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Spectrum Curves for Sturm–Liouville Problem with Integral Boundary Condition*

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Abstract. We consider Sturm–Liouville problem with one integral type nonlocal boundary condition depending on three parameters γ (multiplier in nonlocal condition), ξ_1, ξ_2 ($[\xi_1, \xi_2]$ is a domain of integration). The distribution of zeroes, poles, and constant eigenvalue points of Complex Characteristic Function is presented. We investigate how Spectrum Curves depend on the parameters of nonlocal boundary conditions. In this paper we describe the behaviour of Spectrum Curves and classify critical points of Complex-Real Characteristic function. Phase Trajectories of critical points in Phase Space of the parameters ξ_1, ξ_2 are investigated. We present the results of modelling and computational analysis and illustrate those results with graphs.

Keywords: Sturm–Liouville problem, characteristic function, spectrum curves, critical point, integral boundary condition.

AMS Subject Classification: 34B24, 34B09, 34B15.

1 Introduction

While applying mathematical modelling to various phenomena of physics [8,10], biology and ecology [14] there often arise problems with non-classical boundary conditions, which relate the values of unknown function on the boundary and inside of the given domain. Boundary conditions of such type are called *nonlocal boundary conditions* (NBC). Differential problems with nonlocal conditions are quite a widely investigated area of mathematics. Differential problems with nonlocal conditions are not yet completely and properly investigated, as it is a wide research area.

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The first paper, devoted to the second-order partial differential equations with nonlocal integral conditions goes back to Cannon [3]. The problems with NBCs were investigated for parabolic equations [11,12], for elliptic equations [2], and for hyperbolic equations [9].

Investigations of the spectrum structure is rather important for the analysis of the existence and uniqueness of the solutions of differential and discrete problems, for finding solutions of finite difference scheme (FDS) using iterative methods, for the stability analysis of difference schemes of nonstationary equations. Some necessary and sufficient existence and uniqueness conditions for solving stationary differential and discrete problems were obtained in [5, 6, 28]. Investigation of the spectrum and other similar problems for differential equations with nonlocal Bitsadze–Samarskii or multipoint boundary conditions are analyzed in papers [23, 24, 27], and integral conditions in [15]. The newest and most relevant articles are presented in a review [29]. Green's functions for the discrete the second order problem with NBC were investigated in [16, 22].

We investigate critical points of Characteristic Function (CF) and calculate the behavior real and complex parts of a spectrum. In [1] WKB (Wentzel–Kramers–Brillouin) analysis of $\mathcal{PT}$-symmetric Sturm–Liouville problems were considered. The novelty is that a $\mathcal{PT}$ eigenvalue problem on a infinite domain typically exhibits a sequence of critical points at which pairs of eigenvalues cease to be real and become complex conjugates of one another. WKB analysis is used to calculate the asymptotic behaviors of the real eigenvalues and the locations of the critical points. WKB method is a method for finding approximate solutions to linear differential equations with spatially varying coefficients. It is typically used for a semiclassical calculation in quantum mechanics.

B. Chanane in paper [4] use the regularized sampling method introduced recently to compute the eigenvalues of Sturm–Liouville problems with nonlocal conditions

$$-y'' + q(x)y = \lambda y,\ x \in [0,1], \quad \chi_0(y) = 0,\ \chi_1(y) = 0,$$

where $q \in L^1$ and χ_0 and χ_1 are continuous linear functionals defined by

$$\chi_0(y) = \int_0^1 [y(t)\mathrm{d}\psi_1(t) + y'(t)d\psi_2(t)], \quad \chi_1(y) = \int_0^1 [y(t)\mathrm{d}\phi_1(t) + y'(t)d\phi_2(t)]$$

and ψ_1, ψ_2, ϕ_1 and ϕ_2 are functions of bounded variations. Integration is in the sense of Riemann–Stieltjes. Two numerical examples have been presented to illustrate the effectiveness of the method and comparisons have been made with the exact eigenvalues.

In this paper we investigate special case ($q = 0$) of this problem with one integral NBC, and functionals defined by the formulas

$$y(0) = 0, \qquad y(1) = \gamma \int_{\xi_1}^{\xi_2} y(t)\ \mathrm{d}t,$$

with parameters $\gamma \in \mathbb{R}$ and $\boldsymbol{\xi} \in S_{\boldsymbol{\xi}} := \{(\xi_1, \xi_2) \in [0,1]^2, \xi_1 \leqslant \xi_2\}$. The cases $\boldsymbol{\xi} = (0,1)$ and $\boldsymbol{\xi} = (1/4, 3/4)$ were investigated in [7]. Such problem has been investigated in [13, 17] and some new results were obtained. We note that in [13] complex eigenvalues are investigated only for special cases of $\boldsymbol{\xi}$ with rational components. In our paper we are extending those investigations. Our main goal is to investigate the influence of parameters γ, ξ_1, ξ_2 for the spectrum of Sturm–Liouville problem and a behavior of the critical points of Complex-Real Characteristic Function (CF). CF method was described in [30] for problem with one Bitsadze–Samarskii type NBC. Critical points of the CF are important for investigation of multiple eigenvalues. These points are connected with bifurcations points in Phase Space $S_{\boldsymbol{\xi}}$ of parameter $\boldsymbol{\xi} = (\xi_1, \xi_2)$. The limit cases ($\boldsymbol{\xi} = (0, \xi)$ and $\boldsymbol{\xi} = (\xi, 1)$, $\xi \in [0, 1]$), were investigated in [20, 25]. The special case $\boldsymbol{\xi} = (\xi, 1-\xi)$, $\xi \in [0, 1/2]$), is presented in [26]. Real CF and real critical points were studied for problems with one two-points NBC [19]. Negative critical points for problems with two-point or integral NBC's with one parameter ξ were investigated in paper [21], too.

2 Problem formulation

Let us analyze the Sturm–Liouville problem (SLP)

$$-u'' = \lambda u, \quad t \in (0, 1), \tag{2.1}$$

$\lambda \in \mathbb{C}_\lambda := \mathbb{C}$, with one classical boundary condition

$$u(0) = 0, \tag{2.2}$$

and another integral type NBC:

$$u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\,\mathrm{d}t \tag{2.3}$$

with parameters $\gamma \in \mathbb{R}$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in S_{\boldsymbol{\xi}}$.

For the case $\gamma = 0$ (classical one) eigenvalues are well known:

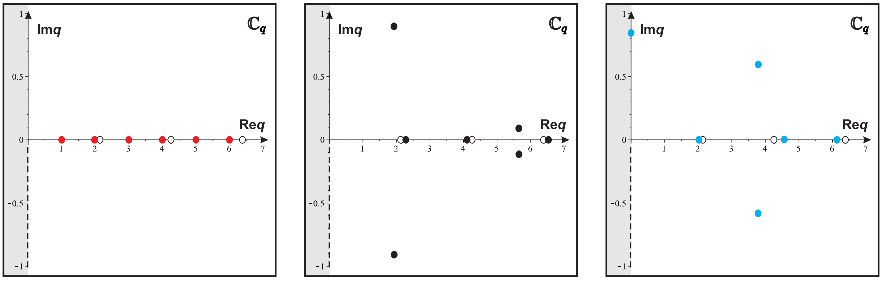
$$\lambda_k = (k\pi)^2, \quad v_k(t) = \sin(k\pi t), \quad k \in \mathbb{N},$$

where $\mathbb{N} := \{1, 2, 3, \ldots\}$. Note that the same classical problem is obtained in the limit case $\xi_1 = \xi_2$.

We use notation $\pi\mathbb{N} := \{\pi k\colon k \in \mathbb{N}\}$, $\mathbb{N}_o$ for odd, $\mathbb{N}_e$ for even, and $\mathbb{Q}$ for rational numbers. Notation $\gcd(n; m)$ defines the *greatest common divisor* of two integers n and m.

