(C)) = \psi_\Lambda(C)$ for any $\xi \in \Lambda$. We also emphasize that in that work the integrating measure ν (as in Equation (12)) was only considered to be a d-fold stochastic measure μ_A, for some copula A.

For simplicity of notation, we will consider the following linear transformation of a bivariate copula C, involving only reflections:

$$\begin{aligned} \Diamond C(u,v) &:= C(u,v) - \sigma_1^*(C)(u,v) - \sigma_2^*(C)(u,v) + \tau^*(C)(u,v) \\ &= C(u,v) - (v - C(1-u,v)) - (u - C(u,1-v)) \\ &\quad + (u + v - 1 + C(1-u,1-v)) \\ &= C(u,v) + C(1-u,v) + C(u,1-v) + C(1-u,1-v) - 1. \end{aligned} \tag{15}$$

The function $\Diamond C$ is not a copula, yet the map $C \mapsto \Diamond C$ is increasing in concordance order, which will prove to be handy later:

$$\begin{aligned} C_1 \prec C_2 \quad &\Longleftrightarrow \quad C_1(u,v) \le C_2(u,v),\ \forall (u,v) \in \mathbb{I}^2 \\ &\Longrightarrow \quad \Diamond C_1(u,v) \le \Diamond C_2(u,v),\ \forall (u,v) \in \mathbb{I}^2. \end{aligned}$$

Additionally, observe that $\pi^*(\Diamond C) = \Diamond(\pi^*(C))$.

We also define a π^*-invariant version of $\Diamond$, given by

$$\blacklozenge C := \frac{1}{2}\Big(\Diamond C + \Diamond(\pi^*(C))\Big) = \frac{1}{2} \sum_{\xi \in D_4} (-1)^{|\xi|} \xi^*(C), \qquad C \in \mathcal{C}. \tag{16}$$

By construction and using $\pi^*(\sigma_i^*(C)) = \sigma_{3-i}^*(\pi^*(C))$,

$$\Diamond(\sigma_i^*(C)) = -\Diamond C, \qquad \blacklozenge(\sigma_i^*(C)) = -\blacklozenge C, \quad i = 1, 2. \tag{17}$$

It is straightforward to check using the definitions (15), (16), and the properties $\tau^*(M) = M$ and $\sigma_i^*(M) = W$, for $i = 1, 2$, that

$$\Diamond\Pi = \blacklozenge\Pi \equiv 0 \quad \text{and} \quad \Diamond M = \blacklozenge M = 2(M - W). \tag{18}$$

Next we establish, under certain conditions, which are less stringent than in Theorem 1 due to acquired symmetries of the average inside the integrand, that $\hat{\kappa}$ is also a concordance measure, so inner averages as in (12) can also give concordance measures.

Conceptually similar are the results of Fuchs [15], where unsigned averages over several subgroups of Γ are considered and the symmetries of the integrand are complimented by those of the integrator to obtain concordance measures. Our result is different as it uses a more general function ϕ (not only $\phi(x) \equiv x$) and also considers *signed* averages.

Theorem 2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing, odd function and such that $\phi(0) = 0$. Fix an arbitrary $A \in \mathcal{C}$. Then, the map $\rho_{\phi,A} : \mathcal{C} \to [-1, 1]$ given by*

$$\rho^{\blacklozenge}_{\phi,A}(C) = \frac{[\phi(\blacklozenge C), A]}{[\phi(2(M - W)), A]}, \qquad C \in \mathcal{C}, \tag{19}$$

defines a concordance measure. If, in addition, A is π^*-invariant, then also the mapping

$$\rho^{\Diamond}_{\phi,A}(C) = \frac{[\phi(\Diamond C), A]}{[\phi(2(M - W)), A]}, \qquad C \in \mathcal{C}, \tag{20}$$

is a concordance measure.

Remark 2. *Due to $\Diamond\Pi = \blacklozenge\Pi \equiv 0$, in the setting of Theorem 2, we have*

$$[\phi(\Diamond\Pi), A] = [\phi(\Diamond\Pi), \Pi] = [\phi(\blacklozenge\Pi), A] = [\phi(\blacklozenge\Pi), \Pi] = \phi(0) = 0,$$

so $\rho^{\blacklozenge}_{\phi,A}$ and $\rho^{\Diamond}_{\phi,A}$ are essentially of the same form as the measures in (13) and (14), albeit for signed averages.

