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DISCRETE UNIVERSALITY THEOREMS FOR THE RIEMANN AND HURWITZ ZETA-FUNCTIONS

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VILNIAUS UNIVERSITETAS

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DISKREČIOSIOS UNIVERSALUMO TEOREMOS RYMANO IR HURVICO DZETA FUNKCIJOMS

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Introduction

The objects investigated in the thesis belong to analytic number theory. Universality properties of the Riemann zeta-function and Hurwitz zeta-function are considered. The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

The Hurwitz zeta-function $\zeta(s, \alpha)$ with parameter α, $0 < \alpha \leq 1$, is also defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and, as the function $\zeta(s)$, has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Clearly, $\zeta(s, 1) = \zeta(s)$ and

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s). \tag{0.1}$$

Thus, the Hurwitz zeta-function is a generalization of the Riemann zeta-function.

The main difference between the functions $\zeta(s)$ and $\zeta(s, \alpha)$ is a fact that the function $\zeta(s)$ has the Euler product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \ \sigma > 1,$$

while the function $\zeta(s, \alpha)$, except for the values $\alpha = 1$ and $\alpha = \frac{1}{2}$, has no such a product. This difference has a great influence for analytic properties of the functions $\zeta(s)$ and $\zeta(s, \alpha)$.

The function $\zeta(s)$ was already known to L. Euler, however, he considered $\zeta(s)$ as a function of a real variable s. B. Riemann was the first who began to study $\zeta(s)$ with a complex variable s, and applied it for the investigation of prime numbers in the set $\mathbb{N}$, more precisely, for the asymptotics of the function

$$\pi(x) = \sum_{p \leq x} 1$$

as $x \to \infty$. B. Riemann proposed [53] a famous idea, but his further arguments were not mathematically strict. Only in 1896, C. J. de la Vallée Poussin [59] and J. Hadamard [16], using Riemann's ideas, completely proved the asymptotic distribution law of prime numbers in the form

$$\pi(x) = \int_2^x \frac{du}{\log u}(1 + o(1)), \ x \to \infty. \tag{0.2}$$

The function $\zeta(s, \alpha)$ was introduced by A. Hurwitz [19]. It has no a direct relation to prime numbers, however, plays an important role in the theory of Dirichlet L-functions which are the main tool for the investigation of prime numbers in arithmetical progressions. Since the function $\zeta(s, \alpha)$ depends on the parameter α, the arithmetic nature of this parameter influences also its analytic properties.

In general, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ are very interesting and important mathematical objects which appear in solving various problems not only in mathematics but also in other sciences, for example, in physics.

Now return to universality of zeta-functions. In general, universality in mathematics has a similar sense as in practice, namely, one object has a certain influence for a wide class of other objects. In analysis, this influence in usually realized by various types of approximation. Roughly speaking, the universality, for example, of the function $\zeta(s)$ means that a wide class of analytic in some region functions can be approximated with a given accuracy by shifts $\zeta(s + i\tau)$, where τ runs a certain set of real numbers. If τ takes arbitrary real values, then we have the continuous universality, while if τ takes values from the set $\{kh : k \in \mathbb{N}_0\}$ with a fixed $h > 0$, then we speak on the discrete universality.

Aims and problems

The aims of the thesis are discrete universality theorems for the functions $\zeta(s)$ and $\zeta(s, \alpha)$. The problems are the following.

1. Prove discrete universality theorems for composite functions of the Riemann zeta-function.
2. Prove discrete universality theorems for composite functions of the Hurwitz zeta-function.
3. Obtain information for the number of zeros of some composite analytic functions.
4. Prove a discrete universality theorem for a new class of Hurwitz zeta-functions.

Actuality

Universality of zeta-functions is a very interesting and important phenomenon in mathematics. Universality theorems of zeta-functions have a lot of theoretical and practical applications. Universality theorems are used for the proof of various denseness results and of the functional independence type theorems, they are used to obtain a certain information on the number of zeros of some functions related to zeta-functions, finally, universality is closely connected to the self–approximation, and, thus to the Riemann type hypotheses. Practical applications of universality are, of course, related to approximation problems of complicated analytic functions. Moreover, for practical applications, the discrete universality of zeta-functions is more convenient than the continuous one, see, for example, [4]. On the other hand, the works on the discrete universality are not numerous to compare with those on the continuous universality. To remove this gap between continuous and discrete universality of zeta-functions, the thesis is devoted to the discrete universality of zeta-functions, and it continues the investigations of A. Reich, B. Bagchi, A. Laurinčikas, R. Garunkštis, K. Matsumoto and J. Steuding.

Methods

For the proof of discrete universality theorems, the probabilistic approach based on limit theorems on weakly convergent probability measures in the space of analytic functions is applied. This approach is combined with continuous operator theory as well as with the Mergelyan theorem on the approximation of analytic functions by polynomials.

Novelty

All results of the thesis are new. Discrete universality theorems for composite functions of the Riemann and Hurwitz zeta-functions have been never studied. Also, the above discrete theorems are applied, for the first time, for the characterization of the zero-distribution. A discrete universality theorem for the Hurwitz zeta-function is obtained for a new class of parameters.

History of the problem and the results

The first universality result in analysis belongs to M. Fekete, see [49], who proved the existence of a real power series

$$\sum_{m=1}^{\infty} a_m x^m$$

such that, for every continuous function $g(x)$, $x \in [-1, 1]$, $g(0) = 0$, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that the partial sum

$$\sum_{m=1}^{m_k} a_m x^m$$

converges to $g(x)$ as $k \to \infty$, uniformly on the interval $[-1, 1]$.

An interesting universality result for entire functions was obtained by G. D. Birkhoff. He proved [3] that there exists an entire function $f(s)$ such that, for arbitrary given entire function $g(s)$, there exists a sequence of complex numbers $\{a_n : n \in \mathbb{N}\}$ such that

$$\lim_{n\to\infty} f(s + a_n) = g(s)$$

uniformly on compact subsets of $\mathbb{C}$.

Later, many analytic objects were discovered, however, all they were not explicitly given, only their existence was proved. Only in 1975, S. M. Voronin found [60] the first explicitly given analytic object, and turned out that this object is the Riemann zeta-function $\zeta(s)$. The initial statement of the Voronin theorem is of the following form.

Theorem A. *Suppose that $0 < r < \frac{1}{4}$. Let $f(s)$ be a continuous non-vanishing function on the disc $|s| \le r$ which is analytic in the interior of this disc. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\le r} \left| \zeta(s + \frac{3}{4} + i\tau) - f(s) \right| < \varepsilon$$

holds.

Theorem A shows that the set of values of the function $\zeta(s)$ is very rich. This property of $\zeta(s)$ was already observed by H. Bohr who proved [5] that the function $\zeta(s)$ takes every non-zero value in the strip $\{s \in \mathbb{C} : 1 < \sigma < 1 + \varepsilon\}$ with arbitrary $\varepsilon > 0$ infinitely many times. Much more interesting is a result of H. Bohr and R. Courant [6] that, for $\frac{1}{2} < \sigma < 1$, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.

Since the space of analytic functions is infinite-dimensional one, Theorem A is an infinite-dimensional generalization of the Bohr-Courant theorem.

Theorem A is a very deep result in the theory of the function $\zeta(s)$, therefore, it was soon observed by the mathematical community, and improved in the following sense. First, the disc $|s| \le r$ was replaced by compact sets, moreover, it was obtained that the set of shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, approximating a given analytic function is infinite. For the latest version of the Voronin theorem, we need some notation. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let meas$A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, in [24], [58] the following version of Theorem A can be found.

Theorem B. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\} > 0.$$

By Theorem B, we have that the set of shifts $\zeta(s+i\tau)$ approximating the function $f(s)$ with accuracy ε has a positive lower density, thus, it is infinite.

Theorems A and B are of the continuous type. A discrete analogue of Theorem B is contained in the next theorem.

Theorem C. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\} > 0.$$

Theorem C with slightly different assumptions on the set K was obtained in [1].

It is not difficult to see that some functions $F(\zeta(s))$ also preserve the universality property. For example, define $\log\zeta(s)$ in the strip D by continuous variation from $\log\zeta(2) \in \mathbb{R}$ along the line segments $[2, 2+it]$ and $[2+it, \sigma+it]$ provided that the path does not pass a possible zero or pole at $s = 1$. If this does, then we take

$$\log\zeta(\sigma+it) = \lim_{\varepsilon\to+0} \log\zeta(\sigma+i(t+\varepsilon)).$$

Denote by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Then Theorem A holds for $\log\zeta(s)$ with $f(s) \in H(K)$. By the way, in [22], the universality of the function $\zeta(s)$ is derived from that of $\log\zeta(s)$. Also, a simple application of the Cauchy integral formula leads to the universality of the derivative $\zeta'(s)$ with $f(s) \in H(K)$. Therefore, a problem arises to describe some class of operators F such that the function $F(\zeta(s))$ remains universal in the above sense. In [29], [30], this was done in the continuous case. Denote by $H(G)$ the space of analytic functions on the region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacta. We give one example from [29]. Let

$$S = \{g \in H(D) : g(s) \ne 0 \text{ or } g(s) \equiv 0\}.$$

Theorem D. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \bigcap S$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0, T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon\} > 0.$$

In Chapter 1 of the thesis, discrete analogues of Theorem D and other allied theorems are obtained.

Theorem 1.1. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the intersection $(F^{-1}G) \bigcap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then,* *for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon\} > 0.$$

The statement of Theorem 1.1 is theoretical, it is difficult to check the hypothesis that the set $(F^{-1}G) \bigcap S \neq \varnothing$ for every open set $G \subset H(D)$. The latter hypothesis can be replaced by a stronger but simpler one.

Theorem 1.2. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \bigcap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 1.1 is true.*

The hypothesis that $(F^{-1}\{p\}) \bigcap S \neq \varnothing$ is related to the non-vanishing of the preimage $F^{-1}\{p\}$ for each polynomial p. It is clear that if the absolute value of the constant term of a polynomial is sufficiently large, then this polynomial has no roots in a bounded region. This observation leads to the following simplification of Theorem 1.2.

