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On the equivalence of discrete Sturm–Liouville problem with nonlocal boundary conditions to the algebraic eigenvalue problem*

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Abstract. We consider the finite difference approximation of the second order Sturm–Liouville equation with nonlocal boundary conditions (NBC). We investigate the condition when the discrete Sturm–Liouville problem can be transformed to an algebraic eigenvalue problem and denote this condition as solvability condition. The examples of the solvability for the most popular NBCs are provided.

Keywords: nonlocal boundary conditions, Sturm–Liouville problem, finite difference scheme, solvability.

Introduction

As a result of technological progress during the last couple decades, there has been an interest investigating problems with rather complicated nonclassical conditions modeling natural, physical, chemical and other processes [4]. In connection with this fact it is natural to investigate whether the problem is well-posed. To understand the behaviour of real processes it is natural to investigate solvability condition on the stationary problems. The solvability results for various type differential problems with nonlocal conditions can be found in [1].

In the present paper, we investigate the solvability of the discrete Sturm–Liouville problem with two nonlocal boundary conditions (NBC) of the general form. We investigate the condition when the discrete Sturm–Liouville problem can be transformed to an algebraic eigenvalue problem. We also provide the examples of the solvability conditions for the most popular nonlocal boundary conditions.

1 Sturm–Liuoville problem with two NBC

In this article in the interval $[0, L]$ we consider grids:

$$\overline{\omega}^h := \{x_i \colon x_i = ih, i = \overline{0, N}\}, \quad h = L/N; \qquad \omega^h := \{x_1, \dots, x_{N-1}\}.$$

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We use notation $[\cdot,\cdot]$ and $(\cdot,\cdot)$ for the inner products in the Hilbert spaces $\overline{H} := H(\overline{\omega}^h)$ and $H := H(\omega^h)$ accordingly.

We consider discrete Sturm–Liouville operator

$$\mathcal{L}U := -\delta(P\delta U) + QU = \lambda U, \quad x_i \in \omega^h, \tag{1}$$

where P, Q are real functions and

$$\bigl(\delta(P\delta U)\bigr)_i := \frac{P_{i+1/2}(U_{i+1} - U_i) - P_{i-1/2}(U_i - U_{i-1})}{h^2},$$

with two nonlocal boundary conditions of general form

$$[k_0, U] = \gamma_0[n_0, U], \qquad [k_1, U] = \gamma_1[n_1, U], \tag{2}$$

where $[k_i, U]$ is the classical part and $[n_i, U]$ is a nonlocal part of boundary conditions, $i = 0, 1$.

Now we investigate the condition when problem (1)–(2) can be transformed to the algebraic eigenvalue problem. The algebraic problem is degenerate if its determinant equals to zero. We rewrite boundary conditions (2) in the following form

$$\bigl[k_0 - \gamma_0 n_0, \delta^0\bigr]U_0 + \bigl[k_0 - \gamma_0 n_0, \delta^N\bigr]U_1 = (\gamma_0 n_0 - k_0, U), \tag{3}$$

$$\bigl[k_1 - \gamma_1 n_1, \delta^0\bigr]U_0 + \bigl[k_1 - \gamma_1 n_1, \delta^N\bigr]U_1 = (\gamma_1 n_1 - k_1, U), \tag{4}$$

where $\delta^s := \delta^s_s$ is the Kronecker delta

$$\delta^s_i = \begin{cases} 0 & \text{if } s \neq i, \\ 1 & \text{if } s = i. \end{cases}$$

Equations (3)–(4) form a system of linear equations respect to boundary values of the function U

$$\begin{pmatrix} [k_0 - \gamma_0 n_0, \delta^0] & [k_0 - \gamma_0 n_0, \delta^N] \\ [k_1 - \gamma_1 n_1, \delta^0] & [k_1 - \gamma_1 n_1, \delta^N] \end{pmatrix} \begin{pmatrix} U_0 \\ U_N \end{pmatrix} = \begin{pmatrix} (\gamma_0 n_0 - k_0, U) \\ (\gamma_1 n_1 - k_1, U) \end{pmatrix}. \tag{5}$$