Proof. To verify the correctness of the definition of $\rho^{\blacklozenge}_{\phi,A}$ and $\rho^{\Diamond}_{\phi,A}$, observe that there is a constant $a \in (0, 1/2)$ such that $V_A\big([a, 1-a] \times [a, 1-a]\big) > 0$, and so

$$\begin{aligned}
[\phi(\Diamond M), A] = [\phi(\blacklozenge M), A] = [\phi(2(M - W)), A] &= \int_{\mathbb{I}^2} \phi(2(M - W))dA \\
&\geq \int_{[a,1-a]\times[a,1-a]} \phi(2(M - W))dA \\
&\geq \phi(2a)V_A\big([a, 1-a] \times [a, 1-a]\big) > 0,
\end{aligned}$$

since ϕ is assumed to be strictly increasing and $\phi(0) = 0$. Thus, the denominators in (19) and (20) do not vanish for any $A \in \mathcal{C}$.

Additionally, it is clear from (18) that $\rho^{\blacklozenge}_{\phi,A}(M) = \rho^{\Diamond}_{\phi,A}(M) = 1$ and $\rho^{\blacklozenge}_{\phi,A}(\Pi) = \rho^{\Diamond}_{\phi,A}(\Pi) = 0$, so Axioms ($\kappa5$) and ($\kappa5'$) hold.

Axiom ($\kappa2$) holds for $\rho^{\blacklozenge}_{\phi,A}$ because the mapping $C \mapsto \blacklozenge C$ is π^*-invariant. In the second case, i.e., when A is π^*-invariant, $C \mapsto [\phi(\Diamond C), A]$ is π^*-invariant by the change of variable formula for integrals, verifying Axiom ($\kappa2$) for $\rho^{\Diamond}_{\phi,A}$.

Axiom ($\kappa3$) holds since the biconvex form (for bivariate copulas) in (3) is increasing in its first argument with respect to concordance order, and we have emphasized that also $C \mapsto \Diamond C$ is increasing with respect to this order; the same is true for $C \mapsto \blacklozenge C$.

Regarding Axiom ($\kappa 6$), using (17), for any $C \in \mathcal{C}$, we have

$$[\phi(\lozenge\sigma_i^*(C)), A] = [\phi(-\lozenge C), A] = -[\phi(\lozenge C), A],$$
$$[\phi(\blacklozenge\sigma_i^*(C)), A] = [\phi(-\blacklozenge C), A] = -[\phi(\blacklozenge C), A], \qquad i = 1, 2,$$

for any odd function ϕ. Therefore,

$$\rho^{\lozenge}_{\phi,A}(\sigma_i^*(C)) = -\rho^{\lozenge}_{\phi,A}(C), \qquad \rho^{\blacklozenge}_{\phi,A}(\sigma_i^*(C)) = -\rho^{\blacklozenge}_{\phi,A}(C),$$

for $i = 1, 2$.

Axiom ($\kappa 4$) holds due to Axioms ($\kappa 3$) and ($\mu 6$), and due to the fact that $\lozenge W = \blacklozenge W = \lozenge\sigma_i^*(M) = \blacklozenge\sigma_i^*(M) = -\lozenge M = -\blacklozenge M$:

$$-1 = \rho^{\lozenge}_{\phi,A}(W) \le \rho^{\lozenge}_{\phi,A}(C) \le \rho^{\lozenge}_{\phi,A}(M) = 1,$$
$$-1 = \rho^{\blacklozenge}_{\phi,A}(W) \le \rho^{\blacklozenge}_{\phi,A}(C) \le \rho^{\blacklozenge}_{\phi,A}(M) = 1.$$

Finally, if a sequence of copulas C_n converges uniformly to a copula C, as $n \to \infty$, then the also $\lozenge C_n$ and $\blacklozenge C_n$ uniformly converge to $\lozenge C$ and $\blacklozenge C$, respectively, and so

$$\phi(\lozenge C_n) \to \phi(\lozenge C) \quad \text{and} \quad \phi(\blacklozenge C_n) \to \phi(\blacklozenge C) \tag{21}$$

pointwise on $\mathbb{I}^2$ as $n \to \infty$, due to the continuity of ϕ. Thus, $\rho^{\lozenge}_{\phi,A}(C_n) \to \rho^{\lozenge}_{\phi,A}(C)$ and $\rho^{\blacklozenge}_{\phi,A}(C_n) \to \rho^{\blacklozenge}_{\phi,A}(C)$ as $n \to \infty$, by a standard application of Lebesgue's dominated convergence theorem, completing the proof. □