Let V be an arbitrary positive number. Define the sets

$$D_V = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1,\ |t| < V \right\}$$

and

$$S_V = \{g \in H(D_V) :\ g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 1.3. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$, and let $V > 0$ be such that $K \subset D_V$. Suppose that $F : H(D_V) \to H(D_V)$ is a continuous operator such that, for each polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \bigcap S_V$ is non-empty. Then the assertion of Theorem 1.1 is true.*

It is not difficult to present an example of the operator F in Theorem 1.3. Let $F : H(D_V) \to H(D_V)$ be given by the formula

$$F(g) = c_1 g' + \cdots + c_r g^{(r)},\ g \in H(D_V),\ c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\},$$

and $g^{(k)}$ denotes the k-th derivative of the function g. In view of the Cauchy integral formula, the operator F is continuous. Moreover, it is easy to check that, for every polynomial $p = p(s)$, there exists a polynomial $q = q(s)$ such that $q \in F^{-1}\{p\}$ and $q(s) \neq 0$ for $s \in D_V$. Then, by Theorem 1.3, the function

$$c_1 \zeta'(s) + \cdots + c_r \zeta^{(r)}(s)$$

is universal in the sense of Theorem 1.1.

In Chapter 1, one more class of operators $F : H(D) \to H(D)$ is investigated. For $a_1, \dots, a_r \in \mathbb{C}$, let

$$H_{F;a_1,\dots,a_r}(D) = \{g \in H(D) :\ (g(s) - a_j)^{-1} \in H(D),\ j = 1, \dots, r\} \bigcup \{F(0)\}.$$

Theorem 1.4. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{F;a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$, and the function $f(s)$ be continuous and $\neq a_1$ on K, and analytic in the interior of K. For $r \geq 2$, let K be an arbitrary compact subset of D, and $f(s) \in H_{F;a_1,\dots,a_r}(D)$. Then the assertion of Theorem 1.1 is true.*

From Theorem 1.4, the discrete universality for the functions $\zeta^N(s)$, $N \in \mathbb{N}$, and $\sin(\zeta(s))$, $\cos(\zeta(s))$, $\sinh(\zeta(s))$, $\cosh(\zeta(s))$ follows.

We note that Theorems 1.1–1.4 are the corresponding discrete analogues of continuous universality theorems for the Riemann zeta-function obtained in [29] and [30].

Theorems 1.1–1.4 were obtained in [51], however, with the restriction that the number $\exp\left\{\frac{2\pi k}{h}\right\}$ is irrational for all $k \in \mathbb{Z} \setminus \{0\}$. In the thesis, we remove this requirement for the number $h > 0$.

The proofs of Theorems 1.1–1.4 are probabilistic, based on limit theorems on weakly convergent probability measures in the space of analytic functions as well as on Theorem C and the Mergelyan theorem [43].

After Voronin's work [60], it was observed that not only the function $\zeta(s)$, but also other zeta and L-functions are universal in the above sense. S. M Voronin himself obtained [60] the universality of all Dirichlet L-functions $L(s, \chi)$ which are defined, for $\sigma > 1$, by the Dirichlet series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

where χ is a Dirichlet character, and can be continued meromorphically to the whole complex plane. Also, zeta-functions $\zeta(s, F)$ of Hecke eigen cusp forms F of weight κ are universal [20], [35], [36], [37]. They are entire functions defined, for $\sigma > \frac{\kappa+1}{2}$, by the series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

where $c(m)$ are the Fourier coefficients of the form F. In [26], the universality of zeta-functions attached to Abelian groups was obtained. All these examples of zeta-functions have Euler products over primes.

The another group of universal zeta-functions have no Euler product over primes. The simplest member of this group is the already mentioned above Hurwitz zeta-function $\zeta(s, \alpha)$. Its generalizations are the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}, \ \sigma > 1,$$

with parameters $\lambda \in \mathbb{R}$ and α, $0 < \alpha \leq 1$, and the periodic Hurwitz zeta-function

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \ \sigma > 1,$$

where $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers, and α is the same parameter as in the definition of the functions $\zeta(s, \alpha)$ and $L(\lambda, \alpha, s)$.

The continuous universality of the Hurwitz zeta-function is contained in the following theorem. We recall that the number α is transcendental if it is not a root of any polynomial $p(s) \not\equiv 0$ with rational coefficients.

Theorem E. *Suppose that α is transcendental or rational number $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\} > 0.$$

Theorem E, for rational α, was obtained independently by S. M. Voronin [61], [62], S. M. Gonek [15] and B. Bagchi [1]. The case of transcendental α can be found in the monograph [13]. The universality of the function $\zeta(s, \alpha)$ with algebraic irrational parameter α is an open problem. We recall that the number α is algebraic if it is a root of a certain polynomial $p(s) \not\equiv 0$ with rational coefficients. For example, the number $\sqrt{2}$ is algebraic irrational because it is a root of the polynomial $s^2 - 2$. Obviously, all rational numbers are algebraic.

The cases of rationals $\alpha = 1$ or $\alpha = \frac{1}{2}$ in Theorem E are excluded because of the equalities $\zeta(s, 1) = \zeta(s)$ and (0.1). In these cases, the function $\zeta(s, \alpha)$ remains universal in the above sense, however, the approximated function $f(s)$ must be non-vanishing on K.

For the function $\zeta(s, \alpha)$, also a discrete university theorem is known.

Theorem F. *Suppose that the parameter α, the set K and the function $f(s)$ are as in Theorem E. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon\} > 0.$$

For rational α, Theorem F with slightly different assumptions on the set K, is given in [1]. It can be easily seen that those assumptions, in view of the Mergelyan theorem [43], can be replaced by the assumptions of Theorem F. By a different method, the case of rational α was treated in [54]. For transcendental α, Theorem F follows from a similar theorem proven in [34] for the periodic Hurwitz zeta-function.

In [31], Theorem E was generalized for composite functions $F(\zeta(s,\alpha))$ with certain operators $F : H(D) \to H(D)$. We present only one example from [31].

Theorem G. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \bigcap H(D)$ is non-empty. Let the parameter α, the set K and the function $f(s)$ be as in Theorem E. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau,\alpha)) - f(s)| < \varepsilon\} > 0.$$

In Chapter 2 of the thesis, discrete analogues of theorems obtained in [31] are proved. For a sufficiently wide class of operators F, the discrete universality of $F(\zeta(s,\alpha))$ can be deduced directly from Theorem F. We say that the operator $F : H(D) \to H(D)$ belongs to the class $\mathrm{Lip}(\beta)$, $\beta > 0$, if the following conditions are satisfied:

1° for each polynomial $p = p(s)$, there exists an element $g \in F^{-1}\{p\} \subset H(D)$;

2° for every set $K \in \mathcal{K}$, there exist a positive constant c and a set $K_1 \in \mathcal{K}$ such that

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \le c \sup_{s\in K_1} |g_1(s) - g_2(s)|^\beta$$

for all $g_1, g_2 \in H(D)$.

The first theorem of Chapter 2 is of the following form.

Theorem 2.1. *Suppose that the numbers α and h, the set K and the function $f(s)$ are as in Theorem F, and that $F \in Lip(\beta)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon\} > 0.$$

It easily follows from the Cauchy integral formula that the function $F(g) = g'$ belongs to the class $\mathrm{Lip}(1)$. Therefore, the assertion of Theorem 2.1 is true for the function $\zeta'(s,\alpha)$.

Now we state discrete universality theorems for $F(\zeta(s,\alpha))$ for other classes of operators F.

Theorem 2.2. *Suppose that the numbers α and h, the set K and the function $f(s)$ are as in Theorem F. Let $F : H(D) \to H(D)$ be a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. The the assertion of Theorem 2.1 is true.*

The assumption of Theorem 2.2 on the operator F can be replaced by a simpler one.

Theorem 2.3. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Then with α, h, K and $f(s)$ as in Theorem F the assertion of Theorem 2.1 is true.*

For $g \in H(D)$, let

$$F(g) = c_1 g' + \cdots + c_r g^{(r)},\ c_1, \ldots, c_r \in \mathbb{C}\backslash\{0\}.$$

Then, clearly, for each polynomial $p = p(s)$, there exists a polynomial $q = q(s)$ such that $q \in F^{-1}\{p\}$. Therefore, by Theorem 2.3, the function

$$c_1\zeta'(s,\alpha) + \cdots + c_r\zeta^{(r)}(s,\alpha)$$

has a discrete universality property with rational $\alpha \neq 1, \frac{1}{2}$ (in this case, $h > 0$ can be chosen arbitrarily) and transcendental α (in this case, the number $\exp\left\{\frac{2\pi}{h}\right\}$ must be rational).

In the next theorem, the functions from a certain subset of $H(D)$ are approximated by shifts $F(\zeta(s+ikh,\alpha))$. For $a_1,\dots,a_r \in \mathbb{C}$, denote

$$H_{a_1,\dots,a_r}(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1,\dots,r\}.$$

Theorem 2.4. *Suppose that the numbers α and h are as in Theorem F, and that $F : H(D) \to H(D)$ is a continuous operator such that $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$, and let $f(s) \neq a_1$ be a continuous function on K, which is analytic in the interior of K. For $r \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{a_1,\dots,a_r}(D)$. Then the assertion of Theorem 2.1 is true.*

For example, if $r = 1$ and $a_1 = 0$, the discrete universality of $\zeta^n(s,\alpha)$, $n \in \mathbb{N} \setminus \{0\}$, follows. If $r = 2$ and $a_1 = -1$, $a_2 = 1$, we have the same property for the functions $\sin(\zeta(s,\alpha))$, $\cos(\zeta(s,\alpha))$, $\sinh(\zeta(s,\alpha))$ and $\cosh(\zeta(s,\alpha))$.

The next theorem shows that the functions from $F(H(D))$ can be always approximated by discrete shifts of $F(\zeta(s,\alpha))$.

Theorem 2.5. *Suppose that the numbers α and h are as in Theorem F, and that $F : H(D) \to H(D)$ is an arbitrary continuous operator. Let $K \subset D$ be a compact subset, and $f(s) \in F(H(D))$. Then the assertion of Theorem 2.1 is true.*

Theorems 2.1–2.5 are published in [40].