System (5) is degenerate if

$$\begin{vmatrix} [k_0 - \gamma_0 n_0, \delta^0] & [k_0 - \gamma_0 n_0, \delta^N] \\ [k_1 - \gamma_1 n_1, \delta^0] & [k_1 - \gamma_1 n_1, \delta^N] \end{vmatrix} = 0,$$

or in the expanded form

$$\gamma_0\gamma_1 \mathrm{D}(n_0, n_1) + \gamma_0 \mathrm{D}(n_0, k_1) + \gamma_1 \mathrm{D}(n_1, k_0) + \mathrm{D}(k_0, k_1) = 0, \tag{6}$$

where

$$\mathrm{D}(n_0, n_1) = \begin{vmatrix} [n_0, \delta^0] & [n_0, \delta^N] \\ [n_1, \delta^0] & [n_1, \delta^N] \end{vmatrix}, \qquad \mathrm{D}(k_1, n_0) = \begin{vmatrix} [k_1, \delta^0] & [k_1, \delta^N] \\ [n_0, \delta^0] & [n_0, \delta^N] \end{vmatrix},$$

$$\mathrm{D}(n_1, k_0) = \begin{vmatrix} [n_1, \delta^0] & [n_1, \delta^N] \\ [k_0, \delta^0] & [k_0, \delta^N] \end{vmatrix}, \qquad \mathrm{D}(k_0, k_1) = \begin{vmatrix} [k_0, \delta^0] & [k_0, \delta^N] \\ [k_1, \delta^0] & [k_1, \delta^N] \end{vmatrix}. \tag{7}$$

Table 1. Classification of the Degeneration Curves.

Curve in plane	Case	Matrix A	Curve in plane	Case	Matrix A
Whole plane	1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Two lines	4a	$\begin{pmatrix} a_{00} & 0 \\ 0 & 0 \end{pmatrix}$
Empty set	2	$\begin{pmatrix} 0 & 0 \\ 0 & a_{11} \end{pmatrix}$		4b	$\begin{pmatrix} a_{00} & a_{01} \\ 0 & 0 \end{pmatrix}$
Line	3a	$\begin{pmatrix} 0 & a_{01} \\ 0 & 0 \end{pmatrix}$		4c	$\begin{pmatrix} a_{00} & 0 \\ a_{10} & 0 \end{pmatrix}$
	3b	$\begin{pmatrix} 0 & 0 \\ a_{10} & 0 \end{pmatrix}$		4d	$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$, $\det \mathrm{A} = 0$
	3c	$\begin{pmatrix} 0 & a_{01} \\ 0 & a_{11} \end{pmatrix}$	Hyperbola	5a	$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$, $\det \mathrm{A} \neq 0$
	3d	$\begin{pmatrix} 0 & 0 \\ a_{10} & a_{11} \end{pmatrix}$		5b	$\begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix}$
	3e	$\begin{pmatrix} 0 & a_{01} \\ a_{10} & 0 \end{pmatrix}$		5c	$\begin{pmatrix} a_{00} & a_{01} \\ 0 & a_{11} \end{pmatrix}$
	3f	$\begin{pmatrix} 0 & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$		5d	$\begin{pmatrix} a_{00} & 0 \\ a_{10} & a_{11} \end{pmatrix}$
				5e	$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & 0 \end{pmatrix}$

In general case Eq. (6) describe a second degree algebraic curve on the plane (γ_0, γ_1). The classification of the curves of such type is given in [3]. We call a set of points (γ_0, γ_1), satisfying Eq. (6), the *Degeneration Curve* for the problem (1)–(2).

Analogously as in [3] we denote matrix

$$\mathrm{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} D(n_0, n_1) & D(n_0, k_1) \\ D(k_0, n_1) & D(k_0, k_1) \end{pmatrix}. \tag{8}$$

Each matrix A corresponds to one of the five types of Degeneration Curves. More detailed classification is shown in Table 1. We have 16 types of matrices overall and one type is split into two cases ($\det \mathrm{A} = 0$ and $\det \mathrm{A} \neq 0$). So, the next lemma is valid for the Degeneration Curve (as well as for the Characteristic Curve in [3]).

