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KĘSTUTIS JANULIS

MIXED JOINT UNIVERSALITY FOR DIRICHLET L - FUNCTIONS AND HURWITZ TYPE ZETA - FUNCTIONS

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VILNIAUS UNIVERSITETAS

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DIRICHLĖ L FUNKCIJŲ IR HURVICO TIPO DZETA FUNKCIJŲ MIŠRUS JUNGTINIS UNIVERSALUMAS

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Contents

Introduction

In this thesis, collections consisting from Dirichlet L - functions and Hurwitz or periodic Hurwitz zeta - functions are considered.

Aims and problems

The aims of the thesis are mixed joint universality theorems for Dirichlet L - functions and Hurwitz type zeta - functions, i.e., theorems on simultaneous approximation of a collection of analytic functions by shifts of Dirichlet L - functions which have Euler's product over primes and by shifts of Hurwitz type zeta - functions which have no the Euler product. The problems are the following:

1. A mixed joint universality theorem for Dirichlet L -functions and Hurwitz zeta - functions;
2. A mixed joint universality theorem of Dirichlet L - functions and periodic Hurwitz zeta - functions;
3. The universality of composite functions of a collection of Dirichlet L - functions and Hurwitz zeta - functions;
4. The universality of composite functions of a collection of Dirichlet L - functions and periodic Hurwitz zeta - functions.

We remind the definitions of functions studied in the thesis. Let χ be a Dirichlet character modulo $q \in \mathbb{N}$. The Dirichlet L - function $L(s,\chi)$, $s=\sigma+i\tau$, is defined, for $\sigma>1$, by the series

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}.$$

If χ is a non - principal character, then the function $L(s,\chi)$ is entire, while, for the principal character χ_0, the function $L(s,\chi_0)$ has analytic continuation to the whole complex plane, except for a simple pole at he point $s=1$ with residue

$$\prod_{p|q}\left(1-\frac{1}{p}\right).$$

Moreover, the function $L(s,\chi)$, for $\sigma > 1$, has the Euler product over primes

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Let α, $0 < \alpha \leq 1$, be a fixed parameter. The Hurwitz zeta - function $\zeta(s,\alpha)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. We have that $\zeta(s,1)$ is the Riemann zeta - function $\zeta(s)$, and

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s).$$

Since, for $\sigma > 1$,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

the function $\zeta(s,\alpha)$ has the Euler product over primes in the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$, only.

Now let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. Then the periodic Hurwitz zeta - function $\zeta(s,\alpha;\mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s,\alpha,\mathfrak{a}) = \frac{1}{k^s} \sum_{m=0}^{k-1} a_m \zeta\left(s, \frac{\alpha+m}{k}\right).$$

Therefore, the properties of the Hurwitz zeta - function show that the function $\zeta(s,\alpha;\mathfrak{a})$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$a \stackrel{\text{def}}{=} \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

If $a = 0$, then the function $\zeta(s,\alpha;\mathfrak{a})$ is entire. Clearly, $\zeta(s,\alpha;\{1\}) = \zeta(s,\alpha)$.

Actuality

The zeta and L - functions are the principal objects of analytic number theory. They are used not only for solving many problems but also are widely studied themselves. Universality of zeta - functions discovered by S.M. Voronin in 1975 is a very interesting and useful phenomenon having a series of theoretical and practical applications. From universality theorems, the hypertranscendence of zeta -

functions conjectured [14] by D. Hilbert in 1900 follows, [10], [31], [24], [59], [77], [79]. In the case of the function $\zeta(s)$, this means that if given continuous function $F_0, F_1, \ldots, F_N : \mathbb{C}^{n+1} \to \mathbb{C}$ are not all identically zero,then

$$\sum_{m=0}^{N} s^m F_m(\zeta(s), \zeta'(s), \ldots, \zeta^{(n)}(s)) \neq 0$$

for some $s \in \mathbb{C}$. Universality of zeta - functions without Euler's product can be applied for the investigation of zero - distribution of these functions [44], [46], [54], [59], [77]. Also, there exists a relation between the universality and zeros of multiple zeta - functions [65],[66], [68]. Universality can be applied for the class number problem [63]. We remind that the class number $h(d)$ of a quadratic number flied $\mathbb{Q}(\sqrt{d})$ of discriminant $d < 0$ is equal to number of reduced binary quadratic forms of discriminant d. Let Λ^{-1} be set of negative discriminants. Then the universality results of [63] imply the density in $\mathbb{R}_+$ of the set $\left\{\frac{h(d)}{\sqrt{d}} : d \in \Lambda^{-1}\right\}$. The practical applications of universality of zeta - functions are related to a direct approximation of complicated analytic functions. For example, the universality of $\zeta(s)$ is applied [4] to path integrals in quantum mechanics. Universality is closely related to the self - approximation, and thus, to the Riemann hypothesis (RH) that $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$ It is known [77] that RH is equivalent to the assertion that the function $\zeta(s)$ can be approximated uniformly on compact subsets by shifts $\zeta(s + i\tau)$. Deep results in this direction were obtained by T. Nakamura [67], T. Nakamura and Ł. Pańkowski [69], [70], Ł. Pańkowski [71], as well as by R. Garunkštis [8], R. Garunkštis [8] and E. Karikovas [9], and by E. Karikovas and Ł. Pańkowski [25]. Theses and other examples show that the attention to universality of zeta - functions has a deep motivation. The schools of universality in various countries (Lithuania, Japan, Germany, Canada, France, South Korea, Poland) also clearly confirm the importance of the universality property for zeta - functions. Therefore, the study of universality of zeta and L - functions is an urgent problem of the contemporary analytic number theory.

Methods

In the thesis, the probabilistic approach for the proof of universality theorems based on limit theorem of weakly convergent probability measures is developed. This method includes elements of the measure and ergodic theory. Also, the results of approximation of analytic functions, in particular, the Mergelyan theorem, are applied.

Novelty

All results of the thesis are new. The mixed joint universality theorem for Dirichlet L - functions and Hurwitz type functions were not considered.

History of the problem and results

Universality problem for zeta and L - functions comes back to S.M. Voronin who proved [78], [80] the universality of the Riemann zeta - function. The initial form of the Voronin theorem is contained in the following theorem.

Theorem A. *Suppose that $0 < r < \frac{1}{4}$, the function $f(s)$ is continuous and non - vanishing on the disc $|s| \leq r$, and analytic in the interior of this disc. Then, for any $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s|\leq r} |\zeta(s + \frac{3}{4} + i\tau) - f(s)| < \varepsilon.$$

The function $\zeta(s)$ is called universal since suitable its shifts $\zeta(s + \frac{3}{4} + i\tau)$ uniformly approximate any analytic target function satisfying the hypotheses of Theorem A.

The Voronin theorem was slightly improved by various authors. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K), K \in \mathcal{K}$, the class of non - vanishing continuous functions on K which are analytic in the interior of K. Moreover, let $meas\ A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then a modern version of the Voronin theorem is of the form, see, for example,[31],[77].

Theorem B. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\} > 0.$$

Thus, Theorem B extends Theorem A in two directions. First, analytic functions are uniformly approximated by shifts $\zeta(s + i\tau)$ not only on discs but on more general compact sets of the class $\mathcal{K}$. Moreover, Theorem B shows that there exist infinitely many shifts $\zeta(s + i\tau)$ approximating a given analytic function, the set of such shifts has a positive lower density.

It turned out that some other zeta and L - functions are also universal in the Voronin sense. S.M. Voronin himself observed [78], see also, [24], [80], that all Dirichlet L - functions are also universal. Thus, the following result is true.

Theorem C. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{s\in K} |L(s + i\tau, \chi) - f(s)| < \varepsilon\} > 0.$$

Now it is known that wide classes of zeta and L - functions having Euler's product over primes are universal in the above sense. Universality for some classes of Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

with multiplication coefficients $(a(mn) = a(m)a(n)$ for all $m, n \in \mathbb{N}$, $(m, n) = 1)$ was obtained in [26] - [30] and [57]. The case of zeta - functions of certain cusp forms was considered in [22],[50],[48] and [51]. In the monograph [77], the universality property was extended to the famous Selberg class [76] of Dirichlet series.

Some zeta - functions without Euler's product are also universal in a similar sense. Denote by $H(K), K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. The simplest and most important of zeta - functions without Euler's product is the Hurwitz zeta - function $\zeta(s, \alpha)$. At the moment, we have the following statement.

Theorem D. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\} > 0.$$

In a slightly different form, Theorem D for rational α was obtained independently by S.M. Gonek [13] and B. Bagchi [1]. The case of transcendental α can be found in [47].

The universality property in Theorem D, sometimes is called a strong universality because the shifts $\zeta(s + i\tau, \alpha)$ approximate functions from a wider class $H(K) \supset H_0(K)$.

A generalization of the Hurwitz zeta - function is the Lerch zeta - function $L(\lambda, \alpha, s)$ which is defined, for $\sigma > 1$, by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere with a fixed $\lambda \in \mathbb{R}$. The universality if the function $L(\lambda, \alpha, s)$ was considered in [32], [33] and [47]. The second generalization of the function $\zeta(s, \alpha)$ is the periodic Hurwitz zeta - function $\zeta(s, \alpha; \mathfrak{a})$. The latter function was introduced in [34]. The universality of the function $\zeta(s, \alpha; \mathfrak{a})$ for transcendental α was proved in [19] and [20].

A more complicated and interesting is the joint universality of zeta and L - functions when a collection of given analytic functions simultaneously are approximated by shifts of zeta or L - functions. The first result in this direction also belongs to S.M. Voronon. In [79], he in a not explicit form obtained the joint universality of Dirichlet L - functions. More precisely, he considered the joint functional independence of Dirichlet L - function, but his arguments are also applicable to their joint universality. Before the statement of a joint universality theorem for Dirichlet L - functions, we recall some notation and definitions.

A non - principal Dirichlet character $\chi(m), m \in \mathbb{N}$, modulo q is called primitive if, for $(m, q) = 1$, the number q is the smallest period of $\chi(m)$. If a character $\chi(m)$ modulo q is non - primitive, then there exists $q_1, q_1 < q$, and a primitive character $\chi_1(m)$ modulo q_1 such that

$$\chi(m) = \begin{cases} \chi_1(m) & \text{if } (m, q) = 1, \\ 0 & \text{if } (m, q) > 1. \end{cases}$$

In this case, we say that the character $\chi(m)$ is generated by a primitive character $\chi_1(m)$. Two Dirichlet characters are said to be equivalent if they are generated by the same primitive character.

Now we state a joint universality theorem for Dirichlet L - functions.

Theorem E. *Suppose that $\chi_1, \ldots, \chi_r$ are pairwise non - equivalent Dirichlet characters. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon\} > 0.$$

Proof of Theorem E can be found in [40]. In [1],[2] and [24], Theorem E was obtained in a slightly different form. It is clear that, in joint universality theorems, the approximating functions would be independent in a certain sense. In Theorem E, this independence is realized by a pairwise non - equivalence of Dirichlet characters.

The joint universality of Hurwitz zeta - functions has been considered in [7],[35], [62] and [64]. We state a result obtained on [35]. For $j = 1, \ldots, r$, let $0 < \alpha_j \le 1$, and

$$L(\alpha_1, \ldots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r\}.$$

Theorem F[35]. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{1 \le j \le r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\} > 0.$$

Theorem F with a stronger hypothesis that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ has been proved in [64]. We recall that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ if there exist no polynomials with rational coefficients $p \not\equiv 0$ such that $p(\alpha_1, \ldots, \alpha_r) = 0$.

In [7], the extension of Theorem F was given.

Many works are devoted to joint universality of periodic Hurwitz zeta - functions. The first results of such a kind for periodic Hurwitz zeta - functions with the same parameter α were obtained in [34]. For $j = 1, \dots, r$, let $0 < \alpha_j \leq 1$, and let $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period k_j. Let $k = [k_1, \dots, k_r]$ denote the least common multiple of the periods $k_1, \dots, k_r$. Define

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr} \end{pmatrix}.$$

Then in [21], the following theorem was proved.

Theorem G. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $rank(A) = r$. Let, for $j = 1, \dots, r$, $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon\} > 0.$$

A more general result has been obtained in [52], where a joint universality theorem is free from a rank condition.

The most general joint universality theorems for periodic Hurwitz zeta - functions are given in [36],[37] and [53]. For $j = 1, \dots, r$, let α_j, $0 < \alpha_j \leq 1$, be a fixed parameter, $l_j \in \mathbb{N}$, $\mathfrak{a}_{jl} = \{a_{kjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$. Then the joint universality for the functions

$$\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}) \tag{0.1}$$

can be considered. Let $k_j = [k_{j1}, \dots, k_{jl_j}]$ denote the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, and let

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \cdots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \cdots & a_{2jl_j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_jj1} & a_{k_jj2} & \cdots & a_{k_jjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Then in [53] we find the following result.

Theorem H. *Suppose that set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that $rank(A_j) = l_j, j = 1, \dots .r$. For $j = 1, \dots, r$, $l = 1, \dots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl}(s) \in H(K_{jl})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq k_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\} > 0.$$

In [61], H. Mishou began to study the joint universality for zeta - functions having and having no Euler products over primes. He proved a joint universality theorem for the Riemann zeta - function and Hurwitz zeta - function with transcendental parameter α. Now, joint universality theorems of such a kind are called mixed universality theorems. The Mishou mixed theorem is of the form.

Theorem I. *Suppose that the number α is transcendental. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$ and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon$$

$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\} > 0.$$

In [23], the Mishou theorem was generalized for a periodic zeta - function and a periodic Hurwitz zeta - function. Let $\mathfrak{a} = \{l_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers. We remind that the periodic zeta - function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere, except for a possible simple pole at the point $s = 1$.

In [12], the Riemann zeta - function was added to the system of functions (0.1), and, for an obtained collection of zeta - functions, a mixed joint universality was proved. In the papers [49],[55],[58], [73] the function $\zeta(s)$ was replaced by zeta - functions of certain cusp forms. Namely, the paper [73] is devoted to a mixed joint universality theorem for zeta - function $\zeta(s,F)$ attached to a normalized Hecke eigen cusp form F and the functions (0.1), in [58], the function $\zeta(s,F)$ was replaced by a zeta - function of a newform, and in [55] the case of a zeta - function of a cusp form with respect to the Hecke subgroup with Dirichlet character was considered.

Chapter 1 of the thesis is devoted to a mixed joint universality theorem for Dirichlet L - functions and Hurwitz zeta - functions. The main result of the chapter is contained in the following statement [18].

Theorem 1.1. *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non - equivalent Dirichlet characters, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \dots, r_1$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, while, for $j = 1, \dots, r_2$, let $\widehat{K}_j \in \mathcal{K}$ and $\widehat{f}_j(s) \in H(\widehat{K}_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{f}_j(s)| < \varepsilon\} > 0.$$

We see that, in Theorem 1.1, differently from Theorem G, any rank hypothesis is not needed.

Proof of Theorem 1.1 is probabilistic. Denote by $H(G)$ the space of analytic functions on region G equipped with the topology of uniform convergence on compacta, and by $\mathcal{B}(S)$ the σ - field of Borel sets of the space S. More precisely, the proof of Theorem 1.1 is based on the weak convergence of the probability measure

$$\frac{1}{T} meas\{\tau \in [0;T] : \Xi(s+i\tau) \in A\}, A \in \mathcal{B}(H^{r_1+r_2}(D)),$$

where

$$\Xi(s) = (L(s,\chi_1), \dots, L(s,\chi_{r_1}), \zeta(s,\alpha_1), \dots, \zeta(s,\alpha_{r_2})).$$

In Chapter 2 of the thesis, the mixed joint universality of Dirichlet L - functions and periodic Hurwitz zeta - functions is investigated, and the following statement is proved. The notation of Theorem H is used.

