mathematics

Article

Joint Universality in Short Intervals with Generalized Shifts for the Riemann Zeta-Function

Antanas Laurinčikas

Faculty of Mathematics and Informatics, Institute of Mathematics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania; antanas.laurincikas@mif.vu.lt

Abstract: In the paper, the simultaneous approximation of a tuple of analytic functions in the strip $\{s = \sigma + it \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts $(\zeta(s + i\varphi_1(\tau)), \ldots, \zeta(s + i\varphi_r(\tau)))$ of the Riemann zeta-function $\zeta(s)$ with a certain class of continuously differentiable increasing functions $\varphi_1, \ldots, \varphi_r$ is considered. This class of functions $\varphi_1, \ldots, \varphi_r$ is characterized by the growth of their derivatives. It is proved that the set of mentioned shifts in the interval $[T, T + H]$ with $H = o(T)$ has a positive lower density. The precise expression for H is described by the functions $(\varphi_j(\tau))^{1/3}(\log \varphi_j(\tau))^{26/15}$ and derivatives $\varphi'_j(\tau)$. The density problem is also discussed. An example of the approximation by a composition $F(\zeta(s + i\varphi_1(\tau)), \ldots, \zeta(s + i\varphi_r(\tau)))$ with a certain continuous operator F in the space of analytic functions is given.

Keywords: joint universality; Mergelyan theorem; Riemann zeta-function; weak convergence

MSC: 11M06

Citation: Laurinčikas, A. Joint Universality in Short Intervals with Generalized Shifts for the Riemann Zeta-Function. *Mathematics* **2022**, *10*, 1652. https://doi.org/10.3390/math10101652

Academic Editor: Sitnik Sergey

Received: 31 March 2022
Accepted: 9 May 2022
Published: 12 May 2022

1. Introduction

As usual, let $\zeta(s)$, $s = \sigma + it$ denote the Riemann zeta-function. Recall that, in the half-plane $\sigma > 1$, the function $\zeta(s)$ is defined by the Dirichlet series or infinite product over prime numbers

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p-\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and has analytic continuation to the whole complex plane, except for the point $s = 1$, a simple pole with residue 1. The function $\zeta(s)$ is not only the main object of analytic number theory but also has applications in other regions of mathematics and even physics. Therefore, it is not surprising that much attention is devoted to investigating the function $\zeta(s)$. One of the most interesting properties of the Riemann zeta-function is its universality discovered by S.M. Voronin in [1]; see also [2]. He observed that a wide class of analytic functions can be approximated to the desired accuracy by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. More precisely, Voronin proved that, for every continuous non-vanishing in the disc $|s| \leqslant r$, $0 < r < 1/4$, and analytic in $|s| < r$ function $f(s)$, and every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Voronin himself applied the universality property of $\zeta(s)$ to investigate the denseness of the set of its values, and also used it for the proof of its functional independence [3]. Physicists obtained [4] estimates for integrals over analytic curves used in quantum mechanics. Other applications of the universality of zeta-functions and some related problems can be found in a survey paper [5].

Attention to the universality of zeta-functions has not stopped for almost half a century. Currently, the Voronin universality theorem has a more general form. Let $D = \{s \in \mathbb{C} :$

$1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and $H_0(K)$ with $K \in \mathcal{K}$ be the class of continuous non-vanishing functions on K that are analytic in the interior of K. Denote by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the following assertion is valid; see [6,7].

Theorem 1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

In place of shifts $\zeta(s+i\tau)$, generalized shifts $\zeta(s+i\varphi(\tau))$ with a certain function $\varphi(\tau)$ can be used. For example, in [8], the function $\varphi(\tau) = \tau^\alpha \log^\beta \tau$ with some $\alpha, \beta \in \mathbb{R}$ was applied. In [9], the increasing differentiable function $\varphi(\tau)$, such that $\varphi(2\tau)\max_{\tau\leqslant u\leqslant 2\tau}(\varphi'(\tau))^{-1} \ll \tau$, $\tau \to \infty$, was used. Here, and in sequel, the classical Landau notation, $a \ll b$, $b > 0$, means that there exists a constant $C > 0$ such that $|a| \leqslant Cb$. More generally, $a \ll_\theta b$ means that the constant C depends on θ, a and b can depend or not on θ. For example, the famous Lindelöf hypothesis asserts that, for every $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon t^\varepsilon, \quad t > t_0 > 0.$$

More general joint universality theorems on the simultaneous approximation of a tuple of analytic functions by shifts of zeta or L-functions are also known. The first theorem of this kind was obtained in [2] for Dirichlet L-functions $L(s,\chi_1),\dots,L(s,\chi_r)$ with pairwise non-equivalent Dirichlet characters $\chi_1,\dots,\chi_r$. The modern version of this theorem is given in [10]. Joint universality theorems for more general functions can be found in [11–13]. In addition, some works on joint approximation of a tuple of analytic functions by shifts $(\zeta(s+i\varphi_1(\tau)),\dots,\zeta(s+i\varphi_r(\tau)))$ with some functions $\varphi_1(\tau),\dots,\varphi_r(\tau)$ are known. In the joint case, the shifts $\zeta(s+i\varphi_1(\tau)),\dots,\zeta(s+i\varphi_r(\tau))$ must be independent in a certain sense. Thus, the functions $\varphi_1(\tau),\dots,\varphi_r(\tau)$ must satisfy some requirements. For example, in [8], the functions $\tau^{\alpha_j}\log^{\beta_j}\tau$, $j = 1,\dots,r$, with reals $\alpha_j \neq \alpha_k$ of $\beta_j \neq \beta_k$ for $j \neq k$ were used. In [14], a joint universality theorem on approximation by shifts $(\zeta(s+ia_1\tau),\dots,\zeta(s+ia_r\tau))$ was obtained, where $a_1,\dots,a_r$ are real algebraic numbers, linearly independent over the field of rational numbers $\mathbb{Q}$.

