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ANTANAS LAURINCİKAS

RENATA MACAITIENE

DARIUS SIAUCIUNAS

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## Universality of an absolutely convergent Dirichlet series with modified shifts

Antanas LAURINČIKAS<sup>1</sup> , Renata MACAITIENĖ<sup>2</sup> , Darius ŠIAUČIŪNAS<sup>2,\*</sup> 

<sup>1</sup>Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Vilnius, Lithuania

<sup>2</sup>Institute of Regional Development, Šiauliai Academy, Vilnius University, Šiauliai, Lithuania

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**Abstract:** In the paper, a theorem on approximation of a wide class of analytic functions by generalized shifts  $\zeta_{u_T}(s + i\varphi(\tau))$  of an absolutely convergent Dirichlet series  $\zeta_{u_T}(s)$  which in the mean is close to the Riemann zeta-function is obtained. Here  $\varphi(\tau)$  is a monotonically increasing differentiable function having a monotonic continuous derivative such that  $\varphi(2\tau) \max_{\tau \leq t \leq 2\tau} \frac{1}{\varphi'(t)} \ll \tau$  as  $\tau \rightarrow \infty$ , and  $u_T \rightarrow \infty$  and  $u_T \ll T^2$  as  $T \rightarrow \infty$ .

**Key words:** Haar measure, Mergelyan theorem, Riemann zeta-function, universality, weak convergence

### 1. Introduction

The Riemann zeta-function  $\zeta(s)$ ,  $s = \sigma + it$ , is defined for  $\sigma > 1$  by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where  $p$  runs over the set of all prime numbers and has the meromorphic continuation to the whole complex plane with unique simple pole at the point  $s = 1$  and residue 1.

It is well known that the function  $\zeta(s)$  has a good approximation property in the space of analytic functions, i.e. its shifts  $\zeta(s + i\tau)$ ,  $\tau \in \mathbb{R}$ , approximate a wide class of analytic functions. The latter property of  $\zeta(s)$  was discovered by Voronin [13] and is called the universality. Let  $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ ,  $\mathcal{K}$  be the class of compact subsets of the strip  $D$  with connected complements, and let  $H_0(K)$  with  $K \in \mathcal{K}$  be the class of continuous nonvanishing functions on  $K$  that are analytic in the interior of  $K$ . Let  $\text{meas}A$  denote the Lebesgue measure of a measurable set  $A \subset \mathbb{R}$ . Then the modern version of the Voronin universality theorem is the following statement, see [1, 4, 5, 9, 12].

**Theorem 1.1** *Let  $K \in \mathcal{K}$  and  $f(s) \in H_0(K)$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Using more general shifts  $\zeta(s + i\varphi(\tau))$  with a certain function  $\varphi(\tau)$  in place of  $\zeta(s + i\tau)$  is also possible. In [11], the function  $\varphi(\tau) = \tau^\alpha \log^\beta \tau$  with some class of real numbers  $\alpha$  and  $\beta$  was applied. The paper [8]

\*Correspondence: [darius.siauciunas@sa.vu.lt](mailto:darius.siauciunas@sa.vu.lt)

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deals with the class  $U(T_0)$  of monotonically increasing differentiable functions  $\varphi(\tau)$  on  $[T_0, \infty)$ ,  $T_0 > 0$ , having a monotonic continuous derivative such that

$$\varphi(2\tau) \max_{\tau \leq u \leq 2\tau} \frac{1}{\varphi'(u)} \ll \tau, \quad \tau \rightarrow \infty.$$

More precisely, the following theorem is valid.

**Theorem 1.2** *Suppose that  $\varphi(\tau) \in U(T_0)$ . Let  $K \in \mathcal{K}$  and  $f(s) \in H_0(K)$ . Then, for every  $\varepsilon > 0$ ,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\varphi(\tau)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta(s + i\varphi(\tau)) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many  $\varepsilon > 0$ .

The aim of this paper is the approximation of functions of the class  $H_0(K)$  by generalized shifts of a certain absolutely convergent Dirichlet series generated by the function  $\zeta(s)$ . Let  $\theta > 0$  be a fixed number. For  $m \in \mathbb{N}$  and  $u > 0$ , define

$$v_u(m) = \exp \left\{ - \left( \frac{m}{u} \right)^\theta \right\},$$

where  $\exp\{a\} = e^a$ . Then the series

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}$$

is absolutely convergent for  $\sigma > \sigma_0$  with arbitrary finite  $\sigma_0$ . The first universality theorem for the function  $\zeta_u(s)$  has been obtained in [6]. In [7], the latter theorem was extended for short intervals. Finally, in [3], a joint universality theorem for the function  $\zeta_u(s)$  was proven in short intervals. We will consider the approximation by shifts  $\zeta_u(s + i\varphi(\tau))$ .