Theorem 2.1. *Suppose that $\chi_1, \dots, \chi_d$ are pairwise non - equivalent Dirichlet characters, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $rank(A_j) = l_j, j = 1, \dots . r$. For $j = 1, \dots, d$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, and, for $j = 1, \dots, r$, $l = 1, \dots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl}(s) \in H(K_{jl})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le d} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r} \sup_{1\le l\le k_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\} > 0.$$

Proof of Theorem 2.1, as of Theorem 1.1, is probabilistic and based on a multidimensional limit theorem for a collection of functions

$$L(s,\chi_1), \dots, L(s,\chi_d), \zeta(s,\alpha_1;\mathfrak{a}_{11}), \dots, \zeta(s,\alpha_1;\mathfrak{a}_{1l_1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{r1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{rl_r}).$$

Universality of functions plays an important role in analytic number theory, and in approximation theory of analytic functions. Therefore, there exists a problem to extend the class of universal functions. One of ways for solving this problem is the investigation of universality of composite functions. The first results in this direction were obtained in [38] and [42], where the universality of composite functions $F(\zeta(s))$ for some operators $F : H(D) \to H(D)$ was considered. For example, the following theorem takes place. Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Theorem J. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon\} > 0.$$

In [46], the ideas of the paper [38] were applied for the Hurwitz zeta - function $\zeta(s,\alpha)$. For example, in [46], the following generalization of Theorem D was proved.

Theorem K. *Suppose that the number α is transcendental, and that $F : H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\} > 0.$$

The paper [43] is devoted to the universality if the functions $F(\zeta(s), \zeta(s,\alpha))$ for some operators $F : H^2(D) \to H(D)$. One of examples is contained in Theorem L.

Theorem L. *Suppose that the number α is transcendental, and that $F : H^2(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S \times H(D))$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau), \zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\} > 0.$$

Finally, in [43], the universality of the composite functions $F(\zeta(s,\alpha_1), \dots, \zeta(s,\alpha_r))$ has been considered. We state one theorem from [43].

Theorem M. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that $F : H^r(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\zeta(s+i\tau,\alpha_1), \dots, \zeta(s+i\tau,\alpha_r)) - f(s)| < \varepsilon\} > 0.$$

Chapter 3 of the thesis is devoted to a generalization of Theorem 1.1. Here the universality of the function $F(L(s,\chi_1), \dots, L(s,\chi_{r_1}), \zeta(s,\alpha_1), \dots, \zeta(s,\alpha_{r_2}))$ is discussed, and several theorems are obtained.

Theorem 3.1. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S^{r_1} \times H^{r_2}(D))$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(L(s+i\tau,\chi_1), \dots, L(s+i\tau,\chi_{r_1}),$$

$$\zeta(s+i\tau,\alpha_1), \dots, \zeta(s+i\tau,\alpha_{r_2})) - f(s)| < \varepsilon\} > 0.$$

Theorem 3.1 is rather general, however, its hypothesis is difficultly checked. The next theorem is a simple modification of Theorem 3.1.

Theorem 3.2. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^{r_1} \times H^{r_2}(D))$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$, the assertion of Theorem 3.1 is true.*

We give an example of Theorem 3.2.

Corollary 3.3. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1. Let $\{j_1, \dots, j_r\} \neq \varnothing$ be a an arbitrary subset of $\{1, \dots, r_1\}$, and $\{l_1, \dots, l_k\} \neq \varnothing$ be a an arbitrary subset of $\{1, \dots, r_2\}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |L(s+i\tau,\chi_{j_1}) \dots L(s+i\tau,\chi_{j_r}) \times$$

$$\times\, \zeta(s+i\tau,\alpha_{l_1}) \dots \zeta(s+i\tau,\alpha_{l_k}) - f(s)| < \varepsilon\} > 0.$$

The non - vanishing of a polynomial in a bounded region can be controlled by its constant term. Therefore, in some cases, it is more convenient to consider the space of analytic functions on a bounded region. For $V > 0$, let $D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\}$, and

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Then we have the following modification of Theorem 3.2.

Theorem 3.4. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, $K \in \mathcal{K}$ and $f(s) \in H(K)$, and let $V > 0$ be such that $K \subset D_V$.*

Let $F : H^{r_1}(D_V) \times H^{r_2}(D) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V^{r_1} \times H^{r_2}(D))$ is non - empty.Then the assertion of Theorem 3.1 is true.

In the next two theorems, we approximate analytic functions from a certain subclass of $H(D)$. For arbitrary distinct complex numbers $a_1, \dots, a_k$, define

$$H_k(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D), j = 1, \dots, k\}.$$

Theorem 3.5. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator such that $F(S^{r_1} \times H^{r_2}(D)) \supset H_r(D)$. If $k = 1$, then let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. If $k \geq 2$, then let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_r(D)$. Then the assertion of Theorem 3.1 is true.*

For example, Theorem 3.5 implies the universality of the functions $\sin(L(s, \chi_1)+L(s, \chi_2)+\zeta(s, \alpha_1)+\zeta(s, \alpha_2))$ and $\cos(L(s, \chi_1) + L(s, \chi_2) + \zeta(s, \alpha_1) + \zeta(s, \alpha_2))$ with non - equivalent character χ_1 and χ_2, and algebraically independent α_1 and α_2.

The next theorem approximates the functions form the set $F(S^{r_1} \times H^{r_2}(D))$.

Theorem 3.6. *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator. Let K be a compact subset of the strip D, and $f(s) \in F(S^{r_1} \times H^{r_2}(D))$. Then the assertion of Theorem 3.1 is true.*

The results of Chapter 3 were obtained in [15] and [17].

In Chapter 4 of the thesis, the universality of the composite function $F(L(s, \chi_1), \dots, L(s, \chi_d)$, $\zeta(s, \alpha_1; \mathfrak{a}_{11})$, $\dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}))$ for some classes of operators F is investigated. First the approximation of functions from the space $H(D)$ is discussed. Let $v = d + l_1 + \dots + l_r$. The operator $F : H^v(D) \to H(D)$ belongs to the class $Lip(\beta_1, \dots, \beta_v)$, $\beta_1 > 0, \dots, \beta_v > 0$, if the following hypotheses are satisfied:

1^o For every polynomial $p = p(s)$ and all sets $K_1, \dots, K_d \in \mathcal{K}$, there exists an element $(g_1, \dots, g_d$, $g_{11}, \dots, g_{1l_1}, g_{r1}, \dots, g_{rl_r}) \in F^{-1}\{p\} \subset H^v(D)$ such that $g_j(s) \neq 0$ on $K_j, j = 1, \dots, d$;

2^o For all $K \in \mathcal{K}$, there exist a constant $c > 0$ and sets $K_1, \dots, K_v \in \mathcal{K}$ such that, for all $(g_{j1}, \dots, g_{jv}) \in H^v(D), j = 1, 2$,

$$\sup_{s \in K} |F(g_{11}(s), \dots, g_{1v}(s)) - F(g_{21}(s), \dots, g_{2v}(s)| \leq c \sup_{1 \leq j \leq v} \sup_{s \in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}.$$

Theorem 4.1 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and $F \in Lip(\beta_1, \dots, \beta_v)$.*

Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} meas\{\tau \in [0;T] : \sup_{s\in K} |F(L(s+i\tau,\chi_1),\ldots,L(s+i\tau,\chi_d),$$

$$\zeta(s+i\tau,\alpha_1;\mathfrak{a}_{11}),\ldots,\zeta(s+i\tau,\alpha_1;\mathfrak{a}_{1l_1}),\ldots,$$

$$\zeta(s+i\tau,\alpha_r;\mathfrak{a}_{r1}),\ldots,\zeta(s+i\tau,\alpha_r;\mathfrak{a}_{rl_r})) - f(s)| < \varepsilon\} > 0.$$

For example, the operator $F(g_1,\ldots,g_d,g_{11},\ldots,g_{1l_1},g_{r1},\ldots,g_{rl_r}) = c_1g_1^{(n_1)} + \ldots + c_dg_d^{(n_d)} + \ldots + c_{11}g_{11}^{(n_{11})} + \ldots + c_{1l_1}g_{1l_1}^{(n_{1l_1})} + \ldots + c_{r1}g_{r1}^{(n_{r1})} + \ldots + c_{rl_r}g_{rl_r}^{(n_{rl_r})}$, where $c_1,\ldots,c_d,c_{11},\ldots,c_{1l_1},\ldots,c_{r1}$, $\ldots,c_{rl_r} \in \mathbb{C}\setminus\{0\}$, $n_1,\ldots,n_d,n_{11},\ldots,n_{1l_1},\ldots,n_{r1},\ldots,n_{rl_r} \in \mathbb{N}$, and $f^{(n)}$ denotes the n th derivative of the function f, belongs to the class $Lip(1,\ldots,1)$.

Other theorems of Chapter 4 are analogues of those from Chapter 3. Let

$$v_1 = \sum_{j=1}^{r} l_j.$$

Theorem 4.3 *Suppose that the Dirichlet characters $\chi_1,\ldots,\chi_d$, the numbers $\alpha_1,\ldots,\alpha_r$ and the sequences $\mathfrak{a}_{11},\ldots,\mathfrak{a}_{1l_1},\ldots,\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G)\cap(S^d\times H^{v_1}(D))$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 4.1 is true.*

Theorem 4.3 implies the following modification of Theorem 4.1.

Theorem 4.4 *Suppose that the Dirichlet characters $\chi_1,\ldots,\chi_d$, the numbers $\alpha_1,\ldots,\alpha_r$ and the sequences $\mathfrak{a}_{11},\ldots,\mathfrak{a}_{1l_1},\ldots,\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{rl_r}$ satisfy the hypothesis of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D))$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 4.1 is true.*

We note that hypothesis 2^0 of the class $Lip(\beta_1,\ldots,\beta_v)$ implies the continuity of the operator F. However, hypothesis 1^0 is weaker than the requirement that $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$.

Sometimes, in Theorem 4.3, it is more convenient to use the space $H^v(D_V,D) = H^d(D_V)\times H^{v_1}(D)$ in place of $H^v(D)$. In notation of Theorem 3.4, the following version of Theorem 4.3 is true.

Theorem 4.5 *Suppose that the Dirichlet characters $\chi_1,\ldots,\chi_d$, the numbers $\alpha_1,\ldots,\alpha_r$ and the sequences $\mathfrak{a}_{11},\ldots,\mathfrak{a}_{1l_1},\ldots,\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, $K \in \mathcal{K}$ and $f(s) \in H(K)$, and $V > 0$ is such that $K \subset D_V$. Let $F : H^v(D_V,D) \to H(D_V)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V^d \times H^{v_1}(D))$ is non - empty. Then the assertion of Theorem 4.1 is true.*

Theorem 4.5 can be applied for the operator

$$F(g_1, \dots, g_v) = c_1 g_1 + \dots + c_d g_d, \quad c_1, \dots, c_d \in \mathbb{C} \setminus \{0\}.$$

Other theorems of Chapter 4 are devoted to the approximation of analytic functions from the image of the set $S^d \times H^{v_1}(D)$ of the operator $F : H^v(D) \to H(D)$.

Theorem 4.8 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact subset and $f(s) \in F(S^d \times H^u(D))$. Then the assertion of Theorem 4.1 is true.*

It is not easy to describe the set $F(S^d \times H^{v_1}(D))$. The next theorem of Chapter 4 is an example with sufficiently simple set contained in $F(S^d \times H^{v_1}(D))$. Namely, the following analogue of Theorem 3.5 is valid.

Theorem 4.9 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that $F(S^d \times H^u(D)) \supset H_k(D)$. For $k = 1$, let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $k \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_k(D)$.Then the assertion of Theorem 4.1 is true.*

A part of the results of Chapter 4 are contained in [16].

Approbation

The results of the thesis were presented at the MMA (Mathematical Modeling and Analysis) conferences (MMA 2012, June 6 - 9, 2012, Tallinn, Estonia), (MMA 2013, May 27 - 30, 2013, Tartu, Estonia), (MMA 2014, May 26 - 39,2014, Druskininkai, Lithuania), at International Conference Algebra, Number Theory and Discrete Geometry: Modern Problems and Applications (May 25 - 30, 2015, Tula, Russia), at the Conferences of Lithuanian Mathematical Society (2013, 2014, 2015), as well as at the Number Theory seminars of Vilnius University.

Principal publications

1. K. Janulis, Remarks on the joint universality of Dirichlet L - functions and Hurwitz zeta - functions, Šiauliai Math. Semin. **9** (**17**) (2014), 61 - 70.
2. K. Janulis, Mixed joint universality of Dirichlet L - functions and Hurwitz type zeta - functions, in: Materialy XVIII mezh. konf. Algebra, teorija chisel i disk. geom: sovremennye problemy u prilosheniya, Tula 2015, Uzd. Tul. goc. neg. usuv. im. L. N. Tolstogo, Tula, 2015, pp: 210 - 213.
3. K. Janulis and A. Laurinčikas, Joint universality of Dirichlet L - functions and Hurwitz zeta - functions, Ann. Univ. Sci. Budapest., Sect. Comp. **39** (2013), 203 - 214.
4. K. Janulis, A. Laurinčikas, R. Macaitienė and D. Šiaučiūnas, Joint universality of Dirichlet L - functions and periodic Hurwitz zeta - functions, Math. Modelling and Analysis **17** (2012), 673 - 685.
5. K. Janulis, D. Jurgaitis, A. Laurinčikas, R. Macaitienė, Universality of some composite functions, (approved).

Conference abstracts

1. K. Janulis, Joint universality of Dirichlet L - functions and Hurwitz zeta-functions, 17th International Conference on Mathematical Modelling and Analysis, Abstracts, Tallinn University of Technology, 2012, p. 58.
2. K. Janulis, On joint universality of Dirichlet L-functions and periodic Hurwitz zeta - functions, 18th International Conference: Mathematical Modelling and Analysis and 4th International Conference: Approximation Methods and Orthogonal Expansions, Abstracts, Institute of Mathematics of the University of Tartu, 2013, p. 50.
3. K. Janulis, Universality of some functions related to Dirichlet L - functions and Hurwitz zeta - functions, 19th International Conference Mathematical Modelling and Analysis, Abstracts, Technika, Vilnius, 2014, p. 29.

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Chapter 1

Joint universality of Dirichlet L - functions and Hurwitz zeta - functions

In this chapter, we prove a joint universality theorem on the simultaneous approximation of a collection of given analytic functions by shifts $L(s+i\tau, \chi_j)$ of Dirichlet L - functions

$$L(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)}{m^s}, \quad \sigma > 1, \quad j = 1, \dots, r_1,$$

and shifts $\zeta(s+i\tau, \alpha_j)$ of Hurwitz zeta - functions

$$\zeta(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha_j)^s}, \quad \sigma > 1, \quad j = 1, \dots, r_2.$$

Since Dirichlet L - functions have the Euler product over primes

$$L(s, \chi_j) = \prod_p \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}, \quad \sigma > 1, \quad j = 1, \dots, r_1,$$

and Hurwitz zeta - functions have no such a product if $\alpha_j \neq 1, \frac{1}{2}$, the theorem of this chapter is a mixed joint universality theorem, and it connects a joint universality theorem for Dirichlet L - functions and a joint universality theorem for Hurwitz zeta - functions. The functions of the collection

$$(L(s, \chi_1), \dots, L(s, \chi_{r_1}), \zeta(s, \alpha_1), \dots, \zeta(s, \alpha_{r_2})) \tag{1.1}$$

must be in some sense independent, therefore, we use additional hypotheses on the characters $\chi_1, \dots, \chi_{r_1}$ and parameters $\alpha_1, \dots, \alpha_{r_2}$.

We recall that $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, $\mathcal{K}$ is the class of compact subsets of the strip D with connected complements, $H(K), K \in \mathcal{K}$, is the class of continuous functions on K which are analytic in the interior of K, and $H_0(K), K \in \mathcal{K}$, is the subclass of $H(K)$ of non-vanishing functions on K.

The main result of this chapter is the following theorem.

Theorem 1.1. *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters, and that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \dots, r_1$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, while, for $j = 1, \dots, r_2$, let $\widehat{K}_j \in \mathcal{K}$ and $\widehat{f}_j(s) \in H(\widehat{K}_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\leq j\leq r_1} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\leq j\leq r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau, \alpha_j) - \widehat{f}_j(s)| < \varepsilon\} > 0.$$

We see that the independence of the functions of the collection (1.1) is expressed by the non-equivalence of the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$ and by the algebraical independence of parameters $\alpha_1, \dots, \alpha_{r_2}$.

Theorem 1.1 has a series of corollaries on the universality of composite functions which will be presented in Chapter 3.

Proof of Theorem 1.1 is based on probabilistic limit theorems on weakly convergent probability measures in the space of analytic functions.