If $\lambda = 0$, then all the functions $u(t) = Cu_0(t)$, where $u_0(t) := t$, satisfy the equation (2.1)–(2.2). By substituting this solution into NBC (2.3) we derive, that the nontrivial solution ($C \neq 0$) exists if $1 = \gamma(\xi_2^2 - \xi_1^2)/2$. So, eigenvalue $\lambda = 0$ exists if and only if $\gamma = 2/(\xi_2^2 - \xi_1^2)$.

In the case $\lambda \neq 0$ we define entire function $u_q(t) := \sin(qt)/q$. Functions $u(t) = Cu_q(t)$ satisfy equation (2.1) with $\lambda = q^2$, $q \neq 0$, and boundary condition (2.2). If $q = x + \imath y \in \mathbb{C}_q := \{q \in \mathbb{C} : x = 0, y \geqslant 0 \text{ or } x > 0\}$, then a map $\lambda = q^2$ is a bijection between $\mathbb{C}_q$ and $\mathbb{C}_\lambda$. Note, that $q = 0$ corresponds to $\lambda = 0$ in this bijection and $u_0 = \lim\limits_{q\to 0} u_q(x)$.



(a) • – zero ($\gamma = 0$); ∘ – pole (b) • $\gamma = -17$ (c) • $\gamma = +17$

Figure 1. A part of the spectrum for SLP (2.1)–(2.3), $\boldsymbol{\xi} = (0.32, 0.61)$.

Remark 1. We use a scaled variable $\boldsymbol{q} := q/\pi$ for plotting the graphs of functions defined on $\mathbb{C}_q$. The spectrum coincides with $\mathbb{N}$ for such variable in the classical case $\gamma = 0$, i.e. $\boldsymbol{q}_k = k$, $k \in \mathbb{N}$. We use the same notation for corresponding points and functions in both domains $\mathbb{C}_q$ and $\mathbb{C}_{\boldsymbol{q}}$.

A nontrivial solution of the problem (2.1)–(2.3) exists if q is a root of the equation

$$u_q(1) = \gamma \int_{\xi_1}^{\xi_2} u_q(t)dt. \tag{2.4}$$

For NBC (2.3) we introduce two entire functions

$$Z(z) := \frac{\sin z}{z}; \quad P_{\boldsymbol{\xi}}(z) := 2\frac{\sin(z(\xi_1 + \xi_2)/2)}{z} \cdot \frac{\sin(z(\xi_2 - \xi_1)/2)}{z}.$$

Zeroes of these functions are important for description of the spectrum. Zeroes of the function $Z(q)$, $q \in \mathbb{C}_q$, coincide with eigenvalue points in the classical case $\gamma = 0$. We can rewrite equality (2.4) in the form:

$$Z(q) = \gamma P_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}_q. \tag{2.5}$$

In Figure 1, one can see the roots (not all) of this equation for $\gamma = -17, 0, +17$ in the case $\boldsymbol{\xi} = (0.32, 0.61)$. There exist complex roots for $\gamma = -17, +17$.

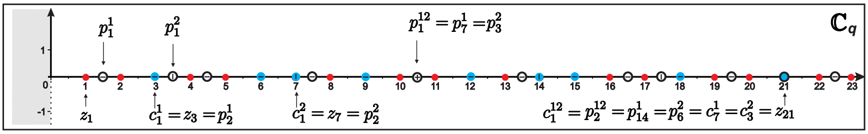
We define the *constant eigenvalue* as the eigenvalue that does not depend on parameter γ. For any constant eigenvalue $\lambda = q^2$ there exists the *constant eigenvalue point* (CE point) $q \in \mathbb{C}_q$ [30]. For NBC (2.3) we can find CE points as solutions of the following system

$$Z(q) = 0, \quad P_{\boldsymbol{\xi}}(q) = 0,$$

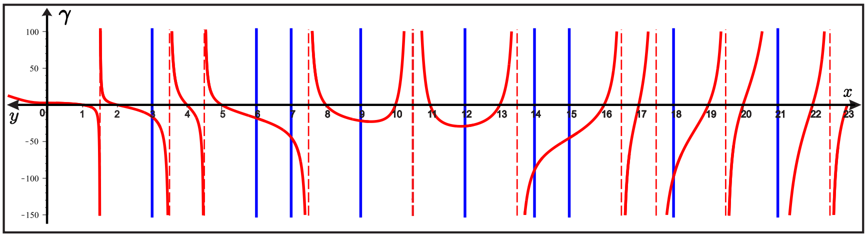
i.e. CE point $c \in \pi\mathbb{N}$ and $P_{\boldsymbol{\xi}}(c) = 0$. The notation $\mathcal{C}$ is used for the set of all CE points.

If $q \notin \pi\mathbb{N}$, i.e. $Z(q) \neq 0$, and q satisfies equation $P_{\boldsymbol{\xi}}(q) = 0$, then the equality (2.5) is not valid for all γ and such point q is *a pole point.* Notation of the pole point is connected with meromorphic function

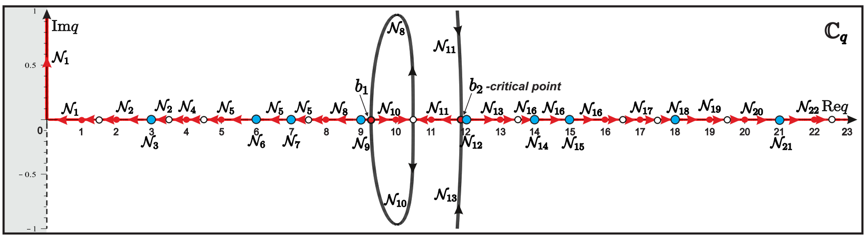
$$\gamma_c(z) = \frac{Z(z)}{P_{\boldsymbol{\xi}}(z)}, \quad z \in \mathbb{C}. \tag{2.6}$$



(a) ● – zero; ○ – pole; ● – CE point; ● – critical point.



(b) Real CF



(c) projection Complex-Real CF onto $\mathbb{C}_q$.

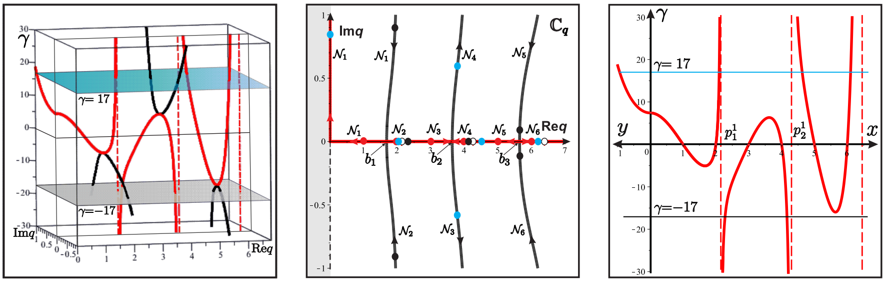
Figure 2. Zeroes, poles, CE points for SLP (2.1)–(2.3), $\boldsymbol{\xi} = (8/21, 20/21)$.

This function is obtained by expressing γ from equation (2.5). If the denominator has a zero at $z = p$ and the numerator does not, then the value of the function will be infinite and we have a pole. If both parts have a zero at $z = p$, then one must compare the multiplicities of these zeroes. For our problem all zeroes $z_k = \pi k, k \in \mathbb{N}$ of function $Z(z)$ are simple and positive if $z \in \mathbb{C}_q$. It follows that function $P_{\boldsymbol{\xi}}(z) = 2P^1_{\boldsymbol{\xi}}(z)P^2_{\boldsymbol{\xi}}(z)$, where

$$P^1_{\boldsymbol{\xi}}(z) := \sin(z(\xi_1 + \xi_2)/2)/z, \quad P^2_{\boldsymbol{\xi}}(z) := \sin(z(\xi_2 - \xi_1)/2)/z. \tag{2.7}$$

Zeroes of the functions $P^1_{\boldsymbol{\xi}}$, $P^2_{\boldsymbol{\xi}}$ in the domain $\mathbb{C}_q$ are simple and positive, too. So, zeroes of function $P_{\boldsymbol{\xi}}$ can be simple or the second order. The restriction of meromorphic function γ_c on $\mathbb{C}_q$ we call *Complex Characteristic Function* (Complex CF) [30]. We define the value of this function at point p, $P_{\boldsymbol{\xi}}(p) = 0$ as a limit $\gamma_c(p) := \lim_{q\to p} Z(q)/P_{\boldsymbol{\xi}}(q)$. This limit is finite $\gamma_c(p) = \frac{Z'(p)}{P'_{\boldsymbol{\xi}}(p)} \neq 0$ (removable singularity) if $p \in \pi\mathbb{N}$ is the first order zero of function $P_{\boldsymbol{\xi}}$ and limit is infinite (function γ_c has the first order pole) if $p \in \pi\mathbb{N}$ is the second order zero of function $P_{\boldsymbol{\xi}}$ or $p \notin \pi\mathbb{N}$. For example, in Figure 2(a) we see such points in the case $\boldsymbol{\xi} = (8/21, 20/21)$.