Denote by $M(D)$ the space of meromorphic functions on D endowed with the topology of uniform convergence on compacta. Clearly, $H(D) \subset M(D)$. We note that some analogues of Theorems 2.2–2.5 can be also obtained for continuous operators $F : H(D) \to M(D)$.

In Chapter 3 of the thesis, the zero-distribution of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ is briefly discussed.

The zero-distribution of zeta and L-functions is the central problem of analytic number theory, and the results in the field allow to solve many other important problems. For example, the location of non-trivial zeros of the Riemann zeta-function $\zeta(s)$ has a direct relation to the distribution of prime numbers. We remind that the zeros $s = -2m$, $m \in \mathbb{N}$, of $\zeta(s)$ are called trivial and they are implied by the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the Euler gamma-function. The trivial zeros do not play an important role in the theory of $\zeta(s)$. Moreover, it is known that the function $\zeta(s)$ has infinitely many the so called non-trivial zeros lying in the critical strip $\{s \in \mathbb{C} : 0 \le \sigma \le 1\}$. The famous Riemann hypothesis (RH) asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$, or that $\zeta(s) \ne 0$ for $\sigma > \frac{1}{2}$.

For the proof of relation (0.2), it is sufficient to know that $\zeta(s) \ne 0$ on the line $\sigma = 1$. It is known [23] that the estimate

$$\pi(x) = \int\limits_2^x \frac{du}{\log u} + O(x^{\alpha+\varepsilon}),$$

where $\varepsilon > 0$ is and arbitrary number , is equivalent to the non-vanishing of $\zeta(s)$ for $\sigma > \alpha$. In particular, the RH is equivalent to the estimate [11]

$$\pi(x) = \int\limits_2^x \frac{du}{\log u} + O(\sqrt{x}\log x).$$

The best known result on the location of zeros of $\zeta(s)$ asserts [22] that $\zeta(s) \ne 0$ in the region

$$\sigma > 1 - \frac{c}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \; |t| \ge t_0 > 0,$$

where $c > 0$ is an absolute constant. This was obtained by H. E. Richert (unpublished). G. H. Hardy proved [17] that infinitely many zeros lie on the critical line. A. Selberg was the first proving [56] that a positive proportion of all nontrivial zeros of $\zeta(s)$ lies on the line $\sigma = \frac{1}{2}$. Now it is know [7] that at least 41 percent of non-trivial zeros of $\zeta(s)$ in the sense of density are located on the line $\sigma = \frac{1}{2}$. All computer calculations also support the RH. For example, it is known [48] that the 10^{22} nd zero of $\zeta(s)$ and 10 billion of its neighbours lie on the critical line.

Without the mentioned above in terms of the function $\pi(x)$, several other equivalents of the RH are known. One of them is stated in terms of self-approximation and is connected to the universality of $\zeta(s)$. Namely, B. Bagchi proved [1] that the RH is equivalent to the assertion that, for every compact set $K \subset D$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\} > 0.$$

Some interesting approximations to this equivalent were made by T. Nakamura [45], T. Nakamura and Ł. Pańkowski [46], and R. Garunkštis [14]. They considered the inequality

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s+id\tau)| < \varepsilon\} > 0.$$

Now it is known that the latter inequality holds for irrational d and rational non-zero $d = \frac{a}{b}$ with $(a,b) = 1$, $|a-b| \ne 1$. To prove RH, it suffices to show that the above inequality holds with $d = 0$.

S. M. Voronin was the first who applied the universality of zeta-functions for estimation of the number of zeros. In [61], he proved the following assertion.

Theorem H. *Let $a, b \in \mathbb{N}$, $1 \leq a < b$, $b \neq 2$, $(a, b) = 1$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(a, b, \sigma_1, \sigma_2) > 0$ such that the function $\zeta\left(s, \frac{a}{b}\right)$, for sufficiently large T, has more than cT zeros lying in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, \ 0 < t < T\}.$$

Generalizations of Theorem H for linear combinations of various zeta and L-functions can be found in [25] and [47]. An analogue of Theorem H for $\zeta'(s, F)$, where $\zeta(s, F)$ is the zeta-function attached to a Hecke eigen cups form F, was obtained in [27]. The estimates for the number of zeros of composite functions of $\zeta(s)$ is discussed in [32].

In Chapter 3 of the thesis, discrete versions of the above mentioned results on the estimates of the number of zeros for composite functions are given.

Theorem 3.1. *Suppose that the operator F is as in one of Theorems 1.1–1.3. Then, for arbitrary σ_1 and σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, F, h) > 0$ such that, for sufficiently large $N \in \mathbb{N}$, the function $F(\zeta(s + ikh))$ has a zero in the disc*

$$|s - \widehat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}, \ \widehat{\sigma} = \frac{\sigma_1 + \sigma_2}{2},$$

for more than cN numbers k, $0 \leq k \leq N$.

Theorem 3.1 is published in [52].

Now we pass to zeros of the Hurwitz zeta-function $\zeta(s, \alpha)$. We have already mentioned that the properties of $\zeta(s, \alpha)$ depend on the arithmetical nature of the parameter α. H. Davenport and H. Heilbronn observed [13] that, for transcendental or rational $\alpha \neq 1, \frac{1}{2}$, the function $\zeta(s, \alpha)$, differently from $\zeta(s)$, has zeros in the region $\sigma > 1$. J. W. S. Cassels extended [9] the latter result for algebraic irrational parameter α. Theorem H shows that, for rational $\alpha \neq 1, \frac{1}{2}$, the function $\zeta(s, \alpha)$ has zeros in the critical strip $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. A similar assertion to Theorem H, for transcendental α, was obtained in [33], Theorem 8.4.7. In the paper [42], the latter results were extended for $F(\zeta(s, \alpha))$ with some operators F defined on $H(D)$.

Chapter 3 of the thesis contains theorems on the number of zeros of the function $\zeta(s + ikh, \alpha)$, and, more generally, of $F(\zeta(s + ikh, \alpha))$, $k \in \mathbb{N}_0$, for some classes of operators F, where $h > 0$ is a fixed number.

Theorem 3.3. *Suppose that α is transcendental or rational number $\neq 1, \frac{1}{2}$. In the case of rational α, let $h > 0$ be an arbitrary fixed number, while in the case of transcendental α, let $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, h) > 0$ such that, for sufficiently large $N \in \mathbb{N}$, the function $\zeta(s + ikh, \alpha)$ has a zero in the disc*

$$|s - \widehat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}, \ \widehat{\sigma} = \frac{\sigma_1 + \sigma_2}{2}, \tag{0.3}$$

for more than cN numbers k, $0 \le k \le N$.

The next theorems are analogues of Theorem 3.3 for $F(\zeta(s,\alpha))$.

Theorem 3.4. *Suppose that the numbers α and h are as in Theorem 3.3, and that $F : H(D \to H(D))$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty, or $s - a \in F(H(D))$ for all $a \in \left(\frac{1}{2}, 1\right)$. Then, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, h, F) > 0$ such that, for sufficiently large $N \in \mathbb{N}$, the function $F(\zeta(s+ikh,\alpha))$ has a zero in the disc (0.3) for more than cN numbers k, $0 \le k \le N$.*

Let $F : H(D) \to H(D)$ be given by the formula

$$F(g) = gg',\ g \in H(D).$$

Then it is not difficult to see that $s - a \in F(H(D))$ for all $a \in \left(\frac{1}{2}, 1\right)$.

Let $H_{a_1,\dots,a_r}(D)$ by the same set as in Theorem 2.4.

Theorem 3.5. *Suppose that the numbers α and h are as in Theorem 3.3, and that $F : H(D) \to H(D)$ is a continuous operator such that $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$, where $Re a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, r$. Then the assertion of Theorem 3.4 is true.*

For example, we can take $F(g) = \sin g$ or $F(g) = \cos g$ in Theorem 3.5.

Theorems 3.3–3.5 are published in [41].

The last, Chapter 4, of the thesis is devoted to the extension of Theorem F and Theorem 2.3.

Let, as usual, $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{Q}_1^+$ be the set of positive rational numbers $\neq 1$, and define, for $q \in \mathbb{Q}_1^+$, the set

$$L(\alpha, q) = \{(\log(m+\alpha) :\ m \in \mathbb{N}_0), \log q\}.$$

Theorem 4.1. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over the field $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N :\ \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\} > 0.$$

We note that if α is transcendental, then the set $L(\alpha, q)$ is linearly independent over $\mathbb{Q}$. On the other hand, if α is algebraic irrational, then it is known [9] that at least 51 persent of elements of the set

$$L(\alpha) = \{\log(m+\alpha) :\ m \in \mathbb{N}_0\}$$

in the sense of density are linearly independent over $\mathbb{Q}$. Thus, it is conceivable that, for some algebraic irrational α, the set $L(\alpha, q)$ is also linearly independent over $\mathbb{Q}$. Unfortunately, examples of algebraic irrational α with linearly independent sets $L(\alpha)$ and $L(\alpha, q)$ at the moment are not known.

Theorem 4.1 can be generalized for composite functions $F(\zeta(s,\alpha))$ with some operators $F: H(D) \to H(D)$. As example, we give only a generalization of Theorem 2.3.

Theorem 4.2. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$, $F: H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the preimage $F^{-1}\{p\}$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon\} > 0.$$

Theorems 4.1 and 4.2 are published in [8].

For the proofs of all universality theorems obtained in the thesis, the probabilistic method based on weakly convergent probability measures in the space $H(D)$ is applied.

Approbation

The main results of the thesis were presented at the 8th International Algebraic Conference in Ukraine dedicated to the memory of Professor Vitaliy Mikhaylovich Usenko, July 5-12, 2011, Lugansk, Ukraine, at the MMA (Mathematical Modelling and Analysis) conferences (MMA 2011, May 25-28, 2011, Sigulda, Latvia; MMA 2012, June 6-9, 2012, Tallinn, Estonia; MMA 2013, May 27-30, 2013, Tartu, Estonia), at the Fifth International Conference in Honour of J. Kubilius, September 4-10, 2011, Palanga, at the International Conference on Number Theory in Honour of A. Laurinčikas, September 9-12, 2013, Šiauliai, at the Conferences of Lithuanian Mathematical Society (2011, 2012, 2013), at the seminars of Vilnius University and Šiauliai University.