Denote by  $\mathcal{B}(\mathbb{X})$  the Borel  $\sigma$ -field of the space  $\mathbb{X}$ , and define the set

$$\Omega = \prod_p \gamma_p,$$

where  $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$  for all primes  $p$ . With the product topology and pointwise multiplication, the torus  $\Omega$  is a compact topological Abelian group. Therefore, on  $(\Omega, \mathcal{B}(\Omega))$ , the probability Haar measure  $m_H$  exists, and we have the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ . Denote by  $\omega(p)$  the  $p$ th component of an element  $\omega \in \Omega$ , and on the probability space  $(\Omega, \mathcal{B}(\Omega), m_H)$ , define the  $H(D)$ -valued random element

$$\zeta(s, \omega) = \prod_p \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}.$$

Recall that the latter product, for almost all  $\omega \in \Omega$ , is uniformly convergent on compact subsets of the strip  $D$ , see, for example, [5]. Here  $H(D)$  is the space of analytic functions on  $D$  endowed with the topology of uniform convergence on compacta.

The main result of the paper is the following statement.

**Theorem 1.3** *Suppose that  $\varphi(\tau) \in U(T_0)$ , and  $u_T \rightarrow \infty$  and  $u_T \ll T^2$  as  $T \rightarrow \infty$ . Let  $K \in \mathcal{K}$  and  $f(s) \in H_0(K)$ . Then the limit*

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\varphi(\tau)) - f(s)| < \varepsilon \right\} = m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \omega) - f(s)| < \varepsilon \right\} > 0$$

*exists for all but at most countably many  $\varepsilon > 0$ .*

For example, the function  $\log \Gamma(\tau)$ , where  $\Gamma(\tau)$  is the Euler gamma-function, satisfies the hypotheses of the class  $U(T_0)$ .

Theorem 1.3 implies that there exists  $\widehat{T} = \widehat{T}(f, \varepsilon, K, \varphi) > 0$  such that, for  $T > \widehat{T}$ , there are infinitely many shifts  $\zeta_{u_T}(s + i\varphi(\tau))$  approximating a given function  $f(s) \in H_0(K)$ .

Since the function  $\zeta_{u_T}(s)$  is given by a rapidly absolutely convergent series, Theorem 1.3, in some sense, is more convenient than Theorem 1.2.

A proof of Theorem 1.3 is based on results of probabilistic type in the space  $H(D)$ .

## 2. Estimates in the mean

In this section, we will consider the distance between shifts  $\zeta(s + i\varphi(\tau))$  and  $\zeta_{u_T}(s + i\varphi(\tau))$  in the mean. We start with a mean square estimates for  $\zeta(s + i\varphi(\tau))$ .

**Lemma 2.1** *Suppose that  $\varphi(\tau) \in U(T_0)$ , and  $1/2 < \sigma < 1$  is fixed. Then, for all  $t \in \mathbb{R}$  and  $T > T_0 + 1$ ,*

$$\int_{T_0}^T |\zeta(\sigma + it + i\varphi(\tau))|^2 d\tau \ll_{\sigma, \varphi} T(1 + |t|).$$

**Proof** It is well known that, for fixed  $1/2 < \sigma < 1$ ,

$$\int_{-T}^T |\zeta(\sigma + it)|^2 dt \ll_{\sigma} T. \tag{2.1}$$

Let  $X > T_0$ . Then the properties of the class  $U(T_0)$  imply

$$\begin{aligned} \int_X^{2X} |\zeta(\sigma + it + i\varphi(\tau))|^2 d\tau &= \int_X^{2X} \frac{1}{\varphi'(\tau)} |\zeta(\sigma + it + i\varphi(\tau))|^2 d\varphi(\tau) \\ &= \int_X^{2X} \frac{1}{\varphi'(\tau)} d \left( \int_{T_0}^{t+\varphi(\tau)} |\zeta(\sigma + iv)|^2 dv \right) \\ &\ll \max_{X \leq \tau \leq 2X} \frac{1}{\varphi'(\tau)} \int_X^{2X} d \left( \int_{T_0}^{t+\varphi(\tau)} |\zeta(\sigma + iv)|^2 dv \right) \\ &\ll \max_{X \leq \tau \leq 2X} \frac{1}{\varphi'(\tau)} \int_{-|t|-\varphi(2X)}^{|t|+\varphi(2X)} |\zeta(\sigma + iv)|^2 dv \\ &\ll_{\sigma} \varphi(2X) \max_{X \leq \tau \leq 2X} \frac{1}{\varphi'(\tau)} + |t| \ll_{\sigma, \varphi} X(|t| + 1) \end{aligned}$$