The proof can easily be adapted to accommodate the case of a more general integrating Borel measure ν. We have

Corollary 1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing, odd function and such that $\phi(0) = 0$. Let ν be a finite Borel measure with support* $\operatorname{supp}\nu \subset (0,1)^2$. *Then, the map $\rho_{\phi,\nu} : \mathcal{C} \to [-1,1]$ given by*

$$\rho^{\blacklozenge}_{\phi,\nu}(C) = \frac{\int_{(0,1)^2} \phi(\blacklozenge C) d\nu}{\int_{(0,1)^2} \phi(2(M-W)) d\nu}, \qquad C \in \mathcal{C}, \tag{22}$$

defines a concordance measure. If, in addition, ν is π-invariant (i.e., $\nu \circ \pi^{-1} = \nu$), then the mapping

$$\rho^{\lozenge}_{\phi,\nu}(C) = \frac{\int_{(0,1)^2} \phi(\lozenge C) d\nu}{\int_{(0,1)^2} \phi(2(M-W)) d\nu}, \qquad C \in \mathcal{C}, \tag{23}$$

is a concordance measure.

Proof. First, we show that the denominators in (22) and (23) are strictly positive and finite, so that both maps are well defined for all $C \in \mathcal{C}$. Indeed, letting $A_n := (1/n, 1-1/n) \times (1/n, 1-1/n)$, $n \ge 2$, we get $A_n \uparrow (0,1)^2$ as $n \to \infty$, so that there is some $n_0 \in \mathbb{N}$ for which $\operatorname{supp}\nu \cap A_{n_0} \neq \emptyset$, which by definition of the support of a measure means that $\nu(\operatorname{supp}\nu \cap A_{n_0}) > 0$. However, then

$$\int_{(0,1)^2} \phi(2(M-W)) d\nu \ge \int_{\operatorname{supp}\nu \cap A_{n_0}} \phi(2(M-W)) d\nu \ge \phi(2/n_0)\nu(\operatorname{supp}\nu \cap A_{n_0}) > 0.$$

Since the function $\phi(2(M-W))$ is bounded on $\mathbb{I}^2$, and ν is assumed finite, the denominators in (22) and (23) are finite. The rest of the proof is essentially the same as the proof of Theorem 2, hence omitted. □

5. Applications and Examples

Our starting observation that led to this study was the fact that the biconvex form in Equation (3) is defined for more general functions—namely, one can consider more general maps $\phi : \mathcal{C} \to C(\mathbb{I}^2)$ (where $C(\mathbb{I}^2)$ denotes the set of continuous functions on the unit square $\mathbb{I}$). We wondered whether concordance measures of similar form as in (13) and (14) could still be obtained. The second motivation to step away from the requirement of having $\phi(C) \in \mathcal{C}$ was the desire to construct examples of concordance measures $\kappa = \kappa_\phi$ such that the map in Equation (4) gives a polynomial of degree $k > 2$ when ϕ is chosen appropriately. Edwards and Taylor ([22] Questions 1 and 2) (see also ([3] Questions 2 and 3, p. 235)) formulated a question about the characterization of bivariate (and multivariate) concordance measures of degree $m \geq 2$. Still, examples of concordance measures of degree $m > 2$ were not provided, leaving one to wonder if such a question is non-vacuous.

The main family of examples we had in mind when starting the study was $\phi(x) = \phi_k(x) = x^k$, $k \in \mathbb{N}$. However, we soon realized that additional restrictions on ϕ were needed, in particular, the oddness of ϕ appeared as in Theorem 1. On the other hand, the multiplicativity property allows taking also $\phi_\alpha(x) = \operatorname{sgn}(x)|x|^\alpha$ for $\alpha > 0$. Thus, to illustrate Theorem 1, we consider these two families of functions ϕ.