Principal publications

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2. A. Laurinčikas and J. Rašytė, Generalizations of a discrete universality theorem for Hurwitz zeta-functions, Lith. Math. J. **52**(2012), No. 2, 172-180.

3. A. Laurinčikas and J. Rašytė, On zeros of some composite functions, in: Analytic Methods of Analysis and Diferential Equations: AMADE 2012, S. V. Rogosin and M. V. Dubatovshaya (Eds), Cambridge Scientific Publishers, Cambridge, 2013, pp. 99-105.

4. J. Rašytė, On discrete universality of composite functions, Math. Modell. and Analysis, **17**(2012), No. 2, 271-280.

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Conference abstracts

1. J. Rašytė, Discrete universality of the composite functions, Book of abstracts of the 8th International Algebraic Conference in Ukraine, Yu. A. Drozd et al (Eds), Lugansk Taras Shevchenko National University, Lugansk, 2011, p. 45.

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4. J. Rašytė, Zeros of some analytic functions, Abstracts of MMA 2013, Institute of Mathematics of the University of Tartu, Tartu, Estonia, 2013, p. 102.

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Chapter 1

Discrete universality theorems for composite functions of the Riemann zeta-function

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. In this chapter, we obtain discrete universality theorems for the functions $F(\zeta(s))$, where $\zeta(s)$ is the Riemann zeta-function and $F : H(D) \to H(D)$, $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, is a certain operator.

We observe that the space $H(G)$ is metrisable one. It is well known, see, for example, [10], that there exists a sequence of compact subsets $\{K_l : l \in \mathbb{N}\} \subset D$ such that

$$G = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset G$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(G)$, define

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_l} |g_1(s) - g_2(s)|}. \tag{1.1}$$

Clearly, ϱ is a metric of the space $H(G)$. Moreover, ϱ induces the uniform convergence on compacta. Indeed, let $\{g_n(s) : n \in \mathbb{N}\} \subset H(G)$ and $g(s)$. Suppose that $g_n(s)$ converges to $g(s)$ as $n \to \infty$. Then, for every compact subset $K \subset D$,

$$\sup_{s \in K} |g_n(s) - g(s)| \xrightarrow[n \to \infty]{} 0. \tag{1.2}$$

Therefore, for every $l \in \mathbb{N}$,

$$\sup_{s \in K_l} |g_n(s) - g(s)| \xrightarrow[n \to \infty]{} 0. \tag{1.3}$$

Hence, in view of (1.1), $\varrho(g_n, g) \to 0$ as $n \to \infty$.

Now let $\varrho(g_n, g) \to 0$ as $n \to \infty$. Then, for every $l \in \mathbb{N}$, (1.3) holds. Suppose that K is an arbitrary compact subset of D. Then there exists $l_0 \in \mathbb{N}$ such that $K \subset K_{l_0}$. Therefore,

$$\sup_{s \in K} |g_n(s) - g(s)| \le \sup_{s \in K_{l_0}} |g_n(s) - g(s)| \xrightarrow[n \to \infty]{} 0.$$

Since K is arbitrary, we have that relation (1.2) is true for all compact subsets $K \subset D$.

1.1. Statement of the results

We recall that

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

An operator $F : H(G) \to H(G)$ is said to be continuous at $g_0 \in H(D)$ if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\varrho(F(g), F(g_0)) < \varepsilon$$

for all $g \in H(G)$ satisfying

$$\varrho(g, g_0) < \delta.$$

Moreover, we remind that $\mathcal{K}$ is the class of compact subsets of the strip D with connected complements, and that $H(K)$, $K \in \mathcal{K}$, is the class of continuous functions on K which are analytic in the interior of K.

The first discrete universality theorem for the composite function $F(\zeta(s))$ uses a hypothesis on the operator F stated in terms of open sets of the space $H(D)$. Denote by $F^{-1}G$ the preimage of the set G, $h > 0$ is a fixe number.

Theorem 1.1. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the intersection $(F^{-1}G) \bigcap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s \in K} |F(\zeta(s + ikh)) - f(s)| < \varepsilon\} > 0.$$

The next theorem, in place of the hypothesis $(F^{-1}G) \bigcap S \neq \varnothing$, apply a hypothesis related to all polynomials. As it will be shown in the proof, the hypothesis of such a kind is stronger, i.e., it implies the hypothesis of Theorem 1.1, however, it is more convenient, because it is not easy to describe all open sets of the space $H(D)$. Denote by $F^{-1}\{p\}$ the preimage of a polynomial $p = p(s)$.

Theorem 1.2. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \bigcap S$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 1.1 is true.*

Usually it is expected that the preimage of a polynomial is again a polynomial. The region $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ is not bounded one. Therefore, though a polynomial has only a finite number of roots, it is not a guarantee that at least one of roots lies in the strip D. This observation suggests an idea to consider the space of analytic function in a bounded region, because a polynomial with sufficiently large modulus of the constant term can't take the value zero in such a region.

For an arbitrary $V > 0$, define the analogues of D and S by

$$D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1,\ |t| < V\}$$

and

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \,\text{or}\, g(s) \equiv 0\}.$$

Then we have the following analogue of Theorem 1.2.

Theorem 1.3. *Let $K \in \mathcal{K}$ and $f(s) \in H(K)$, and let $V > 0$ be such that $K \subset D_V$. Suppose that $F : H(D_V) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the intersection $(F^{-1}\{p\}) \bigcap S_V$ is non-empty. Then the assertion of Theorem 1.1 is true.*

We give a simple example. Denote by $g^{(k)}$ the k th derivative of the function $g \in H(D_V)$, and define the operator $F : H(D_V) \to H(D_V)$ by the formula

$$F(g) = c_1 g' + \cdots + c_r g^{(r)},\ g \in H(D_V),\ c_1, \ldots, c_r \in \mathbb{C}\backslash\{0\}.$$

For the proof of the continuity of F, we apply the classical Cauchy integral formula, see, for example, [55]: Let G be a region in $\mathbb{C}$, $f(s)$ be an analytic function on G, Γ be a simple contour contained with its interior in G, and $s_0 \in \text{int}\Gamma$. Then, for $n = 0, 1, 2, \ldots$,

$$f^{(n)}(s_0) = \frac{n!}{2\pi i} \int\limits_{\Gamma} \frac{f(s)}{(s - s_0)^{n+1}} ds.$$

We use an equivalent of the continuity of operator in terms of sequences: F is continuous at the point g if, for every sequence $\{g_n\}$, $g_n \xrightarrow[n\to\infty]{} g$, we have that $F(g_n) \xrightarrow[n\to\infty]{} F(g)$. Let K be an arbitrary compact subset of the rectangle D_V, $G \supset K$ be an open set lying in D_V, and let K_1 be a compact subset such that $G \subset K_1 \subset D_V$. We take a simple closed contour L lying in $K_1 \setminus G$ and enclosing the set G. Then, by the Cauchy integral formula, we find that

$$\sup_{s\in K} |F(g_n(s)) - F(g(s))| \leq \frac{1}{2\pi}\sum_{j=1}^{r} j!|c_j| \int_L \frac{|g_n(z) - g(z)|}{|z-s|^{j+1}}|dz| \leq$$
$$\leq \frac{1}{2\pi}\sum_{j=1}^{r} \frac{j!|c_j||L|}{\delta^{j+1}} \sup_{s\in K_1} |g_n(s) - g(s)|, \tag{1.4}$$

where δ is the distance of the contour L from the set K, and $|L|$ is the length of L. Now if $g_n(s) \xrightarrow[n\to\infty]{} g(s)$ in $H(D_V)$, then

$$\sup_{s\in K_1} |g_n(s) - g(s)| \xrightarrow[n\to\infty]{} 0.$$

This together with (1.4) shows that

$$\sup_{s\in K} |F(g_n(s)) - F(g(s))| \xrightarrow[n\to\infty]{} 0.$$

Therefore, $F(g_n) \xrightarrow[n\to\infty]{} F(g)$ in the space $H(D_V)$, thus, the operator F is continuous.

Now we check the hypothesis that the preimage $F^{-1}\{p\}$ of an arbitrary polynomial $p = p(s)$ contains an element g such that $g(s) \neq 0$ on D_V. We have to prove that, for each polynomial $p = p(s)$, there exists an other polynomial $q = q(s)$ such that $q \in F^{-1}\{p\}$, and $q(s) \neq 0$ for $s \in D_V$. Thus, we take a polynomial

$$p(s) = a_k s^k + \cdots + a_1 s + a_0,\ a_k \neq 0,$$

of degree k, and search for a polynomial $q(s)$ of degree $k+1$ in the form

$$q(s) = b_{k+1}s^{k+1} + \cdots + b_1 s + b_0,\ b_{k+1} \neq 0.$$

First suppose that $r \leq k+1$. Then we find that

$$\begin{cases} q'(s) = (k+1)b_{k+1}s^k + \cdots + 2b_2 s + b_1, \\ q''(s) = (k+1)kb_{k+1}s^{k-1} + \cdots + 2b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q^{(r)}(s) = (k+1)k\dots(k-r+2)b_{k+1}s^{k-r+1} + \cdots + r!b_r. \end{cases}$$

Since $F(q) = p$, hence we obtain that

$$(k+1)c_1 s^k + \cdots + 2c_1 b_2 s + c_1 b_1 + (k+1)kc_2 b_{k+1}s^{k-1} + \cdots +$$
$$2c_2 b_2 + \cdots + (k+1)k\dots(k-r+2)c_r b_{k+1}s^{k-r+1} + \cdots + r!c_r b_r =$$
$$a_k s^k + \cdots + a_1 s + a_0.$$

Therefore, we have a system of equations

$$\begin{cases} (k+1)c_1 b_{k+1} = a_k, \\ (k+1)kc_2 b_{k+1} + kc_1 b_k = a_{k-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ r!c_r b_r + \dots + 2c_2 b_2 + c_1 b_1 = a_0 \end{cases}$$

for the coefficients b_j, $j = k+1, \dots, 1$, of the polynomial $q(s)$.