in view of (2.1). Now, taking  $X = T2^{-l}$  and summing over  $l \in \mathbb{N}$ , we obtain the estimate of the lemma.  $\square$

Recall the distance in the space  $H(D)$ . There exists a sequence  $\{K_l : l \in \mathbb{N}\} \subset D$  of compact embedded subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and every compact set  $K \subset D$  lies in some  $K_l$ . For example, we can take closed embedded rectangles. Then

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric in  $H(D)$  inducing the topology of uniform convergence on compacta.

In the sequel, the integral representation for the function  $\zeta_u(s)$  will be useful. Define

$$l_u(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) u^s,$$

where  $\theta$  is from the definition of  $v_u(m)$ .

**Lemma 2.2** *Suppose that  $\widehat{\theta} > 1/2$ . Then, for  $s \in D$ , the representation*

$$\zeta_u(s) = \frac{1}{2\pi i} \int_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta(s+z) \frac{l_u(z)}{z} dz$$

holds.

**Proof** The Mellin formula

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s} ds = e^{-b}, \quad a, b > 0,$$

implies the equality

$$v_u(m) = \frac{1}{2\pi i} \int_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} m^{-z} \frac{l_u(z)}{z} dz.$$

Therefore,

$$\zeta_u(s) = \frac{1}{2\pi i} \int_{\hat{\theta}-i\infty}^{\hat{\theta}+i\infty} \sum_{m=1}^{\infty} \frac{1}{m^{s+z}} \frac{l_u(z)}{z} dz = \frac{1}{2\pi i} \int_{\hat{\theta}-i\infty}^{\hat{\theta}+i\infty} \zeta(s+z) \frac{l_u(z)}{z} dz,$$

since  $\text{Re}(s+z) > 1$  for  $s \in D$  and  $\text{Re}z = \hat{\theta}$ . □

Now we are ready to prove an important lemma on the distance between  $\zeta(s+i\varphi(\tau))$  and  $\zeta_{u_T}(s+i\varphi(\tau))$ .

**Lemma 2.3** *Suppose that  $\varphi(\tau) \in U(T_0)$ , and  $u_T \rightarrow \infty$  and  $u_T \ll T^2$  as  $T \rightarrow \infty$ . Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \int_{T_0}^T \rho(\zeta(s+i\varphi(\tau)), \zeta_{u_T}(s+i\varphi(\tau))) d\tau = 0.$$

**Proof** It suffices to show that, for arbitrary compact set  $K \subset D$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} |\zeta(s+i\varphi(\tau)) - \zeta_{u_T}(s+i\varphi(\tau))| d\tau = 0. \tag{2.2}$$

By the integral representation of Lemma 2.2, for  $s \in D$ ,

$$\zeta_{u_T}(s) = \frac{1}{2\pi i} \int_{\hat{\theta}-i\infty}^{\hat{\theta}+i\infty} \zeta(s+z) \frac{l_{u_T}(z)}{z} dz. \tag{2.3}$$

Let  $K \subset D$  be arbitrary compact set. Fix  $\varepsilon > 0$  such that, for  $s = \sigma + it \in K$ , the inequalities  $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$  would be satisfied. The integration function in (2.3) has simple poles at  $z = 0$  and  $z = 1 - s$ . Therefore, taking

$$\theta_1 = \frac{1}{2} + \varepsilon - \sigma < 0 \quad \text{and} \quad \hat{\theta} = 1/2 + \varepsilon,$$

by the residue theorem, we find

$$\zeta_{u_T}(s) - \zeta(s) = \frac{1}{2\pi i} \int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z) \frac{l_{u_T}(z)}{z} dz + \frac{l_{u_T}(1-s)}{1-s}.$$