In the present paper, we will prove a joint universality theorem in short intervals for the Riemann zeta-function on the approximation of analytic functions by generalized shifts with certain differentiable functions $\varphi_j(\tau)$. We say that $(\varphi_1,\dots,\varphi_r) \in U_r$ if the following hypotheses are satisfied:

1° $\varphi_1(\tau),\dots,\varphi_r(\tau)$ are increasing to $+\infty$ functions in $[T_0,\infty]$, $T_0 > 0$;

2° $\varphi_1(\tau),\dots,\varphi_r(\tau)$ have continuous derivatives such that

$$\varphi_j'(\tau) = \widehat{\varphi}_j(\tau)(1+o(1)), \quad \tau \to \infty,\ j = 1,\dots,r,$$

where $\widehat{\varphi}_j(\tau)$ are monotonic functions compared with respect to their growth. Without loss of generality, we require that, for $\tau \to \infty$,

$$\widehat{\varphi}_j(\tau) = o(\widehat{\varphi}_{j+1}(\tau)), \quad j = 1,\dots,r-1.$$

3° the estimates

$$\begin{cases} \dfrac{\widehat{\varphi}_j(2\tau)}{\widehat{\varphi}_j(\tau)} \ll 1 & \text{if } \widehat{\varphi}_j(\tau) \text{ is increasing,} \\ \dfrac{\widehat{\varphi}_j(\tau)}{\widehat{\varphi}_j(2\tau)} \ll 1 & \text{if } \widehat{\varphi}_j(\tau) \text{ is decreasing, } j = 1,\dots,r, \end{cases}$$

are valid.

For $(\varphi_1, \dots, \varphi_r) \in U_r$, define

$$\psi_j(\tau) = (\varphi_j(\tau))^{1/3}(\log \varphi_j(\tau))^{26/15},$$

$$\widehat{H}_j(\tau) = \begin{cases} \frac{\psi_j(\tau)}{\widehat{\varphi}_j(2\tau)} & \text{if } \widehat{\varphi}_j(\tau) \text{ is increasing,} \\ \frac{\psi_j(\tau)}{\widehat{\varphi}_j(\tau)} & \text{if } \widehat{\varphi}_j(\tau) \text{ is decreasing, } j = 1, \dots, r, \end{cases}$$

$$\widehat{\widehat{H}}_j(\tau) = \begin{cases} \frac{\varphi_j(\tau)}{2\widehat{\varphi}_j(2\tau)} & \text{if } \widehat{\varphi}_j(\tau) \text{ is increasing,} \\ \frac{\varphi_j(\tau)}{2\widehat{\varphi}_j(\tau)} & \text{if } \widehat{\varphi}_j(\tau) \text{ is decreasing, } j = 1, \dots, r, \end{cases}$$

and $\widehat{H}(\tau) = \max_{1\leqslant j\leqslant r} \widehat{H}_j(\tau)$ and $\widehat{\widehat{H}}(\tau) = \min_{1\leqslant j\leqslant r} \widehat{\widehat{H}}_j(\tau)$.Then, the following statement is valid.

Theorem 2. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{ \tau \in [T, T+H] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\varphi_j(\tau)) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

Theorem 2 is an example of a universality theorem in short intervals because the length of the interval $[T, T + H]$ is $o(T)$ as $T \to \infty$ for $\widehat{\widehat{H}}(T) \ll T$. This type of universality theorem is one of the ways of their effectivization. In short intervals, it is easier to detect a shift with the approximation property. The first one-dimensional universality theorem was obtained in [15] for shifts $\zeta(s + i\tau)$, and in [16] for generalized shifts.

Approximation of analytic functions is also possible by some compositions of generalized shifts. Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta,

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r},$$

and $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$, $S^r = \underbrace{S \times \cdots \times S}_{r}$. Then, it is possible to approximate the functions defined on $H(D)$ by shifts $F(\zeta(s + i\varphi_1(\tau)), \dots, \zeta(s + i\varphi_r(\tau)))$ for some classes of operators $F : H^r(D) \to H(D)$. For results of this type, see, for example, [17,18]. We will give only one example of such compositions, and other results will be given in a subsequent paper. Let $H(K)$ with $K \in \mathcal{K}$ be the class of functions continuous on K that are analytic in the interior of K. Thus, $H_0(K) \subset H(K)$.

Theorem 3. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r$, $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$, and $F : H^r(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{ \tau \in [T, T+H] : \sup_{s\in K} |F(\zeta(s + i\varphi_1(\tau)), \dots, \zeta(s + i\varphi_r(\tau))) - f(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

For example, the tuples of functions $(\tau \log \tau, \dots, \tau^r \log \tau)$ and $(\tau + 1, \tau^2 + \tau + 1, \dots, \tau^r + \tau^{r-1} + \cdots + 1)$ satisfy the hypotheses for the class U_r.

Unfortunately, it is not easy to present an example of the operator F satisfying the conditions of Theorem 3. In [15], it was observed that a continuous operator $F : H^r(D) \to H(D)$ such that the set $F(S^r)$ being dense in $H(D)$ satisfies the hypotheses of Theorem 3.

For $V > 0$, let $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1, \ |t| < V\}$. In place of $H(D)$, the space $H(D_V)$ of analytic in $H(D_V)$ functions can be studied. Then, S is replaced by $S_V = \{g \in H(D_V) : g(s) \neq 0, \text{ or } g(s) \equiv 0\}$. Suppose that V is such that $K \subset D_V$, and $F : H^r(D_V) \to H(D_V)$ is a continuous operator such that, for every polynomial p, the set $(F^{-1}\{p\}) \cap S_V^r \neq \varnothing$. Then, the assertion of Theorem 3 remains valid. Since the non-vanishing of a polynomial in a bounded region can be controlled by its constant term, for example, the operator $F(g_1, \ldots, g_r) = g_1' + \cdots + g_r'$, $g_j \in H(D_V)$, $j = 1, \ldots, r$, satisfies the condition $(F^{-1}\{p\}) \cap S_V^r \neq \varnothing$.

Note that a polynomial appears in the above hypothesis because of the application of the Mergelyan theorem on approximation of analytic functions by polynomials, see Lemma 6.

Theorems 2 and 3 are derived from weak convergence of some probability measures in the space of analytic functions.

2. Estimate for the Second Moment

It is well known that estimates for the second moments play an important role in the proof of the universality of the Dirichlet series. We need the estimate for

$$I(T, H, \sigma, t) \stackrel{def}{=} \int_T^{T+H} |\zeta(\sigma + i\varphi(\tau) + it)|^2 \, d\tau$$

for fixed $1/2 < \sigma < 1$, $t \in \mathbb{R}$ and $\varphi(\tau)$ satisfying the hypotheses 1° and 2° of the class U_r.

Lemma 1. *Suppose that $\widehat{H}(T)$ and $\widehat{\widehat{H}}(T)$ correspond $\varphi(\tau)$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$. Then,*

$$I(T, H, \sigma, t) \ll_\sigma H(1 + |t|).$$

Proof. It is well known that, for fixed $1/2 < \sigma < 1$, uniformly in H, $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$,

$$\int_{T-H}^{T+H} |\zeta(\sigma + it)|^2 \, dt \ll_\sigma H. \tag{1}$$

This is implied by an analogous estimate of Theorem 7.1 from [19] for $T^{(\kappa+\lambda+1)/2(\kappa+1)} \times (\log T)^{(2+\kappa)/(\kappa+1)} \leqslant H \leqslant T$ and $1 + \lambda - \kappa \geqslant 2\sigma$ with application of the exponential pair $(\kappa, \lambda) = (4/11, 6/11)$.