1.1. A limit theorem

Denote by γ the unit circle on the complex plane, i.e.,

$$\gamma = \{s \in \mathbb{C} : |s| = 1\},$$

and define two infinite - dimensional tori

$$\Omega = \prod_p \gamma_p$$

and

$$\widehat{\Omega} = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori Ω and $\widehat{\Omega}$ with the product topology and operation of pointwise multiplication, by the Tikhonov theorem, see, for example [72], are compact topological Abelian groups. Let $\kappa = 1 + r_2$, and

$$\Omega^\kappa = \Omega \times \widehat{\Omega}_1 \times \dots \times \widehat{\Omega}_{r_2},$$

where $\widehat{\Omega}_j = \widehat{\Omega}$ for $j = 1, \dots, r_2$. Then again, by the Tikhonov theorem, Ω^κ is a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(X)$ the Borel σ - field of the space X, we have that, on

$(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$, the probability Haar measure m_H^κ can be defined. This gives the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$. We note that the measure m_H^κ is the product of the Haar measures m_H and $\widehat{m}_{1H}, \dots, \widehat{m}_{r_2 H}$ on $(\Omega, \mathcal{B}(\Omega))$ and $(\widehat{\Omega}_1, \mathcal{B}(\widehat{\Omega}_1)), \dots, (\widehat{\Omega}_{r_2}, \mathcal{B}(\widehat{\Omega}_{r_2}))$, respectively. For elements of Ω^κ, we use the notation $\underline{\omega} = (\omega, \widehat{\omega}_1, \dots, \widehat{\omega}_{r_2})$, where $\omega \in \Omega$ and $\widehat{\omega}_j \in \widehat{\Omega}_j, j = 1, \dots, r_2$. Moreover, let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathcal{P}$, and let $\widehat{\omega}_j(m)$ denote the projection of $\widehat{\omega}_j \in \widehat{\Omega}_j$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. For brevity, we put $r = r_1 + r_2$. Now, on the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the $H^r(D)$ - valued random element $\Xi(s, \underline{\omega})$ by the formula

$$\Xi(s, \underline{\omega}) = (L(s, \omega, \chi_1), \dots, L(s, \omega, \chi_{r_1}), \zeta(s, \alpha_1, \widehat{\omega}_1), \dots, \zeta(s, \alpha_{r_2}, \widehat{\omega}_{r_2})),$$

where

$$L(s, \omega, \chi_j) = \prod_p \left(1 - \frac{\chi_j(p)\omega(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r_1,$$

and

$$\zeta(s, \alpha_j, \widehat{\omega} j) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r_2.$$

Let P_Ξ be the distribution of the random element $\Xi(s, \underline{\omega})$, i.e.,

$$P_\Xi(A) = m_H^\kappa(\underline{\omega} \in \Omega^\kappa : \Xi(s, \underline{\omega}) \in A), A \in \mathcal{B}(H^r(D)).$$

Moreover, we will use the notation

$$\Xi(s) = (L(s, \chi_1), \dots, L(s, \chi_{r_1}), \zeta(s, \alpha_1), \dots, \zeta(s, \alpha_{r_2})),$$

and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\{\tau \in [0; T] : \Xi(s + i\tau) \in A\}.$$

Let P and $P_n, n \in \mathbb{N}$, be probability measures on $(X, \mathcal{B}(X))$. We recall that P_n converges weakly to P as $n \to \infty$ if, for every real continuous bounded function g on X,

$$\lim_{n \to \infty} \int_X g dP_n = \int_X g dP.$$

We will consider the weak convergence of P_T to P, when T is a continuous parameter, $T \to \infty$. By the definition, this means that, for every real continuous bounded function g on X,

$$\lim_{T \to \infty} \int_X g dP_T = \int_X g dP. \tag{1.2}$$

If the function g is fixed, then the relation (1.2) is true if and only if

$$\lim_{n \to \infty} \int_X g dP_{T_n} = \int_X g dP$$

for every sequence $\{T_n\}, T_n \xrightarrow[n\to\infty]{} \infty$. Consequently, P_T converges weakly to P if and only if P_{T_n} converges weakly to P for any sequence $\{T_n\}$, $T_n \xrightarrow[n\to\infty]{} \infty$. Therefore, the case of continuous parameter does not differ essentially.

Then we have the following limit theorem.

Theorem 1.2. *Suppose that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then P_T converges weakly to P_Ξ as $T \to \infty$.*

Proof. The theorem follows from Theorem 2 of [37]. For $j = 1, \dots, r_1$, let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$, and let $\zeta(s; \mathfrak{a}_j)$ be the corresponding periodic zeta - function. Analogically, for $j = 1, \dots, r_2$, let $\mathfrak{b}_j = \{b_{jm} : m \in \mathbb{N}_0\}$ be an another periodic sequence of complex numbers with minimal period $l_j \in \mathbb{N}$, and let $\zeta(s, \alpha_j; \mathfrak{b}_j), 0 < \alpha_j \leq 1$, be the periodic Hurwitz zeta - function. In [37], under certain additional hypotheses, the joint universality of the functions $\zeta(s; \mathfrak{a}_1), \dots, \zeta(s; \mathfrak{a}_{r_1}), \zeta(s, \alpha_1; \mathfrak{b}_1), \dots, \zeta(s, \alpha_{r_2}; \mathfrak{b}_{r_2})$ was considered, and, first of all, a joint limit theorem in the space $H^r(D)$ was obtained. Let, for brevity, $\underline{\alpha} = (\alpha_1, \dots, \alpha_{r_2}), \underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1}), \underline{\mathfrak{b}} = (\mathfrak{b}_1, \dots, \mathfrak{b}_{r_2})$ and

$$\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}) = (\zeta(s; \mathfrak{a}_1), \dots, \zeta(s; \mathfrak{a}_{r_1}), \zeta(s, \alpha_1; \mathfrak{b}_1), \dots, \zeta(s, \alpha_{r_2}; \mathfrak{b}_{r_2})).$$

Then Theorem 2 of [37] asserts that if the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_{r_1}$ are multiplicative, and that the numbers $\alpha_1, \dots, \alpha_{r_1}$ are algebraically independent over $\mathbb{Q}$, then

$$\frac{1}{T} meas\{\tau \in [0; T] : \underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}) \in A\}, A \in \mathcal{B}(H^r(D)),$$

converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$, where $P_{\underline{\zeta}}$ is the distribution of the $H^r(D)$ - valued random element

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, \underline{\mathfrak{b}}) = (\zeta(s, \omega; \mathfrak{a}_1), \dots, \zeta(s, \omega; \mathfrak{a}_{r_1}), \zeta(s, \widehat{\omega}_1, \alpha_1; \mathfrak{b}_1), \dots, \zeta(s, \widehat{\omega}_{r_2}, \alpha_{r_2}; \mathfrak{b}_{r_2})),$$

where

$$\zeta(s, \omega; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} \omega(m)}{m^s}, j = 1, \dots, r_1,$$

and

$$\zeta(s, \alpha_j; \widehat{\omega}_j; \mathfrak{b}_j) = \sum_{m=0}^{\infty} \frac{b_{jm} \widehat{\omega}_j(m)}{(m + \alpha_j)^s}, j = 1, \dots, r_2.$$

It is not difficult to see that Theorem 1.2 satisfies the hypotheses of Theorem 2 of [37]. Indeed, the characters $\chi_1, \dots, \chi_{r_1}$ are periodic completely multiplicative functions. Therefore, Dirichlet L - functions $L(s, \chi_1), \dots, L(s, \chi_{r_1})$ are partial case of the periodic zeta - functions. Moreover, obviously, the Hurwitz zeta - functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_{r_2})$ are partial case of the periodic Hurwitz zeta - functions $\zeta(s, \alpha_1; \mathfrak{b}_1), \dots, \zeta(s, \alpha_{r_2}; \mathfrak{b}_{r_2})$. In this case, $b_{j1} = \dots = b_{jr_2} \equiv 1$.

1.2. Support of P_Ξ

Let X be a separable metric space, and P be a probability measure on $(X, \mathcal{B}(X))$. We remind that a minimal closed set $S_P \in \mathcal{B}(X)$ is called the support of the measure P if $P(S_P) = 1$. The set S_P consists of all elements $x \in X$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied.

For the proof of Theorem 1.1, the explicit form of the support of the measure P_Ξ is needed. We will use the following lemmas. Let, for brevity, $\underline{\chi} = (\chi_1, \dots, \chi_{r_1})$, $\underline{\widehat{\omega}} = (\widehat{\omega}_1, \dots, \widehat{\omega}_{r_2})$,

$$\underline{L}(s, \omega, \underline{\chi}) = (L(s, \omega, \chi_1), \dots, L(s, \omega, \chi_{r_1}))$$

and

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\widehat{\omega}}) = (\zeta(s, \alpha_1, \omega_1), \dots, \zeta(s, \alpha_{r_2}, \omega_{r_2})).$$

Moreover, let $P_{\underline{L}}$ denote the distribution of the random element $\underline{L}(s, \omega, \underline{\chi})$, i.e.,

$$P_{\underline{L}}(A) = m_H(\omega \in \Omega : \underline{L}(s, \omega, \underline{\chi}) \in A), A \in \mathcal{B}(H^{r_1}(D)),$$

where m_H is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$, and let $P_{\underline{\zeta}}$ denote the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\widehat{\omega}})$, i.e.,

$$P_{\underline{\zeta}}(A) = \widehat{m}_H(\underline{\widehat{\omega}} \in \widehat{\Omega}_1 \times \ldots \times \widehat{\Omega}_{r_2} : \underline{\zeta}(s, \underline{\alpha}, \underline{\widehat{\omega}}) \in A), A \in \mathcal{B}(H^{r_2}(D)),$$

where $\widehat{m}_H$ is the probability Haar measure on $(\widehat{\Omega}_1 \times \ldots \times \widehat{\Omega}_{r_2}, \mathcal{B}(\widehat{\Omega}_1 \times \ldots \times \widehat{\Omega}_{r_2}))$. Define the set $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Lemma 1.3. *Suppose that $\chi_1, \dots \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters. Then the support of the measure $P_{\underline{L}}$ is the set S^{r_1}.*

Proof of the lemma is given in [40], Lemma 12.

Lemma 1.4. *Suppose that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $H^{r_2}(D)$.*

Proof. Let $L(\alpha_1, \dots, \alpha_{r_2}) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r_2\}$. Then in [35], Theorem 11, it was proved that if the set $L(\alpha_1, \dots, \alpha_{r_2})$ is linearly independent over $\mathbb{Q}$, then the support of $P_{\underline{\zeta}}$ is the set $H^{r_2}(D)$. However, if the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$, then the set $L(\alpha_1, \dots, \alpha_{r_2})$ is linearly independent over $\mathbb{Q}$. Indeed, suppose, on the contrary, that this set is linearly dependent over $\mathbb{Q}$, i.e., there exist integers $k_1, \dots, k_v$, not all zeros, such that

$$k_1 \log(m_1 + \alpha_1) + \ldots + k_v \log(m_v + \alpha_v) = 0.$$

Suppose that $k_1 > 0, \dots, k_u > 0$ and $k_{u+1} < 0, \dots, k_v < 0$. Then

$$(m_1 + \alpha_1)^{k_1} \dots (m_u + \alpha_u)^{k_u} = (m_{u+1} + \alpha_{u+1})^{k_{u+1}} \dots (m_v + \alpha_v)^{k_v}.$$

However, this contradicts the algebraic independence of the numbers $\alpha_1, \dots, \alpha_v$.

Therefore, the lemma follows from Theorem 11 of [35].

Theorem 1.5. *Suppose tat $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters, and that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the support of P_Ξ is the set $S^{r_1} \times H^{r_2}(D)$.*

Proof. The spaces H^{r_1} and H^{r_2} are separable, therefore [13]

$$\mathcal{B}(H^{r_1+r_2}(D)) = \mathcal{B}(H^{r_1}(D)) \times \mathcal{B}(H^{r_2}(D)).$$

Therefore, it suffices to consider P_Ξ on the set $A = A_1 \times A_2$, where $A_1 \in \mathcal{B}(H^{r_1}(D))$ and $A_2 \in \mathcal{B}(H^{r_2}(D))$. Moreover, we observe that the measure m_H^κ is the product of the measures m_H and $\widehat{m}_H$ in the above notation, i.e.,

$$m_H^\kappa(A_1 \times A_2) = m_H(A_1)\widehat{m}_H(A_2).$$

Therefore,

$$P_\Xi = m_H^\kappa(\underline{\omega} \in \Omega^\kappa : \Xi(s, \underline{\omega}) \in A) =$$

$$= m_H(\omega \in \Omega : L(s, \omega, \underline{\chi}) \in A_1) \times$$

$$\widehat{m}_H(\underline{\widehat{\omega}} \in \widehat{\Omega}_1 \times \dots \times \widehat{\Omega}_{r_2} : \underline{\zeta}(s, \underline{\alpha}, \underline{\widehat{\omega}}) \in A_2).$$

This shows that $P_\Xi(A) = 1$ if and only if the equalities

$$m_H(\omega \in \Omega : L(s, \omega, \underline{\chi}) \in A_1) = 1 \tag{1.3}$$

and

$$\widehat{m}_H(\underline{\widehat{\omega}} \in \widehat{\Omega}_1 \times \dots \times \widehat{\Omega}_{r_2} : \underline{\zeta}(s, \underline{\alpha}, \underline{\widehat{\omega}}) \in A_2) = 1 \tag{1.4}$$

hold. However, in view of Lemma 1.3, the minimal set A_1 satisfying (1.3) is $S^{r_1}(D)$, and, by Lemma 1.4, the minimal set A_2 satisfying (1.4) is $H^{r_2}(D)$. Therefore, the minimal closed set A such that $P_\Xi(A) = 1$ is the set $S^{r_1} \times H^{r_2}(D)$.

1.3. Proof of the universality theorem

We start with the statement of the Mergelyan theorem on the approximation of analytic functions by polynomials. We state it as a separate lemma.

Lemma 1.6. *Suppose that $K \subset D$ is a compact subset with connected complement, and that $f(s)$ is a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

The proof of lemma is given in [60], see also [81].

We also need an equivalent of the weak convergence of probability measures in terms of open sets. Taking into account a remark before the statement of Theorem 1.2, we state this equivalent for a sequence of probability measures.

Lemma 1.7. *Let P and $P_n, n \in \mathbb{N}$, be probability measures on $(X, \mathcal{B}(X))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every open set G of X,*

$$\liminf_{n \to \infty} P_n(G) \geq P(G).$$

The lemma is a part of Theorem 2.1 from [3], and there its proof given.

Proof of Theorem 1.1. By Lemma 1.6, there exist polynomials $p_1(s), \dots, p_{r_1}(s)$ and $\widehat{p}_1(s), \dots, \widehat{p}_{r_2}(s)$ such

$$\sup_{1 \leq j \leq r_1} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \tag{1.5}$$

and

$$\sup_{1 \leq j \leq r_2} \sup_{s \in \widehat{K}_j} |\widehat{f}_j(s) - \widehat{p}_j(s)| < \frac{\varepsilon}{2}. \tag{1.6}$$

Define

$$G = \{g_1, \dots, g_{r_1}, \widehat{g}_1, \dots, \widehat{g}_{r_2}) \in H^r(D) : \sup_{1 \leq j \leq r_1} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2},$$

$$\sup_{1 \leq j \leq r_2} \sup_{s \in \widehat{K}_j} |\widehat{g}_j(s) - \widehat{p}_j(s)| < \frac{\varepsilon}{2}\}.$$

Then G is an open set in $H^r(D)$. Moreover, by Theorem 1.5, the collection

$$(e^{p_1(s)}, \dots, e^{p_{r_1}(s)}, \widehat{p}_1(s), \dots, \widehat{p}_{r_2}(s)) \in H^r(D)$$

is an element of the support of the measure P_Ξ. This means that the set G is an open neighbourhood of an element of the support of P_Ξ. Therefore, the properties of the support imply the inequality $P_\Xi(G) > 0$. From this, using Theorem 1.2 and Lemma 1.7, we find that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \Xi(s + i\tau) \in G\} \geq P_\Xi(G) > 0,$$

or, by the definition of the set G and vector Ξ,

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - e^{p_j(s)}| < \frac{\varepsilon}{2},$$

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{p}_j(s)| < \frac{\varepsilon}{2}\} > 0. \tag{1.7}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - e^{p_j(s)}| < \frac{\varepsilon}{2}.$$

Then it follows from (1.5) that, for such τ,

$$\sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| \le \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - e^{p_j(s)}| +$$

$$+ \sup_{1\le j\le r_1} \sup_{s\in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, inequality (1.6) shows that, for $\tau \in \mathbb{R}$ satisfying the inequality

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{p}_j(s)| < \frac{\varepsilon}{2},$$

the inequality

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon$$

is valid. From these remarks, it follows that

$$\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - e^{p_j(s)}| < \frac{\varepsilon}{2},$$

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{p}_j(s)| < \frac{\varepsilon}{2}\} \subset$$

$$\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon\}.$$

Therefore, taking into account inequality (1.7), we obtain that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{f}_j(s)| < \varepsilon\} > 0.$$

The theorem is proved.