If $r > k+1$, then we obtain a similar system of equations. In this case, $q^{(j)}(s) = 0$ for $j \geq k+2$.

Having the coefficients $b_{k+1}, \dots, b_1$, we take b_0 to be $|b_0|$ sufficiently large, then $q(s) \neq 0$ for $s \in D_V$.

In the next theorem, we approximate by shifts $F(\zeta(s+it))$ the functions from some subsets of $H(D)$. For different $a_1, \dots, a_r \in \mathbb{C}$, and $F : H(D) \to H(D)$, define the set

$$H_{F;a_1,\dots,a_r}(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1, \dots, r\} \bigcup \{F(0)\}.$$

Theorem 1.4. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{F;a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$, and the function $f(s)$ be continuous and $\neq a_1$ on K and analytic in the interior of K. For $r \geq 2$, let K be an arbitrary compact subset of D, and $f(s) \in H_{F;a_1,\dots,a_r}(D)$. Then the assertion of Theorem 1.1 is true.*

If $r = 1$ and $a_1 = 0$, Theorem 1.4 implies the universality of the function $\zeta^N(s)$, $N \in \mathbb{N}$. In the case $r = 2$, $a_1 = 1$, $a_2 = -1$, we obtain the universality for the functions $\sin(\zeta(s))$, $\cos(\zeta(s))$, $\sinh(\zeta(s))$, $\cosh(\zeta(s))$. We remind that

$$\sin s = \frac{e^{is} - e^{-is}}{2i}, \quad \cos s = \frac{e^{is} + e^{-is}}{2}$$

and

$$\sinh s = \frac{e^{s} - e^{-s}}{2}, \quad \cosh s = \frac{e^{s} + e^{-s}}{2}.$$

We consider only the case of $\sin(\zeta(s))$. Other functions are considered analogically. We solve the equation

$$\frac{e^{ig} - e^{-ig}}{2i} = f$$

with respect to g. Putting $e^{ig} = y$, we have the equation

$$y^2 - 2ify - 1 = 0.$$

Hence,

$$y = fi \pm \sqrt{1-f^2},$$

and

$$g = \frac{1}{i} \log \left(fi \pm \sqrt{1-f^2} \right).$$

If $f \in H_{F;1,-1}(D)$, i.e., $f(s) \neq 1$ and $f(s) \neq -1$ on D, the function $fi + \sqrt{1-f^2}$ is analytic on D, and $fi + \sqrt{1-f^2} \neq 0$ on D. Therefore, there exists a branch of logarithm which is analytic and non-vanishing on D. Thus, $g \in S$, and we have that

$$F(S) = \sin(S) \supset H_{F;1,-1}(D).$$

Therefore, by Theorem 1.4, for arbitrary compact set $K \subset D$, $f(s) \in H(D)$ and $f(s) \neq 1, -1$ on D, and every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\sin(\zeta(s+ikh)) - f(s)| < \varepsilon \right\} > 0.$$

1.2. Limit theorems

In this section, we present probabilistic limit theorems which are applied for the proof of universality. Let $\mathcal{B}(X)$ be the σ-field of Borel sets of the space X. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\} \stackrel{def}{=} \gamma$ is the unit circle for all primes p. By the Tikhonov theorem, see [50], the infinite-dimensional torus Ω, with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the unique probability Haar measure m_H exists. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\mathbb{P}$ be the set of all prime numbers. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$, and on the space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D)$-valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s} \right)^{-1}.$$

We note that the latter product, for almost all $\omega \in \Omega$ with respect to the measure m_H, converges uniformly on compact subsets of the strip D. All above statements can be found in [24]. Let P_ζ be the distribution of the random element $\zeta(s, \omega)$, i,e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A),\ A \in \mathcal{B}(H(D)).$$

We recall that if P_n, $n \in \mathbb{N}$, and P are probability measures on $(X, \mathcal{B}(X))$, then we say that P_n converges weakly to P as $n \to \infty$ if, for every real bounded continuous function f on X

$$\lim_{n\to\infty} \int\limits_X f dP_n = \int\limits_X f dP.$$

The proof of Theorem B is based on the assertion that

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\},\ A \in \mathcal{B}(H(D)),$$

converges weakly to the measure P_ζ as $T \to \infty$, and that the support of P_ζ is the set S.

The probabilistic background for Theorem C is more complicated, and we need some additional definitions. A fixed number $h > 0$ is called of type 1 if the number $\exp\left\{\frac{2\pi m}{h}\right\}$ is irrational for all $m \in \mathbb{N}$, and h is of type 2 if it is not of type 1.

Suppose that $h > 0$ is of type 2. Then there exists the smallest $m_0 \in \mathbb{N}$ such that the number $\exp\left\{\frac{2\pi m_0}{h}\right\}$ is rational. Let $m = um_0 + v$, $u, v \in \mathbb{N}_0$, $0 \le v < n$, and the number $\exp\left\{\frac{2\pi m}{h}\right\}$ is rational. Since the number $\exp\left\{\frac{2\pi m_0}{h}\right\}$ is rational, the number $\exp\left\{\frac{2\pi u m_0}{h}\right\}$ is rational as well. Moreover,

$$\exp\left\{\frac{2\pi m}{h}\right\} = \exp\left\{\frac{2\pi u m_0}{h}\right\}\exp\left\{\frac{2\pi v}{h}\right\}.$$

Hence, we find that the number

$$\exp\left\{\frac{2\pi v}{h}\right\} = \exp\left\{\frac{2\pi m}{h}\right\}\Big/\exp\left\{\frac{2\pi u m_0}{h}\right\}.$$

is also rational. However, $0 \le v < m_0$, and m_0 is the smallest with property that the number $\exp\left\{\frac{2\pi m}{h}\right\}$ is rational. This shows that $v = 0$, and we have that $m = um_0$, i.e., m is a multiple of m_0.

Suppose that

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{a}{b}$$

with $a, b \in \mathbb{N}$, $(a, b) = 1$. Consider the set $\mathbb{P}_0 \subset \mathbb{P}$,

$$\mathbb{P}_0 = \left\{p \in \mathbb{P} : \alpha_p \ne 0 \text{ in } \frac{a}{b} = \prod_{p\in\mathbb{P}} p^{\alpha_p}\right\}.$$

Obviously, we have that the set $\mathbb{P}_0$ is finite one.

Denote by Ω_h the closed subgroup of the group Ω generated by p^{-ih}, $p \in \mathbb{P}$. Then again Ω_h is a compact topological Abelian group, therefore, on $(\Omega_h, \mathcal{B}(\Omega_h))$, the probability Haar measure m_H^h can be defined, and we obtain a new probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. The group was considered in [1], Lemma 4.2.2, and it was proved that $\Omega_h = \Omega$ if h is of type 1, and

$$\Omega_h = \{\omega \in \Omega : \omega(a) = \omega(b)\}$$

if the number h is of type 2.

On the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$, define the $H(D)$-valued random element $\zeta_h(s, \omega_h)$ by the formula

$$\zeta_h(s, \omega_h) = \prod_p \left(1 - \frac{\omega_h(p)}{p^s}\right)^{-1}.$$

Let $P_{\zeta,h}$ be the distribution of the random element $\zeta_h(s, \omega_h)$, i.e.,

$$P_{\zeta,h}(A) = m_H^h\{\omega_h \in \Omega_h : \zeta(s, \omega_h) \in A\},\ A \in \mathcal{B}(H(D)).$$

Clearly, if h is of type 1, then $P_{\zeta,h}$ coincide with P_ζ.

Theorem 1.5. *For $N \to \infty$,*

$$P_{N,h}(A) \stackrel{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : \zeta(s + ikh) \in A\},\ A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\zeta,h}$.

Proof. The case of h of type 1 is a partial case of a discrete theorem for Matsumoto zeta-functions obtained in [21].

The case of h of type 2 is based on the following assertions proved in [39]. Let

$$Q_{N,h}(A) \stackrel{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : (p^{-ikh};\ p \in \mathbb{P}) \in A\},\ A \in \mathcal{B}(\Omega_h).$$

Then $Q_{N,h}$ converges weakly to the Haar measure m_H^h as $N \to \infty$.

The second assertion is related to the ergodic theory. For brevity, we put $a_h = (p^{-ih} : p \in \mathbb{P})$. Then, clearly, $a_h \in \Omega_h$. Define the transformation $\varphi_h(\omega_h)$ of Ω_h by

$$\varphi_h(\omega_h) = a_h\omega_h,\ \omega \in \Omega_h.$$

Since the Haar measure m_H^h is invariant with respect to translations by points from Ω_h, we have that φ_h is a measurable measure preserving transformation on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_H^h)$. We recall that a set $A \in \mathcal{B}(\Omega_h)$ is called invariant with respect to the transformation φ_h if the sets A and $\varphi_h(A)$ can differ one from another at most by a set of m_H^h-measure zero. The transformation φ_h is ergodic if the σ-field of invariant sets consists only from the sets A with $m_H^h(A) = 1$ or $m_H^h(A) = 0$. In [21], it is proved that h is of type 2, then the transformation φ_h is ergodic.

Now the proof for h of type 2 runs in a standard way. First the weak convergence of $Q_{N,h}$ allows to prove limit theorems for certain absolutely convergent Dirichlet series related to $\zeta(s)$. The ergodicity of the transformation φ_h together with the Gallagher lemma, Lemma 1.4 of [44], ensures the approximation in the mean for the functions $\zeta(s)$ and $\zeta(s, \omega_h)$ by absolutely convergent Dirichlet series. From this, using a standard method, limit theorems for $\zeta(s)$ and $\zeta(s,\omega_h)$ follow. Finally, an application of the classical Birkhoff-Khintchine ergodic theorem together with ergodicity of the transformation φ_h shows that the measure $P_{N,h}$ converges weakly to $P_{\zeta,h}$ as $N\to\infty$.