Example 1 (Generalized Blomqvist's beta)**.** *By taking $\phi(x) = \phi_{2\ell+1}(x)$, $\ell \in \mathbb{N} \cup \{0\}$, and $\mu = \delta_{(1/2,1/2)}$, we have*

$$\int_{(0,1)^2} \phi(M - \Pi) d\mu = \left(\frac{1}{2} - \frac{1}{4}\right)^{2\ell+1}, \qquad \ell = 0, 1, \ldots,$$

and so in this case

$$\kappa_{\phi,\mu} = 4^{2\ell+1}(C(1/2, 1/2) - 1/4)^{2\ell+1} = (\beta(C))^{2\ell+1},$$

which generalizes Blomqvist's beta.

Remark 1. *The above power-construction can be generalized: indeed, for any bivariate concordance measure ρ and any $\ell \in \mathbb{N} \cup \{0\}$, $\rho^{2\ell+1}$ is again a concordance measure as all the axioms clearly hold. In particular, if $\rho = \tau_K$ is Kendall's tau, then $\tau_K^{2\ell+1}$ provides examples of degree $4\ell + 2$ polynomial-type concordance measures.*

Example 2. *As another example, consider $\phi(x) = \phi_\alpha(x) = \operatorname{sgn}(x)|x|^\alpha$; μ, being any Borel measure satisfying the conditions of Theorem 1; and, $\mu\big((0,1)^2\big) < +\infty$. Then, $\kappa_{\phi_\alpha,\mu}$ provides examples of non-polynomial-type concordance measures if $\alpha > 0$ is not an integer, or if $\alpha = 2\ell$ for $\ell \in \mathbb{N}$. When $\alpha = 2\ell + 1$, an odd integer, we recover the function $\phi_{2\ell+1}$ from the previous example.*

Observe, furthermore, that the extreme case $\alpha = 0$ does not produce a concordance measure. Indeed, if

$$\rho(C) := \gamma \int_{(0,1)^2} \operatorname{sgn}(C - \Pi) d\mu,$$

where

$$\gamma^{-1} = \int_{(0,1)^2} \operatorname{sgn}(M - \Pi) d\mu = \mu\big((0,1)^2\big),$$

then

$$\rho(C) = \frac{\mu\big(\{(x,y) \in \mathbb{I}^2 : C(x,y) > xy\}\big) - \mu\big(\{(x,y) \in \mathbb{I}^2 : C(x,y) < xy\}\big)}{\mu\big((0,1)^2\big)},$$

which does not satisfy Axiom (κ7). Indeed, considering $C_t = tM + (1-t)\Pi$, $t \in \mathbb{I}$, we have $C_t \to C_0 = \Pi$ pointwise as $t \to 0+$, but $\rho(C_t) = 1 \not\to 0 = \rho(C_0)$. In particular, this shows that the set of bivariate concordance measures is not closed as a subset of the space of continuous functions on $\mathcal{C}$, $C(\mathcal{C})$, with $\mathcal{C}$ considered as a compact and convex subset of $C(\mathbb{I}^2)$!

Example 3 (Generalized Gini's gamma)**.** *Consider a function $\phi(x)$ which satisfies the conditions of Theorem 1, and this time take $\mu = \mu_{(M+W)/2}$. Then,*

$$\begin{aligned}\int_{(0,1)^2} \phi(C-\Pi)d\mu_{(M+W)/2} &= \frac{1}{2}\int_{(0,1)^2} \phi(C-\Pi)dM + \frac{1}{2}\int_{(0,1)^2} \phi(C-\Pi)dW \\ &= \frac{1}{2}\int_0^1 \phi(C(t,t)-t^2)dt + \frac{1}{2}\int_0^1 \phi(C(t,1-t)-t(1-t))dt.\end{aligned}$$

As for the normalizing constant, we obtain

$$\begin{aligned}(\gamma_{\phi,\mu_{(M+W)/2}})^{-1} &= \int_{(0,1)^2} \phi(M-\Pi)d\mu_{(M+W)/2} \\ &= \frac{1}{2}\int_0^1 \phi(t-t^2)dt + \frac{1}{2}\int_0^1 \phi(t\wedge(1-t)-t(1-t))dt,\end{aligned}$$

thus

$$\kappa_{\phi,\mu_{(M+W)/2}}(C) = \frac{\int_0^1 \phi(C(t,t)-t^2)dt + \int_0^1 \phi(C(t,1-t)-t(1-t))dt}{\int_0^1 \phi(t-t^2)dt + \int_0^1 \phi(t\wedge(1-t)-t(1-t))dt}.$$

In particular, letting $\phi(x) \equiv x$, for $x \in \mathbb{I}$, we recover Gini's gamma.