Chapter 2

Joint universality of Dirichlet L - functions and periodic Hurwitz zeta - functions

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$, and let α, $0 < \alpha \leq 1$, be a fixed parameter. We remind that the periodic Hurwitz zeta - function $\zeta(s, \alpha; \mathfrak{a})$ is of the form

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \sigma > 1,$$

and has meromorphic continuation to the whole complex plane. For $j = 1, \dots, r$, let α_j, $0 < \alpha_j \leq 1$, be a fixed parameter, $l_j \in \mathbb{N}$, $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$. In this chapter, we obtain a joint universality theorem for the functions

$$L(s, \chi_1), \dots, L(s, \chi_d), \zeta(s, \alpha_1, \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \mathfrak{a}_{rl_r}). \tag{2.1}$$

Periodic Hurwitz zeta - functions, as classical Hurwitz zeta - functions, also have no Euler product over primes. Therefore, the joint universality of the functions (2.1) is also of mixed type. Now we state the main result of the chapter.

For $j = 1, \dots, r$, denote by k_j the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, and define the matrix

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \cdots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \cdots & a_{2jl_j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_jj1} & a_{k_jj2} & \cdots & a_{k_jjl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Theorem 2.1. *Suppose that $\chi_1, \dots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that rank$(A_j) = l_j$, $j = 1 \dots, r$. For $j = 1, \dots, d$, let $K_j \in \mathcal{K}$, and $f_j(s) \in H_0(K_j)$, and, for $j = 1, \dots, r$, $l = 1, \dots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl}(s) \in H(K_{jl})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le d} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+it, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\} > 0.$$

For the proof of Theorem 2.1, a limit theorem in the space of analytic functions for the functions (2.1) is applied.

2.1. An extended multidimensional limit theorem

We use the same infinite - dimensional tori Ω and $\widehat{\Omega}$, however, it is more convenient, in this chapter, to use the notation Ω_1 in place of $\widehat{\Omega}$. Define

$$\Omega^\kappa = \Omega \times \Omega_{11} \times \ldots \times \Omega_{1r},$$

where $\Omega_{1j} = \Omega_1$ for all $j = 1, \dots, r$, and $\kappa = 1+r$. Then we have the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, where m_H^κ is the probability Haar measure on $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$. Let $\Omega_1^r = \Omega_{11} \times \ldots \times \Omega_{1r}$. Then the measure m_H^κ is the product of probability Haar measures m_H and m_H^r on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega^r, \mathcal{B}(\Omega^r))$, respectively.

Let

$$u = \sum_{j=1}^{r} l_j,$$

and

$$H_{d,u} = H_{d,u}(D) = H^d(D) \times H^u(D).$$

On the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the $H_{d,u}$ - valued random element using the following notation. We denote by $\omega(p)$ the projection of $\omega \in \Omega$ to $\gamma_p, p \in \mathcal{P}$, and by brevity, $\underline{\omega} = (\omega, \omega_1, \dots, \omega_r)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\chi} = (\chi_1, \dots, \chi_d)$ and $\mathfrak{a} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$. Let the $H_{d,u}$ - valued random element $\Xi(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ be given by

$$\Xi(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = (L(s, \omega, \chi_1), \dots, L(s, \omega, \chi_d), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots,$$

$$\zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_1,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_1,\omega_r;\mathfrak{a}_{rl_r})),$$

where

$$L(s,\omega,\chi_j)=\prod_p\left(1-\frac{\chi_j(p)\omega(p)}{p^s}\right)^{-1},\quad j=1,\dots,d,$$

and

$$\zeta(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega(p)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\quad l=1,\dots,l_j.$$

Denote by P_Ξ the distribution of the random element $\Xi(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$, i.e., for $A\in\mathcal{B}(H_{d,u})$,

$$P_\Xi(A)=m_H^\kappa(\underline{\omega}\in\Omega^\kappa:\Xi(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A).$$

Moreover, let

$$\Xi(s,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})=(L(s,\chi_1),\dots,L(s,\chi_d),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,$$

$$\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{rl_r})).$$

In this section, we consider the weak convergence of

$$P_T(A)\stackrel{\text{def}}{=}\frac{1}{T}meas\{\tau\in[0;T]:\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})\in A\},A\in\mathcal{B}(H_{d,u}),$$

as $T\to\infty$.

Theorem 2.2. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then P_T converges weakly to P_Ξ as $T\to\infty$.*

We note that Theorem 2.2 is more general than Theorem 1.2, therefore, we will give its proof.

Lemma 2.3. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then*

$$P_T(A)\stackrel{def}{=}\frac{1}{T}meas\{\tau\in[0;T]:((p^{-i\tau}:p\in\mathcal{P}),((m+\alpha_1)^{-i\tau}:m\in\mathbb{N}_0),$$

$$\dots,((m+\alpha_r)^{-i\tau}:m\in\mathbb{N}_0))\in A\},A\in\mathcal{B}(\Omega^\kappa),$$

converges weakly to the Haar measure m_H^κ as $T\to\infty$.

The proof of the lemma is given in [37], Theorem 3.

Let $\sigma_0>\frac{1}{2}$ be a fixed parameter number, and

$$v_n(m)=\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_0}\right\},\quad m,n\in\mathbb{N}.$$

$$v_n(m,\alpha)=\exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_0}\right\},\quad m\in\mathbb{N}_0,n\in\mathbb{N}.$$

Define

$$L_n(s,\chi_j)=\sum_{m=1}^{\infty}\frac{\chi_j(m)v_n(m)}{(m)^s},\quad j=1,\dots,d,$$

$$\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\quad l=1,\dots,l_j,$$

and, for $\underline{\omega}_0=(\omega_0,\omega_{10},\dots,\omega_{r0})\in\Omega^\kappa$, put

$$L_n(s,\chi_j,\omega_0)=\sum_{m=1}^{\infty}\frac{\chi_j(m)\omega_0(m)v_n(m)}{m^s},\quad j=1,\dots,d,$$

$$\zeta_n(s,\alpha_j,\omega_{0j};\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_{0j}(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\quad l=1,\dots,l_j.$$

It is known [19], [40] that all latter series are absolutely convergent for $\sigma>\frac{1}{2}$. Here the function $\omega(p)$ is extended to the set $\mathbb{N}$ by the formula

$$\omega(m)=\prod_{\substack{p^k|m\\ p^{k+1}\nmid m}}\omega^k(p),\quad m\in\mathbb{N}.$$

Let, for brevity,

$$\Xi_n(s,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})=(L_n(s,\chi_1),\dots,L_n(s,\chi_d),\zeta_n(s,\alpha_1;\mathfrak{a}_{11}),\dots,$$

$$\zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta_n(s,\alpha_1;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_1;\mathfrak{a}_{rl_r})),$$

and

$$\Xi_n(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})=(L_n(s,\omega,\chi_1),\dots,L_n(s,\omega,\chi_d),\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11}),\dots,$$

$$\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\zeta_n(s,\alpha_1,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_1,\omega_r;\mathfrak{a}_{rl_r})).$$

In the sequel, the following property of the weak convergence of probability measures will be useful. Let X_1 and X_2 be two metric spaces. The mapping $h:X_1\to X_2$ is called $(\mathcal{B}(X_1),\mathcal{B}(X_2))$ - measurable if $h^{-1}A\in\mathcal{B}(X_1)$ for every $A\in\mathcal{B}(X_2)$, in other words, $h^{-1}\mathcal{B}(X_2)\subset\mathcal{B}(X_1)$. It is well known, see, for example, [3], that every continuous mapping $h:X_1\to X_2$ is $(\mathcal{B}(X_1),\mathcal{B}(X_2))$ - measurable. If h is $(\mathcal{B}(X_1),\mathcal{B}(X_2))$ - measurable, than every probability measure P on $(X_1,\mathcal{B}(X_1))$ induces on $(X_2,\mathcal{B}(X_2))$ the unique probability measure Ph^{-1} given by the formula

$$Ph^{-1}(A)=P(h^{-1}A),\quad A\in\mathcal{B}(X_2).$$

Moreover, the following statement is true.

Lemma 2.4. *Suppose that $P_n, n\in\mathbb{N}$, and P are probability measures on $(X_1,\mathcal{B}(X_1))$, and that $h:X_1\to X_2$ is a continuous mapping. If P_n, as $n\to\infty$, converges weakly to P, then also P_nh^{-1}, as $n\to\infty$, converges weakly to Ph^{-1}.*

The lemma is partial case of theorem 5.1 from [3].

Lemma 2.5. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, on $(H_{d,u}, \mathcal{B}(H_{d,u}))$, there exists a probability measure P_n such that*

$$P_{T,n}(A) \overset{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : \Xi_n(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\}, \quad A \in \mathcal{B}(H_{d,u}),$$

and

$$P_{T,n,\underline{\omega}_0}(A) \overset{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : \Xi_n(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}}) \in A\}, \quad A \in \mathcal{B}(H_{d,u}),$$

both converges weakly to P_n as $T \to \infty$.

Proof. Define the function $h_n : \Omega^\kappa \to H_{d,u}$ by the formula

$$h_n(\underline{\omega}) = \Xi_n(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}), \underline{\omega} \in \Omega^\kappa.$$

In view of the absolute convergence of the series for the functions $L_n(s, \chi_j, \omega)$ and $\zeta_n(s, \alpha_j, \omega_j; \mathfrak{a}_{jl})$, the function h_n is continuous. Moreover,

$$h_n((p^{-i\tau} : p \in \mathcal{P}), ((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) = \Xi_n(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Therefore, we have that $P_{T,n} = Q_T h_n^{-1}$. This, the continuity of the function h_n, and Lemmas 2.3 and 2.4 show that $P_{T,n}$ converges weakly to $P_n = m_H^\kappa h_n^{-1}$ as $T \to \infty$.

Let $h : \Omega^\kappa \to \Omega^\kappa$ be given by formula $h(\underline{\omega}) = \underline{\omega}\underline{\omega}_0, \underline{\omega} \in \Omega^\kappa$, and $h_{n,\underline{\omega}_0} : \Omega^\kappa \to H_{d,u}$ be defined by the formula

$$h_{n,\underline{\omega}_0}(\underline{\omega}) = \Xi_n(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}\underline{\omega}_0; \underline{\mathfrak{a}}), \underline{\omega} \in \Omega^\kappa.$$

Then, repeating the above arguments, we find that $P_{T,n,\underline{\omega}_0}$ converges weakly to $P_{n,\underline{\omega}_0} = m_H^\kappa h_{n,\underline{\omega}_0}^{-1}$ as $T \to \infty$. However, we have that $h_{n,\underline{\omega}_0}(\underline{\omega}) = h_n(h(\underline{\omega}))$. Since the Haar measure m_H^κ is invariant with respect to translations by points from Ω^κ, from this it follows that

$$m_H^\kappa h_{n,\underline{\omega}_0}^{-1} = m_H^\kappa (h_n h)^{-1} = (m_H^\kappa h^{-1}) h_n^{-1} = m_H^\kappa h_n^{-1}.$$

Thus, $P_{T,n,\underline{\omega}_0}$ also converges weakly to the measure $P_n = m_H^\kappa h_n^{-1}$ as $T \to \infty$. The lemma is proved.

For the proof of Theorem 2.2, it remains to pass from the measure $P_{T,n}$ to P_T. For this, the metric on $H_{d,u}$ is needed. Let $\{K_v : v \in \mathbb{N}\}$ be a sequence of compact subsets of the strip D such that

$$D = \bigcup_{v=1}^{\infty} K_v,$$

$K_v \subset K_{v+1}$ for all $v \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_v$ for some v. The existence of such a sequence $\{K_v : v \in \mathbb{N}\}$ is proved in [5]. For $g_1, g_2 \in H(D)$, define

$$\varrho(g_1, g_2) = \sum_{v=1}^{\infty} 2^{-v} \frac{\sup_{s\in K_v} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_v} |g_1(s) - g_2(s)|}.$$

Then ϱ is a metric on the space $H(D)$ inducing the topology of uniform convergence on compacta. For $\underline{g}_j = (g_{j1}, \dots, g_{j,d+u}) \in H_{d,u}, \quad j = 1, 2$, we put

$$\varrho_{d+u}(\underline{g}_1, \underline{g}_2) = \max_{1\le l\le d+u} \varrho(g_{1l}, g_{2l}).$$

Then ϱ_{d+u} is a desired metric on $H_{d,u}$. Let ϱ_d and ϱ_u be analogical metrics on $H^d(D)$ and $H^u(D)$, respectively.

Lemma 2.6. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho_{d+u}(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}), \Xi_n(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}))dt = 0$$

holds.

Proof. In [40], it was proved that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho_d(\underline{L}(s + i\tau, \underline{\chi}), \underline{L}_n(s + i\tau, \underline{\chi})dt = 0, \tag{2.2}$$

where

$$\underline{L}(s, \underline{\chi}) = (L(s, \chi_1), \dots, L(s, \chi_d))$$

and

$$\underline{L}_n(s, \underline{\chi}) = (L_n(s, \chi_1), \dots, L_n(s, \chi_d)).$$

Moreover, in [53], it was obtained that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho_u(\underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}))dt = 0, \tag{2.3}$$

where

$$\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}))$$

and

$$\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r})).$$

Thus, the assertion of the lemma follows from (2.2), (2.3) and the definition of the metric ϱ_{d+u}.

Analogically, from [40] and [53], we deduce

Lemma 2.7. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho_{d+u}(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}), \Xi_n(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}))dt = 0$$

holds for almost all $\underline{\omega} \in \Omega^\kappa$.

In what follows, we will use the notions of the relative compactness and tightness of families of probability measures. Let $\{P\}$ be a family of probability measures on $(X, \mathcal{B}(X))$. We say that the family $\{P\}$ is relatively compact if each sequence $\{P_n\} \subset \{P\}$ contains a weakly convergent subsequence $\{P_{n_k}\}$ to a certain probability measure P as $k \to \infty$. The family $\{P\}$ is tight if, for every $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon) \subset X$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$. The notions of the relative compactness and tightness are related by the Prokhorov theorem which is the following assertion.

Lemma 2.8. *If the family* $\{P\}$ *is tight, then it is relatively compact.*

The lemma is Theorem 6.1 of [3].

Lemmas 2.5 and 2.6 are sufficient for the proof of the weak convergence for P_T. However, the identification of the limit measure requires one more lemma. Define one more measure

$$P_{T,\underline{\omega}}(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\}, \quad A \in \mathcal{B}(H_{d,u}).$$

Denote by $\xrightarrow{\mathcal{D}}$ convergence in distribution. For the proof of Theorem 2.2, the following statement will be very useful.

Lemma 2.9. *Let* $(X, \widehat{\varrho})$ *be a separable metric space, and let* $Y_n, X_{1n}, X_{2n}, \dots$ *be the* X *- valued random elements defined on a certain probability space* $(\widehat{\Omega}, \mathcal{A}, \mu)$. *Suppose that* $X_{kn} \xrightarrow{\mathcal{D}} X_k$ *as* $n \to \infty$, *and also* $X_k \xrightarrow{\mathcal{D}} X$ *as* $k \to \infty$. *If, for every* $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu(\widehat{\varrho}(X_{kn}, Y_n) \geq \varepsilon) = 0,$$

then $Y_n \xrightarrow{\mathcal{D}} X$ *as* $n \to \infty$.

The lemma is 4.2 Theorem from [3].

Lemma 2.10. *Suppose that the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then, on* $(H_{d,u}, \mathcal{B}(H_{d,u}))$, *there exists a probability measure* P *such that the measures* P_T *and* $P_{T,\underline{\omega}}$ *both converge weakly to* P *as* $T \to \infty$.