From

$$\frac{1}{1+a} = 1 - \frac{a}{1+a}, \quad a \neq -1,$$

it follows that

$$\frac{1}{1+o(1)} = 1 + o(1). \tag{2}$$

Suppose that the function $\widehat{\varphi}(\tau)$ corresponding $\varphi(\tau)$ is increasing. Then, in virtue of the mean value theorem,

$$
\begin{aligned}
\int_T^{T+H} |\zeta(\sigma+i\varphi(\tau)+it)^2\,\mathrm{d}\tau &= \int_T^{T+H} \tfrac{1}{\varphi'(\tau)}|\zeta(\sigma+i\varphi(\tau)+it)^2\,\mathrm{d}\varphi(\tau) \\
&= \int_T^{T+H} \tfrac{1}{\widehat{\varphi}(\tau)}(1+o(1))|\zeta(\sigma+i\varphi(\tau)+it)^2\,\mathrm{d}\varphi(\tau) \\
&\ll \int_T^{T+H} \tfrac{1}{\widehat{\varphi}(\tau)}\,\mathrm{d}\Big(\int_T^{\varphi(\tau)+t} |\zeta(\sigma+iu)|^2\,\mathrm{d}u\Big) \\
&= \tfrac{1}{\widehat{\varphi}(T)}\int_T^{\xi}\mathrm{d}\Big(\int_T^{\varphi(\tau)+t} |\zeta(\sigma+iu)|^2\Big) \\
&= \tfrac{1}{\widehat{\varphi}(T)}\int_{\varphi(\tau)+t}^{\varphi(\xi)+t} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \\
&\leqslant \tfrac{1}{\widehat{\varphi}(T)}\int_{\varphi(T)-|t|}^{\varphi(T+H)+|t|} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \\
&= \tfrac{1}{\widehat{\varphi}(T)}\int_{\varphi(T)-|t|}^{\varphi(T)+H\varphi'(\xi)+|t|} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \\
&\leqslant \tfrac{1}{\widehat{\varphi}(T)}\int_{\varphi(T)-2H\widehat{\varphi}(2T)-|t|}^{\varphi(T)+2H\widehat{\varphi}(2T)+|t|} |\zeta(\sigma+iu)|^2\,\mathrm{d}u,
\end{aligned} \tag{3}
$$

where $T \leqslant \xi \leqslant T+H$. The definitions of H and U_r show that

$$
2H\widehat{\varphi}(2T)+|t| \geqslant 2\psi(T)\frac{\widehat{\varphi}(2T)}{\widehat{\varphi}(T)}+|t| \geqslant (\varphi(T))^{1/3}(\log\varphi(T))^{26/15}.
$$

Thus, we can apply (1), and, for $2H\widehat{\varphi}(2T)+|t| \leqslant \varphi(T)$, we find, in view of (3),

$$
\begin{aligned}
I(T,H,\sigma,t) &\ll_\sigma \frac{H\widehat{\varphi}(2T)+|t|}{\widehat{\varphi}(T)} \ll_\sigma H+\frac{|t|}{\widehat{\varphi}(T)} \ll_\sigma H\left(1+\frac{|t|}{H\widehat{\varphi}(T)}\right) \\
&\ll_\sigma H\left(1+\frac{|t|\widehat{\varphi}(2T)}{\psi(T)\widehat{\varphi}(T)}\right) \ll_\sigma H\left(1+\frac{|t|}{\psi(T)}\right) \ll_\sigma H(1+|t|).
\end{aligned}
$$

If $2H\widehat{\varphi}(2T)+|t| > \varphi(T)$, then $\varphi(T)+2H\widehat{\varphi}(2T)+|t| < 4(H\widehat{\varphi}(2T)+|t|)$ and $\varphi(T)-2H\widehat{\varphi}(2T)-|t| > -4(H\widehat{\varphi}(2T)+|t|)$. Therefore, the classical estimate

$$
\int_{-T}^{T} |\zeta(\sigma+it)|^2\,\mathrm{d}t \ll_\sigma T \tag{4}
$$

and (3) imply

$$
I(T,H,\sigma,t) \ll \frac{1}{\widehat{\varphi}(T)}\int_{-4(H\widehat{\varphi}(2T)+|t|)}^{4(H\widehat{\varphi}(2T)+|t|)} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \ll_\sigma \frac{H\widehat{\varphi}(2T)+|t|}{\widehat{\varphi}(T)} \ll_\sigma H(1+|t|).
$$

Let $\widehat{\varphi}(T)$ be decreasing. Then, similarly as above, we have

$$
\begin{aligned}
I(T,H,\sigma,t) &\ll \tfrac{1}{\widehat{\varphi}(T+H)}\int_{\xi}^{T+H}\mathrm{d}\Big(\int_T^{\varphi(\tau)+t} |\zeta(\sigma+iu)|^2\,\mathrm{d}u\Big) \\
&\ll \tfrac{1}{\widehat{\varphi}(2T)}\int_{\varphi(\xi)+t}^{\varphi(T+H)+t} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \\
&\ll \tfrac{1}{\widehat{\varphi}(2T)}\int_{\varphi(T)-2H\widehat{\varphi}(T)-|t|}^{\varphi(T)+2H\widehat{\varphi}(T)+|t|} |\zeta(\sigma+iu)|^2\,\mathrm{d}u.
\end{aligned} \tag{5}
$$

Since, by the definition of H,

$$
2H\widehat{\varphi}(T)+|t| \geqslant (\varphi(T))^{1/3}(\log\varphi(T))^{26/15},
$$

this and (1) show that, for $2H\widehat{\varphi}(T)+|t| \leqslant \varphi(T)$,

$$
I(T,H,\sigma,t) \ll_\sigma \frac{1}{\widehat{\varphi}(2T)}(H\widehat{\varphi}(T)+|t|) \ll_\sigma H(1+|t|).
$$

If $2H\widehat{\varphi}(T)+|t| > \varphi(T)$, then, in view of (5) and (4),

$$
I(T,H,\sigma,t) \ll \frac{1}{\widehat{\varphi}(2T)}\int_{-4(H\widehat{\varphi}(T)+|t|)}^{4(H\widehat{\varphi}(T)+|t|)} |\zeta(\sigma+iu)|^2\,\mathrm{d}u \ll_\sigma \frac{H\widehat{\varphi}(T)+|t|}{\widehat{\varphi}(2T)} \ll_\sigma H(1+|t|).
$$

□

3. Absolutely Convergent Series

In this section, we will show that there exists a collection of absolutely convergent Dirichlet series that approximates in the mean the collection $(\zeta(s+i\varphi_1(\tau)),\dots,\zeta(s+i\varphi_r(\tau)))$ with $(\varphi_1,\dots,\varphi_r)\in U_r$.