Example 4. *To illustrate Theorem 2, we consider $A = \Pi$. Then,*

$$\begin{aligned}[\phi(\lozenge M),\Pi] = [\phi(\blacklozenge M),\Pi] &= \int_{\mathbb{I}^2} \phi(2(M-W))d\Pi \\ &= \int_{\mathbb{I}^2} \phi(2(x\wedge y-(x+y-1)^+))dxdy \\ &= \sum_{i=1}^{4}\int_{A_i} \phi(2(x\wedge y-(x+y-1)^+))dxdy,\end{aligned}$$

where

$$\begin{aligned}A_1 &:= \{(x,y)\in\mathbb{I}^2 : x\geq y, x+y\leq 1\}, \quad A_2 := \{(x,y)\in\mathbb{I}^2 : x\geq y, x+y> 1\}, \\ A_3 &:= \{(x,y)\in\mathbb{I}^2 : x< y, x+y\geq 1\}, \quad A_4 := \{(x,y)\in\mathbb{I}^2 : x< y, x+y< 1\}.\end{aligned}$$

Observing that, for any $i = 1,\ldots,4$,

$$\int_{A_i} \phi(2(x\wedge y-(x+y-1)^+))dxdy = \int_0^{1/2} \phi(2t)(1-2t)dt,$$

we get

$$[\phi(\lozenge M),\Pi] = [\phi(\blacklozenge M),\Pi] = 4\int_0^{1/2} \phi(2t)(1-2t)dt = 2\int_0^1 \phi(u)(1-u)du.$$

In particular, for $\phi(x) = \phi_{2\ell+1}(x)$,

$$\begin{aligned}[\phi_{2\ell+1}(\lozenge M),\Pi] = [\phi_{2\ell+1}(\blacklozenge M),\Pi] &= 2\int_0^1 t^{2\ell+1}(1-t)dt \\ &= 2\Big(\frac{1}{2\ell+2}-\frac{1}{2\ell+3}\Big) = \frac{1}{(\ell+1)(2\ell+3)},\end{aligned}$$

and so

$$\kappa^{\lozenge}_{\phi_{2\ell+1},\Pi}(C) = (\ell+1)(2\ell+3)[(\lozenge C)^{2\ell+1},\Pi],$$
$$\kappa^{\blacklozenge}_{\phi_{2\ell+1},\Pi}(C) = (\ell+1)(2\ell+3)[(\blacklozenge C)^{2\ell+1},\Pi].$$

Choosing $\ell = 0$, we get

$$\kappa^{\lozenge}_{\phi_1,\Pi}(C) = 3[\lozenge C,\Pi] = 3([C,\Pi] - [\sigma_1^*(C),\Pi] - [\sigma_2^*(C),\Pi] + [\tau^*(C),\Pi]) = \rho_S(C),$$

where $\rho_S(C)$ denotes Spearman's ρ for a copula C. Similar computations also yield $\kappa^{\blacklozenge}_{\phi_1,\Pi}(C) = \rho_S(C)$ as Π is, in particular, π^*-invariant. Thus, the families

$$\{\kappa^{\lozenge}_{\phi,A} : \phi \text{ satisfies the conditions of Theorem 2},\ A \in \mathcal{C}\},$$
$$\{\kappa^{\blacklozenge}_{\phi,A} : \phi \text{ satisfies the conditions of Theorem 2},\ A \in \mathcal{C}\}$$

generalize Spearman's ρ.

Additionally, observe that for $C_1, C_2 \in \mathcal{C}$,

$$\kappa^{\lozenge}_{\phi_{2\ell+1},A}(tC_1 + (1-t)C_2) \quad \text{and} \quad \kappa^{\blacklozenge}_{\phi_{2\ell+1},A}(tC_1 + (1-t)C_2)$$

are polynomials of (odd) degree $2\ell + 1$ in $t \in \mathbb{I}$, so that Edward and Taylor's question mentioned earlier is non-vacuous at least for odd degrees. Unfortunately, this still leaves open the question about even degree (≥ 4) examples, generalizing Kendall's τ. To give more details, consider, for example, the mapping

$$\rho_\phi(C) := \frac{[\phi(\lozenge C), C]}{[\phi(2(M-W), M]},$$

with

$$[\phi(2(M-W), M] = \int_{\mathbb{I}^2} \phi(2(M-W))dM = \int_0^1 \phi(2(t - \max\{2t-1, 0\}))dt$$
$$= 2\int_0^{1/2} \phi(2t)dt = \int_0^1 \phi(t)dt.$$