In the sequel, we will use several times the following well-known fact. Let X_1 and X_2 be two metric spaces, and $u: X_1\to X_2$ be a $(\mathcal{B}(X_1),\mathcal{B}(X_2))$-measurable function, i.e.,

$$u^{-1}\mathcal{B}(X_2)\subset\mathcal{B}(X_1).$$

Then every probability measure P on $(X_1,\mathcal{B}(X_1))$ induces on $(X_2,\mathcal{B}(X_2))$ the unique probability measure Pu^{-1} defined by the formula

$$Pu^{-1}(A)=P(u^{-1}A),\ A\in\mathcal{B}(X_2).$$

Lemma 1.6. *Let P_n, $n\in\mathbb{N}$, and P be probability measures on $(X,\mathcal{B}(X))$ and $u: X_1\to X_2$ be a continuous function. Suppose that P_n converges weakly to P as $n\to\infty$. Then P_nu^{-1} also converges weakly to Pu^{-1} as $n\to\infty$.*

The lemma is a partial case of Theorem 5.1 from [2].

Lemma 1.7. *Suppose that $F: H(D)\to H(D)$ is a continuous operator. Then*

$$P_{N,h,F}(A)\stackrel{def}{=}\frac{1}{N+1}\#\{0\le k\le N: F(\zeta(s+ikh))\in A\},\ A\in\mathcal{B}(H(D)),$$

converges weakly to the measure $P_{\zeta,h}F^{-1}$.

Proof. The lemma is an immediate corollary of Theorem 1.5 and Lemma 1.6. Really, we have that, for $A\in\mathcal{B}(H(D))$,

$$P_{N,h,F}(A)=\frac{1}{N+1}\#\{0\le k\le N:\zeta(s+ikh)\in F^{-1}A\}=P_{N,h}(F^{-1}A).$$

Thus, $P_{N,h,F}=P_{N,h}F^{-1}$. Since the operator F is continuous, Theorem 1.5 and Lemma 1.6 imply the weak convergence of $P_{N,h,F}$ to $P_{\zeta,h}F^{-1}$ as $N\to\infty$.

For $V>0$, we denote by $P_{N,h,V}$ and $P_{\zeta,h,V}$ the restrictions to the space $(H(D_V),\mathcal{B}(H(D_V)))$ for the measures $P_{N,h}$ and $P_{\zeta,h}$, respectively.

Lemma 1.8. *For every $V>0$, $P_{N,h,V}$ converges weakly to $P_{\zeta,h,V}$ as $N\to\infty$.*

Proof. Obviously, $D_V\subset D$. Therefore, the function $u: H(D)\to H(D_V)$ given by the formula $u(g(s))=g(s)|_{s\in D_V}$, $g\in H(D)$, is continuous. Thus, the lemma follows from Lemmas 1.7 and 1.6.

Lemma 1.9. *Suppose that the operator $F: H(D_V)\to H(D_V)$ is continuous. Then*

$$P_{N,h,F,V}(A)\stackrel{def}{=}\frac{1}{N+1}\#\{0\le k\le N: F(\zeta(s+ikh))\in A\},\ A\in\mathcal{B}(H(D_V)),$$

converges weakly to the measure $P_{\zeta,h,V}F^{-1}$ as $N \to \infty$.

Proof. We use Lemmas 1.9 and 1.6, and repeat the proof of Lemma 1.7.

1.3. Supports

In this section, we consider the supports of the limit measures in Lemmas 1.7 and 1.9. For this, we will use the properties of operators F in Theorems 1.1-1.4. We remind that if X a separable metric space, and P is a probability measure on $(X, \mathcal{B}(X))$, then a minimal closed set S_P such that $P(S_P) = 1$ is called the support of the measure P.

Lemma 1.10. *The support of the measure $P_{\zeta,h}$ is the set*

$$S = \{g \in H(D) : g(s) \neq 0 \; or \, g(s) \equiv 0\}.$$

Proof. In the case of h of type 1, the proof of the lemma is given in [33], Lemma 6.5.5. In the case of h of type 2, we repeat the arguments of h of type 1 because the random elements $\zeta(s,\omega)$, $\omega \in \Omega$, and $\zeta(s,\omega_h)$, $\omega_h \in \Omega_h$, have the same form. Moreover, by a different method, the lemma for h of type 2 is proved in [1], Theorem 5.3.2.

Lemma 1.11. *The support of the measure $P_{\zeta,h,V}$ is the set*

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \; or \, g(s) \equiv 0\}.$$

Proof of the lemma completely coincides with that of Lemma 1.10.

Lemma 1.12. *Suppose that the operator F satisfies the hypotheses of Theorem 1.1. Then the support of the measure $P_{\zeta,h}F^{-1}$ is the whole of $H(D)$.*

Proof. Let g be an arbitrary element of $H(D)$, and G be any open neighbourhood of g. Since the operator F is continuous, the set $F^{-1}G$ is open as well. Moreover, by the hypothesis of Theorem 1.1, the set $(F^{-1}G) \bigcap S$ is non-empty. Therefore, there exists an element $g_1 \in S$ which is also an element of the set $F^{-1}G$. Thus, $F^{-1}G$ is an open neighbourhood of the element g_1. However, the support of the measure $P_{\zeta,h}$ consists of all elements g_1 such that, for every open neighbourhood G_1 of g_1, the inequality $P_{\zeta,h}(G_1) > 0$ is satisfied. Therefore, by Lemma 1.10,

$$P_{\zeta,h}F^{-1}(G) = P_{\zeta,h}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the lemma.

For the investigation of supports of other limit measures, we will apply the Mergelyan theorem on the appriximation of analytic functions by polynomials. We state this theorem as a separate lemma.

Lemma 1.13. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma is given in [43], see also [63].

Lemma 1.14. *Suppose that the operator F satisfies the hypotheses of Theorem 1.2. Then the support of the measure $P_{\zeta,h}F^{-1}$ is the whole of $H(D)$.*

Proof. We will prove that the operator F satisfies the hypotheses of Theorem 1.1. Then the lemma will follow from Lemma 1.12.

Let $\varepsilon > 0$ be an arbitrary fixed number. We fix $l_0 \in \mathbb{N}$ such that

$$\sum_{l > l_0} 2^{-l} < \frac{\varepsilon}{2}. \tag{1.5}$$

Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact subsets of the strip D which occurs in the definition of the metric ϱ in the space $H(D)$. Suppose that, for $f, g \in H(D)$,

$$\sup_{s \in K_{l_0}} |f(s) - g(s)| < \frac{\varepsilon}{2}.$$

Then, in view of the relation $K_l \subset K_{l+1}$, $l \in \mathbb{N}$, we find that

$$\sup_{s \in K_l} |f(s) - g(s)| < \frac{\varepsilon}{2}.$$

for all $l = l, \dots, l_0 - 1$. Thus, in virtue of (1.5),

$$\varrho(f,g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s \in K_l} |f(s) - g(s)|}{1 + \sup\limits_{s \in K_l} |f(s) - g(s)|} \leq$$
$$\leq \sum_{l=1}^{l_0} 2^{-l} \frac{\sup\limits_{s \in K_l} |f(s) - g(s)|}{1 + \sup\limits_{s \in K_l} |f(s) - g(s)|} + \sum_{l > l_0} 2^{-l} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that, in the space $H(D)$, the function g approximates a function f with a given accuracy if g approximates f with a suitable accuracy uniformly on K_l for sufficiently large l. Clearly, the sets K_l can be chosen to be with connected complements. For example, we can take closed rectangles.

Therefore, in the space $H(D)$, we can limit ourselves by uniform approximation on compact subsets with connected complements.

Let g be an arbitrary element of $H(D)$, and G be an open neighbourhood of g. Then the continuity of F implies that the set $F^{-1}G$ is also open. We will prove that the set $(F^{-1}G)\bigcap S$ is non-empty.

Let $K \subset D$ be a compact subset with connected complement. Then, by Lemma 1.13, there exists a polynomial $p = p(s)$ which approximates the function g uniformly on K with desired accuracy. Therefore, since $g \in G$, we may find a polynomial $p(s)$ such that $p \in G$, too. By the hypothesis of Theorem 1.2, we have that $(F^{-1}\{p\})\bigcap S \neq \varnothing$. Thus, $(F^{-1}G)\bigcap S \neq \varnothing$, and the lemma follows from Lemma 1.12.

Lemma 1.15. *Suppose that the operator F satisfies the hypotheses of Theorem 1.3. Then the support of the measure $P_{\zeta,h,V}F^{-1}$ is the whole of $H(D_V)$.*

Proof. Let g be an arbitrary element of $H(D_V)$, and G be any open neighbourhood of g. Then the set $F^{-1}G$ is open as well. Repeating the proof of Lemma 1.14, we obtain that $(F^{-1}G)\bigcap S_V \neq \varnothing$. Therefore, there exists an element $g_1 \in S_V$ which also belongs to $F^{-1}G$. Thus, $F^{-1}G$ is an open neighbourhood of element g_1. Therefore, by Lemma 1.11, $P_{\zeta,h,V}(F^{-1}G) > 0$. Hence,

$$P_{\zeta,h,V}F^{-1}(G) = P_{\zeta,h,V}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the lemma.

Lemma 1.16. *Suppose that the operator F satisfies the hypotheses of Theorem 1.4. Then the support of the measure $P_{\zeta,h}F^{-1}$ contains the closure of the set $H_{F;a_1,\dots,a_r}(D)$.*

Proof. By the hypotheses for the operator F, we have that, for each element $f \in H_{F;a_1,\dots,a_r}(D)$, there exists an element $g \in S$ such that $F(g) = f$. Therefore, for every open neighbourhood G of f, in view of Lemma 1.10,

$$P_{\zeta,h}F^{-1}(G) = P_{\zeta,h}(F^{-1}G) > 0.$$

This shows that f is an element of the support of the measure $P_{\zeta,h}F^{-1}$. Hence, it follows that the set $H_{F;a_1,\dots,a_r}(D)$ is a subset of the support of $P_{\zeta,h}F^{-1}$. Since the support is a closed set, we have that the closure of the set $H_{F;a_1,\dots,a_r}(D)$ belongs to the support of $P_{\zeta,h}F^{-1}$.

1.4. Proof of universality theorems

Proof of Theorems 1-4 uses the corresponding limit theorems, explicit forms of supports of the limit measures in them, as well as Lemma 1.13 (the Mergelyan theorem).

Additionally, we remind an equivalent of the weak convergence of probability measures in terms of open sets.

Lemma 1.17. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

The lemma is a part of Theorem 2.1 from [2].