All *nonconstant eigenvalues* (which depend on the parameter γ) are γ-points of *Complex-Real Characteristic Function* (Complex-Real CF) [30]. In Figure 3(a) one can see a Complex-Real CF graph in the case $\boldsymbol{\xi} = (0.32, 0.61)$. Complex-Real CF $\gamma(q)$ is the restriction of function $\gamma_c(q)$ on a set $\mathcal{N}^\gamma := \{q \in \mathbb{C}_q\colon \mathrm{Im}\gamma_c(q) = 0\}$. *Real CF* $\gamma(q)$ is defined on the domain $\{q \in \mathbb{C}_q\colon \lambda = q^2 \in \mathbb{R}\}$

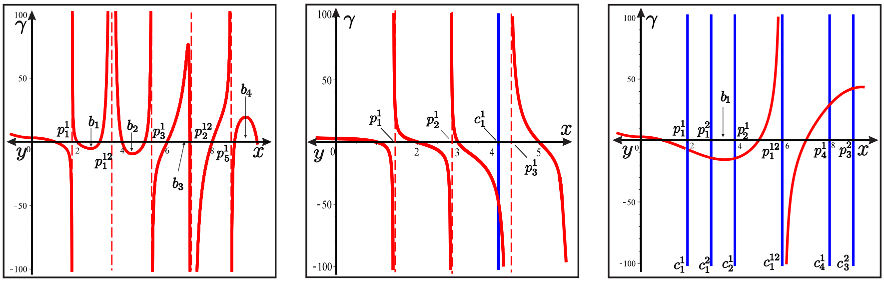


(a) Complex-Real CF (b) Spectrum Domain (c) Real CF

Figure 3. CF for $\boldsymbol{\xi} = (0.32, 0.61)$ and its projections.

and describes only real eigenvalues ($\lambda = q^2$, where $q = x$, $x \geq 0$ and $q = y\imath$, $y \geq 0$). We plot the graph of Real CF for positive eigenvalue points $x > 0$ in the right half plane and $y > 0$ in the left half plane. The γ-axis corresponds to point $q = 0$. One can see the Real CF graph in Figure 2(b) for $\boldsymbol{\xi} = (8/21, 20/21)$ and in Figure 3(c) for $\boldsymbol{\xi} = (0.32, 0.61)$. In the case $\boldsymbol{\xi} = (8/21, 20/21)$ the vertical lines are added at the CE points.

Spectrum Domain is the set $\mathcal{N} = \mathcal{N}^{\gamma} \cup \mathcal{C}$. Function γ_c has real values on $\mathcal{N}$ except pole points. Example of the Spectrum Domain one can see in Figure 3(b). We also add the eigenvalue points ($\gamma = -17, 0, +17$) from Figure 1 and pole points ($\gamma = \infty$). Eigenvalue points for $\gamma \in \mathbb{R}$ are in this domain only. Spectrum Domain is symmetric with respect the real axis for $\operatorname{Re} q > 0$. Complex-Real CF $\gamma(q)$ describes the value of the parameter γ at point $q \in \mathcal{N}^{\gamma}$ (see Figure 3(a)) such that there exists the eigenvalue $\lambda = q^2$. For each $\gamma_0 \in \mathbb{R}$ a set $\mathcal{N}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all eigenvalue points for nonconstant eigenvalues. So, Spectrum Domain $\mathcal{N} = \cup_{\gamma \in \mathbb{R}} \mathcal{N}(\gamma) \cup \mathcal{C}$. For example, $\mathcal{N}(0) \cup \mathcal{C}$ corresponds to a spectrum for the classical case. If $q \in \mathcal{N}^{\gamma}$ and $\gamma_c'(q) \neq 0$ (q is not a critical point of CF) then $\mathcal{N}(\gamma)$ is continuous parametric curve $\mathcal{N} \colon \mathbb{R} \to \mathbb{C}_q$ and we can add arrow on this curve. The arrows indicate the direction in which γ is increasing (from $-\infty$ to $+\infty$). So, eigenvalues point is moving along this curve when parameter γ is increasing. If $\gamma = 0$ then the eigenvalue points are $q = z_k = \pi k$, $k \in \pi\mathbb{N}$. So, we can numerate the part of $\mathcal{N}(\gamma)$ for this point by the classical case $\mathcal{N}_k(0) = z_k$, $k \in \mathbb{N}$. For every CE point $c_j = \pi j$ we define $\mathcal{N}_j = \{c_j\}$, i.e. every such $\mathcal{N}_j$ has one point only (see Figure 2(c), Figure 3(b)). We call every $\mathcal{N}_k$, $k \in \mathbb{N}$, as a *Spectrum Curve*. Spectrum Domain $\mathcal{N}$ is a countable union of Spectrum Curves $\mathcal{N}_k$. Different Spectrum Curves may have a common point. For example, CE point may be on other $\mathcal{N}_k$ or few Spectrum Curves are intersect at the critical point b. If p_1 and p_2 are two neighbouring poles of the Complex–Real CF and $\gamma'(x) \neq 0$, $x \in (p_1, p_2)$, then exist $z_k \in (p_1, p_2)$ and $\mathcal{N}_k = (p_1, p_2)$. If there exists $b \in (p_1, p_2)$ such that $\gamma'(b) = 0$, then b is a critical point of real CF, such that eigenvalue point leaves or enters the real axis symmetrically [30]. In some sense, two eigenspectrum points "intersect" at the critical point $q = b$ and after that turn right (we observe the "*right hand*" rule). For the $\gamma \to \pm\infty$ Spectrum Curve $\mathcal{N}_k(\gamma)$ approaches a pole point or a point $x_k \pm \imath\infty$. Pole point does not belong to $\mathcal{N}$, and we can interpret pole as a point where $\mathcal{N}_k$ can change the



(a) $\boldsymbol{\xi} = (\sqrt{2}/5, 3\sqrt{2}/5)$, $p_1^{12} = p_2^1 = p_1^2$

(b) $\boldsymbol{\xi} = (\frac{2\sqrt{2}-1}{4}, \frac{2\sqrt{2}+1}{4})$, $p_1^2 = c_1^1$

(c) $\boldsymbol{\xi} = (1/6, 5/6)$, $c_3^1 = c_2^2 = c_1^{12}$

Figure 4. Real CF and CE for different $\boldsymbol{\xi}$ values.

index k to k_1. For the investigation of the Spectrum Curves we must know zeroes, poles and CE point of CF.

3 Zeroes, poles and constant eigenvalues points of the Complex Characteristic Function

We use notation: $\xi = \xi_1/\xi_2$, $\xi_+ = \xi_1 + \xi_2$, $\xi_- = \xi_2 - \xi_1$.