Let $\theta>0$ be a fixed number,

$$v_n(m)=\exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\},\quad m,n\in\mathbb{N},$$

where $\exp\{a\}=\mathrm{e}^a$. Define the series

$$\zeta_n(s)=\sum_{m=1}^{\infty}\frac{v_n(m)}{m^s}.$$

Since the coefficients of the latter series decrease exponentially, this series is absolutely convergent in the half-plane $\sigma>\sigma_0$ with an arbitrary finite σ_0.

Recall the metric in $H^r(D)$. There exists a sequence of embedded compact subsets $\{K_l : l\in\mathbb{N}\}\subset D$ such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

and every compact set $K\subset D$ lies in some K_l. For $g_1,g_2\in H(D)$, define

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then, ρ is a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Putting

$$\underline{\rho}(\underline{g}_1,\underline{g}_2)=\max_{1\leqslant j\leqslant r}\rho(g_{1j},g_{2j}),\quad \underline{g}_k=(g_{k1},\dots,g_{kr})\in H^r(D),\ k=1,2,$$

gives a metric in $H^r(D)$ inducing the product topology.

Define

$$\underline{\zeta}(s+i\underline{\varphi}(\tau))=(\zeta(s+i\varphi_1(\tau)),\dots,\zeta(s+i\varphi_r(\tau)))$$

and

$$\underline{\zeta}_n(s+i\underline{\varphi}(\tau))=(\zeta_n(s+i\varphi_1(\tau)),\dots,\zeta_n(s+i\varphi_r(\tau))).$$

We will prove the approximation in the mean of $\underline{\zeta}(s+i\underline{\varphi}(\tau))$ by $\underline{\zeta}_n(s+i\underline{\varphi}(\tau))$

Lemma 2. *Suppose that $(\varphi_1,\dots,\varphi_r)\in U_r$ and $\widehat{H}(T)\leqslant H\leqslant\widehat{\widehat{H}}(T)\leqslant T$. Then, the equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{H}\int_T^{T+H}\underline{\rho}\Big(\underline{\zeta}(s+i\underline{\varphi}(\tau)),\underline{\zeta}_n(s+i\underline{\varphi}(\tau))\Big)\,\mathrm{d}\tau=0$$

holds.

Proof. In view of the definition of metrics $\underline{\rho}$ and ρ, it suffices to show that, for every compact set $K\subset D$, $\widehat{H}_j(T)\leqslant H\leqslant\widehat{\widehat{H}}_j(T)\leqslant T$ and all $j=1,\dots,r$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{H}\int_T^{T+H}\sup_{s\in K}\big|\zeta(s+i\varphi_j(\tau))-\zeta_n(s+i\varphi_j(\tau))\big|\,\mathrm{d}\tau=0. \tag{6}$$

Let $\varphi(\tau)$ be one of the functions $\varphi_1(\tau), \dots, \varphi_r(\tau)$. We will prove (6) for the function $\varphi(\tau)$. We will use the representation, see, for example, [6]

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta(s+z) \frac{l_n(z)}{z} \, dz, \tag{7}$$

where $\widehat{\theta} > 1/2$ is a fixed number,

$$l_n(s) = \frac{s}{\theta} \Gamma\Big(\frac{s}{\theta}\Big) n^s,$$

θ is from definition of $v_n(m)$, and $\Gamma(s)$ denotes the Euler gamma-function. Let $K \subset D$ be an arbitrary compact set. Fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s = \sigma + it \in K$. For such σ, we have $\theta_1 \overset{def}{=} 1/2 + \varepsilon - \sigma < 0$. Let $\widehat{\theta} = 1/2 + \varepsilon > 1/2$. Then, (7) and the residue theorem imply, for $s \in K$,

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z) \frac{l_n(z)}{z} dz + \frac{l_n(1-s)}{1-s}.$$

Thus, for $s \in K$,

$$\begin{aligned}
&\zeta_n(s+i\varphi(\tau)) - \zeta(s+i\varphi(\tau)) \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta\Big(\frac{1}{2} + \varepsilon + it + i\varphi(\tau) + iv\Big) \frac{l_n(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv} \, dv + \frac{l_n(1-s-i\varphi(\tau))}{1-s-i\varphi(\tau)} \\
&\ll \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Big|\zeta\Big(\frac{1}{2} + \varepsilon + i\varphi(\tau) + iv\Big)\Big| \sup_{s\in K} \Big| \frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} \Big| \, dv \\
&\quad + \sup_{s\in K} \Big| \frac{l_n(1-s-i\varphi(\tau))}{1-s-i\varphi(\tau)} \Big|
\end{aligned}$$

after a shift $t + v \to v$. Hence,

$$\begin{aligned}
&\tfrac{1}{H} \textstyle\int_T^{T+H} \sup_{s\in K} |\zeta(s+i\varphi(\tau)) - \zeta_n(s+i\varphi(\tau))| \, d\tau \\
&\ll \textstyle\int_{-\infty}^{\infty} \Big(\tfrac{1}{H} \int_T^{T+H} \Big| \zeta\Big(\tfrac{1}{2} + \varepsilon + i\varphi(\tau) + iv\Big) \Big| \, d\tau \Big) \sup_{s\in K} \Big| \tfrac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} \Big| \, dv \\
&\quad + \tfrac{1}{H} \textstyle\int_T^{T+H} \sup_{s\in K} \Big| \tfrac{l_n(1-s-i\varphi(\tau))}{1-s-i\varphi(\tau)} \Big| =^{def} J_1 + J_2.
\end{aligned} \tag{8}$$

Lemma 1 implies the estimate

$$\begin{aligned}
\tfrac{1}{H} \textstyle\int_T^{T+H} \Big| \zeta\Big(\tfrac{1}{2} + \varepsilon + i\varphi(\tau) + iv\Big) \Big| \, d\tau \quad &\leqslant \Big(\tfrac{1}{H} \textstyle\int_T^{T+H} \Big| \zeta\Big(\tfrac{1}{2} + \varepsilon + i\varphi(\tau) + iv\Big) \Big|^2 \, d\tau \Big)^{1/2} \\
&\ll_\varepsilon (1+|v|)^{1/2}.
\end{aligned} \tag{9}$$