Proof of Theorem 1.1. By Lemma 1.13, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.6}$$

Define the set

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since the set G is open, Lemmas 1.7 and 1.17 imply the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \leq k \leq N : F(\zeta(s + ikh)) \in G\} \geq P_{\zeta,h}F^{-1}(G). \tag{1.7}$$

In virtue of Lemma 1.12, the polynomial $p(s)$ is an element of the support of the measure $P_{\zeta,h}F^{-1}$. Since the set G is an open neighbourhood of the polynomial $p(s)$, the properties of the support imply the inequality $P_{\zeta,h}F^{-1}(G) > 0$. This together with (1.7) shows that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - p(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{1.8}$$

It remains to replace the polynomial $p(s)$ by the function $f(s)$. Taking into account (1.6), for k satisfying

$$\sup_{s\in K} |F(s + ikh) - p(s)| < \frac{\varepsilon}{2},$$

we find that

$$\sup_{s\in K} |F(s + ikh) - f(s)| \leq \sup_{s\in K} |F(s + ikh) - p(s)| + \sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Therefore,

$$\left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - p(s)| < \frac{\varepsilon}{2} \right\} \subset \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - f(s)| < \varepsilon \right\}.$$

This and (1.8) shows that

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - f(s)| < \varepsilon\} > 0.$$

The theorem is proved.

Proof of Theorem 1.2. We repeat the arguments of the proof of Theorem 1.1, and, in place of Lemma 1.12, we apply Lemma 1.14.

Proof of Theorem 1.3. We argue with obvious changes similarly to the proof of Theorem 1.1, and, in place of Lemmas 1.7 and 1.12, we use Lemmas 1.9 and 1.15.

Proof of Theorem 1.4. We begin with the case $r = 1$. Using Lemma 1.13, we fix a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{1.9}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well if ε is small enough. Thus, we can define a continuous branch of $\log(p(s) - a_1)$ which will be an analytic function in the interior of K. Again, in view if Lemma 1.13, there exists a polynomial $p_1(s)$ such that

$$\sup_{s\in K} |p(s) - a_1 - e^{p_1(s)}| < \frac{\varepsilon}{4}. \tag{1.10}$$

For brevity, we put $h_{a_1}(s) = e^{p_1(s)} + a_1$. Then we have that $h_{a_1}(s) \in H(D)$, and, obviously, $h_{a_1}(s) \neq a_1$. Therefore, by Lemma 1.16, $h_{a_1}(s)$ is an element of the support of the measure $P_{\zeta,h}F^{-1}$. Moreover, the inequalities (1.9) and (1.10) imply the inequality

$$\sup_{s\in K} |f(s) - h_{a_1}(s)| \leq \sup_{s\in K} |f(s) - p(s)| + \sup_{s\in K} |p(s) - h_{a_1}(s)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{1.11}$$

Let define the set

$$G_1 = \left\{ g \in H(D) : \sup_{s\in K} |h_{a_1}(s) - g(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G_1 an open neighbourhood of the element $h_{a_1}(s)$ of the support of $P_{\zeta,h}F^{-1}$. Therefore, $P_{\zeta,h}F^{-1}(G) > 0$, thus, by Lemmas 1.7 and 1.17,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - h_{a_1}(s)| < \frac{\varepsilon}{2} \right\} > P_{\zeta,h}F^{-1}(G) > 0.$$

This and (1.11) give the assertion of the theorem.

Now let $r \geq 2$. Define the set

$$G_2 = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Since $f(s) \in H_{F:a_1,\ldots,a_r}(D)$, Lemma 1.16 shows that $f(s)$ is an element of the support of the measure $P_{\zeta,h}F^{-1}$. Since G_2 is an open set, hence we have that $P_{\zeta,h}F^{-1}(G) > 0$. Therefore, Lemmas 1.7 and 1.17 given the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh)) - f(s)| < \varepsilon \right\} > P_{\zeta,h}F^{-1}(G) > 0.$$

The theorem is proved.

Chapter 2

Discrete universality theorems for composite functions of the Hurwitz zeta-function

Let α, $0 \leq \alpha \leq 1$, be a fixed parameter. We recall that the Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

This chapter is devoted to the discrete universality of the functions $F(\zeta(s,\alpha))$ for some classes of operators $F : H(D) \to H(D)$. It is easy to give an example of such operators. Suppose that $F(g) = e^g$ for $g \in H(D)$, $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $h > 0$ is as in Theorem F. Then we can define a continuous branch of the function $\log f(s)$ on K, which will be analytic in the interior of K. Suppose that $k \in \mathbb{N}_0$ satisfies the inequality

$$\sup_{s \in K} |\zeta(s + ikh, \alpha) - \log f(s)| < \frac{\varepsilon}{eM_K}, \tag{2.1}$$

where $M_K = \max\left(\sup\limits_{s \in K} |f(s)|, 1\right)$. Using the inequality

$$|e^s - 1| \leq |s|e^{|s|}$$

which is valid for all $s \in \mathbb{C}$, we obtain that, for $k \in \mathbb{N}_0$ satisfying (2.1) with $0 < \varepsilon < 1$,

$$\sup_{s \in K} |e^{\zeta(s+ikh,\alpha)} - f(s)| = \sup_{s \in K} |f(s)||e^{\zeta(s+ikh,\alpha) - \log f(s)} - 1| \leq$$

$$\leq \sup_{s\in K} |f(s)||\zeta(s+ikh,\alpha) - \log f(s)|e^{|\zeta(s+ikh,\alpha)-\log f(s)|} < \varepsilon. \tag{2.2}$$

However, by Theorem F, the set of $m \in \mathbb{N}_0$ satisfying (2.1) has a positive lower density. Therefore, in view of (2.2), we obtain that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} \left| e^{\zeta(s+ikh,\alpha)} - f(s) \right| < \varepsilon \right\} > 0.$$

The later example shows that the composite functions $F(\zeta(s,\alpha)$ can preserve the discrete universality property for the Hurwitz zeta-function. In this chapter, we describe some classes of operators F for which any analytic function can be approximated by discrete shifts $F(\zeta(s+ikh,\alpha))$. In other words, we give generalizations of the results of Chapter 1 for the function $\zeta(s,\alpha)$.

2.1. Lipschitz class

For a sufficiently wide class of operators $F : H(D) \to H(D)$, the discrete universality of $F(\zeta(s,\alpha))$ can be deduced directly from Theorem F. We say that the operator $F : H(D) \to H(D)$ belongs to the class $\mathrm{Lip}(\beta)$, $\beta > 0$, if the following hypotheses are satisfied:

1° for each polynomial $p = p(s)$, there exists an element $g \in F^{-1}\{p\} \subset H(D)$;

2° for every set $K \in \mathcal{K}$, there exists a positive constant c and a set $K_1 \in \mathcal{K}$ such that

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leq c \sup_{s\in K_1} |g_1(s) - g_2(s)|^\beta$$

for all $g_1, g_2 \in H(D)$.

We observe that hypotheses 2° of the class $\mathrm{Lip}(\beta)$ is similar to the classical Lipschitz condition.

Theorem 2.1. *Suppose that the numbers α and h, the set K and the function $f(s)$ are as in Theorem F, and that $F \in Lip(\beta)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon \right\} > 0.$$

It is easy to see that the operator $F : H(D) \to H(D)$ given by the formula $F(g) = g'$, $g \in H(D)$, belongs to the class $\mathrm{Lip}(1)$. Obviously, each polynomial has a preimage in $H(D)$. Therefore, it remains to check hypothesis 2° of the class $\mathrm{Lip}(\beta)$.

Let K_1 and Γ be the same as in Chapter 1, page 24. Then, by the Cauchy integral formula, we have that for $g_1, g_2 \in H(D)$ and $s \in K$,

$$F(g_1(s)) - F(g_2(s)) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g_1(z) - g_2(z)}{(z-s)^2} dz.$$

Hence,

$$\sup_{s\in K}|F(g_1(s))-F(g_2(s))|\le \frac{|L|}{2\pi\delta^2}\sup_{s\in K_1}|g_1(s)-g_2(s)|=c\sup_{s\in K_1}|g_1(s)-g_2(s)|,$$

where $c=\frac{|L|}{2\pi\delta^2}$, and the quantities $|L|$ and δ are the same as in Chapter 1, page 24. Thus, hypothesis 2° of the class Lip(1) is also satisfied, and we have that $F\in \mathrm{Lip}(1)$.

Proof of Theorem 2.1. By Lemma 1.13, there exists a polynomial $p=p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}. \tag{2.3}$$

Hypothesis 1° of the class Lip(β) implies the existence of $g(s)\in F^{-1}\{p\}\subset H(D)$. Let $k\in\mathbb{N}_0$ satisfy the inequality

$$\sup_{s\in K_1}|\zeta(s+ikh,\alpha)-g(s)|<c^{-\frac{1}{\beta}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}, \tag{2.4}$$

where K_1 and c are from hypothesis 2° of the class Lip(β). Then, in view of hypothesis 2°, for k satisfying (2.4),

$$\begin{aligned}\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-p(s)|&=\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-F(g(s))|\le\\ &\le c\sup_{s\in K_1}|\zeta(s+ikh,\alpha)-g(s)|^\beta<c\left(c^{-\frac{1}{\beta}}\frac{\varepsilon^{\frac{1}{\beta}}}{2}\right)^\beta=\frac{\varepsilon}{2}.\end{aligned} \tag{2.5}$$

However, by Theorem F,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\le k\le N:\sup_{s\in K_1}|\zeta(s+ikh,\alpha)-g(s)|<c^{-\frac{1}{\beta}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\right\}>0.$$

Thus, by (2.5),

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\le k\le N:\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-p(s)|<\frac{\varepsilon}{2}\right\}>0. \tag{2.6}$$

From (2.3), we have that

$$\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-f(s)|\le\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-p(s)|+\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that

$$\left\{0\le k\le N:\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-p(s)|<\frac{\varepsilon}{2}\right\}\subset\left\{0\le k\le N:\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-f(s)|<\varepsilon\right\}.$$

Therefore, in view of (2.6), we find that

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\le k\le N:\sup_{s\in K}|F(\zeta(s+ikh,\alpha))-f(s)|<\varepsilon\right\}>0.$$

The theorem is proved.