Even though such a ρ_ϕ satisfies most of the axioms of concordance measures, it is not clear if Axiom ($\kappa 3$) holds. The main reason for this is that $\phi(\lozenge C)$ is no longer a copula, so the symmetry of the biconvex form, i.e., $[C, D] = [D, C]$, valid for bivariate copulas C, D (see [25] Theorem 3.3), can no longer be applied without additional arguments. (While this paper was being reviewed, we made some progress in this direction and also found a way to generalize Kendall's τ. The details will appear in a follow-up publication [27].)

One can wonder, as one of the referees did, if the concordance measures considered in this paper could be computed for popular elliptic or Archimedean families of copulas. This could definitely be achieved in the setting of Examples 1 and 3, for generalized Blomqvist's β and Gini's γ. Regarding generalizations of Spearman's ρ, unfortunately, we do not have a general formula (even for the usual Spearman's ρ) in terms, e.g., of the generator of an Archimedean copula, as opposed to the case of Kendall's τ. Thus, computations, in general, need to be done on a case by case basis. To illustrate that this can be achieved, we present the final example of this section.

Example 5 (Generalized Spearman's ρ for Marshall–Olkin copulas)**.** *Let $\phi(x) = x^{2\ell+1}$, $\ell \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{I}$, and consider Marshall–Olkin (see, e.g., ([8] (3.1.3))) family of copulas:*

$$C_{\alpha,\beta}(u,v) = \begin{cases} u^{1-\alpha}v, & \text{if } u^{\alpha} \geq v^{\beta}; \\ uv^{1-\beta}, & \text{if } u^{\alpha} < v^{\beta}, \end{cases} \qquad 0 < \alpha, \beta < 1.$$

Then,

$$\int_{(0,1)^2} \phi(C_{\alpha,\beta}(u,v) - uv)dudv = \int_0^1 (u^{1-\alpha} - u)^{2\ell+1}\left(\int_0^{u^{\alpha/\beta}} v^{2\ell+1}dv\right)du + \int_0^1 u^{2\ell+1}\left(\int_{u^{\alpha/\beta}}^1 (v^{1-\beta} - v)^{2\ell+1}dv\right)du = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &:= \int_0^1 (u^{1-\alpha} - u)^{2\ell+1}\left(\int_0^{u^{\alpha/\beta}} v^{2\ell+1}dv\right)du \\ &= \frac{1}{2\ell+2}\int_0^1 u^{2\ell+1+(2\ell+2)\alpha/\beta}(u^{-\alpha} - 1)^{2\ell+1}du \\ &= \frac{1}{2\ell+2}\int_0^1 u^{2\ell+1+(2\ell+2)\alpha/\beta}\sum_{k=0}^{2\ell+1}(-1)^k\binom{2\ell+1}{k}u^{-\alpha(2\ell+1-k)}du \\ &= \frac{1}{2\ell+2}\sum_{k=0}^{2\ell+1}\frac{(-1)^k\binom{2\ell+1}{k}}{(2\ell+2)(1+\alpha/\beta) - \alpha(2\ell+1-k)} \end{aligned}$$

and

$$\begin{aligned} J_2 &:= \int_0^1 u^{2\ell+1}\left(\int_{u^{\alpha/\beta}}^1 (v^{1-\beta} - v)^{2\ell+1}dv\right)du \\ &= \int_0^1 u^{2\ell+1}\left(\int_{u^{\alpha/\beta}}^1 v^{2\ell+1}\sum_{k=0}^{2\ell+1}(-1)^k\binom{2\ell+1}{k}v^{-\beta(2\ell+1-k)}dv\right)du \\ &= \sum_{k=0}^{2\ell+1}(-1)^k\binom{2\ell+1}{k}\int_0^1 u^{2\ell+1}\left(\int_{u^{\alpha/\beta}}^1 v^{2\ell+1-\beta(2\ell+1-k)}dv\right)du \\ &= \sum_{k=0}^{2\ell+1}\frac{(-1)^k\binom{2\ell+1}{k}}{2\ell+2-\beta(2\ell+1-k)}\left(\frac{1}{2\ell+2} - \frac{1}{(2\ell+2)(1+\alpha/\beta) - \alpha(2\ell+1-k)}\right). \end{aligned}$$