All zeroes of the functions Z, $P_{\boldsymbol{\xi}}^1$, $P_{\boldsymbol{\xi}}^2$ (see (2.7)) in $\mathbb{C}_q$ are simple (of the first order), real and positive:

$$z_k = k\pi,\ k \in \mathbb{N}, \quad p_k^1 = \tfrac{2}{\xi_+}k\pi,\ k \in \mathbb{N}, \quad p_k^2 = \tfrac{2}{\xi_-}k\pi,\ k \in \mathbb{N}. \tag{3.1}$$

We denote the corresponding sets of points as $\overline{\mathcal{Z}}$, $\overline{\mathcal{Z}}_{\boldsymbol{\xi}}^1$, $\overline{\mathcal{Z}}_{\boldsymbol{\xi}}^2$. Then a set $\mathcal{Z}_{\boldsymbol{\xi}} = \mathcal{Z}_{\boldsymbol{\xi}}^1 + \mathcal{Z}_{\boldsymbol{\xi}}^2 + \mathcal{Z}_{\boldsymbol{\xi}}^{12}$ describes all zeroes of the function $P_{\boldsymbol{\xi}}$, where $\mathcal{Z}_{\boldsymbol{\xi}}^1 := \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^1 \setminus \mathcal{Z}_{\boldsymbol{\xi}}^{12}$ and $\mathcal{Z}_{\boldsymbol{\xi}}^2 := \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^2 \setminus \mathcal{Z}_{\boldsymbol{\xi}}^{12}$ are two families of the first order zeroes, $\mathcal{Z}_{\boldsymbol{\xi}}^{12} := \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^1 \cap \overline{\mathcal{Z}}_{\boldsymbol{\xi}}^2$ is family of the second order zeroes. If $\xi \notin \mathbb{Q}$, then the second order zeroes do not exist, i.e. $\mathcal{Z}_{\boldsymbol{\xi}}^{12} = \varnothing$. If $\xi = m/n \in \mathbb{Q}$, where $m,\ n \in \mathbb{N}$, then a set $\mathcal{Z}_{\boldsymbol{\xi}}^{12}$ describes the second order zeroes:

$$p_k^{12} = 2n/(\xi_2 d_p)k\pi = 2m/(\xi_1 d_p)k\pi,\ k \in \mathbb{N}, \quad d_p = \gcd(n-m; n+m). \tag{3.2}$$

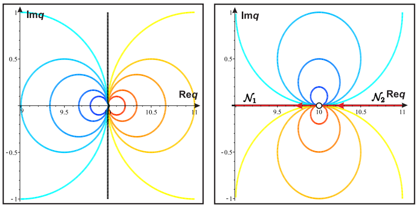
If $\xi \in \mathbb{Q}$ then $\xi_1, \xi_2 \in \mathbb{Q}$ or $\xi_1, \xi_2 \notin \mathbb{Q}$. If $\xi \notin \mathbb{Q}$ then $\xi_1 \notin \mathbb{Q}$ or $\xi_2 \notin \mathbb{Q}$, or both $\xi_1, \xi_2 \notin \mathbb{Q}$.

For (real) CF we consider the following sets: a set of poles $\mathcal{P}_{\boldsymbol{\xi}} := \mathcal{P}_{\boldsymbol{\xi}}^1 + \mathcal{P}_{\boldsymbol{\xi}}^2 + \mathcal{P}_{\boldsymbol{\xi}}^{12}$, where $\mathcal{P}_{\boldsymbol{\xi}}^1 := \mathcal{Z}_{\boldsymbol{\xi}}^1 \setminus \overline{\mathcal{Z}}$ and $\mathcal{P}_{\boldsymbol{\xi}}^2 := \mathcal{Z}_{\boldsymbol{\xi}}^2 \setminus \overline{\mathcal{Z}}$ are two families of the poles of the first order, a set of the second order poles $\mathcal{P}_{\boldsymbol{\xi}}^{12} := \mathcal{Z}_{\boldsymbol{\xi}}^{12} \setminus \overline{\mathcal{Z}}$; a set of the CE points $\mathcal{C}_{\boldsymbol{\xi}} := \mathcal{C}_{\boldsymbol{\xi}}^1 + \mathcal{C}_{\boldsymbol{\xi}}^2 + \mathcal{C}_{\boldsymbol{\xi}}^{12}$, where $\mathcal{C}_{\boldsymbol{\xi}}^1 := \mathcal{Z}_{\boldsymbol{\xi}}^1 \cap \overline{\mathcal{Z}}$ and $\mathcal{C}_{\boldsymbol{\xi}}^2 := \mathcal{Z}_{\boldsymbol{\xi}}^2 \cap \overline{\mathcal{Z}}$ are sets of the points with removable singularity, $\mathcal{C}_{\boldsymbol{\xi}}^{12} := \mathcal{Z}_{\boldsymbol{\xi}}^{12} \cap \overline{\mathcal{Z}}$ is the set of the points with the first order pole, too; a set of zeroes $\mathcal{Z}_{\boldsymbol{\xi}} := \overline{\mathcal{Z}} \setminus \mathcal{C}_{\boldsymbol{\xi}}$.

Remark 2. The sets $\mathcal{P}_{\boldsymbol{\xi}}^1$, $\mathcal{P}_{\boldsymbol{\xi}}^2$, $\mathcal{P}_{\boldsymbol{\xi}}^{12}$, $\mathcal{C}_{\boldsymbol{\xi}}^1$, $\mathcal{C}_{\boldsymbol{\xi}}^2$, $\mathcal{C}_{\boldsymbol{\xi}}^{12}$, $\mathcal{Z}_{\boldsymbol{\xi}}$ points have form $q_k = \alpha k$, $k \in \mathbb{N}$, $\alpha \geqslant 1$, or can be empty. So, nonempty sets are described by the first point ($k = 1$). Since $\pi < p_1^1 < p_1^2 \le p_1^{12}$, the set $\mathcal{Z}_{\boldsymbol{\xi}} \neq \varnothing$. We note, that $2\pi < p_1^2$, too.

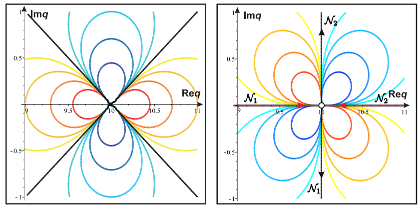
Table 1. Zeroes, poles and CE points of CF (special cases), $m, l, m_1, m_2, n_1, n_2 \in \mathbb{N}$. "+" means that the set above is nonempty, "−" means that the set above is empty.