Proof. Let θ be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mathbb{P})$ and uniformly distributed on the interval $[0,1]$. Define the $H_{d,u}$ - valued random element

$$\underline{X}_{T,n} = X_{T,n}(s) = (X_{T,n,1}(s), \dots, X_{T,n,d}(s), X_{T,n,1,1}(s), \dots,$$

$$X_{T,n,1,l_1}(s), \dots, X_{T,n,r,1}(s), \dots, X_{T,n,r,l_r}(s)) = \Xi_n(s + i\theta T, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then, by Lemma 2.5, we have that

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n, \tag{2.4}$$

where

$$\underline{X}_n = (X_{n,1}, \ldots, X_{n,d}, X_{n,1,1}, \ldots, X_{n,1,l_1}, \ldots, X_{n,r,1}, \ldots, X_{n,r,l_r})$$

is a $H_{d,u}$ - valued random element having the distribution P_n, and P_n is the limit measure in Lemma 2.5. We will prove that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight.

Let K_v be the set from the definition of the metric ϱ. Since the series for $L_n(s, \chi_j)$ converges absolutely for $\sigma > \frac{1}{2}$, we have that, for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |L_n(\sigma + it, \chi_j)|^2 dt = \sum_{m=1}^{\infty} \frac{|\chi_j(m) v_n(m)|^2}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}}.$$

From this, by a standard way, we deduce that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_v} |L_n(s + i\tau, \chi_j)| dt \le C_v R_{jv}, \tag{2.5}$$

where

$$R_{jv} = \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_v}} \right)^{\frac{1}{2}}, \quad j = 1, \ldots, d,$$

with some $C_v > 0$ and $\sigma_v > \frac{1}{2}$, $v \in \mathbb{N}$. Similarly, the absolute convergence of the series for $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ leads to the estimate

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_v} |\zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_{jl})| dt \le \widehat{C}_v R_{jlv}, \tag{2.6}$$

where

$$R_{jv} = \left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m + \alpha_j)^{2\widehat{\sigma}_v}} \right)^{\frac{1}{2}}, \quad j = 1, \ldots, r, l = 1, \ldots, l_j,$$

with some $\widehat{C}_v > 0$ and $\widehat{\sigma}_v > \frac{1}{2}$, $v \in \mathbb{N}$.

Now let $\varepsilon > 0$ be arbitrary fixed number,

$$M_{jv} = M_{jv}(\varepsilon) = C_v R_{jv} 2^{v+1} d\varepsilon^{-1}, j = 1, \ldots, d, v \in \mathbb{N},$$

and

$$M_{jlv} = M_{jlv}(\varepsilon) = \widehat{C}_v R_{jlv} 2^{v+1} u\varepsilon^{-1}, j = 1, \ldots, r, l = 1, \ldots, l_j, v \in \mathbb{N}.$$

Then the bounds (2.5) and (2.6) imply

$$\limsup_{T\to\infty} \mathbb{P}(\exists j=1,\dots,d : \sup_{s\in K_v} |X_{T,n,j}(s)| > M_{jv} \quad \text{or}$$

$$\exists (j,l), j=1,\dots,r, l=1,\dots,l_j : \sup_{s\in K_v} |X_{T,n,j,l}(s)| > M_{jlv})$$

$$\leq \sum_{j=1}^{d} \limsup_{T\to\infty} \mathbb{P}(\sup_{s\in K_v} |X_{T,n,j}(s)| > M_{jv})$$

$$+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}(\sup_{s\in K_v} |X_{T,n,j,l}(s)| > M_{jlv})$$

$$= \sum_{j=1}^{d} \limsup_{T\to\infty} \frac{1}{T} meas\{\tau\in[0,T] : \sup_{s\in K_v} |L_n(s+i\tau,\chi_j)| > M_{jv}\}$$

$$+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \frac{1}{T} meas\{\tau\in[0,T] : \sup_{s\in K_v} |\zeta_n(s+i\tau,\alpha_j);\mathfrak{a}_{jl}| > M_{jlv}\}$$

$$\leq \sum_{j=1}^{d} \frac{1}{M_{jv}} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_v} |L_n(s+i\tau,\chi_j)| dt$$

$$+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{1}{M_{jlv}} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_v} |\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})| dt <$$

$$< \frac{\varepsilon}{2^{v+1}} + \frac{\varepsilon}{2^{v+1}} = \frac{\varepsilon}{2^v}, v\in\mathbb{N}.$$

This together with (2.4), for all $n\in\mathbb{N}$, gives

$$\mathbb{P}(\exists j=1,\dots,d : \sup_{s\in K_v} |X_{n,j}(s)| > M_{jv} \quad \text{or}$$

$$\exists (j,l), j=1,\dots,r, l=1,\dots,l_j : \sup_{s\in K_v} |X_{n,j,l}(s)| > M_{jlv})$$

$$< \frac{\varepsilon}{2^v}, v\in\mathbb{N}. \tag{2.7}$$

Define the set

$$K_{d,u} = K_{d,u}(\varepsilon) = \{(g_1, \dots, g_d, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H_{d,u} :$$

$$\sup_{s \in K_v} |g_j(s)| \le M_{jv}, j = 1, \dots, d,$$

$$\sup_{s \in K_v} |g_{jl}(s)| \le M_{jlv}, j = 1, \dots, r, l = 1, \dots, l_j, v \in \mathbb{N}\}.$$

Then, by the compactness principle, $K_{d,u}$ is a compact subset in the space $H_{d,u}$. Moreover, in view of (2.7), we find that for, all $n \in \mathbb{N}$,

$$P(\underline{X}_n \in K_{d,u}) = 1 - \mathbb{P}(\underline{X}_n \notin K_{d,u}) \ge 1 - \varepsilon \sum_{v=1}^{\infty} \frac{1}{2^v} = 1 - \varepsilon,$$

or, by the definition of the random element $\underline{X}_n$, we obtain that, for all $n \in \mathbb{N}$,

$$P_n(K_{d,u}(\varepsilon)) \ge 1 - \varepsilon.$$

Thus, we have proved that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Hence, by Lemma 2.8, it is relatively compact. Therefore, there exists a sequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H_{d,u}, \mathcal{B}(H_{d,u}))$ as $k \to \infty$. This also can be written in the form

$$\underline{X}_n \xrightarrow[k \to \infty]{\mathcal{D}} P. \tag{2.8}$$

Let

$$\underline{X}_T = X_T(s) = \Xi(s + iT\theta, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})$$

be one more $H_{d,u}$ - valued random element on the probability space $(\widehat{\Omega}, \mathcal{A}, \mathbb{P})$. By Lemma 2.6, we have that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}(\varrho_{d+u}(\underline{X}_T(s), \underline{X}_{T,n}(s)) \ge \varepsilon)$$

$$= \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \varrho_{d+u}(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}), \Xi_n(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) \ge \varepsilon\}$$

$$\le \lim_{n \to \infty} \limsup_{T \to} \frac{1}{T\varepsilon} \int_0^T \varrho_{d+u}(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}), \Xi_n(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}))dt = 0.$$

Along with (2.4), (2.8) and Lemma 2.9, the last equality implies the relation

$$\underline{X}_T \xrightarrow[T \to \infty]{\mathcal{D}} P \tag{2.9}$$

which is equivalent to the weak convergence of P_T to P as $T \to \infty$. Moreover, (2.9) shows that the measure P is independent of the choice of subsequence $\{P_{n_k}\}$. Therefore, taking into account the relative compactness of the family $\{P_n : n \in \mathbb{N}\}$, we obtain that

$$\underline{X}_n \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{2.10}$$

It remains to prove the weak convergence of the measure $P_{T,\underline{\omega}}$. We put

$$X_{T,n,\underline{\omega}} = X_{T,n,\underline{\omega}}(s) = \Xi_n(s + iT\theta, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}))$$

and

$$X_{T,\underline{\omega}} = X_{T,\underline{\omega}}(s) = \Xi_n(s + iT\theta, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})).$$

Then, repeating the arguments used above for the random elements $X_{T,n,\underline{\omega}}$ and $X_{T,\underline{\omega}}$, and using (2.10) and Lemma 2.7, we find, as in the case of P_T, that the measure $P_{T,\underline{\omega}}$ also converges weakly to P as $T \to \infty$. The lemma is proved.

Lemma 2.10 gives the weak convergence of P_T to a certain probability measure P as $T \to \infty$. It remains to show that the measure P coincides with P_{Ξ}. For this, some elements of ergodic theory are applied.

For $\tau \in \mathbb{R}$, let

$$\underline{a}_\tau = \{(p^{-i\tau} : p \in \mathcal{P}), ((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)\}.$$

Then $\{\underline{a}_\tau : \tau \in \mathbb{R}\}$ is a one - parameter group. Define a family of transformations $\{\Phi_\tau : \tau \in \mathbb{R}\}$ on Ω^κ by putting $\Phi_\tau(\underline{\omega}) = \underline{a}_\tau\underline{\omega}$, $\underline{\omega} \in \Omega^\kappa$. Then $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is the one - parameter group of measurable measure preserving transformations on Ω^κ. We recall that a set $A \in \mathcal{B}(\Omega^\kappa)$ is said to be invariant with respect to the group $\{\Phi_\tau : \tau \in \mathbb{R}\}$ if, for every $\tau \in \mathbb{R}$, the sets A and $A_\tau = \Phi_\tau(A)$ may differ one from another at most by m_H^κ - measure zero. The group $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is called ergodic if its σ - field of invariant sets consists only of the sets of m_H^κ - measure zero or one.

Lemma 2.11. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the group $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma is given in [37], Lemma 7, and uses the linear independence over $\mathbb{Q}$ for the set

$$\{(\log p : p \in \mathcal{P}), (\log(m+\alpha_1) : p \in \mathbb{N}_0), \dots, (\log(m+\alpha_r) : p \in \mathbb{N}_0)\}.$$

Now we recall the classical Birkhoff-Khinchin ergodic theorem.

Lemma 2.12. *Let a process $X(\tau,\omega)$ be ergodic, $\mathbb{E}|X(\tau,\omega)| < \infty$, and let its simple paths be integrable almost surely in the Riemann sense over every finite interval. Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(\tau,\omega)dt = \mathbb{E}X(\tau,\omega)$$

almost surely.

The above definitions and proof of the lemma are given in [6], see also [31].

For the proof Theorem 2.2, we will apply one equivalent of the weak convergence of probability measures.

Let P be a probability measure $(X, \mathcal{B}(X))$. $A \in \mathcal{B}(X)$ is called a continuity set of P, if $P(\partial A) = 0$, where ∂A denotes the boundary of A.

Lemma 2.13. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every continuity set A of P,*

$$\lim_{T\to\infty} P_n(A) = P(A).$$

The lemma is a part of the Theorem 2.1 from [3].

Now we are ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. We have mentioned above that it is sufficient to show that the limit measure P in Lemma 2.10 coincides with P_Ξ.

Let A be a fixed continuity set of the measure P. Then, by Lemmas 2.10 and 2.13, we have that

$$\lim_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\} = P(A). \tag{2.11}$$

On the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the random variable

$$\xi(\underline{\omega}) = \begin{cases} 0 & \text{if } \Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 1 & \text{otherwise.} \end{cases}$$

It is easily seen that the expectation $\mathbb{E}\xi$ of ξ is of the form

$$\mathbb{E}\xi = \int_\Omega^\kappa \xi(\underline{\omega}) dm_H^\kappa = m_H^\kappa(\underline{\omega} \in \Omega^\kappa : \Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A). \tag{2.12}$$

In view of Lemma 2.11, the random process $\xi(\Phi_\tau(\underline{\omega}))$ is ergodic. Therefore, Lemma 2.12 implies the equality

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega}))dt = \mathbb{E}\xi. \tag{2.13}$$

On the other hand, the definitions of ξ and Φ_τ yield

$$\frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega}))dt = \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\}.$$

This, (2.12) and (2.13) show that

$$\lim_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:\Xi(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A\}=P_{\Xi}(A).$$

Hence, in view of (2.11), we find that $P(A)=P_{\Xi}(A)$. Since A was an arbitrary continuity set of the measure P, the latter equality holds for all continuity sets of P. It is well known that continuity sets constitute a determining class, see, for example, [3]. Consequently, $P(A)=P_{\Xi}(A)$ for all $A\in\mathcal{B}(H_{d,u})$, i.e., $P=P_{\Xi}$. The theorem is proved.

2.2. Support of the limit measure in Theorem 2.1

This section is devoted to the explicit form of the support $S_{P_{\Xi}}$ of the limit measure P in Theorem 2.1. We preserve the notation of Section 1.2 for the set S.

Theorem 2.14. *Suppose that $\chi_1,\dots,\chi_d$ are pairwise non - equivalent Dirichlet characters, the parameters $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $rank(A_j)=l_j, j=1,\dots,r$. Then the support of P_{Ξ} is the set $S^d\times H^u(D)$.*

Proof. The space $H_{d,u}$ is separable one, therefore [3]

$$\mathcal{B}(H_{d,u})=\mathcal{B}(H^d(D))\times\mathcal{B}(H^u(D)).$$

Thus, it suffices to consider $P_{\Xi}(A\times B)$, where $A\in\mathcal{B}(H^d(D))$ and $B\in\mathcal{B}(H^u(D))$. Let

$$\underline{L}(s,\chi,\omega)=(L(s,\chi_1,\omega),\dots,L(s,\chi_d,\omega))$$

and

$$\underline{\zeta}(s,\underline{\alpha}\,\underset{\sim}{\omega};\underline{\mathfrak{a}})=(\zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})),$$

where $\underset{\sim}{\omega}=(\omega_1,\dots,\omega_r)$. Since the Haar measure m_H^{κ} is the product of the Haar measures m_H and m_H^r, we have that

$$P_{\Xi}(A\times B)=m_H^{\kappa}(\underline{\omega}\in\Omega^{\kappa}:\Xi(s,\underline{\chi},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A\times B)=$$

$$m_H(\omega\in\Omega:\underline{L}(s,\underline{\chi},\omega)\in A)m_H^r(\underset{\sim}{\omega}\in\Omega^r:\underline{\zeta}(s,\underline{\alpha},\underset{\sim}{\omega};\mathfrak{a})\in B). \tag{2.14}$$

In [40], it was obtained that S^d is a minimal closed set such that

$$m_H(\omega\in\Omega:\underline{L}(s,\underline{\chi},\omega)\in S^d)=1,$$

and in [53], it was proved that $H^u(D)$ is a minimal closed set such that

$$m_H^r(\underset{\sim}{\omega}\in\Omega^r:\underline{\zeta}(s,\underline{\alpha},\underset{\sim}{\omega};\mathfrak{a})\in H^u(D))=1.$$

These equalities, the minimality of the support and (2.14) prove the theorem.

2.3. Proof of Theorem 2.1

The proof of Theorem 2.1 is similar to that of Theorem 1.1, and is based on the Mergelyan theorem and Theorems 2.2 and 2.14.

Proof of Theorem 2.1. By Lemma 1.6, there exist polynomials $p_j(s), j = 1, \dots, d$, and $p_{jl}(s), j = 1, \dots, r, l = 1, \dots, l_j$, such that

$$\sup_{1 \le j \le d} \sup_{s \in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{4} \tag{2.15}$$

and

$$\sup_{1 \le j \le r} \sup_{1 \le j \le l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2}. \tag{2.16}$$

Since $f_j(s) \neq 0$ on K_j, we have that $p_j(s) \neq 0$ on K_j if ε is small enough, $j = 1, \dots, d$. Therefore, there exists a continuous branch of $\log p_j(s)$ which is analytic in the interior of K_j, $j = 1, \dots, d$. Applying Lemma 1.6 once more, we find polynomials $q_j(s), j = 1, \dots, d$, such that

$$\sup_{1 \le j \le d} \sup_{s \in K_j} |p_j(s) - e^{q_j(s)}| < \frac{\varepsilon}{4}.$$

Combining this with (2.15) gives

$$\sup_{1 \le j \le d} \sup_{s \in K_j} |f_j(s) - e^{q_j(s)}| < \frac{\varepsilon}{2}. \tag{2.17}$$

By Theorem 2.14,

$$(e^{q_1(s)}, \dots, e^{q_d(s)}, p_{11}(s), \dots, p_{1l_1}(s), \dots, p_{r1}(s), \dots, p_{rl_r}(s))$$

is an element of the support of the measure P_Ξ. Therefore, setting

$$G = \{\underline{g} \in H_{d,r} : \sup_{1 \le j \le d} \sup_{s \in K_j} |g_j(s) - e^{q_j(s)}| < \frac{\varepsilon}{2},$$

$$\sup_{1 \le j \le r} \sup_{1 \le j \le l_j} \sup_{s \in K_{jl}} |g_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2}\},$$

we obtain, by Theorem 2.2, Lemma 1.7 and properties of the support, that

$$\lim_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in G\} \ge P_\Xi(G) > 0. \tag{2.18}$$

Clearly, in view of (2.16) and (2.17),

$$\{\tau \in [0; T] : \sup_{1 \le j \le d} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - e^{q_j(s)}| < \frac{\varepsilon}{2},$$

$$\sup_{1 \le j \le r} \sup_{1 \le j \le l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)| < \frac{\varepsilon}{2}\} \subset$$

$$\{\tau \in [0;T] : \sup_{1\le j\le d} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{1\le j\le r} \sup_{1\le j\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\}.$$

Therefore, the definition of G and (2.18) complete the proof of the theorem.