Lemma 1. *A Degeneration Curve for problem* (1)–(2) *in the plane* $\mathbb{R}^2$ *can be one of the following five types*:

1. *If* $D(n_0, n_1) = D(k_0, k_1) = D(n_0, k_1) = D(k_0, n_1) = 0$ *then the curve is whole plane*;
2. *If* $D(n_0, n_1) = D(n_0, k_1) = D(k_0, n_1) = 0$, $D(k_0, k_1) \neq 0$ *then the curve is empty set*;
3. *If* $D(n_0, n_1) = 0$, $D(n_0, k_1) \neq 0$ *or* $D(n_0, n_1) = 0$, $D(k_0, n_1) \neq 0$ *then the curve is line*;
4. *If* $D(n_0, n_1) \neq 0$ *and* $\det \mathrm{A} = 0$ *then the curve is union of vertical and horizontal lines*;
5. *If* $D(n_0, n_1) \neq 0$ *and* $\det \mathrm{A} \neq 0$ *then the curve is hyperbola.*

Remark 1. We see, that Degeneration Curve in the plane $\mathbb{R}^2$ cannot be algebraic curve such as ellipse, parabola, point, parallel lines, double line.

Remark 2. If $\det \mathrm{A} \neq 0$ then the line (Case 3) is neither vertical nor horizontal (see Cases 3e, 3f in Table 1), otherwise we have single vertical or single horizontal line (see Cases 3a–3d in Table 1).

Remark 3. Investigated problem can be easily extended from plane $\mathbb{R}^2$ to the cone $\mathbb{T}^2$ analogously as it was done in [3].

2 Applications

Example 1 [Nonlocal integral boundary conditions]. We consider Dirichlet integral boundary conditions with weights. We use the following inner products corresponding to the trapezoid rule and defined as (see e.g. [2]):

$$[U,V] := \frac{U_0 V_0 h}{2} + \sum_{i=1}^{N-1} U_i V_i h + \frac{U_N V_N h}{2}, \quad U,V \in \overline{H},$$

$$(U,V) := \sum_{i=1}^{N-1} U_i V_i h, \quad U,V \in H.$$

The boundary conditions (2) are of the form

$$\left[\delta^0, U\right] = \gamma_0 \left[B^0, U\right], \qquad \left[\delta^N, U\right] = \gamma_1 \left[B^1, U\right], \tag{9}$$

where B^0 and B^1 are the weight functions. In the general case we have the following nondegeneracy condition:

$$\begin{vmatrix} 1 - h\gamma_1 B^1_N/2 & h\gamma_1 B^0_N/2 \\ h\gamma_0 B^1_0/2 & 1 - h\gamma_0 B^0_0/2 \end{vmatrix} \neq 0.$$

The degeneration curve is of the following form

$$\frac{h^2}{4} \begin{vmatrix} B^0_0 & B^0_N \\ B^1_0 & B^1_N \end{vmatrix} \gamma_0\gamma_1 - \frac{h}{2} B^0_0 \gamma_0 - \frac{h}{2} B^1_N \gamma_1 + 1 = 0.$$

If $B^0 \equiv 1$ and $B^1 \equiv 1$, then the full integral which was investigated in [2]. The degeneration curve is of the following form

$$-\frac{h}{2}(\gamma_0 + \gamma_1) + 1 = 0.$$

In the case of classical boundary conditions $B^0 \equiv 0$ and $B^1 \equiv 0$,the degeneration curve is a whole plane (see Table 1, Case 1).

Example 2 [Bicadze–Samarskii NBC]. We consider boundary conditions of the Bicadze–Samarskii form

$$\left[\delta^0, U\right] = \gamma_0 [\delta^{s_0}, U], \qquad \left[\delta^N, U\right] = \gamma_1 \left[\delta^{s_1}, U\right], \tag{10}$$

where $[\cdot,\cdot]$ is the classical inner product

$$[U,V] := \sum_{i=0}^{N} U_i V_i, \quad U,V \in \overline{H}. \tag{11}$$

In this case the degeneration curve is of the form

$$\begin{vmatrix} \delta_0^{s_0} & \delta_N^{s_0} \\ \delta_0^{s_1} & \delta_N^{s_1} \end{vmatrix} \gamma_0\gamma_1 - \delta_0^{s_0}\gamma_0 - \delta_N^{s_1}\gamma_1 + 1 = 0. \tag{12}$$

As one can see from Eq. (12) the classification for the degeneration curves in the case of Bicadze–Samarskii nonlocal boundary condition is the same as for the integral conditions, except the coefficients. For the investigated case the classifications depends on whether the nonlocal point is inner or boundary.