Case	Example	Poles			CE points				Remarks
subcase	$\boldsymbol{\xi} = (\xi_1, \xi_2)$	$\mathcal{P}^1_{\boldsymbol{\xi}}$	$\mathcal{P}^2_{\boldsymbol{\xi}}$	$\mathcal{P}^{12}_{\boldsymbol{\xi}}$	$\mathcal{C}^1_{\boldsymbol{\xi}}$	$\mathcal{C}^2_{\boldsymbol{\xi}}$	$\mathcal{C}^{12}_{\boldsymbol{\xi}}$	$\mathcal{Z}_{\boldsymbol{\xi}}$	
$\xi \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$, $l > 1$:									
$\xi \neq \frac{l-1}{l+1}$	$\left(\frac{\sqrt{2}}{20}, \frac{\sqrt{2}}{5}\right)$	+	+	+	−	−	−	+	$p_1^2 < p_1^{12}$
$\xi = \frac{l-1}{l+1}$	$\left(\frac{\sqrt{2}}{5}, \frac{3\sqrt{2}}{5}\right)$	+	−	+	−	−	−	+	$p_1^2 = p_1^{12}$
$\xi \notin \mathbb{Q}$, $l > 1$, $m > 2$:									
$\xi_+, \xi_- \notin \mathbb{Q}$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$; $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$	+	+	−	−	−	−	+	$p_1^1 < p_1^2$
$\xi_+ \in \mathbb{Q}$, $\xi_+ \neq \frac{2}{l}$, $\xi_- \notin \mathbb{Q}$	$\left(\frac{3-\sqrt{2}}{8}, \frac{3+\sqrt{2}}{8}\right)$	+	+	−	+	−	−	+	$p_1^1 < c_1^1$
$\xi_- \in \mathbb{Q}$, $\xi_- \neq \frac{2}{m}$, $\xi_+ \notin \mathbb{Q}$	$\left(\frac{5\sqrt{2}-7}{24}, \frac{5\sqrt{2}+7}{24}\right)$	+	+	−	−	+	−	+	$p_1^2 < c_1^2$
$\xi_+ = \frac{2}{l}$, $\xi_- \notin \mathbb{Q}$	$\left(\frac{2-\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}\right)$	−	+	−	+	−	−	+	$p_1^1 = c_1^1$
$\xi_- = \frac{2}{m}$, $\xi_+ \notin \mathbb{Q}$	$\left(\frac{2\sqrt{2}-1}{4}, \frac{2\sqrt{2}+1}{4}\right)$	+	−	−	−	+	−	+	$p_1^2 = c_1^2$
$\xi_1 = m_1/n_1, \xi_2 = m_2/n_2 \in \mathbb{Q}$ (a)–(d) $p_1^k < c_1^k, k = 1,2,12$, $p_1^k < p_1^{12}, k = 1,2$, (m)–(q) $n = n_1 = n_2 = m_1 + m_2$:									
(a)	$\left(\frac{8}{21}, \frac{20}{21}\right)$	+	+	+	+	+	+	+	$c_1^1, c_1^2 < c_1^{12}$
(b)	$\left(\frac{10}{27}, \frac{25}{27}\right)$	+	+	+	−	+	+	+	$c_1^2 < c_1^1 = c_1^{12}$
(c)	$\left(\frac{2}{25}, \frac{18}{25}\right)$	+	+	+	+	−	+	+	$c_1^1 < c_1^2 = c_1^{12}$
(d)	$\left(\frac{6}{17}, \frac{15}{17}\right)$	+	+	+	−	−	+	+	$c_1^1 = c_1^2 = c_1^{12}$
(e)	$\left(\frac{4}{11}, \frac{10}{11}\right)$	+	+	−	−	−	+	+	$p_1^{12} = c_1^{12}$, $p_1^1 < c_1^1 = c_1^2$
(f)	$\left(\frac{4}{9}, \frac{8}{9}\right)$	+	−	+	+	−	+	+	$p_1^2 = p_1^{12}$, $p_1^1 < c_1^1 < c_1^{12}$
(g)	$\left(\frac{2}{7}, \frac{6}{7}\right)$	+	−	+	−	−	+	+	$p_1^2 = p_1^{12}$, $p_1^1 < c_1^1 = c_1^{12}$
(h)	$\left(\frac{5}{12}, \frac{11}{12}\right)$	+	−	−	+	+	+	+	$p_1^2 = c_1^2 < c_1^{12}$, $p_1^1 < c_1^1$
(i)	$\left(\frac{1}{2}, \frac{5}{6}\right)$	+	−	−	+	−	+	+	$p_1^2 = c_1^2 = c_1^{12}$, $p_1^1 < c_1^1 < c_1^{12}$
(k)	$\left(\frac{1}{5}, \frac{13}{15}\right)$	+	−	−	−	+	+	+	$p_1^2 = c_1^2 < c_1^{12}$, $c_1^1 = c_1^{12}$
(l)	$\left(\frac{1}{5}, \frac{3}{5}\right)$	+	−	−	−	−	+	+	$p_1^2 = c_1^{12} = c_1^1$
(m)	$\left(\frac{1}{10}, \frac{9}{10}\right)$	−	+	−	+	+	+	+	$p_1^1 = c_1^1$,
(n)	$\left(\frac{1}{7}, \frac{6}{7}\right)$	−	+	−	+	−	+	+	$p_1^1 = c_1^1$, $p_1^2 < c_1^2 = c_1^{12}$
(p)	$\left(\frac{1}{6}, \frac{5}{6}\right)$	−	−	−	+	+	+	+	$p_1^1 = c_1^1$, $c_1^1 < c_1^2 = p_1^2$
(q)	$\left(\frac{1}{4}, \frac{3}{4}\right)$	−	−	−	+	−	+	+	$p_1^1 = c_1^1$, $p_1^2 = c_1^{12}$



(a) real part (b) image part

Figure 5. The first order pole.



(a) real part (b) image part

Figure 6. The second order pole.

The case $\xi = m/n \in \mathbb{Q}$, $\xi_1, \xi_2 \notin \mathbb{Q}$. In this case ξ_+, $\xi_- \notin \mathbb{Q}$, i.e. $\mathcal{C}_{\boldsymbol{\xi}} = \varnothing$. So, CF has two families of the first order poles in $\mathcal{P}^1_{\boldsymbol{\xi}}$ and $\mathcal{P}^2_{\boldsymbol{\xi}}$, respectively, and the second order poles in $\mathcal{P}^{12}_{\boldsymbol{\xi}}$ (see formulae (3.1)–(3.2) for calculation p^1_k, p^2_k, p^{12}_k). Note, that $\mathcal{P}^2_{\boldsymbol{\xi}} = \varnothing$ for $\xi = (l-1)/(l+1)$, $1 < l \in \mathbb{N}$. In this special case $p^2_1 = p^{12}_1$ (see Figure 4(a) and Table 1).

The case $\xi \notin \mathbb{Q}$. In this case at least one number ξ_+ or ξ_- is irrational (and at least one number ξ_1 or ξ_2 is irrational). If $\xi_+ \notin \mathbb{Q}$ and $\xi_- \notin \mathbb{Q}$ then CF has two families of the first order poles $\mathcal{P}^1_{\boldsymbol{\xi}}$ and $\mathcal{P}^2_{\boldsymbol{\xi}}$, respectively, and $\mathcal{P}^{12}_{\boldsymbol{\xi}} = \mathcal{C}_{\boldsymbol{\xi}} = \varnothing$. If $\xi_+ = m_+/n_+ \in \mathbb{Q}$ then $\mathcal{C}^1_{\boldsymbol{\xi}} \neq \varnothing$:

$$c^1_k = p^1_{m_+/d_1 k} = z_{2n_+/d_1 k} = 2n_+/d_1 k\pi, \quad k \in \mathbb{N}, \quad d_1 = \gcd(2n_+; m_+). \tag{3.3}$$

If $\xi_- = m_-/n_- \in \mathbb{Q}$ then $\mathcal{C}^2_{\boldsymbol{\xi}} \neq \varnothing$:

$$c^2_k = p^2_{m_-/d_2 k} = z_{2n_-/d_2 k} = 2n_-/d_2 k\pi, \quad k \in \mathbb{N}, \quad d_2 = \gcd(2n_-; m_-). \tag{3.4}$$

CF has removable singularities in these CE points (there is one family of such points) and the first order poles in the set $\mathcal{P}^1_{\boldsymbol{\xi}} + \mathcal{P}^2_{\boldsymbol{\xi}}$. The set $\mathcal{P}^1_{\boldsymbol{\xi}} = \varnothing$ for $\xi_+ = 2/l$, $1 < l \in \mathbb{N}$, because $p^1_1 = c^1_1$. The set $\mathcal{P}^2_{\boldsymbol{\xi}} = \varnothing$ for $\xi_- = 2/m$, $2 < m \in \mathbb{N}$, because $p^2_1 = c^2_1$ (see Figure 4(b) and Table 1).