As for the normalizing constant,

$$\begin{aligned} (\gamma_{\phi,\mu_\Pi})^{-1} &= \int_{(0,1)^2}(u \wedge v - uv)^{2\ell+1}dudv = 2\int_0^1 (1-u)^{2\ell+1}\left(\int_0^u v^{2\ell+1}dv\right)du \\ &= \frac{2B(2\ell+3, 2\ell+2)}{2\ell+2} = \frac{2((2\ell+1)!)^2}{(4\ell+4)!}, \end{aligned}$$

so that

$$\kappa_{\phi,\mu_\Pi}(C_{\alpha,\beta}) = \frac{(4\ell+4)!}{2((2\ell+1)!)^2}(J_1 + J_2).$$

In particular, if $\ell = 0$, we recover a well-known expression (see, e.g., [8] Example 5.7(b)) for Spearman's ρ,

$$\rho(C_{\alpha,\beta}) = \frac{3\alpha\beta}{2\alpha - \alpha\beta + 2\beta}.$$

6. Closing Remarks and Directions for Further Research

In this paper, we have presented two construction methods and several examples for generating bivariate polynomial-type concordance measures of odd and certain even degree, and of non-polynomial-type concordance measures. More general constructions for even-degree measures will be discussed in a follow-up paper [27]. Our research was based on the ideas and results by Edwards, Mikusiński and Taylor, and of Fuchs, and should be extended further. For example, the full characterization of higher degree polynomial-type concordance measures is lacking, as is its generalization to the multivariate case. To be precise, we still do not know if the construction of the integrating measure μ, given in [22], in the case $\phi(x) \equiv x$, is extendable to the case of more general multiplicative functions ϕ. It would also be interesting to explore the connections of our findings to the recent results of Borroni [9], where yet another characterization of degree-one measures is provided and even some second-degree measures, generalizing Kendall's τ, are constructed.

We believe that our work, especially the examples constructed, showed that the questions of Edwards, Mikusiński, and Taylor about the characterization of polynomial-type concordance measures are justified and deserve further research efforts. It would indeed be interesting to further develop this theory to get a clearer picture about the diversity of the set of concordance measures. This way, we hope to get more information about the set of copulas and their properties from the (not only linear) functionals defined on them—a common theme in functional analysis and other branches of mathematics.

Yet another direction, not touched in this work, would be to explore statistical applications of such measures to the analysis of data. We have mentioned two recent works by Derumigny and Fermanian [20] and Denuit et al. [21], but, of course, there is much more research on applications of copulas and concordance measures in statistics. We wonder what advantage, if any, a more general multiplicative function ϕ as in Theorem 1 has over the simplest linear one, or what other properties of concordance measures are attractive to statisticians who select one concordance measure and not some other. Among such properties one naturally stumbles upon effective computability, easiness of application, etc. Thus, perhaps the recent results of Dalessandro and Peters [19] could be useful in this direction. Only the future will tell.

Funding: This research was funded, in part, by the Research Council of Lithuania (project No. S-MIP-20-16).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank three anonymous referees for their suggestions that led to several improvements of this paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Scarsini, M. On measures of concordance. *Stochastica* **1984**, *8*, 201–218.
2. Taylor, M. Multivariate measures of concordance. *Ann. Inst. Stat. Math.* **2007**, *59*, 789–806. [CrossRef]
3. Taylor, M. Multivariate measures of concordance for copulas and their marginals. *Depend. Model.* **2016**, *4*, 224–236. [CrossRef]
4. Joe, H. Multivariate concordance. *J. Multivar. Anal.* **1990**, *35*, 12–30. [CrossRef]
5. Nelsen, R. Concordance and Copulas: A Survey. In *Distributions with Given Marginals and Statistical Modelling*; Kluwer: Dordrecht, The Netherlands, 2002; pp. 169–178.
6. Dolati, A.; Úbeda-Flores, M. On measures of multivariate concordance. *J. Prob. Stat. Sci.* **2006**, *4*, 147–164.
7. Rényi, A. On measures of dependence. *Acta Math. Acad. Sci. Hungar.* **1959**, *10*, 441–451. [CrossRef]
8. Nelsen, R.B. *An Introduction to Copulas*, 2nd ed.; Springer: New York, NY, USA, 2006.
9. Borroni, C. Mutual association measure. *Stat. Methods Appl.* **2019**, *28*, 571–591. [CrossRef]
10. Kokol Bukovšek, D.; Košr, T.; Mojškerc, B.; Omladič, M. Relation between non-exchangeability and measures of concordance of copulas. *J. Math. Anal. Appl.* **2020**, *487*, 123951. [CrossRef]