Hence, for all  $s \in K$ ,

$$\begin{aligned} & \zeta_{u_T}(s+i\varphi(\tau)) - \zeta(s+i\varphi(\tau)) \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + it + iv + i\varphi(\tau)\right) \frac{l_{u_T}(1/2 + \varepsilon - \sigma + iv)}{1/2 + \varepsilon - \sigma + iv} dv + \frac{l_{u_T}(1-s-\varphi(\tau))}{1-s-\varphi(\tau)} \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \varepsilon + iv + i\varphi(\tau)\right) \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} dv + \frac{l_{u_T}(1-s-\varphi(\tau))}{1-s-\varphi(\tau)} \\ &\ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \varepsilon + iv + i\varphi(\tau)\right) \right| \sup_{s \in K} \left| \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \right| dv + \sup_{s \in K} \left| \frac{l_{u_T}(1-s-\varphi(\tau))}{1-s-\varphi(\tau)} \right|. \end{aligned}$$

Therefore,

$$\frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} |\zeta(s+i\varphi(\tau)) - \zeta_{u_T}(s+i\varphi(\tau))| d\tau \ll I_1 + I_2, \tag{2.4}$$

where

$$I_1 = \int_{-\infty}^{\infty} \left( \frac{1}{T - T_0} \int_{T_0}^T \left| \zeta \left( \frac{1}{2} + \varepsilon + iv + i\varphi(\tau) \right) \right| d\tau \right) \sup_{s \in K} \left| \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} \right| dv,$$

and

$$I_2 = \frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} \left| \frac{l_{u_T}(1 - s - \varphi(\tau))}{1 - s - \varphi(\tau)} \right| d\tau.$$

It is well known that uniformly in  $\sigma$ ,  $\sigma_1 \leq \sigma \leq \sigma_2$  with arbitrary  $\sigma_1 < \sigma_2$ , the gamma-function satisfies the bound

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}. \tag{2.5}$$

Thus, for all  $s \in K$ ,

$$\begin{aligned} \frac{l_{u_T}(1/2 + \varepsilon - s + iv)}{1/2 + \varepsilon - s + iv} &\ll_{\theta} u_T^{1/2 + \varepsilon - \sigma} \left| \Gamma \left( \frac{1}{\theta} \left( \frac{1}{2} + \varepsilon - \sigma - it + iv \right) \right) \right| \\ &\ll_{\theta} u_T^{-\varepsilon} \exp \left\{ \frac{c}{\theta} |v - t| \right\} \ll_{\theta, K} u_T^{-\varepsilon} \exp\{-c_1|v|\}, \quad c_1 > 0. \end{aligned}$$

Moreover, in virtue of Lemma 2.1,

$$\begin{aligned} \frac{1}{T - T_0} \int_{T_0}^T \left| \zeta \left( \frac{1}{2} + \varepsilon + iv + i\varphi(\tau) \right) \right| d\tau &\leq \left( \frac{1}{T - T_0} \int_{T_0}^T \left| \zeta \left( \frac{1}{2} + \varepsilon + iv + i\varphi(\tau) \right) \right|^2 d\tau \right)^{1/2} \\ &\ll_{\varepsilon, \varphi} (1 + |v|)^{1/2} \ll_{\varepsilon, \varphi} 1 + |v|. \end{aligned}$$

Therefore, the latter two estimates show that

$$I_1 \ll_{\varepsilon, \theta, K, \varphi} u_T^{-\varepsilon} \int_{-\infty}^{\infty} (1 + |v|) \exp\{-c_1|v|\} dv \ll_{\varepsilon, \theta, K, \varphi} u_T^{-\varepsilon}. \tag{2.6}$$

Using (2.5), similarly as above, we find that, for all  $s \in K$ ,

$$\begin{aligned} \frac{l_{u_T}(1 - s - \varphi(\tau))}{1 - s - \varphi(\tau)} &\ll_{\theta} u_T^{1 - \sigma} \left| \Gamma \left( \frac{1}{\theta} (1 - \sigma - it - \varphi(\tau)) \right) \right| \\ &\ll_{\theta} u_T^{1/2 - 2\varepsilon} \exp \left\{ -\frac{c}{\theta} |t + \varphi(\tau)| \right\} \ll_{\theta, K} u_T^{1/2 - 2\varepsilon} \exp\{-c_2\varphi(\tau)\}, \quad c_2 > 0. \end{aligned}$$