The case $\xi_1 = m_1/n_1, \xi_2 = m_2/n_2 \in \mathbb{Q}$. In this case $n_\pm = n_1 n_2$, $m_+ = m_2 n_1 + m_1 n_2$, $m_- = m_2 n_1 - m_1 n_2$, $m = m_1 n_2$, $n = m_2 n_1$. So, we can use expressions (3.1)–(3.4) and get formulae for poles and CE points ($k \in \mathbb{N}$)

$$p^1_k = \frac{2n_1 n_2 k\pi}{m_+}, \qquad p^2_k = \frac{2n_1 n_2 k\pi}{m_-}, \qquad p^{12}_k = \frac{2n_1 n_2 k\pi}{\gcd(m_-; m_+)},$$
$$c^1_k = \frac{2n_1 n_2 k\pi}{\gcd(2n_\pm; m_+)}, \qquad c^2_k = \frac{2n_1 n_2 k\pi}{\gcd(2n_\pm; m_-)}, \qquad c^{12}_k = \frac{2n_1 n_2 k\pi}{\gcd(2n_\pm; m_+; m_-)}$$

of the sets $\mathcal{P}^1_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^2_{\boldsymbol{\xi}} + \mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{P}^{12}_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^1_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^2_{\boldsymbol{\xi}} + \mathcal{C}^{12}_{\boldsymbol{\xi}}$, $\mathcal{C}^{12}_{\boldsymbol{\xi}}$, respectively. The set $\mathcal{C}^{12}_{\boldsymbol{\xi}} \neq \varnothing$ for all $\boldsymbol{\xi}$. $\mathcal{P}^1_{\boldsymbol{\xi}} = \varnothing$ for $p^1_1 = c^1_1$; $\mathcal{P}^2_{\boldsymbol{\xi}} = \varnothing$ for $p^2_1 = p^{12}_1$ or $p^2_1 = c^2_1$ or $p^2_1 = c^{12}_1$; $\mathcal{P}^{12}_{\boldsymbol{\xi}} = \varnothing$ for $p^{12}_1 = c^{12}_1$ or $p^1_1 = c^1_1$ or $p^2_1 = c^2_1$; $\mathcal{C}^1_{\boldsymbol{\xi}} = \varnothing$ for $c^1_1 = c^{12}_1$; $\mathcal{C}^2_{\boldsymbol{\xi}} = \varnothing$ for $c^2_1 = c^{12}_1$; $\mathcal{C}^{12}_{\boldsymbol{\xi}} = \varnothing$ for $c^2_1 = c^1_1$ (see Table 1).

Some information on the first or the second order poles can be presented as contour lines of the functions $(z-10)^{-1}$ and $(z-10)^{-2}$. Real CF in neighbourhood of the first order pole are shown in Figure 5 and Figure 6. In this case

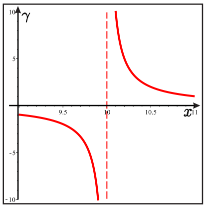
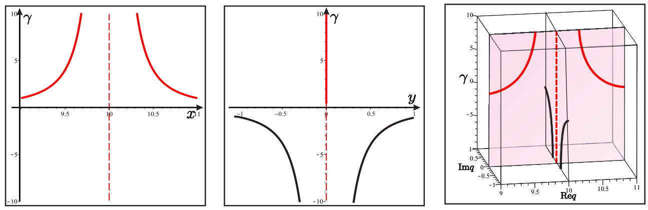


Figure 7. Real CF. A neighbourhood of the first order pole.



(a) Real CF (b) complex part (c) 3D view

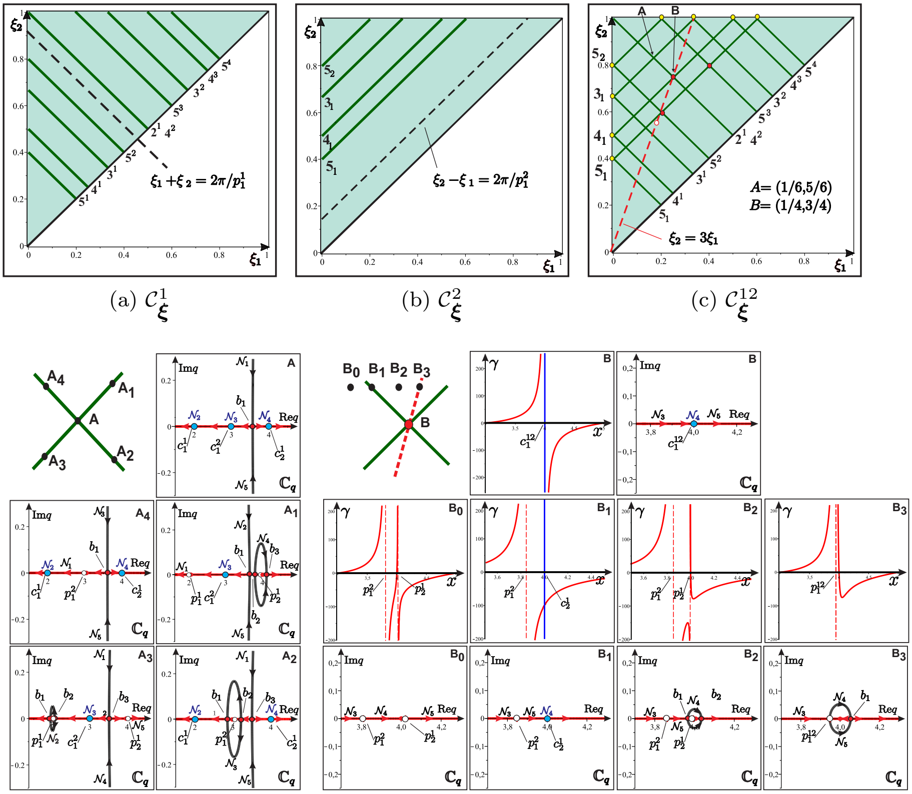
Figure 8. A neighbourhood of the second order pole.

there are two Spectrum Curves $\mathcal{N}_1$ and $\mathcal{N}_2$ on the real axis (see Figure 5(b)). In the neighbourhood of the first order pole there exist only real eigenvalues (see Figure 7). The Spectrum Curves $\mathcal{N}_1$, $\mathcal{N}_2$ and the Complex–Real CF in neighbourhood of the second order poles are presented in Figure 6(b) and Figure 8.

We have two families $\mathcal{C}^1_{\boldsymbol{\xi}}$ and $\mathcal{C}^2_{\boldsymbol{\xi}}$ of CE points (these eigenvalues do not exist if $\xi_1, \xi_2 \notin \mathbb{Q}$, but $\xi \in \mathbb{Q}$). The dependence CE points on NBC parameters ξ_1 and ξ_2 are presented in Figure 9. The CE points of the first family $\mathcal{C}^1_{\boldsymbol{\xi}}$ are in the lines $\xi_1 + \xi_2 = 2k/l$, $l \in \mathbb{N} \setminus \{1\}$, which are perpendicular to the line $\xi_2 = \xi_1$ (see Figure 9(a)). The CE points of the second family $\mathcal{C}^2_{\boldsymbol{\xi}}$ are in the lines $\xi_2 - \xi_1 = 2k/l$, $l \in \mathbb{N} \setminus \{1, 2\}$, which are parallel to the line $\xi_2 = \xi_1$ (see Figure 9(b)). Notation l^k or l_k near the line show that the CE point is $c^1_k = \pi l$ or $c^2_k = \pi l$, accordingly. The intersection points of the CE lines from the different families with the same number l give the set $\mathcal{C}^{12}_{\boldsymbol{\xi}}$ (see Figure 9(c)). We have the first order pole p^1_1 or p^2_1 in the lines $\xi_1 + \xi_2 = 2\pi/p^1_1$ or $\xi_2 - \xi_1 = 2\pi/p^2_1$, too. The double pole is in the line $\xi_2 = n/m \cdot \xi_1$ (see Figure 9(c), $m = 1$, $n = 3$). We analyze two points in Phase Space $S_{\boldsymbol{\xi}}$: $A = (1/6, 5/6)$ and $B = (1/4, 3/4)$. The point A corresponds situation without poles ($p^1_1 = c^1_1$, $p^2_1 = c^2_1$, see Table 1 and Figure 4(c)), point B corresponds situation with first order pole in CE point. If $\boldsymbol{\xi}$ is moving across line (A_2, A_4) or (A_1, A_3) then at the CE point the complex part of Spectrum Curve is arising or disappearing in $\mathbb{C}_q$ (see Figure 9(d)). In this case the complex part of the Spectrum Curve is between two critical points. We have the same situation near point B (see Figure 9(e), $B_0 \to B_1 \to B_2$). At the point B_3 two the first order poles create the second order pole and the complex part of the Spectrum Curve is between a critical point and this pole. All complex parts of the Spectrum Curve are disappeared in the point B.