Chapter 3

Universality of composite functions of Dirichlet L - functions and Hurwitz zeta - functions

In this chapter, we use Theorem 1.1 to extend the class of universal functions. Namely, we consider the universality of functions

$$F(L(s,\chi_1),\dots,L(s,\chi_{r_1}),\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_{r_2}))$$

for some classes of operators $F: H^{r_1+r_2}(D) \to H(D)$.

3.1. Statement of the results

We start with a general theorem.

Theorem 3.1 *Suppose that the Dirichlet characters $\chi_1,\dots,\chi_{r_1}$ and the numbers $\alpha_1,\dots,\alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F: H^{r_1+r_2}(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S^{r_1} \times H^{r_2}(D))$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(L(s,\chi_1),\dots,L(s,\chi_{r_1}),\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_{r_2})) - f(s)| < \varepsilon\} > 0.$$

The main requirement for the operator F that the set $(F^{-1}G)\cap(S^{r_1}\times H^{r_2}(D))$ would be non-empty can be replaced by a stronger but simple one.

Theorem 3.2 *Suppose that the Dirichlet characters* $\chi_1, \dots, \chi_{r_1}$ *and the numbers* $\alpha_1, \dots, \alpha_{r_2}$ *satisfy the hypotheses of Theorem 1.1, and that* $F : H^{r_1+r_2}(D) \to H(D)$ *is a continuous operator such that, for every polynomial* $p = p(s)$*, the set* $(F^{-1}\{p\}) \cap (S^{r_1} \times H^{r_2}(D))$*is non-empty. Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$*. Then the assertion of Theorem 3.1 is true.*

The next corollary gives an example for Theorem 3.2.

Corollary 3.3 *Suppose that the Dirichlet characters* $\chi_1, \dots, \chi_{r_1}$ *and the numbers* $\alpha_1, \dots, \alpha_{r_2}$ *satisfy the hypotheses of Theorem 1.1. Let* $\{j_1, \dots, j_r\} \neq \varnothing$ *be arbitrary subset of* $\{1, \dots, r_1\}$*, and* $\{l_1, \dots, l_k\} \neq \varnothing$ *be arbitrary subset of* $\{1, \dots, r_2\}$*. Let* $K \in \mathcal{K}$ *and* $f \in H(K)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |L(s+i\tau, \chi_{j_1}) \dots L(s+i\tau, \chi_{j_r}) \times$$

$$\times \zeta(s+i\tau, \alpha_{l_1}) \dots \zeta(s+i\tau, \alpha_{l_k}) - f(s)| < \varepsilon\} > 0.$$

The strip D is not bounded, therefore, it is not easy to ensure the non-vanishing of a polynomial. However, the problem of non-vanishing becomes simpler if we consider polynomials on a bounded region. Thus, sometimes it is more convenient to deal with the following modification of Theorem 3.2. For arbitrary fixed $V > 0$, let $D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\}$, and

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 3.4 *Suppose that the Dirichlet characters* $\chi_1, \dots, \chi_{r_1}$ *and the numbers* $\alpha_1, \dots, \alpha_{r_2}$ *satisfy the hypotheses of Theorem 1.1,* $K \in \mathcal{K}$*,* $f(s) \in H(K)$*, and let* $V > 0$ *be such that* $K \subset D_V$*. Let* $F : H^{r_1}(D_V) \times H^{r_2}(D) \to H(D_V)$ *be a continuous operator such that, for every polynomial* $p = p(s)$*, the set* $(F^{-1}\{p\}) \cap (S_V^{r_1} \times H^{r_2}(D))$ *is non-empty. Then the assertion of Theorem 3.1 is true.*

The above universality theorems are rather general, in them functions from the whole space $H(D)$ are approximated. The next theorems are devoted to approximation of analytic function from certain subsets of $H(D)$.

Let $a_1, \dots, a_k$ be arbitrary distinct complex numbers. Define the set of analytic functions $H_k(D)$ by

$$H_k(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D), j = 1, \dots, k\}.$$

Theorem 3.5 *Suppose that the Dirichlet characters* $\chi_1, \dots, \chi_{r_1}$ *and the numbers* $\alpha_1, \dots, \alpha_{r_2}$ *satisfy the hypotheses of Theorem 1.1, and that* $F : H^{r_1+r_2}(D) \to H(D)$ *is a continuous operator such that* $F(S^{r_1} \times H^{r_2}(D)) \supset H_k(D)$*. If* $k = 1$*, then let* $K \in \mathcal{K}$*,* $f(s) \in H(K)$ *and* $f(s) \neq a_1$ *on* K*. If* $k \geq 2$*,* *then let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_k(D)$. Then the assertion of Theorem 3.1 is true.*

It is not difficult to see that from Theorem 3.5 the universality of the functions $\sin(L(s,\chi_1) + L(s,\chi_2) + \zeta(s,\alpha_1) + \zeta(s,\alpha_2))$, $\cos(L(s,\chi_1) + L(s,\chi_2) + \zeta(s,\alpha_1) + \zeta(s,\alpha_2))$, $\sinh(L(s,\chi_1) + L(s,\chi_2) + \zeta(s,\alpha_1) + \zeta(s,\alpha_2))$ and $\cosh(L(s,\chi_1) + L(s,\chi_2) + \zeta(s,\alpha_1) + \zeta(s,\alpha_2))$ follows provided the characters χ_1 and χ_2 are non-equivalent, and the numbers α_1 and α_2 are algebraically independent.

The last theorem of the chapter is of the following form.

Theorem 3.6 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_{r_1}$ and the numbers $\alpha_1, \dots, \alpha_{r_2}$ satisfy the hypotheses of Theorem 1.1, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator. Let K be a compact subset of the strip D, and $f(s) \in F(S^{r_1} \times H^{r_2}(D))$. Then the assertion of Theorem 3.1 is true.*

Theorems 3.1 and 3.4 – 3.6 were obtained in [15], while Theorem 3.2 and Corollary 3.3 were proved in [17].

3.2. Limit theorems

For the proof of universality theorems stated in Section 3.1, we will use the probabilistic method. Therefore, we start with certain limit theorems in the space of analytic functions for composite functions. These theorems are based on Theorem 1.2. We recall that

$$P_T(A) = \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s+i\tau) \in A\}, A \in \mathcal{B}(H^r(D)),$$

where

$$\Xi(s) = (L(s,\chi_1), \dots, L(s,\chi_{r_1}), \zeta(s,\alpha_1), \dots, \zeta(s,\alpha_{r_2})),$$

and $r = r_1 + r_2$. Moreover, P_Ξ is the distribution of the random element $\Xi(s,\underline{\omega})$.

Theorem 3.7 *Suppose that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$, and that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous operator. Then*

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in A\}, A \in \mathcal{B}(H(D)),$$

converges weakly to $P_\Xi F^{-1}$ as $T \to \infty$.

Proof. By the definitions of the measures P_T and $P_{T,F}$, we have that, for $A \in \mathcal{B}(H(D))$,

$$P_{T,F}(A) = \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in F^{-1}A\} = P_T F^{-1}(A).$$

Therefore, the assertion of the theorem follows from Theorem 1.2, Lemma 2.4 and the continuity of the operator F.

Now, for $V > 0$, denote by $P_{T,V}$ and $P_{\Xi,V}$ the restrictions of the measures P_T and P_Ξ, respectively, to the space $(H^{r_1}(D_V) \times H^{r_2}(D), \mathcal{B}(H^{r_1}(D_V) \times H^{r_2}(D)))$. More precisely, for $A \in \mathcal{B}(H^{r_1}(D_V) \times H^{r_2}(D))$,

$$P_{T,V}(A) = \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s+i\tau) \in A\}$$

and

$$P_{\Xi,V}(A) = m_H^\kappa\{\underline{\omega} \in \Omega^\kappa : \Xi(s, \underline{\omega}) \in A\}.$$

Theorem 3.8 *Suppose that the numbers $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the measure $P_{T,V}$ converges weakly to $P_{\Xi,V}$ as $T \to \infty$.*

Proof. The mapping $h_V : H^{r_1+r_2}(D) \to H^{r_1}(D_V) \times H^{r_2}(D)$ given by the formula

$$h_V(\underline{g}_{r_1}(s), \underline{g}_{r_2}(s)) = (\underline{g}_{r_1}(s)|_{s \in D_V}, \underline{g}_{r_2}(s)), (\underline{g}_{r_1}(s), \underline{g}_{r_2}(s)) \in H^{r_1+r_2}(D),$$

is continuous because $D_V \subset D$. Moreover, $P_{T,V} = P_T h_V^{-1}$. Therefore, the theorem, as Theorem 3.7, follows from Theorem 1.2 and Lemma 2.4.

Theorem 3.9 *Suppose that the numbers $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$, and that $F : H^{r_1+r_2}(D_V) \to H(D_V)$ is a continuous operator. Then*

$$P_{T,V,F}(A) \stackrel{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in A\}, A \in \mathcal{B}(H(D_V)),$$

converges weakly to $P_{\Xi,V}F^{-1}$ as $T \to \infty$.

Proof. By the definitions of $P_{T,V}$ and $P_{T,V,F}$, we have that, for $A \in \mathcal{B}(H(D_V))$,

$$P_{T,V,F}(A) = \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in A\} =$$

$$\frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in F^{-1}A\} = P_{T,V}F^{-1}(A).$$

Therefore, the theorem is a corollary of Theorem 3.8 and Lemma 2.4.

3.3. Supports

In this section, we present explicit forms of the supports the limit measures in theorems of Section 3.2.

Theorem 3.10 *Suppose that all hypotheses Theorem 3.1 are satisfied. Then the support of the measure $P_\Xi F^{-1}$ is the whole of $H(D)$.*

Proof. Let $g \in H(D)$ be an arbitrary element, and G be its open neighbourhood. Since the operator F is continuous, the set $F^{-1}G$ is also open, and, in virtue of the hypothesis $(F^{-1}G) \cap (S^{r_1} \times H^{r_2}(D))$, contains at least one element of the set $S^{r_1} \times H^{r_2}(D)$. Thus, the set $F^{-1}G$ is an open neighbourhood of a certain element of the set $S^{r_1} \times H^{r_2}(D)$. Therefore, by Theorem 1.5, we have that $P_{\Xi}(F^{-1}G) > 0$. Hence, $P_{\Xi}F^{-1}(G) > 0$. Since g and G are arbitrary, this inequality proves the theorem.

Theorem 3.11 *Suppose that all hypotheses Theorem 3.2 are satisfied. Then the support of the measure $P_{\Xi}F^{-1}$ is the whole of $H(D)$.*

Proof. Let $g = g(s)$ be an arbitrary element of $H(D)$, and G be any open neighbourhood of g. In [42], it was noted that the approximation on $H(D)$ reduces to that on compact subsets $K \subset D$ with connected complements. Moreover, by Lemma 1.6, for every $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K} |g(s) - f(s)| < \varepsilon.$$

If ε is small enough, we have that the polynomial $p(s)$ belongs to G. Since F is continuous, the set $F^{-1}G$ is open. Moreover, in view of the inequality $(F^{-1}\{p\}) \cap (S^{r_1} \times H^{r_2}(D)) \neq \varnothing$, it follows that $(F^{-1}G) \cap (S^{r_1} \times H^{r_2}(D)) \neq \varnothing$. This shows that $F^{-1}G$ is an open neighbourhood of an element of the set $S^{r_1} \times H^{r_2}(D)$. This and Theorem 1.5 imply the inequality $P_{\Xi}(F^{-1}G) > 0$. Thus, $P_{\Xi}F^{-1}(G) > 0$, and the theorem is proved.

Lemma 3.12 *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters, and that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the support of $P_{\Xi,V}$ is the set $S_V^{r_1} \times H^{r_2}(D))$.*

The proof of the lemma completely coincides with that of Theorem 1.5.

Theorem 3.13 *Suppose that all hypotheses of Theorem 3.4 are satisfied. Then the support of the measure $P_{\Xi,V}F^{-1}$ is the whole of $H(D_V)$.*

Proof. We apply arguments similar to those of the proof of Theorem 3.11. We take an arbitrary $g = g(s) \in H(D_V)$ and an arbitrary open neighbourhood G of g. Let $K \subset D_V$ be a compact subset with connected complement. By Lemma 1.6, there exists a polynomial $p = p(s)$ which approximates on K the element g with a desired accuracy. This means that p approximates g in the space $H(D_V)$. Therefore, we may suppose that p belongs to G. Since the set $(F^{-1}\{p\}) \cap (S_V^{r_1} \times H^{r_2}(D))$ is non-empty, from this it follows that $(F^{-1}G) \cap (S_V^{r_1} \times H^{r_2}(D) \neq \varnothing$. Thus, repeating the proof of Theorem 3.1 and using Lemma 3.12, we obtain that $P_{\Xi}F^{-1}(G) > 0$. This proves the theorem.

Theorem 3.14 *Suppose that all hypotheses of Theorem 3.5 are satisfied. Then the support of the measure $P_{\Xi}F^{-1}$ contains the closure of $H_r(D)$.*

Proof. In virtue of the relation $F(S^{r_1} \times H^{r_2}(D)) \supset H_k(D)$, we have that, for every $g \in H_k(D)$, there exists $g_1 \in S^{r_1} \times H^{r_2}(D)$ such that $F(g_1) = g$. Let G be an arbitrary open neighbourhood of g. Then $F^{-1}G$ is an open neighbourhood of g_1. Therefore, by Theorem 1.5, we find that

$$P_{\Xi}(F^{-1}G) = P_{\Xi}F^{-1}(G) > 0.$$

This shows that g is an element of the support S_F of the measure $P_{\Xi}F^{-1}$. Since g is arbitrary, we have that $H_k(D) \subset S_F$. However, since S_F is a closed set, S_F contains the closure of $H_k(D)$.

Theorem 3.15 *Suppose that all hypotheses of Theorem 3.6 are satisfied. Then the support of the measure $P_{\Xi}F^{-1}$ is the closure of the set $F(S^{r_1} \times H^{r_2}(D))$.*

Proof. Let g be an arbitrary element of the set $F(S^{r_1} \times H^{r_2}(D))$, and G be its arbitrary open neighbourhood. Then there exists $g_1 \in S^{r_1} \times H^{r_2}(D)$ such that $F(g_1) = g$. Thus, $F^{-1}G$ is an open neighbourhood of the element g_1. By Theorem 1.5, $S^{r_1} \times H^{r_2}(D)$ is the support of the measure P_{Ξ}. Therefore, $P_{\Xi}(F^{-1}G) > 0$. Hence, $P_{\Xi}F^{-1}(G) > 0$. Moreover, again by Theorem 1.5,

$$P_{\Xi}F^{-1}(F(S^{r_1} \times H^{r_2}(D))) = P_{\Xi}(S^{r_1} \times H^{r_2}(D)) = 1.$$

Therefore, since the support is a closed set, the support of $P_{\Xi}F^{-1}$ is the closure of $F(S^{r_1} \times H^{r_2}(D))$.