It is well known that, with a certain $c > 0$

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\} \tag{10}$$

uniformly in every interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $\sigma_1 < \sigma_2$. Therefore, for all $s \in K$,

$$\frac{l_n(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} \ll_\theta n^{1/2+\varepsilon-\sigma} \exp\Big\{-\frac{c}{\theta}|v-\sigma|\Big\} \ll_{\theta,K} n^{-\varepsilon} \exp\{-c_1|v|\}, \quad c_1 > 0.$$

Thus, in view of (9),

$$J_1 \ll_{\varepsilon,\theta,K} n^{-\varepsilon} \int_{-\infty}^{\infty} (1+|v|)^{1/2} \exp\{-c_1|v|\} \, dv \ll_{\varepsilon,\theta,K} n^{-\varepsilon}. \tag{11}$$

Applying the estimate (10) once more, we obtain that, for all $s \in K$,

$$\frac{l_n(1-s-i\varphi(\tau))}{1-s-i\varphi(\tau)} \ll_\theta n^{1-\sigma} \exp\Big\{-\frac{c}{\theta}|t+\varphi(\tau)|\Big\} \ll_{\theta,K} n^{1/2-2\varepsilon} \exp\{-c_2\varphi(\tau)\}, \quad c_2 > 0.$$

Thus,

$$J_2 \ll_{\theta,K} n^{1/2-2\varepsilon} \frac{1}{H} \int_T^{T+H} \exp\{-c_2\varphi(\tau)\}\, d\tau \ll_{\theta,K} n^{1/2-2\varepsilon} \exp\{-c_2\varphi(T)\}.$$

This and (11) show that the left-hand side of (8) is estimated as

$$\ll_{\varepsilon,\theta,K} n^{-\varepsilon} + n^{1/2-2\varepsilon} \exp\{-c_2\varphi(T)\}.$$

Since $\varphi(T) \to \infty$ as $T \to \infty$, this proves (6) for the function $\varphi(\tau)$. □

4. Limit Theorems

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$. In this section, we will consider the weak convergence for

$$P_{T,H}(A) \overset{def}{=} \frac{1}{H} \text{meas}\Big\{\tau \in [T, T+H] : \underline{\zeta}(s + i\underline{\varphi}(\tau)) \in A\Big\}, \quad A \in \mathcal{B}(H^r(D))$$

as $T \to \infty$. We divide the study of $P_{T,H}$ into lemmas. Let $\mathbb{P}$ the set of all prime numbers. Define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. The infinite-dimensional torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Then, again, Ω^r is a compact topological Abelian group. Thus, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{T,H}(A) = \frac{1}{H} \text{meas}\Big\{\tau \in [T, T+H] : \Big(\Big(p^{-i\varphi_1(\tau)} : p \in \mathbb{P}\Big), \ldots, \Big(p^{-i\varphi_r(\tau)} : p \in \mathbb{P}\Big)\Big) \in A\Big\}.$$

Lemma 3. *Suppose that $(\varphi_1, \ldots, \varphi_r) \in U_r$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$. Then, $Q_{T,H}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. By $\omega_j(p)$, denote the pth component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \ldots, r$. Then, the Fourier transform $g_{Q_{T,H}}(\underline{k}_1, \ldots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \ldots, r$, is given by

$$g_{Q_{T,H}}(\underline{k}_1, \ldots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^r \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p)\right) dQ_{T,H},$$

where the star "$*$" shows that only a finite number of integers k_{jp} are distinct from zero. Thus, by definition of $Q_{T,H}$,

$$\begin{aligned} g_{Q_{T,H}}(\underline{k}_1, \ldots, \underline{k}_r) &= \tfrac{1}{H} \int_T^{T+H} \Big(\prod_{j=1}^r \prod{}^{*}_{p \in \mathbb{P}} p^{-ik_{jp}} \varphi_j(\tau)\Big)\, d\tau \\ &= \tfrac{1}{H} \int_T^{T+H} \exp\Big\{-i \sum_{j=1}^r \varphi_j(\tau) \sum{}^{*}_{p \in \mathbb{P}} k_{jp} \log p\Big\}\, d\tau. \end{aligned} \tag{12}$$

Obviously,

$$g_{Q_{T,H}}(\underline{0},\ldots,\underline{0})=1. \tag{13}$$

For brevity, let

$$a_j=\sum_{p\in\mathbb{P}}^{*} k_{jp}\log p, \quad j=1,\ldots,r.$$

Now suppose that $(\underline{k}_1,\ldots,\underline{k}_r)\neq(\underline{0},\ldots,\underline{0})$. Hence, for at least one, $j\in\{1,\ldots,r\}$, $\underline{k}_j\neq\underline{0}$. Since the logarithms of prime numbers are linearly independent over $\mathbb{Q}$, $a_j=0$ if and only if $\underline{k}_j=0$. Hence, by the properties of the class U_r, as $\tau\to\infty$,

$$\left(\sum_{j=1}^{r}a_j\varphi_j(\tau)\right)'=\sum_{j=1}^{r}a_j\varphi_j'(\tau)=\sum_{j=1}^{r}a_j\widehat{\varphi}_j(\tau)(1+o(1))=a_{j_0}\widehat{\varphi}_{j_0}(\tau)(1+o(1)),$$

where $j_0=\max(j:a_j\neq 0)$. This together with (2) implies, for $\tau\to\infty$,

$$\left(\left(\sum_{j=1}^{r}a_j\varphi_j(\tau)\right)'\right)^{-1}=\frac{1}{a_{j_0}\widehat{\varphi}_{j_0}(\tau)(1+o(1))}=\frac{1}{a_{j_0}\widehat{\varphi}_{j_0}(\tau)}(1+o(1)). \tag{14}$$

Put

$$b(\tau)=\sum_{j=1}^{r}a_j\varphi_j(\tau).$$

Then, in view of (14),

$$\begin{aligned}\int_T^{T+H}\cos b(\tau)\,\mathrm{d}\tau&=\int_T^{T+H}\frac{1}{b'(\tau)}\,\mathrm{d}\sin b(\tau)\\&=\frac{1}{a_{j_0}}\int_T^{T+H}\frac{1}{\widehat{\varphi}_{j_0}(\tau)}\,\mathrm{d}\sin b(\tau)+\int_T^{T+H}\frac{o(1)}{\widehat{\varphi}_{j_0}(\tau)}\,\mathrm{d}\sin b(\tau).\end{aligned}$$

Since $\widehat{\varphi}_{j_0}(\tau)$ is monotonic, the first integral on the right-hand side of the latter equality has the estimate

$$\ll\begin{cases}\widehat{\varphi}_{j_0}^{-1}(T) & \text{if } \widehat{\varphi}_{j_0}(\tau) \text{ is increasing,}\\ \widehat{\varphi}_{j_0}^{-1}(T+H) & \text{if } \widehat{\varphi}_{j_0}(\tau) \text{ is decreasing.}\end{cases}$$