4 Critical points

If $\gamma_c^{'}(b) = 0$, $b \in \mathbb{C}_q$, then we have a *critical point* b of the Complex CF, and value $\gamma_c(b)$ is a *critical value* of the Complex CF [18, 21]. Critical points of the Complex CF are saddle points of this function. For Real CF critical points can be a half-saddle point ($x = 0$), or maximum, minimum point or inflection (saddle) points. At the critical point the Spectrum Curves change the direction. If the Complex–Real CF at the critical point $b \in \mathbb{C}_q$ satisfies



(a) $\mathcal{C}^1_{\boldsymbol{\xi}}$ (b) $\mathcal{C}^2_{\boldsymbol{\xi}}$ (c) $\mathcal{C}^{12}_{\boldsymbol{\xi}}$

(d) point A (e) point B

Figure 9. CE points ($c_k = \pi k$, $k = 1, \ldots, 5$) in Phase Space $S_{\boldsymbol{\xi}}$ and Spectrum Curves in the points A and B and in their neighbourhood.

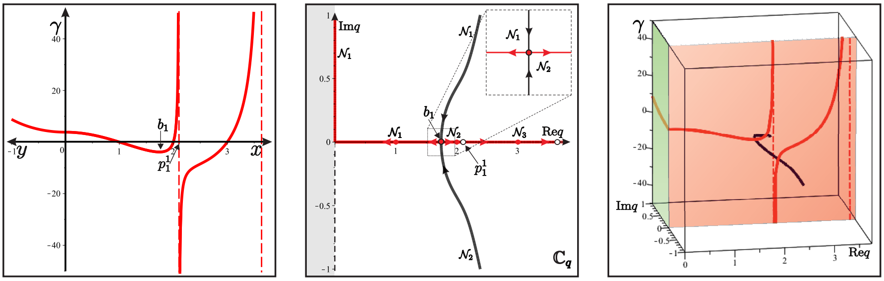
$\gamma_c^{'}(b) = 0, \ldots, \gamma_c^{k}(b) = 0$, $\gamma_c^{(k+1)}(b) \neq 0$, then b is called the *k-order critical point.* At this point Spectrum Curves change direction and the angle between old and the new direction is $\frac{\pi}{k+1}$ (as we note in Section 1, the Spectrum Curve turn to the right).

4.1 The first order critical points

For SLP (2.1)–(2.3) there exist two types the first order critical points. The first type critical point appears for $b \in \mathbb{C}_q$, $b^2 \in \mathbb{R}$. In this case, multiple eigenvalue is real (usually double or triple, where the critical point coincide with CE point). *The first order real critical point* $b \in \mathbb{C}_q$ can be find from the following equation:

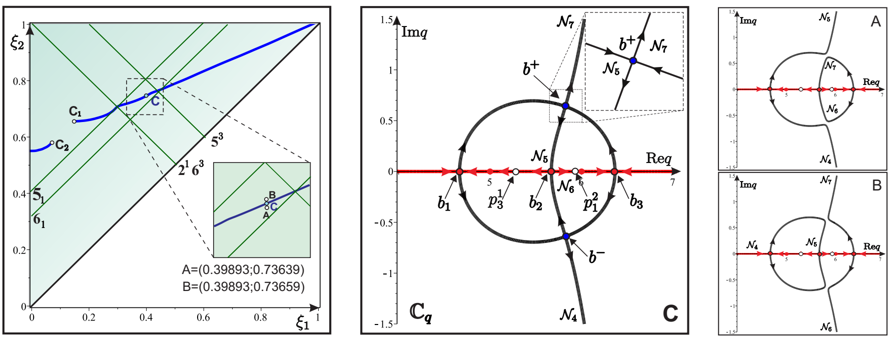
$$\gamma'(b; \boldsymbol{\xi}) = 0, \quad b^2 \in \mathbb{R}.$$

CF in the neighbourhood of the first order real critical point is presented in Figure 10. Real critical point of CF exists in the typical situations : 1) between two zeroes do not exist pole, 2) between two poles do not exist zero. For SLP (2.1)–(2.3) all the first order real critical points are positive.



(a) Real CF (b) Spectrum Curves (c) 3D view

Figure 10. The first order real critical point, $\boldsymbol{\xi} = (0.2, 0.75)$.



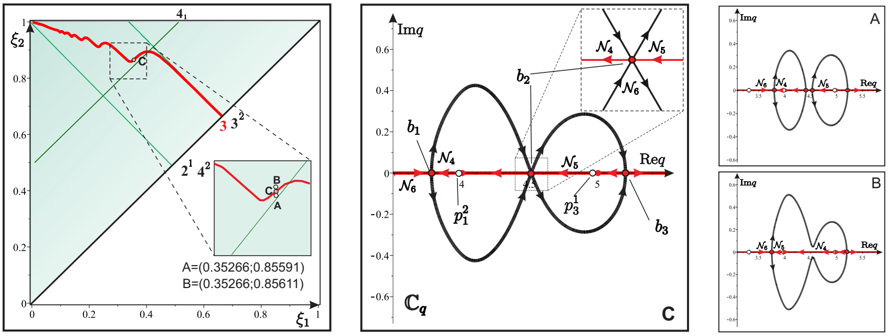
(a) (b) (c)

Figure 11. The first order complex critical point C, $\boldsymbol{\xi} = (0.39893, 0.73649\ldots)$.

The first order complex critical point $b = x + \imath y \in \mathbb{C}_q$ can be calculated solving the system of equations:

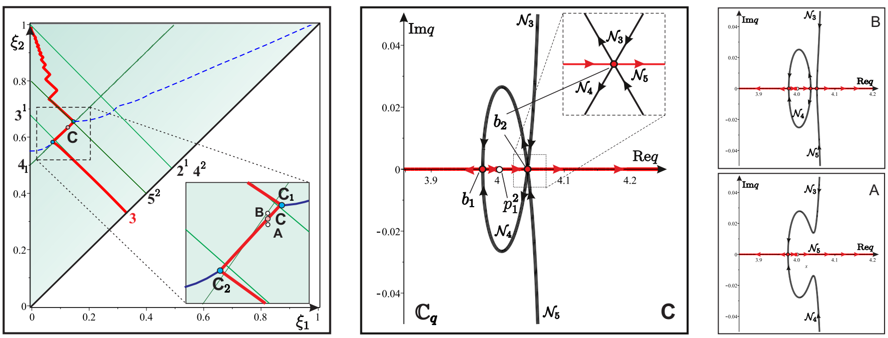
$$\operatorname{Im}\gamma(b;\boldsymbol{\xi}) = 0, \qquad \operatorname{Re}\gamma'(b;\boldsymbol{\xi}) = 0, \qquad \operatorname{Im}\gamma'(b;\boldsymbol{\xi}) = 0, \quad b^2 \notin \mathbb{R}.$$

Spectrum Domain is symmetrical and we have pair complex critical points. The solution of this system is a curve in Phase Space $S_{\boldsymbol{\xi}}$. Two trajectories of the such complex critical points are presented in Figure 11(a). The Spectrum Domain for $\boldsymbol{\xi} = (0.39893, 0.73649...)$ with complex critical point is presented in Figure 11(b). Every point of the trajectory in $S_{\boldsymbol{\xi}}$ has similar Spectrum Curves in the neighbourhood of a critical point. If Phase Point moves across this trajectory, then the view of the Spectrum Curves are qualitative different (see Figure 11(a),(c), points A and B). In this example (see Figure 11(b), point C) we have the both cases the first order critical points: b_1, b_2, b_3 are real critical points and b^-, b^+ are pair complex critical points. This example shows that Spectrum Curve $\mathcal{N}_5$ approach infinity (the point A) or pole (the points C and B) for $\gamma \to +\infty$. The gap between two trajectories (points C_1 and C_2) we explain later on.