Hence,

$$I_2 \ll_{\theta, K} u_T^{1/2 - 2\varepsilon} \frac{1}{T - T_0} \int_{T_0}^T \exp\{-c_2\varphi(\tau)\} d\tau. \tag{2.7}$$

The definition of the class  $U(T_0)$  implies that  $\varphi(\tau) \geq \tau^{c_3}$  with certain  $c_3 > 0$ . Thus, by (2.7), and the estimate  $u_T \ll T^2$

$$I_2 \ll_{\theta, K, \varphi} u_T^{1/2 - 2\varepsilon} \frac{1}{T - T_0} \int_{T_0}^T \exp\{-c_2\tau^{c_3}\} d\tau \ll_{\theta, K, \varphi} u_T^{-\varepsilon}.$$

This, (2.6), and (2.4) prove (2.2). The lemma is proven. □

**3. Limit theorems**

We will derive Theorem 1.3 from a probabilistic limit theorem in the space  $H(D)$  for the function  $\zeta_{u_T}(s)$  which follows from a similar theorem for the function  $\zeta(s)$ .

For  $A \in \mathcal{B}(H(D))$ , define

$$P_T(A) = \frac{1}{T - T_0} \text{meas}\{\tau \in [T_0, T] : \zeta(s + i\varphi(\tau)) \in A\}.$$

**Lemma 3.1** *Suppose that  $\varphi(\tau) \in U(T_0)$ . Then  $P_T$  converges weakly to the distribution of the random element  $\zeta(s, \omega)$ , i.e. to the measure*

$$P_\zeta(A) \stackrel{\text{def}}{=} m_H\{\omega \in \Omega : \zeta(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

as  $T \rightarrow \infty$ . Moreover, the support of  $P_\zeta$  is the set  $S \stackrel{\text{def}}{=} \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$ .

A proof of Lemma 3.1 is given in [8], Theorem 5.

For  $A \in \mathcal{B}(H(D))$ , define

$$Q_T(A) = \frac{1}{T - T_0} \text{meas}\{\tau \in [T_0, T] : \zeta_{u_T}(s + i\varphi(\tau)) \in A\}.$$

**Lemma 3.2** *Suppose that  $\varphi(\tau) \in U(T_0)$ , and  $u_T \rightarrow \infty$  and  $u_T \ll T^2$  as  $T \rightarrow \infty$ . Then  $Q_T$  converges weakly to  $P_\zeta$  as  $T \rightarrow \infty$ .*

**Proof** On a certain probability space with the measure  $\mathbb{P}$ , define the random variable  $\theta_T$  which is uniformly distributed on the interval  $[T_0, T]$ . On the mentioned probability space, define the  $H(D)$ -valued random elements

$$X_T = \zeta(s + i\varphi(\theta_T)) \quad \text{and} \quad Y_T = \zeta_{u_T}(s + i\varphi(\theta_T)).$$

Then we have, for  $A \in \mathcal{B}(H(D))$ ,

$$P_T(A) = \mathbb{P}\{X_T \in A\} \quad \text{and} \quad Q_T(A) = \mathbb{P}\{Y_T \in A\}. \tag{3.1}$$

We will apply the equivalent of weak convergence of probability measures in terms of closed sets. Namely, [2]  $P_n$  converges weakly to  $P$  as  $n \rightarrow \infty$  in the space  $\mathbb{X}$  if and only if, for every closed set  $F \subset \mathbb{X}$ ,

$$\limsup_{n \rightarrow \infty} P_n(F) \leq P(F).$$

Thus, let  $F \subset H(D)$  be an arbitrary closed set, and  $\varepsilon > 0$  be fixed. Denote by  $\rho(g_1, F) = \inf_{g \in F} \rho(g_1, g)$ , and define  $F_\varepsilon = \{g \in H(D) : \rho(g, F) \leq \varepsilon\}$ . Then the set  $F_\varepsilon$  is closed as well. Moreover,

$$\{Y_T \in F\} \subset \{X_T \in F_\varepsilon\} \cup \{\rho(X_T, Y_T) \geq \varepsilon\}.$$

Hence,

$$\mathbb{P}\{Y_T \in F\} \leq \mathbb{P}\{X_T \in F_\varepsilon\} + \mathbb{P}\{\rho(X_T, Y_T) \geq \varepsilon\}. \tag{3.2}$$