3.4. Proof of universality theorems

Proof of Theorem 3.1. By Lemma 1.6, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.1}$$

Define the set

$$G = \{g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\}.$$

Then G is an open neighbourhood of the polynomial $p(s)$. By Theorem 3.10, the polynomial $p(s)$ is an element of the support of the measure $P_{\Xi}F^{-1}$. Thus, $P_{\Xi}F^{-1}(G) > 0$. Hence, in view of Theorem 3.7 and Lemma 1.7, we have that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in G\} \geq P_{\Xi}F^{-1}(G) > 0.$$

From this, using the definition of the set G, we find that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s+i\tau)) - p(s)| < \frac{\varepsilon}{2}\} > 0. \tag{3.2}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s \in K} |F(\Xi(s+i\tau)) - p(s)| < \frac{\varepsilon}{2}.$$

Then, for such τ, inequality (3.1) implies

$$\sup_{s\in K}|F(\Xi(s+i\tau))-f(s)|\le\sup_{s\in K}|F(\Xi(s+i\tau))-p(s)|$$

$$+\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.$$

This shows that

$$\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau))-p(s)|<\frac{\varepsilon}{2}\}\subset$$

$$\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau))-f(s)|<\varepsilon\}.$$

Therefore, taking into account inequality (3.2), we find that

$$\liminf_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau))-p(s)|<\varepsilon\}>0.$$

The theorem is proved.

Proof of Theorem 3.2. We repeat the proof of Theorem 3.1 with Theorem 3.11 in place of Theorem 10.

Proof of Corollary 3.3. We take the operator $F:H^{r_1+r_2}(D_V)\to H(D_V)$ given by

$$F(g_1,\dots,g_{r_1};\widehat{g}_1,\dots,\widehat{g}_{r_2})=g_{j_1}\dots g_{j_r}\widehat{g}_{l_1}\dots\widehat{g}_{l_k}.$$

Then the operator F is continuous. Moreover, for every polynomial $p=p(s)$, we have that

$$F(1,\dots,1;1,\dots,1,\widehat{g}_{l_1},1,\dots,1)=p$$

and

$$(1,\dots,1;1,\dots,1,\widehat{g}_{l_1},1,\dots,1)\in S^{r_1}\times H^{r_2}(D)$$

with $\widehat{g}_{l_1}=p$. Thus, the hypotheses of Theorem 3.2 are satisfied, therefore, we have the assertion of the corollary.

Proof of Theorem 3.4. The proof is similar to that of Theorems 3.1 and 3.2. Define the set

$$G_V=\{g\in H(D_V):\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}\},$$

where the polynomial $p(s)$ satisfies inequality (3.1). Then, by Theorem 3.8 and Lemma 1.7, we obtain that

$$\liminf_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:F(\Xi(s+i\tau))\in G_V\}\ge P_{\Xi,V}(G).$$

Moreover, Theorem 3.13 shows that $P_{\Xi}F^{-1}(G)>0$. These two inequality give inequality (3.2) which together with (3.1) proves the theorem.

Proof of Theorem 3.5. We consider separately the cases $r = 1$ and $r \geq 2$. First let $r = 1$. By Lemma 1.6, we find a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{3.3}$$

Since $f(s) \neq a_1$ on K, we have that also $p(s) \neq 0$ on K provided the number ε is small enough. Therefore, on K there exists a continuous branch of the logarithm $\log(p(s) - a_1)$ which is analytic in the interior of K. Again by Lemma 1.6, there exists a polynomial $q(s)$ such that

$$\sup_{s \in K} |p(s) - a_1 - e^{q(s)}| < \frac{\varepsilon}{4}. \tag{3.4}$$

Clearly, we have that $e^{q(s)} + a_1 \in H(D)$ and $e^{q(s)} + a_1 \neq a_1$ in the strip D. In other words, the function $e^{q(s)} + a_1$ is an element of the set $H_1(D)$. Therefore, in view of Theorem 3.14, $e^{q(s)} + a_1$ is an element of the support of the measure $P_\Xi F^{-1}$. Define the set

$$G_1 = \{g \in H(D) : \sup_{s \in K} |g(s) - e^{q(s)} - a_1| < \frac{\varepsilon}{2}\}.$$

Then, similarly as above, we have that $P_\Xi F^{-1}(G_1) > 0$, and obtain, by Theorem 3.7 and Lemma 1.7, that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s + i\tau) - e^{q(s)} - a_1| < \frac{\varepsilon}{2}\} > 0. \tag{3.5}$$

Inequalities (3.3) and (3.4) show that

$$\sup_{s \in K} |f(s) - e^{q(s)} - a_1| \leq \sup_{s \in K} |f(s) - p(s)| +$$

$$+ \sup_{s \in K} |p(s) - e^{q(s)} - a_1| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This and (3.5) prove the theorem in the case $r = 1$.

Now let $r \geq 2$. Define the set

$$G_2 = \{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon\}.$$

By the hypotheses of the theorem and Theorem 3.14, the function $f(s)$ is an element of the support of the measure $P_\Xi F^{-1}$. Thus, $P_\Xi F^{-1}(G_2) > 0$. This together with Theorem 3.7 and Lemma 1.7 prove the theorem in the case $r \geq 2$.

Now we will prove the universality for the functions defined after the statement of Theorem 3.5. We will consider only the case of the function $\cos(\cdot)$, the cases of other functions are similar. Suppose that $f(s) \in H_2(D)$ with $a_1 = 1$ and $a_2 = -1$. In view of the formula

$$\frac{e^{is} + e^{-is}}{2} = \cos s,$$

we have to solve the equation

$$\frac{e^{ig(s)} + e^{-ig(s)}}{2} = f(s)$$

with respect to function $g(s)$. Denoting $y = e^{ig(s)}$, we obtain the equation

$$y^2 - 2f(s)y + 1 = 0.$$

Hence,

$$y = f(s) \pm \sqrt{f^2(s) - 1}.$$

Therefore, taking the case with the sign "+" and the principal branch of the logarithm, we have that

$$g(s) = \frac{1}{i}\log(f(s) + \sqrt{f^2(s) - 1})$$

is an analytic function on D. Then the function

$$\cos(g_1 + g_2 + g_3 + g_4)$$

with $g_1 = 1, g_2 = -1, g_3 = g(s), g_4 = 0$ satisfies the equality

$$\cos(g_1 + g_2 + g_3 + g_4) = f(s).$$

Moreover, $g_1, g_2 \in S$ and $g_3, g_4 \in H(D)$. This shows that $\cos(g_1 + g_2 + g_3 + g_4) \supset H_2(D)$ with $a_1 = 1$ and $a_2 = -1$. Thus by Theorem 3.5, the function $\cos(L(s,\chi_1) + L(s,\chi_2) + \zeta(s,\alpha_1) + \zeta(s,\alpha_2))$ with non-equivalent Dirichlet characters χ_1 and χ_2, and algebraically independent α_1 and α_2 is universal. For example [11], we can take $\alpha_1 = (\sqrt[3]{2})^{-1}$ and $\alpha_2 = (\sqrt[3]{4})^{-1}$.

Proof of Theorem 3.6. We use Theorem 3.15 and repeat the proof of Theorem 3.5 in the case $r \geq 2$.

Chapter 4

Universality of composite functions of Dirichlet L - functions and periodic Hurwitz zeta - functions

In this chapter, we consider the universality of the function

$$F(L(s,\chi_1),\dots,L(s,\chi_d),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})) \qquad (4.1)$$

for some classes of operators $F : H^v(D) \to H(D)$, where $v = d+l_1+\dots+l_r$. For this, we apply, except for Theorem 4.1, the probabilistic method based on Theorem 2.1. Also, we preserve the notation of Chapter 2.

We divide universality theorems for the function (4.1) into two groups. The first group consists of universality theorems on the approximation of analytic functions from the class $H(K), K \in \mathcal{K}$, thus from space $H(D)$. The second group deals with approximation of analytic functions from the set $(F(S^d \times H^{v_1}(D))$, where $v_1 = l_1 + \dots + l_r$.

4.1.Application of the Lipschitz type inequality

In this section, we consider a sufficiently wide class of operators $F : H^v(D) \to H(D)$ described as follows. We say that an operator F belongs to the class $Lip(\beta_1,\dots,\beta_v)$ with $\beta_1 > 0,\dots,\beta_v > 0$ if the following hypotheses are satisfied:

1^o For each polynomial $p = p(s)$ and all compact subsets $K_1,\dots,K_d \in \mathcal{K}$, there exists an element $\underline{g} = (g_1,\dots,g_d,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r}) \in F^{-1}\{p\} \subset H^v(D)$ such that $g_j(s) \neq 0$ on $K_j, j = 1,\dots,d$;

2^o For all $K \in \mathcal{K}$, there exist a constant $c > 0$ and sets $K_1, \dots, K_v \in \mathcal{K}$ such that, for all $(g_{j1}, \dots, g_{jv}) \in H^v(D), j = 1, 2,$

$$\sup_{s\in K} |F(g_{11}(s), \dots, g_{1v}(s)) - F(g_{21}(s), \dots, g_{2v}(s)| \le c \sup_{1\le j\le v} \sup_{s\in K_j} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}.$$

We note that hypothesis 2^0 is an analogue of the classical Lipschitz condition.

Theorem 4.1 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and $F \in Lip(\beta_1, \dots, \beta_v)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} meas\{\tau \in [0; T] : \sup_{s\in K} |F(L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_d),$$

$$\zeta(s + i\tau, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s + i\tau, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s + i\tau, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s + i\tau, \alpha_r; \mathfrak{a}_{rl_r})) - f(s)| < \varepsilon\} > 0.$$

Thus, Theorem 4.1 belongs to the first group of universality theorems on approximation of analytic functions from the space $H(D)$.

It is not difficult to give an example of application of Theorem 4.1. Define the operator $F : H^v(D) \to H(D)$ by the formula

$$F(g_1, \dots, g_d, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) = c_1 g_1^{(n_1)} + \dots + c_d g_d^{(n_d)} + \dots +$$

$$c_{11} g_{11}^{(n_{11})} + \dots + c_{1l_1} g_{1l_1}^{(n_{1l_1})} + \dots + c_{r1} g_{r1}^{(n_{r1})} + \dots + c_{rl_r} g_{rl_r}^{(n_{rl_r})},$$

where $c_1, \dots, c_d, c_{11}, \dots, c_{1l_1}, \dots, c_{r1}, \dots, c_{rl_r} \in \mathbb{C} \setminus \{0\}, n_1, \dots, n_d, n_{11}, \dots, n_{1l_1}, \dots, n_{r1}, \dots, n_{rl_r} \in \mathbb{N}$, and $g^{(n)}$ denotes the n th derivative. For each polynomial $p = p(s)$ and all sets $K_1, \dots, K_d \in \mathcal{K}$, there exists an element $\underline{g} \in F^{-1}\{p\} \subset H^v(D)$ such that $g_j(s) \ne 0$ on $K_j, j = 1, \dots, d$. Indeed, if

$$p = p(s) = a_k s^k + a_{k-1} s^{k-1} + \dots + a_0, \quad a_k \ne 0,$$

then we can take $\underline{g} = (1, \dots, 1, 1, \dots, 1, 1, \dots, 1, g_{rl_r})$, where

$$g_{rl_r}(s) = \frac{1}{c_{rl_r}} \left(\frac{a_k s^{k+n_{rl_r}}}{(k+1)\dots(k+n_{rl_r})} + \dots + \frac{a_0 s^{n_{rl_r}}}{1\dots n_{rl_r}} \right).$$

Clearly, we have that $g_j(s) = 1 \ne 0$ on $K_j, j = 1, \dots, d,$, and $F(\underline{g}) = p$. Thus, hypothesis 1^0 of the class $Lip(\beta_1, \dots, \beta_v)$ is satisfied.

Hypothesis 2^0 of the class $Lip(\beta_1, \dots, \beta_v)$ follows from the well - known Cauchy integral formula which we state as the following lemma, see, for example [75].

Lemma 4.2 *Let G be a domain in $\mathbb{C}, f(s)$ an analytic function in G. Then, for $s_0 \in intL$ and $n \in \mathbb{N}_0$,*

$$f^{(n)}(s_0) = \frac{n!}{2\pi i}\int_L \frac{f(s)}{(s-s_0)^{n+1}}ds.$$

We write the operator F in a more convenient form

$$F(g_1, \ldots, g_v) = \sum_{j=1}^{v} c_j g_j^{(n_j)}$$

with obvious changes of the notation. Let $K \in \mathcal{K}$, and $K \subset G \subset K_1$, where G is an open set and $K_1 \in \mathcal{K}$. Moreover, let L be a simple closed contour lying in $K_1 \setminus G$ and containing inside the set K. Then Lemma 4.2 shows that, for $(g_{j1}, \ldots, g_{jv}) \in H^v(D), j = 1, 2$ and $s \in K$,

$$|F(g_{11}(s), \ldots, g_{1v}(s)) - F(g_{21}(s), \ldots, g_{2v}(s))| =$$

$$\left|\sum_{j=1}^{v} c_j \frac{n_j!}{2\pi i}\int_L \frac{g_{1j}(z), \ldots, g_{2j}(z)}{(z-s)^{n_j+1}}dz\right| \leq$$

$$\sum_{j=1}^{v} |c_j| C_j \sup_{s\in L} |g_{1j}(s) - g_{2j}(s)| \leq$$

$$c \sup_{1\leq j\leq v} \sup_{s\in K_1} |g_{1j}(s) - g_{2j}(s)|$$

with some constants $C_j > 0, j = 1, \ldots, v$, and $c > 0$. Thus, we have that $F \in Lip(1, \ldots, 1)$. Moreover, in this case $K_1 = \ldots = K_v = K_1$.

Proof of Theorem 4.1. Lemma 1.6 implies the existence of polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.2}$$

Using hypothesis 1^0 of the class $Lip(\beta_1, \ldots, \beta_v)$, we have that, for all sets $K_1, \ldots, K_d \in \mathcal{K}$, there exists an element $(g_1, \ldots, g_d, g_{11}, \ldots, g_{rl_1}, \ldots, g_{r1}, \ldots, g_{rl_r}) \in F^{-1}\{p\}$ such that $g_j(s) \neq 0$ on $K_j, j = 1, \ldots, d$. Suppose that $\tau \in \mathbb{R}$ satisfies the inequalities

$$\sup_{1\leq j\leq d} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - g_j(s)| < c^{-\frac{1}{\beta}} (\frac{\varepsilon}{4})^{\frac{1}{\beta}} \tag{4.3}$$

and

$$\sup_{1\leq j\leq d} \sup_{1\leq l\leq l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - g_{jl}(s)| < c^{-\frac{1}{\beta}} (\frac{\varepsilon}{4})^{\frac{1}{\beta}}, \tag{4.4}$$

where the sets $K_1, \ldots, K_d, K_{11}, \ldots, K_{1l_1}, \ldots, K_{r1}, \ldots, K_{rl_r} \in \mathcal{K}$ correspond the set K in hypothesis 2^0 of the class $Lip(\beta_1, \ldots, \beta_v)$, and $\beta = \min_{1\leq j\leq v} \beta_j$, with notation $K_{1l} = K_{d+l}, l = 1, \ldots, l_1, \ldots, K_{rl} = K_{d+l_1+\ldots+l_{r-1}+l}, l = 1, \ldots, l_r$. Then, in view of Theorem 2.1, the set of $\tau \in \mathbb{R}$ satisfying inequalities

(4.3) and (4.4) has a positive lower density. Moreover, hypothesis 2^0 of the class $Lip(\beta_1, \dots, \beta_v)$ shows that, for the same τ,

$$\sup_{s \in K} |F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) - p(s)| \leq$$

$$c \sup_{1 \leq j \leq d} \sup_{s \in K_j} |L(s+i\tau, \chi_j) - g_j(s)|^{\beta_j} +$$

$$\sup_{1 \leq j \leq d} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - g_{jl}(s)|^{\beta_j} \leq$$

$$2cc^{-\frac{\beta}{\beta}} (\frac{\varepsilon}{4})^{\frac{\beta}{\beta}} = \frac{\varepsilon}{2}. \tag{4.5}$$

We recall that here $\underline{\chi} = (\chi_1, \dots, \chi_d)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$ and

$$\Xi(s, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}}) = L(s, \chi_1), \dots, L(s, \chi_d), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots,$$

$$\zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}),$$

and the notation $\beta_{1l} = \beta_{d+l}$, $l = 1, \dots, l_1, \dots, \beta_{rl} = \beta_{d+l_1+\dots+l_{r-1}+l}$, $l = 1, \dots, l_r$, is used. Thus, by the above remark, inequality (4.5) and monotonicity of the measure, we obtain that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) - p(s)| < \frac{\varepsilon}{2}\} > 0.$$

Combining this with inequality (4.2) proves the theorem.