11. Mroz, T.; Trutschnig, W. A sharp inequality for Kendall's τ and Spearman's ρ of extreme-value copulas. *Depend. Model.* **2018**, *6*, 369–376.
12. Nelsen, R. Nonparametric measures of multivariate association. In *Distribution with Fixed Marginals and Related Topics*; IMS Lecture Notes—Monograph Series; Institute of Mathematical Statistics: Hayward, CA, USA, 1996; Volume 28, pp. 223–232.
13. Úbeda-Flores, M. Multivariate versions of Blomqvist's beta and Spearman's footrule. *Ann. Inst. Stat. Math.* **2005**, *57*, 781–788. [CrossRef]
14. Fuchs, S. Multivariate copulas: Transformations, symmetry, order and measures of concordance. *Kybernetika* **2014**, *50*, 725–743. [CrossRef]
15. Fuchs, S. Copula-induced measures of concordance. *Depend. Model.* **2016**, *4*, 205–214. [CrossRef]
16. Mesfioui, M.; Quessy, J. Concordance measures for multivariate non-continuous random vectors. *J. Multivariate Anal.* **2010**, *101*, 2398–2410. [CrossRef]
17. Joe, H. *Multivariate Models and Dependence Concepts*; Chapman & Hall: London, UK, 1997.
18. Durante, F.; Sempi, C. *Principles of Copula Theory*; CRC Press/Taylor & Francis Group: Boca Raton, FL, USA, 2016.
19. Dalessandro, A.; Peters, G. Efficient and accurate evaluation methods for concordance measures via functional tensor characterization of copulas. *Method. Comput. Appl. Probab.* **2020**, *22*, 1089–1124. [CrossRef]
20. Derumigny, A.; Fermanian, J. On kernel-based estimation of conditional Kendall's tau: Finite distance bounds and asymptotic behavior. *Depend. Model.* **2019**, *7*, 292–321. [CrossRef]
21. Denuit, M.; Mesfioui, M.; Trufin, J. Concordance-based predictive measures in regression models for discrete responses. *Scand. Actuar. J.* **2019**, *2019*, 824–836. [CrossRef]
22. Edwards, H.; Taylor, M. Characterizations of degree one bivariate measures of concordance. *J. Multivariate Anal.* **2009**, *100*, 1777–1791. [CrossRef]
23. Edwards, H.; Mikusiński, P.; Taylor, M. Measures of concordance determined by D_4-invariant measures on $(0,1)^2$. *Proc. AMS* **2005**, *133*, 1505–1513. [CrossRef]
24. Edwards, H.; Mikusiński, P.; Taylor, M. Measures of concordance determined by D_4-invariant copulas. *Int. J. Math. Math. Sci.* **2004**, *2004*, 3867–3875. [CrossRef]
25. Fuchs, S. A biconvex form for copulas. *Depend. Model.* **2016**, *4*, 63–75. [CrossRef]
26. Fuchs, S.; Schmidt, K. Bivariate copulas: Transformations, asymmetry, and measures of concordance. *Kybernetika* **2014**, *50*, 109–125. [CrossRef]
27. Manstavičius, M. Diversity of Bivariate Concordance Measures: Part II. Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Vilnius, Lithuania. 2022, *manuscript in preparation*.
28. Darsow, W.; Nguyen, B.; Olsen, E. Copulas and Markov processes. *Ill. J. Math.* **1992**, *36*, 600–642. [CrossRef]
29. Scarsini, M. Strong measures of concordance and convergence in probability. *Riv. Mat. Sci. Econom. Soc.* **1984**, *7*, 39–44. [CrossRef]
30. Kruskal, W. Ordinal measures of association. *J. Amer. Statist. Soc.* **1958**, *53*, 814–861. [CrossRef]