For the second integral, by (14) again, we have

$$\int_T^{T+H}\frac{o(1)(1+o(1))}{b'(\tau)}\,\mathrm{d}\sin b(\tau)=\int_T^{T+H}o(1)\cos b(\tau)\,\mathrm{d}\tau=o(H).$$

The same estimates remain valid for the integral of the function $\sin b(\tau)$. Therefore, returning to (12), we find that, in the case $(\underline{k}_1,\ldots,\underline{k}_r)\neq(\underline{0},\ldots,\underline{0})$,

$$g_{Q_{T,H}}(\underline{k}_1,\ldots,\underline{k}_r)\ll\begin{cases}\frac{\widehat{\varphi}_{j_0}(2T)}{\widehat{\varphi}_{j_0}(T)\psi_{j_0}(T)} & \text{if } \widehat{\varphi}_{j_0}(\tau) \text{ is increasing,}\\ \frac{\widehat{\varphi}_{j_0}(T)}{\widehat{\varphi}_{j_0}(2T)\psi_{j_0}(T)} & \text{if } \widehat{\varphi}_{j_0}(\tau) \text{ is decreasing}\end{cases}+o(1)\ll\frac{1}{\psi(T)}+o(1)=o(1)$$

as $T\to\infty$. This and (13) show that

$$\lim_{T\to\infty}g_{Q_{T,H}}(\underline{k}_1,\ldots,\underline{k}_r)=\begin{cases}1 & \text{if } (\underline{k}_1,\ldots,\underline{k}_r)=(\underline{0},\ldots,\underline{0}),\\ 0 & \text{if } (\underline{k}_1,\ldots,\underline{k}_r)\neq(\underline{0},\ldots,\underline{0}).\end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma is proved. □

Extend the functions $\omega_j(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N},$$

and define

$$\underline{\zeta}_n(s, \underline{\omega}) = (\zeta_n(s, \omega_1), \dots, \zeta_n(s, \omega_r)),$$

where $\underline{\omega} = (\omega_1, \dots, \omega_r)$ and

$$\zeta_n(s, \omega_j) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega_j(m)}{m^s}, \quad j = 1, \dots, r. \tag{15}$$

The series (15), as $\zeta_n(s)$, are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. Define the mapping $u_n : \Omega^r \to H^r(D)$ by $u_n(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\omega})$. In virtue of the absolute convergence of the series (15), the mapping u_n is continuous.

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,H,n}(A) = \frac{1}{H} \text{meas}\left\{\tau \in [T, T+H] : \underline{\zeta}_n(s + i\underline{\varphi}(\tau)) \in A\right\}$$

and $V_n = m_H u_n^{-1}$, where $V_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A)$.

Lemma 4. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$. Then, $Q_{T,H,n}$ converges weakly to V_n as $T \to \infty$.*

Proof. We have

$$u_n\left(\left(p^{-i\varphi_1(\tau)} : p \in \mathbb{P}\right), \dots, \left(p^{-i\varphi_r(\tau)} : p \in \mathbb{P}\right)\right) = \underline{\zeta}_n(s + i\underline{\varphi}(\tau)).$$

Therefore, for $A \in \mathcal{B}(H^r(D))$,

$$\begin{aligned} &P_{T,H,n}(A) \\ &= \frac{1}{H} \text{meas}\left\{\tau \in [T, T+H] : \left(\left(p^{-i\varphi_1(\tau)} : p \in \mathbb{P}\right), \dots, \left(p^{-i\varphi_r(\tau)} : p \in \mathbb{P}\right)\right) \in u_n^{-1}A\right\} \\ &= Q_{T,H}(u_n^{-1}A) = Q_{T,H}u_n^{-1}(A), \end{aligned}$$

where $Q_{T,H}$ is the measure from Lemma 3. Thus, the lemma is a consequence of the preservation of weak convergence under continuous mappings, see, for example, Theorem 5.1 of [20], continuity of u_n, Lemma 3 and definition of V_n. □

The measure V_n plays an important role in the study of $P_{T,H}$. The measure V_n depends only on the tuple $\underline{\zeta}_n(s)$, and appears in all joint limit theorems for the function $\zeta(s)$. It is proved that the limit measures for the joint distribution of $\zeta(s)$ and V_n coincide. We will use the paper [14]. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element $\underline{\zeta}(s, \underline{\omega})$ by

$$\underline{\zeta}(s, \underline{\omega}) = (\zeta(s, \omega_1), \dots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r.$$

Denote by $P_{\underline{\zeta}}$ the distribution of $\underline{\zeta}(s, \underline{\omega})$, i. e.,

$$P_{\underline{\zeta}}(A) = m_H\left\{\underline{\omega} \in \Omega^r : \underline{\zeta}(s, \underline{\omega}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

In [14], see proofs of Lemma 10 and Theorem 3, and the following assertion was obtained.

Lemma 5. *V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{\zeta}}$ is the set S^r.*

Now, we are ready to prove a limit theorem for $P_{T,H}$.

Theorem 4. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$. Then, $P_{T,H}$ converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. On the probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$, define the random variable $\xi_{T,H}$ uniformly distributed on $[T, T+H]$. Thus, $\xi_{T,H}$ has the density

$$p_{T,H}(\tau) = \begin{cases} 0 & \text{if } \tau < T, \\ \frac{1}{H} & \text{if } T \leqslant \tau \leqslant T+H, \\ 0 & \text{if } \tau > T+H. \end{cases}$$

Denote by X_n the $H^r(D)$-valued random element with the distribution V_n, and define the $H^r(D)$-valued random elements

$$X_{T,H,n} = X_{T,H,n}(s) = \underline{\zeta}_n(s + i\underline{\varphi}(\xi_{T,H}))$$

and

$$Y_{T,H} = Y_{T,H}(s) = \underline{\zeta}(s + i\underline{\varphi}(\xi_{T,H})).$$

Denote $\xrightarrow{\mathcal{D}}$ as the convergence in distribution. Then, by Lemma 4,

$$X_{T,H,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{16}$$

while Lemma 5 gives

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}. \tag{17}$$

The definitions of $X_{T,H,n}$ and $Y_{T,H}$ together with Lemma 2 show that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}(Y_{T,H}, X_{T,H,n}) \geqslant \varepsilon\right\} \\ &\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon H} \int_T^{T+H} \underline{\rho}\Big(\underline{\zeta}(s + i\underline{\varphi}(\tau)), \underline{\zeta}_n(s + i\underline{\varphi}(\tau))\Big) = 0. \end{aligned}$$

Thus, by (16) and (17), we have that all hypotheses of Theorem 4.2 of [20] are satisfied by the random elements X_n, $X_{T,H,n}$ and $Y_{T,H}$, and we obtain that

$$Y_{T,H} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\zeta}}.$$

The latter relation is equivalent to the assertion of the theorem. □

Corollary 1. *Suppose that $(\varphi_1, \dots, \varphi_r) \in U_r$ and $\widehat{H}(T) \leqslant H \leqslant \widehat{\widehat{H}}(T) \leqslant T$ and $F : H^r(D) \to H(D)$ is a continuous operator. Then,*

$$P_{T,H,F}(A) \overset{def}{=} \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : F\Big(\underline{\zeta}(s + i\underline{\varphi}(\tau))\Big) \in A\Big\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{\zeta}}F^{-1}$ as $T \to \infty$.