(a) (b) (c)

Figure 12. The second order critical point C, $\boldsymbol{\xi} = (0.35266, 0.85601\ldots)$.



(a) (b) (c)

Figure 13. The second order critical point C, $\boldsymbol{\xi} = (0.11625, 0.616239\ldots)$, $A = (0.11625, 0.616238\ldots)$, $B = (0.11625, 0.616240\ldots)$, $C_1 = (0.15454\ldots, 0.64970\ldots)$, $C_2 = (0.07331\ldots, 0.57495\ldots)$.

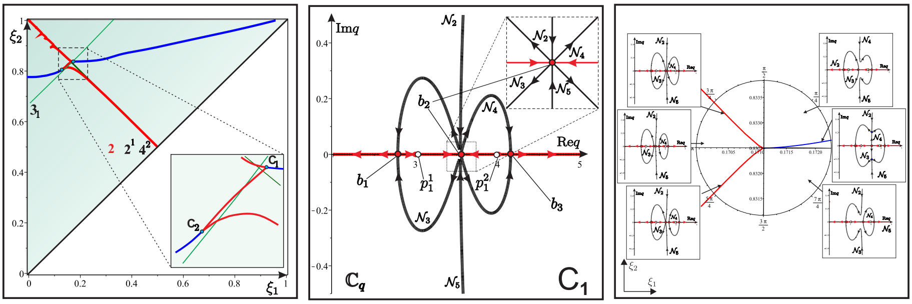
4.2 The second order and the third order critical points

The second order critical point appears when two the first order real critical points coincide in the same point b. The second order critical point can be found from the following equation:

$$\gamma'(b;\boldsymbol{\xi}) = 0, \qquad \gamma''(b;\boldsymbol{\xi}) = 0, \quad b^2 \in \mathbb{R}.$$

For SLP (2.1)–(2.3) all the second order real critical points are positive. Two trajectories of such the second order critical points in Phase Space are shown in Figure 12(a) and Figure 13(a). In the Figure 12(b) and Figure 13(b) Spectrum Curves are presented in the point C which is on corresponding trajectory of the second order critical point and in Figure 12(c) and Figure 13(c) we can see Spectrum Curves in the Phase Points A and B near this trajectory. Points C_1 and C_2 in Figure 13(a) are the same as in Figure 11(a). So, the gap between Phase Points C_1 and C_2 is the part of the second order critical point trajectory.

Numerical calculations show that such gaps exist for $\xi_1 + \xi_2 \lessapprox 1$ (see Figure 11(a), Figure 13(a), Figure 14(a)). The gap boundary points C_1 and C_2



(a) (b) (c)

Figure 14. The third order critical point C_1, $\boldsymbol{\xi} = (0.17122\ldots, 0.83250\ldots)$

are the third order critical points and they can be found from the system:

$$\gamma'(b;\boldsymbol{\xi}) = 0, \qquad \gamma''(b;\boldsymbol{\xi}) = 0, \qquad \gamma'''(b;\boldsymbol{\xi}) = 0, \quad b^2 \in \mathbb{R}.$$

The views of Spectrum Curves in point C_1 and in the neighbourhood of this third order critical point are presented in Figure 14(b)–(c). At this point the trajectory of the second order critical point change direction and pair the first order complex critical points come real ($y = 0$ and positive).

If $\xi_1 + \xi_2 \gtrapprox 1$ (see Figure 12(a), Figure 15(a)) then a trajectory of the second order critical point is "smooth" curve. This trajectory intersects with the first order complex critical point trajectory without the third order critical points, i.e. pair complex critical points do not reach the real axis. Typical Spectrum Curves are presented in Figure 15(c).

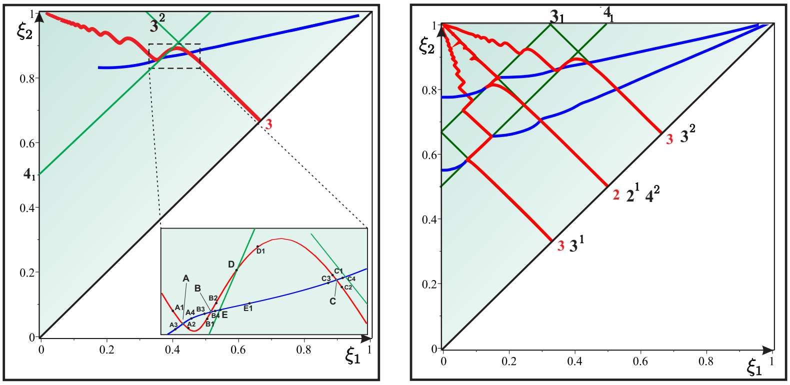
The general behaviour the second and the third order trajectories Phase Space is more complicated. For small x three trajectories are shown in Figure 15(b). The second order trajectories leave points $\boldsymbol{\xi} = (1/3, 1/3)$, $(1/2, 1/2)$, $(2/3, 2/3)$ for $x = 3, 2, 3$, accordingly. All these trajectories approach Phase Point $\boldsymbol{\xi} = (0, 1)$. There is no the second order critical point for integral NBC with $\xi_2 = 1$ or $\xi_1 = 0$. Trajectories of the first order complex critical points start at points $\boldsymbol{\xi} = (0, b)$ and move towards a point which corresponds to the third order critical point and after "gap" these trajectories approach Phase Point $\boldsymbol{\xi} = (1, 1)$.

5 Conclusions

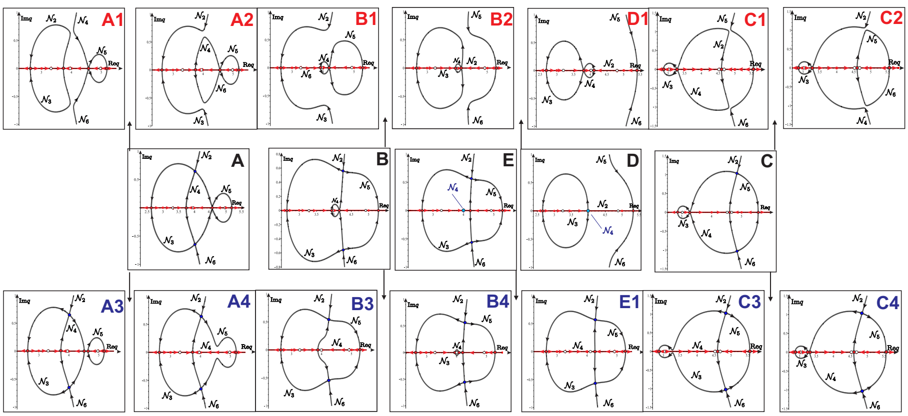
In this paper the spectrum for Sturm–Liouville problem with one integral NBC depending on two parameters was investigated.

Qualitative view of the Spectrum Curves with respect to parameters ξ_1 and ξ_2 in integral BC, the location of the zeroes, poles and CE points of the CF is very important for investigation. In this article we find all such points in the case SLP (2.1)–(2.3).

Critical points of CF are important for investigation of complex eigenvalues and Spectrum Curves in the complex plane. We find trajectories of the first order complex critical points and the second order (real) critical points. Such



(a) (b)



(c)

Figure 15. The trajectories of the third order and the second order critical points and Spectrum Curves, $A = (0.3526\ldots, 0.8560\ldots)$, $B = (0, 3600\ldots, 0.8601\ldots)$, $C = (0.4491\ldots, 0.8771\ldots)$, $D = (0.3660\ldots, 0.8660\ldots)$, $E = (0.3603\ldots, 0.8603\ldots)$.

trajectories can be found only numerically. In this article we describe how Spectrum Curves vary on parameters ξ_1 and ξ_2. We investigate the first order real and complex critical points, trajectories of the first order complex critical points and the second order critical points in the Phase Space $S_{\boldsymbol{\xi}}$, find location of the third order critical points. Some interesting properties of Spectrum Curves were found.

Investigation of the Spectrum Curves gives useful information about the spectrum for problems with NBC.

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