4.2. Approximation of analytic functions from the space $H(D)$

In Theorem 4.1, analytic functions from the class $H(K)$ are approximated. In this section, we develop the latter approximation. We recall that

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\},$$

and that

$$v_1 = \sum_{j=1}^{r} l_j.$$

Theorem 4.3 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S^d \times H^{v_1}(D))$ is non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 4.1 is true.*

We note that Theorem 4.3 is a generalization of Theorem 3.1.

The hypotheses $(F^{-1}G) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$ for every open set $G \subset H(D)$ is general but sufficiently complicated. Obviously, it is satisfied if every $g \in H(D)$ has a preimage in the set $S^d \times H^{v_1}(D)$. On the other hand, Theorem 4.3 implies the following modification of Theorem 4.1.

Theorem 4.4 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D))$ in non - empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 4.1 is true.*

Clearly, hypotheses 2^0 of the class $Lip(\beta_1, \dots, \beta_v)$ implies the continuity of the operator F. However, on the other hand, hypotheses 1^0 of the class $Lip(\beta_1, \dots, \beta_v)$ is weaker than the requirement of Theorem 4.3.

Theorem 4.4 is a generalization of Theorem 3.2.

Non-vanishing of the polynomial $p(s)$ on a bounded region can be controlled by its constant term. Therefore, sometimes it is more convenient, in place of Theorem 4.3, to have a generalization of Theorem 3.4. We preserve the notation used in Theorem 3.4.

Theorem 4.5 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, $K \in \mathcal{K}$ and $f(s) \in H(K)$, and $V > 0$ is such that $K \subset D_V$. Let $H^v(D_V, D) = H^d(D_V) \times H^{v_1}(D)$ and $F : H^v(D_V, D) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V^d \times H^{v_1}(D)$ is non - empty. Then the assertion of Theorem 4.1 is true.*

Before the proof of the stated above theorems, we will give some auxiliary results of probabilistic type. We preserve the notation of Chapter 2.

Theorem 4.6 *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that the operator $F : H^v(D) \to H(D)$ is continuous. Then*

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) \in A\}, A \in \mathcal{B}(H(D)),$$

converges weakly to $P_\Xi F^{-1}$ as $T \to \infty$.

Proof. In our notation, we have that $H_{d,u} = H_{d,u}(D) = H^v(D)$ Therefore, we may use Theorem 2.2, where the measure

$$P_T(A) = \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\}, A \in \mathcal{B}(H_{d,u}),$$

is considered. From the definitions of P_T and $P_{T,F}$, we find that, for all $A \in \mathcal{B}(H^v(D))$,

$$P_{T,F}(A) = \frac{1}{T} meas\{\tau \in [0;T] : \Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}}) \in F^{-1}A\} = P_T(F^{-1}A) = P_T F^{-1}(A).$$

This, Theorem 2.2, the continuity of F and Lemma 2.4 prove the theorem.

For the proof of universality theorems, we also need the explicit form of the support of the measure $P_{\Xi}F^{-1}$. For this, we will apply Theorem 2.14.

Lemma 4.7 *Suppose that the hypotheses of Theorem 2.14 are satisfied, and that the operator* $F : H^v(D) \to H(D)$ *is continuous. Then the support of the measure* $P_{\Xi}F^{-1}$ *is the closure of the set* $F(S^d \times H^u(D))$.

Proof. Let g be an arbitrary element of the set $F(S^d \times H^u(D))$, and G be any open neighbourhood of g. From the continuity of F, we have that the set $F^{-1}G$ is open as well. This means that $F^{-1}G$ is an open neighbourhood of a certain element of the set $S^d \times H^u(D)$. Therefore, Theorem 2.14 and properties of the support imply that $P_{\Xi}(F^{-1}G) > 0$. Hence, $P_{\Xi}F^{-1}(G) > 0$. Moreover, again in virtue of Theorem 2.14,

$$P_{\Xi}F^{-1}(F(S^d \times H^u(D))) = P_{\Xi}(S^d \times H^u(D)) = 1.$$

Since g and G are arbitrary, and the support of $P_{\Xi}F^{-1}$ is a closed set, from this the lemma follows.

Proof of Theorem 4.3. It is not difficult to see that, under hypotheses of Theorem 4.3, the support of the measure $P_{\Xi}F^{-1}$ is the whole of $H(D)$. Indeed, if $(F^{-1}G) \cap (S^d \times H^u(D)) \neq \varnothing$ for every open set $G \subset H(D)$, we have that, for every element $g \in H(D)$ and its any open neighbourhood G, there exists an element of the set $F(S^d \times H^u(D))$ which belongs to the set G. This shows that the set $F(S^d \times H^u(D))$ is dense in $H(D)$. Since, by Lemma 4.7, the support of the measure $P_{\Xi}F^{-1}$ is the closure of the set $F(S^d \times H^u(D))$, from this we obtain that the support of $P_{\Xi}F^{-1}$ is the whole of $H(D)$.

In view of Lemma 1.6, there exists a polynomial $p(s)$ satisfying inequality (4.2). Define the set

$$G = \{g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\}.$$

Obviously, G is an open neighbourhood of the polynomial $p(s)$ which is an element of the support of the measure $P_{\Xi}F^{-1}$. Thus, $P_{\Xi}F^{-1}(G) > 0$. Therefore, Theorem 4.6 and Lemma 1.7 imply the inequality

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) \in G\} \geq P_{\Xi}F^{-1}(G) > 0,$$

or, by the definition of G,

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - p(s)| < \frac{\varepsilon}{2}\} > 0. \tag{4.6}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s \in K} |F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - p(s)| < \frac{\varepsilon}{2}.$$

Then, for this τ, inequality (4.2) shows that

$$\sup_{s \in K} |F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - f(s)| \leq$$

$$\sup_{s\in K}|F(\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}))-p(s)|+$$

$$\sup_{s\in K}|f(s)-p(s)|<\varepsilon.$$

Thus, we have that

$$\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}))-p(s)|<\frac{\varepsilon}{2}\}\subset$$

$$\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}))-f(s)|<\varepsilon\}.$$

Combining this with (4.6) implies the inequality

$$\liminf_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:\sup_{s\in K}|F(\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}}))-p(s)|<\varepsilon\}>0.$$

The theorem is proved.

Proof of Theorem 4.4. In the proof of Theorem 3.2, we have observed that the approximation in the space $H(D)$ reduces to that on compact subsets with connected complements. Using this remark, we will show that the hypotheses of the theorem imply those of Theorem 4.3. Let $G\subset H(D)$ be an arbitrary non - empty open set. Then, in view of Lemma 1.6 and the above remark on the approximation in the space $H(D)$, there exists a polynomial $p=p(s)\in G$. Therefore, the hypothesis

$$(F^{-1}\{p\})\cap(S^d\times H^u(D))\neq\varnothing$$

implies that of Theorem 4.3 that the set

$$(F^{-1}G)\cap(S^d\times H^u(D))$$

is non - empty. Therefore, the theorem is a corollary of Theorem 4.3.

Proof of Theorem 4.5. The approximation in the space $H(D_V)$ also reduces to that on compact subsets with connected complements. Therefore, as in the proof of Theorem 4.4, using of Lemma 1.6 shows that the hypotheses that

$$(F^{-1}\{p\})\cap(S_V^d\times H^u(D))\neq\varnothing$$

for every polynomial $p=p(s)$ implies the hypothesis that

$$(F^{-1}G)\cap(S_V^d\times H^u(D))\neq\varnothing \tag{4.7}$$

for every open set $G\subset H(D_V)$. Thus, it suffices to repeat with corresponding changes the proof of Theorem 4.3.

Denote by $P_{\Xi,V}$ the restriction of the limit measure P_Ξ in Theorem 2.2 to the space $H^v(D_V,D)$. Then it follows from Theorem 2.2 and Lemma 2.4 that

$$\frac{1}{T}meas\{\tau\in[0;T]:\Xi(s+i\tau,\underline{\chi},\underline{\alpha};\underline{\mathfrak{a}})\in A\}, A\in\mathcal{B}(H^v(D_V,D)),$$

converges weakly to $P_{\Xi,V}$ as $T \to \infty$. This together with Lemma 2.4 and the continuity of F leads to the weak convergence of

$$P_{T,V,F}(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) \in A\}, A \in \mathcal{B}(H(D_V)),$$

to $P_{\Xi,V}F^{-1}$ as $T \to \infty$.

Repeating the proof of Theorem 2.14 shows that the support of the measure $P_{\Xi,V}$ is the set $S_V^d \times H(D)$. Now the continuity of the operator F together with (4.7) gives that the support of the measure $P_{\Xi,V}F^{-1}$ is the whole of $H(D_V)$. To obtain this, it suffices to repeat the proof of Lemma 4.7 and the first part of the proof of Theorem 4.3.

The remaining part of the proof is similar to that of Theorem 4.3. We take a polynomial $p(s)$ satisfying (4.2) and define the set

$$G_V = \{g \in H(D_V) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\}.$$

Since $p(s)$ is an element of the support of $P_{\Xi,V}F^{-1}$, we obtain from the weak convergence of the measure $P_{T,V,F}$ that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) \in G_V\} \geq P_{\Xi,V}F^{-1}(G_V) > 0.$$

Hence, by the definition of G_V,

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - p(s)| < \frac{\varepsilon}{2}\} > 0.$$

Combining this with (4.2) shows that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s \in K} |F(\Xi(s+i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - f(s)| < \varepsilon\} > 0.$$

The theorem is proved.

We give an example of an operator satisfying the hypothesis of Theorem 4.5. Suppose that

$$F(g_1, \ldots, g_v) = c_1 g_1 + \ldots + c_d g_d, \quad c_1, \ldots, c_d \in \mathbb{C} \setminus \{0\}.$$

Let $p(s)$ be an arbitrary polynomial. Then, taking

$$g_1(s) = \ldots = g_{d-2}(s) = 1,$$

$$g_{d-1}(s) = -\frac{1}{c_{d-1}} C,$$

$$g_d(s) = \frac{1}{c_d}(p(s) + C),$$

and

$$g_{d+1}(s) = \ldots = g_v(s) = 1,$$

where $|C|$ is sufficiently large, we have that

$$F(g_1, \dots, g_v) = p(s).$$

Moreover, $g_1(s) \neq 0, \dots, g_d(s) \neq 0$ on $H(D_V)$ if $|C|$ is large enough. Thus,

$$(F^{-1}\{p\}) \cap (S_V^d \times H^u(D)) \neq \varnothing.$$

4.3. Approximation of analytic functions from subsets of $H(D)$

In this section, we approximate analytic functions from the set $F(S^d \times H^u(D))$.

Theorem 4.8 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact subset and $f(s) \in F(S^d \times H^u(D))$. Then the assertion of Theorem 4.1 is true.*

Obviously, the set $F(S^d \times H^u(D))$ depends on the operator F, therefore, in general, its structure is not known. However, we can approximate analytic functions from some simple subset of $F(S^d \times H^u(D))$. Thus, we have the following generalization of Theorem 3.5. We recall that, for $a_1, \dots, a_k \in \mathbb{C}$,

$$H_k(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D), j = 1, \dots, k\}.$$

Theorem 4.9 *Suppose that the Dirichlet characters $\chi_1, \dots, \chi_d$, the numbers $\alpha_1, \dots, \alpha_r$ and the sequences $\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r}$ satisfy the hypotheses of Theorem 2.1, and that $F : H^v(D) \to H(D)$ is a continuous operator such that*

$$F(S^d \times H^u(D)) \supset H_k(D).$$

For $k = 1$, let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $k \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_k(D)$.Then the assertion of Theorem 4.1 is true.

Proof of Theorem 4.8. Since $f(s) \in F(S^d \times H^u(D))$, we have, in view of Lemma 4.7, that $f(s)$ is an element of the support of the measure $P_\Xi F^{-1}$. Consequently,$P_\Xi F^{-1}(G) > 0$ for

$$G = \{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon\}.$$

Therefore, from Theorem 4.6 and Lemma 1.7, and the definition of G, it follows that

$$\liminf_{T \to \infty} \frac{1}{T} meas\{\tau \in [0; T] : \sup_{s \in K} |F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - f(s)| < \varepsilon\} > 0.$$

Proof of Theorem 4.9. First suppose that $k = 1$. By Lemma 1.6, we can find a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{4.8}$$

By the hypotheses of the theorem, $f(s) \neq a_1$, on K. Therefore, $p(s) \neq 0$ on K as well if $\varepsilon > 0$ is small enough. Hence, on K there exists a continuous branch of the logarithm $\log(p(s) - a_1)$ which will be an analytic function on the interior of K. Applying Lemma 1.6 once more, we can find a polynomial $q(s)$ such that

$$\sup_{s\in K} |p(s) - a_1 - e^{q(s)}| < \frac{\varepsilon}{4}. \tag{4.9}$$

Let, for brevity, $g_{a_1}(s) = a_1 + e^{q(s)}$. Then $g_{a_1}(s) \in H(D)$ and $g_{a_1}(s) \neq a_1$ on D. This shows that $g_{a_1}(s) \in H_1(D)$. In view of Theorem 4.7, the support of the measure $P_{\Xi}F^{-1}$ contains the closure of the set $H_1(D)$. Therefore, the function $g_{a_1}(s)$ is an element of the support of the measure $P_{\Xi}F^{-1}$. Hence, $P_L F^{-1}(G) > 0$, where

$$G = \{g \in H(D) : \sup_{s\in K} |g(s) - g_{a_1}(s)| < \frac{\varepsilon}{2}\}.$$

This together with Theorem 4.6 and Lemma 1.7 shows that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\Xi(s + i\tau, \underline{\chi}, \underline{\alpha}; \underline{\mathfrak{a}})) - g_{a_1}(s)| < \varepsilon\} > 0. \tag{4.10}$$

It is easily seen that inequalities (4.8) and (4.9) imply the inequality

$$\sup_{s\in K} |f(s) - g_{a_1}(s)| < \frac{\varepsilon}{2}.$$

This and (4.10) yield the assertion of theorem in the case $k = 1$.

The case $k \geq 2$ is contained in Theorem 4.8.

Conclusions

In the thesis, the following universality results were proved:

1. Collections consisting from Dirichlet L - functions with pairwise non - equivalent characters and from Hurwitz zeta - functions with algebraically independent parameters are jointly universal.

2. Collections consisting from Dirichlet L - functions with pairwise non - equivalent characters and from periodic Hurwitz zeta functions with algebraically independent parameters are jointly universal.

3. Some classes of composite functions of collections described in 1. are universal.

4. Some classes of composite functions of collections described in 2. are universal.

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Notation

$\mathcal{P}$	set of all prime numbers
$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non - negative integers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all positive real numbers
$\mathbb{C}$	set of all complex numbers
$i = \sqrt{-1}$	imaginary unity
$s = \sigma + it$, $\sigma, t \in \mathbb{R}$	complex variable
$H(G)$	space of analytic functions on G
$\mathcal{B}(X)$	Borel σ - field of the space X
$\chi(m)$	Dirichlet character
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\zeta(s)$	Riemann zeta-function defined, for $\sigma > 1$, by $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere
$L(s, \chi)$	Dirichlet L - function defined, for $\sigma > 1$, by $L(s, \chi) = \sum\limits_{m=1}^{\infty} \frac{\chi(m)}{m^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha)$	Hurwitz zeta - function defined, for $\sigma > 1$, by for $\sigma > 1$, by $\zeta(s, \alpha) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha; \mathfrak{a})$	periodic Hurwitz zeta - function defined, for $\sigma > 1$, by for $\sigma > 1$, by $\zeta(s, \alpha; \mathfrak{a}) = \sum\limits_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}$, and by analytic continuation elsewhere

$\Gamma(s)$	Euler gamma-function defined, for $\sigma > 1$, by $\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du$, and by analytic continuation elsewhere
$measA$	Lebesgue measure of $A \subset \mathbb{R}$
$F^{-1}G$	preimage of a set G
$F^{-1}\{p\}$	preimage of a polynomial p