Proof. The corollary follows from Theorem 4, continuity of F and Theorem 5.1 of [20]. □

5. Proof of Universality

We recall the Mergelyan theorem [21] on the approximation of analytic functions by polynomials.

Lemma 6. *Suppose that $K \subset \mathbb{C}$ is a compact set with a connected complement, and $g(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |g(s) - p(s)| < \varepsilon.$$

In addition, we will use two equivalents of weak convergence of probability measures; see, for example, [20], Theorem 2.1.

Lemma 7. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then, the following statements are equivalent:*

1° *P_n converges weakly to P as $n \to \infty$;*

2° *For every open set G of $\mathbb{X}$,*

$$\liminf_{n \to \infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set A of P, i. e., $P(\partial A) = 0$, ∂A is the boundary of A,*

$$\lim_{n \to \infty} P_n(A) = P(A).$$

Proof of Theorem 2. The case of lower density. Lemma 6 implies the existence of polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{18}$$

By Lemma 5, $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$. Therefore, putting

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\},$$

we have

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{19}$$

Define one more set

$$\widehat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

The inequality (18) shows that $G_\varepsilon \subset \widehat{G}_\varepsilon$. Thus, $P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0$ by (19). Hence, in view of Theorem 4 and 2° of Lemma 7,

$$\liminf_{T \to \infty} P_{T,H}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0,$$

and the definitions of $P_{T,H}$ and $\widehat{G}_\varepsilon$, prove the assertion of the theorem.

For the case of density, the boundary of the set $\widehat{G}_\varepsilon$ lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K} \left| g_j(s) - f_j(s) \right| = \varepsilon \right\}.$$

Therefore, $\partial\widehat{G}_{\varepsilon_1} \cap \partial\widehat{G}_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. Hence, $P_{\underline{\zeta}}(\partial\widehat{G}_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$, i. e., the set $\widehat{G}_\varepsilon$ is a continuity set of $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Thus, Theorem 4 and 3° of Lemma 7 prove the theorem. □

Lemma 8. *Let F satisfy the hypotheses of Theorem 3. Then, the support of the measure $P_{\underline{\zeta}}F^{-1}$ is the whole of $H(D)$.*

Proof. First, we observe that the hypothesis $(F^{-1}\{p\}) \cap S^r \neq \varnothing$ for any polynomial $p = p(s)$ implies that $(F^{-1}G) \cap S^r \neq \varnothing$ for any open set $G \subset H(D)$. Let $\varepsilon > 0$ be such that

$$\sum_{l>l_0} 2^{-l} < \frac{\varepsilon}{2}, \tag{20}$$

and $\{K_l\} \subset D$ be a sequence of compact sets with connected complements from the definition of the metric ρ. We take an arbitrary element $g \in H(D)$ and its open neighborhood G. Since the sets K_l are embedded, by Lemma 6, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K_l} |g(s) - p(s)| < \frac{\varepsilon}{2}, \quad l = 1, \dots, l_0.$$

This and (20) show that $\rho(g, p) < \varepsilon$. Thus, if ε is sufficiently small, then the polynomial $p(s)$ lies in the set G and has a preimage $\widehat{g} \in S^r$ lying in the open set $F^{-1}G$. Thus, $(F^{-1}G) \cap S^r \neq \varnothing$. Since, by Lemma 5, S^r is the support of the measure $P_{\underline{\zeta}}$, the inequality $P_{\underline{\zeta}}(F^{-1}G) > 0$ holds. Hence,

$$P_{\underline{\zeta}}F^{-1}(G) = P_{\underline{\zeta}}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of $P_{\underline{\zeta}}F^{-1}$ is the whole of $H(D)$. □

Proof of Theorem 3. The case of lower density. By Lemma 6, we find a polynomial $p(s)$ satisfying the inequality

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}, \tag{21}$$

and take the set

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then, in view of Lemma 8, the set $\mathcal{G}_\varepsilon$ is an open neighborhood of an element $p(s)$ of the support of the measure $P_{\underline{\zeta}}F^{-1}$. Thus,

$$P_{\underline{\zeta}}F^{-1}(\mathcal{G}_\varepsilon) > 0. \tag{22}$$

This, Corollary 1 and 2° of Lemma 7 imply

$$\liminf_{T\to\infty} P_{T,H,F}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\zeta}}F^{-1}(\mathcal{G}_\varepsilon) > 0,$$

and it remains to use the definitions of $P_{T,H,F}$ and $\mathcal{G}_\varepsilon$.

For the case of density, define the set

$$\widehat{\mathcal{G}}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, $\widehat{\mathcal{G}}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. By (21), we have the inclusion $\mathcal{G}_\varepsilon \subset \widehat{\mathcal{G}}_\varepsilon$. Hence, by (22), $P_{\underline{\zeta}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0$. Therefore, by Corollary 1 and 3° of Lemma 7,

$$\lim_{T\to\infty} P_{T,H,F}(\widehat{\mathcal{G}}_\varepsilon) = P_{\underline{\zeta}}F^{-1}(\widehat{\mathcal{G}}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