Example 3 [Multipoint NBC]. We consider boundary conditions of the following form

$$U(0) = \gamma_0 \sum_{i=0}^{N} \alpha_i^0 U(\xi_i), \qquad U(1) = \gamma_1 \sum_{i=0}^{N} \alpha_i^1 U(\xi_i),$$

where $\alpha^k = \sum_{j=0}^{N} \alpha_j^k \delta^j$, $k = 0,1$; $0 \le \xi_0 < \cdots < \xi_N \le 1$. Using the inner product (11) we rewrite NBC in the following form

$$\left[\delta^0, U\right] = \gamma_0\left[\alpha^0, U\right], \qquad \left[\delta^N, U\right] = \gamma_1\left[\alpha^1, U\right]. \tag{13}$$

The method of investigating multipoint case is similar to the method in Example 2. The form of the degeneration curve is equivalent to the Eq. (12)

$$\begin{vmatrix} \alpha_0^{s_0}\delta_0^{s_0} & \alpha_N^{s_0}\delta_N^{s_0} \\ \alpha_0^{s_1}\delta_0^{s_1} & \alpha_N^{s_1}\delta_N^{s_1} \end{vmatrix} \gamma_0\gamma_1 - \alpha_0^{s_0}\delta_0^{s_0}\gamma_0 - \alpha_N^{s_1}\delta_N^{s_1}\gamma_1 + 1 = 0. \tag{14}$$

Example 4 [Left and right rectangle rules for integral NBC]. We consider boundary conditions (13) with the inner products

$$[U,V]_l := \sum_{i=0}^{N-1} U_i V_i h, \qquad [U,V]_r := \sum_{i=1}^{N} U_i V_i h$$

corresponding to the left and right rectangle rules respectively. So the degeneration curves are of the following forms:

$$hB_0^0\gamma_0 - 1 = 0 \quad \text{for the left rectangle rule,}$$
$$hB_N^1\gamma_1 - 1 = 0 \quad \text{for the right rectangle rule.}$$

Remark 4. Examples 1–4 describe all the cases mentioned in the table 1, except of an empty set (Case 2). This situation is valid when the boundary conditions are of the following form

$$\int_{\xi_0}^{\xi_1} \beta_0(x)U(x,t)\,dx = a_0, \qquad \int_{\xi_2}^{\xi_3} \beta_1(x)U(x,t)\,dx = a_1,$$

where $0 < \xi_0 \le \xi_1 < 1$, $0 < \xi_2 \le \xi_3 < 1$, $a_0, a_1 \in \mathbb{R}$, β_0 and β_1 are weight functions.

Remark 5. The technique investigated in this article is suitable for defining the solvability conditions for different stationary and non-stationary problems with nonlocal boundary conditions. As one can see only the boundary conditions are needed to define the solvability. It is enough to define the operators, corresponding to the classical and nonlocal parts of the boundary conditions. Obtained solvability condition mostly depends only on the values of the operators of the nonlocal parts on the boundaries.

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REZIUMĖ

Nelokaliojo skirtuminio Šturmo ir Liuvilio uždavinio ekvivalentiškumas algebriniam tikrinių reikšmių uždaviniui

Jurij Novickij ir A. Štikonas

Darbe nagrinėjamas skirtuminis Šturmo ir Liuvilio uždavinys su nelokaliosiomis kraštinėmis sąlygomis. Gauta uždavinio ekvivalentiškumo algebriniam tikrinių reikšmių uždaviniui sąlyga bei pateikti pavyzdžiai.

Raktiniai žodžiai: Nelokaliosios sąlygos, išsprendžiamumo sąlyga, Šturmo ir Liuvilio uždavinys.