By the definitions of  $X_T$  and  $Y_T$ , and Lemma 2.3,

$$\lim_{T \rightarrow \infty} \mathbb{P}\{\rho(X_T, Y_T) \geq \varepsilon\} \leq \lim_{T \rightarrow \infty} \frac{1}{\varepsilon(T - T_0)} \int_{T_0}^T \rho(\zeta(s + i\varphi(\tau)), \zeta_{u_T}(s + i\varphi(\tau))) \, d\tau = 0.$$

Therefore, in view of (3.2),

$$\limsup_{T \rightarrow \infty} \mathbb{P}\{Y_T \in F\} \leq \limsup_{T \rightarrow \infty} \mathbb{P}\{X_T \in F_\varepsilon\},$$

or, by (3.1),

$$\limsup_{T \rightarrow \infty} Q_T(F) \leq \limsup_{T \rightarrow \infty} P_T(F_\varepsilon). \tag{3.3}$$

Lemma 3.1 and the mentioned above equivalent of weak convergence in terms of closed sets imply the inequality

$$\limsup_{T \rightarrow \infty} P_T(F_\varepsilon) \leq P_\zeta(F_\varepsilon).$$

Thus, by (3.3),

$$\limsup_{T \rightarrow \infty} Q_T(F) \leq P_\zeta(F_\varepsilon).$$

Now, taking  $\varepsilon \rightarrow +0$ , we obtain that

$$\limsup_{T \rightarrow \infty} Q_T(F) \leq P_\zeta(F),$$

i.e.  $Q_T$  converges weakly to  $P_\zeta$  as  $T \rightarrow \infty$ . □

#### 4. Proof of Theorem 1.3

Theorem 1.3 follows from Lemma 3.2 and the Mergelyan theorem on approximation of analytic functions by polynomials. For convenience, we recall it, see [10].

**Lemma 4.1** *Suppose that  $K \subset \mathbb{C}$  is a compact set with connected complement, and  $g(s)$  is a continuous function on  $K$  that is analytic in the interior of  $K$ . Then, for every  $\varepsilon > 0$ , there exists a polynomial  $p_\varepsilon(s)$  such that*

$$\sup_{s \in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

**Proof** [Proof of Theorem 1.3] Firstly, we prove the existence of the limit. For  $g \in H(D)$ , define

$$h(g) = \sup_{s \in K} |g(s) - f(s)|.$$

Then the function  $h : H(D) \rightarrow \mathbb{R}$  is continuous. Therefore, the property of preservation of weak convergence, see, for example, Theorem 5.1 of [2], and Lemma 3.2 imply that

$$\frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\varphi(\tau)) - f(s)| \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

converges weakly to the measure

$$m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \omega) - f(s)| \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

as  $T \rightarrow \infty$ . It is well known that the weak convergence of probability measures in  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  is equivalent to that of the corresponding distribution functions. Thus, the above remark shows that the distribution function

$$\frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |\zeta_{u_T}(s + i\varphi(\tau)) - f(s)| < \varepsilon \right\} \quad (4.1)$$

converges weakly to the distribution function

$$m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \omega) - f(s)| < \varepsilon \right\} \quad (4.2)$$

as  $T \rightarrow \infty$ . Since the weak convergence of distribution functions means the convergence at all continuity points of the limit function, and each distribution function has at most countable set of discontinuity points, we obtain that (4.1) converges to (4.2) as  $T \rightarrow \infty$  for all but at most countably many  $\varepsilon > 0$ .

It remains to prove the positivity of (4.2). By Lemma 4.1, there exists a polynomial  $p(s)$  such that

$$\sup_{s \in K} \left| f(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}. \quad (4.3)$$

Since, in view of Lemma 3.1, the support of the measure  $P_\zeta$  is the set  $S$ , and  $e^{p(s)} \in S$ , we have by a property of the support

$$m_H \left\{ \omega \in \Omega : \sup_{s \in K} \left| \zeta(s, \omega) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\} > 0. \quad (4.4)$$

However, inequality (4.3) implies the inclusion

$$\left\{ \omega \in \Omega : \sup_{s \in K} \left| \zeta(s, \omega) - e^{p(s)} \right| < \frac{\varepsilon}{2} \right\} \subset \left\{ \omega \in \Omega : \sup_{s \in K} \left| \zeta(s, \omega) - f(s) \right| < \varepsilon \right\}.$$

This and (4.4) prove the positivity of (4.2). The theorem is proven.  $\square$

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