6. Conclusions

Universality theorems for zeta-functions are not effective in the sense that, for example, in the case of the function $\zeta(s)$, we do not know any specific value τ with shift $\zeta(s+i\tau)$ approximating a given function. Clearly, it is impossible to find such a value τ; therefore, the easier problem of finding the interval $[T, T+H]$ containing τ with approximating shifts is considered. The first results in this direction were obtained in [22], where the interval $[T, 2T]$ with explicitly given T was indicated. Denote by $\underline{f} = (f_0, f_1, \dots, f_{n-1})$, $n \in \mathbb{N}$ the vector composed from the Taylor coefficients for f at the point s_0, and let, for $\varepsilon > 0$,

$$A(n, f, \varepsilon) = |\log|f_0|| + \left(\frac{\|\underline{f}\|}{\varepsilon}\right)^{n^2},$$

where

$$\|\underline{f}\| = \sum_{0\leqslant k<n} |f_k(s_0)|.$$

Then, in [22], it was proved that, if $\sigma_0 \in \left(\frac{1}{2}, 1\right)$, $s_0 = \sigma_0 + it_0$, the function $f(s)$ is continuous on the disc $|s - s_0| \leqslant r$, $r > 0$, $f(s_0) \neq 0$, and analytic for $|s - s_0| < r$, then, for any $\varepsilon \in (0, |f(s_0)|)$, there exist real numbers $\tau \in [T, 2T]$, and $\delta = \delta(\varepsilon, f, \tau) > 0$ defined by

$$\max_{|s-s_0|=r} |\zeta(s+i\tau)| \frac{\delta^n}{1-\delta} = \frac{\varepsilon}{3}\left(2 - e^{\delta r}\right)$$

such that

$$\max_{|s-s_0|\leqslant\delta r} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$

where $T = T(f, \varepsilon, \sigma_0)$ satisfies

$$T \geqslant \max\left(r, C(n, \sigma_0)\exp\exp\left\{5A\left(n, f, \frac{\varepsilon}{3}\right)^{8/(1-\sigma_0)+8/(\sigma_0-1/2)}\right\}\right),$$

$C(n, \sigma_0)$ is a positive effectively computable constant, and δ is effectively computable.

The effectivization problem of universality for $\zeta(s)$ consists of the description of the interval as short as possible containing τ satisfying

$$\sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon.$$

This leads to universality in short intervals. The first result in this direction was obtained in [15]. Suppose that $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H}\text{meas}\left\{\tau \in [T, T+H] : \sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

In [16], the latter theorem was extended for generalized shifts $\zeta(s + i\varphi(\tau))$. Suppose for $\tau \geqslant T_0$ that the function increases to $+\infty$ and has a monotonic derivative which satisfies estimates of type 3° of the class U_r. Then, the main result of [16] asserts that, if H_1 coincides with $\widehat{H}_1(T)$ in Theorem 2, $H_1 \leqslant H \leqslant T$, $K \in \mathcal{K}$ and $f(s) \in H_0(K)$; then,

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{ \tau \in [T, T+H] : \sup_{s\in K} |\zeta(s + i\varphi(\tau)) - f(s)| < \varepsilon \right\} > 0.$$

In the present paper (Theorem 2), a joint version of the above theorem is obtained.

In our opinion, researching universality theorems in short intervals for zeta-functions has a good future. It stimulates the investigations of mean squares in short intervals and leads to the effectivization of universality. Therefore, we are planning to continue this direction and obtain a discrete version of the results of this paper on the approximation by shifts $\zeta(s + i\varphi_j(k))$, $k \in [N, N+M]$ as well as by some compositions of generalized shifts.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* **1975**, *9*, 443–453. [CrossRef]
2. Karatsuba, A.A.; Voronin, S.M. *The Riemann Zeta-Function*; Walter de Gruiter: Berlin, Germany; New York, NY, USA, 1992.
3. Voronin, S.M. On the functional independence of Dirichlet *L*-functions. *Acta Arith.* **1975**, *27*, 493–503. (In Russian)
4. Bitar, K.M.; Khuri, N.N.; Ren, H.C. Path integrals and Voronin's theorem on the universality of the Riemann zeta-function. *Ann. Phys.* **1991**, *211*, 172–196. [CrossRef]
5. Matsumoto, K. A survey on the theory of universality for zeta and *L*-functions. In *Number Theory: Plowing and Starring through High Wave Forms, Proceedings of the 7th China-Japan Seminar (Fukuoka 2013), Fukuoka, Japan, 28 October–1 November 2013*; Series on Number Theory and Its Applications; Kaneko, M., Kanemitsu, S., Liu, J., Eds.; World Scientific Publishing Co.: Hackensack, NJ, USA; London, UK; Singapore; Bejing, China; Shanghai, China; Hong Kong, China; Taipei, China; Chennai, India, 2015; pp. 95–144.
6. Laurinčikas, A. *Limit Theorems for the Riemann Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
7. Steuding, J. *Value-Distribution of L-Functions*; Lecture Notes Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2007; Volume 1877.
8. Pańkowski, Ł. Joint universality for dependent *L*-functions. *Ramanujan J.* **2018**, *45*, 181–195. [CrossRef]
9. Laurinčikas, A.; Macaitienė, R.; Šiaučiūnas, D. A generalization of the Voronin theorem. *Lith. Math. J.* **2019**, *59*, 156–168. [CrossRef]
10. Laurinčikas, A. On joint universality of Dirichlet *L*-functions. *Chebyshevskiĭ Sb.* **2011**, *12*, 124–139.
11. Kačinskaitė, R.; Kazlauskaitė, B. Two results related to the universality of zeta-functions with periodic coefficients. *Results Math.* **2018**, *73*, 95. [CrossRef]
12. Kačinskaitė, R.; Matsumoto, K. The mixed joint universality for a class of zeta-functions. *Math. Nachr.* **2015**, *288*, 1900–1909. [CrossRef]
13. Kačinskaitė, R.; Matsumoto, K. Remarks on the mixed joint universality for a class of zeta-functions. *Bull. Aust. Math. Soc.* **2017**, *95*, 187–198. [CrossRef]
14. Laurinčikas, A. On joint universality of the Riemann zeta-function. *Math. Notes* **2021**, *110*, 210–220. [CrossRef]
15. Laurinčikas, A. Universality of the Riemann zeta-function in short intervals. *J. Number Theory* **2019**, *204*, 279–295. [CrossRef]
16. Laurinčikas, A. Approximation by generalized shifts of the Riemann zeta-function in short intervals. *Ramanujan J.* **2021**, *56*, 309–322. [CrossRef]
17. Laurinčikas, A. Universality of composite functions of periodic zeta functions. *Sbornik Math.* **2012**, *203*, 1631–1646. [CrossRef]
18. Šiaučiūnas, D.; Šimėnas, R.; Tekorė, M. Approximation of analytic functions by shifts of certain compositions. *Mathematics* **2021**, *9*, 2583. [CrossRef]
19. Ivič, A. *The Riemann Zeta-Function. Theory and Applications*; Dover Publications: Mineola, NY, USA, 2012.
20. Billingsley, P. *Convergence of Probability Measures*; Wiley: New York, NY, USA, 1968.
21. Mergelyan, S.N. Uniform approximations to functions of a complex variable. In *American Mathematical Society Translations No. 101*; American Mathematical Society: Providence, RI, USA, 1954.
22. Garunkštis, R.; Laurinčikas, A.; Matsumoto, K.; Steuding, J.; Steuding, R. Effective uniform approximation by the Riemann zeta-function. *Publ. Math.* **2010**, *54*, 209–219. [CrossRef]