2.2. Other classes of operators F

In this section, we state universality theorems for the composite functions $F(\zeta(s,\alpha))$ for some other classes of operators $F: H(D \to H(D))$. They are analogues of the corresponding universality theorems of Chapter 1 for the Riemann zeta-function. However, we do not consider the space $H(D_V)$ because the shifts $\zeta(s+ikh,\alpha)$ approximate functions from the class $H(K)$, $K \in \mathcal{K}$, and we do not need the non-vanishing of preimages $F^{-1}\{p\}$ for polynomials $p = p(s)$.

Theorem 2.2. *Suppose that the numbers α and h, the set K and the function $f(s)$ are as in Theorem F. Let $F: H(D) \to H(D)$ be a continuous operator such that, for every open set $G \subset H(H)$, the set $F^{-1}G$ is non-empty. Then the assertion of Theorem 2.1 is true.*

The hypothesis that $F^{-1}G \neq \varnothing$ for every open set $G \subset H(D)$ is not easily checked. Obviously, it is satisfied in the case $F(H(D)) = H(D)$. On the other hand, this hypothesis is quite general, and we can consider its special cases. The next theorem is an example of such a type.

Theorem 2.3. *Suppose that $F: H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Then with α, h, K and $f(s)$ as in Theorem F the assertion of Theorem 2.1 is true.*

The example

$$F(g) = c_1 g' + \cdots + c_r g^{(r)},\ c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\},$$

of operators $F: H(D) \to H(D)$ satisfies the hypothesis $F^{-1}\{p\} \neq \varnothing$ for each polynomial $p = p(s)$. Indeed, always there exists a polynomial $q = q(s)$ such that $F(q) = p$, and this was proved in the example of Theorem 1.3. Unfortunately, other non-trivial examples of operators F, which are not related to derivatives, are not numerous. Obvious, one of examples is an integral operator

$$F(g) = \int_{s_0}^{s} g(z)dz,\ g \in H(D),$$

with $s_0 \in D$.

Now we state an analogue of Theorem 1.4. For this, we define the subset of analytic functions on $H(D)$

$$H_{a_1,\ldots,a_r}(D) = \{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\ j = 1, \ldots, r\}$$

with different complex numbers $a_1, \ldots, a_r$. Clearly, the set $H_{a_1,\ldots,a_r}(D)$ consists of analytic functions on D which do not take the values $a_1, \ldots, a_r$. In the next theorem, we approximate analytic functions from the set $H_{a_1,\ldots,a_r}(D) \subset F(H(D))$ by shifts $F(\zeta(s+ikh,\alpha))$. Clearly, $H_{a_1,\ldots,a_r}(D) \subset H(D)$, therefore, such an universality is weaker than that above theorem, however, it includes the universality of some elementary functions.

Theorem 2.4. *Suppose that the numbers* α *and* h *are as in Theorem F, and that* $F : H(D) \to H(D)$ *is a continuous operator such that* $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$. *For* $r = 1$, *let* $K \in \mathcal{K}$, *and let* $f(s) \neq a_1$ *be a continuous function on* K *which is analytic in the interior of* K. *For* $r \geq 2$, *let* $K \subset D$ *be an arbitrary compact subset, and* $f(s) \in H_{a_1,\dots,a_r}(D)$. *Then the assertion of Theorem 2.1 is true.*

We note that, in the case $r = 1$, the functions $f(s) \in H_{a_1}(D)$ also can be approximated by shifts $F(\zeta(s + ikh, \alpha))$, however, the requirement, of the theorem is weaker than that $f(s) \in H_{a_1}(D)$.

It is easy to check that the operator $F : H(D) \to H(D)$ given by the formula

$$F(g) = g^N, \; g \in H(D), \; N \in \mathbb{N} \setminus \{1\},$$

satisfies the hypotheses of Theorem 2.4. Let $g_n \to g$ in the space $H(D)$. Then, for every compact subset $K \subset D$,

$$\sup_{s \in K} |g_n(s) - g(s)| \xrightarrow[n \to \infty]{} 0.$$

Hence,

$$\sup_{s \in K} |F(g_n(s)) - F(g(s))| = \sup_{s \in K} |g_n^N(s) - g^N(s) = \sup_{s \in K} |g_n(s) - g(s)|||g_n^{N-1}(s) + \cdots + g^{N-1}(s)| \xrightarrow[n \to \infty]{} 0.$$

Therefore, the operator F is continuous.

Now let $f(s) \in H_0(D)$. Then the equation

$$g^N = f$$

has a solution $g = \sqrt[N]{f} \in H(D)$. Therefore, $F(H(D)) \supset H_0(D)$. Thus, by Theorem 2.4, we have that if $K \in \mathcal{K}$, and $f(s) \neq 0$ is a continuous function on K which is analytic in the interior of K, then, for every $\varepsilon > 0$ and $N \in \mathbb{N} \setminus \{1\}$

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta^N(s + ikh, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

The case $N = 1$ corresponds Theorem F. In this case, $f(s)$ is an arbitrary function from the class $H(K)$.

Theorem 2.4 also implies the universality of the functions $F(\zeta(s, \alpha))$ with the operators $F : H(D) \to H(D)$ given by the formulae

$$F(g) = \sin g, \; F(g) = \cos g, \; F(g) = \sinh(g), \; F(g) = \cosh(g).$$

We check the hypotheses of Theorem 2.4 for the operator defined by the hyperbolic cosine. Suppose that $f(s) \in H_{-1,1}(D)$. Consider the equation

$$\cosh(g) = \frac{e^g + e^{-g}}{2} = f.$$

Putting $y = e^g$ gives the equation

$$y^2 - 2fy + 1 = 0.$$

Hence,

$$y = f \pm \sqrt{f^2 - 1},$$

and

$$g = \log(f \pm \sqrt{f^2 - 1}) \in H(D).$$

Therefore, $F(H(D)) \supset H_{-1,1}(D)$, and we have the universality of $\cosh(\zeta(s,\alpha))$.

The last theorem of Chapter 2 extends the approximation of analytic function from the set $F(H(D))$ for all continuous operator $F : H(D) \to H(D)$ by shifts $F(\zeta(s + ikh, \alpha))$.

Theorem 2.5. *Suppose that the numbers α and h are as in Theorem F, and that $F : H(D) \to H(D)$ is an arbitrary continuous operator. Let $K \subset D$ be a compact subset, and $f(s) \in F(H(D))$. Then the assertion of Theorem 2.1 is true.*

Theorem 2.5 is rather general, however, from the other hand, it is difficult to describe the set $F(H(D))$. Therefore, Theorem 2.4 is more convenient for the investigation of concrete universal functions.

2.3. Lemmas

For proving Theorem 2.2-2.5, we will apply as, in Chapter 1, a probabilistic approach based on limit theorems on the weak convergence of probability measures in the space $H(D)$.

Additionally to the torus Ω defined in Section 1.2, we define one more infinite-dimensional torus

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all nonnegative integers m. By the Tikhonov theorem, the torus Ω_1 with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega_1, \mathcal{B}(\Omega_1))$, the probability Haar measure m_{1H} can be defined, and we have the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$. We denote by $\omega_1(m)$ the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_m, $m \in \mathbb{N}_0$.

We recall that $\omega(p)$ is the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$. We extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^r \mid m \\ p^{r} \nmid m}} \omega^r(p), \; m \in \mathbb{N}.$$

Suppose that α is a rational number $\neq 1, \frac{1}{2}$. Thus, $\alpha = \frac{a}{q}$ with some $a, q \in \mathbb{N}$, $1 \leq a \leq q$, $q \geq 3$, $(a, q) = 1$. Then, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, we define the $H(D)$-valued random element $\zeta(s, \omega, \alpha)$ by the formula

$$\zeta(s, \omega, \alpha) = \overline{\omega(q)} q^s \sum_{\substack{m=1 \\ m \equiv a (\mathrm{mod}\, q)}}^{\infty} \frac{\omega(m)}{m^s},$$

where $\overline{\omega(q)}$ means the complex conjugate of $\omega(q)$, and denote by P_ζ the distribution of $\zeta(s, \omega, \alpha)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega, \alpha) \in A), A \in \mathcal{B}(H(D)).$$

If α is transcendental, then, on the probability space $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$, we define the $H(D)$-valued random element $\zeta(s, \omega_1, \alpha)$ by the formula

$$\zeta(s, \omega_1, \alpha) = \sum_{m=0}^{\infty} \frac{\omega_1(m)}{(m + \alpha)^s},$$

and denote by $P_{1\zeta}$ its distribution, i.e.,

$$P_{1\zeta}(A) = m_{1H}(\omega_1 \in \Omega_1 : \zeta(s, \omega_1, \alpha) \in A), A \in \mathcal{B}(H(D)).$$

Lemma 2.6. *Suppose that α is a rational number $\neq 1, \frac{1}{2}$, and that $h > 0$ is an arbitrary number. Then*

$$P_N(A) \stackrel{def}{=} \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta(s + ikh, \alpha) \in A\}, \; A \in \mathcal{B}(H(D),$$

converges weakly to P_ζ as $N \to \infty$. If α is transcendental and $h > 0$ is such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number, then P_N converges weakly to $P_{1\zeta}$ as $N \to \infty$.

For rational α, the lemma was obtained in [1], and, for transcendental α, the lemma is given in [34].

Now we state a limit theorem for composite functions.

Lemma 2.7. *Suppose that $F : H(D) \to H(D)$ is a continuous operator. If α is a rational number $\neq 1, \frac{1}{2}$ and $h > 0$ is an arbitrary number, then*

$$P_{N,F}(A) \stackrel{def}{=} \frac{1}{N+1} \#\{0 \leq k \leq N : F(\zeta(s + ikh, \alpha)) \in A\}, \; A \in \mathcal{B}(H(D),$$

converges weakly to $P_\zeta F^{-1}$ *as* $N \to \infty$. *If* α *is transcendental and* $h > 0$ *is such that* $\exp\left\{\frac{2\pi}{h}\right\}$ *is a rational number, then* $P_{N,F}$ *converges weakly to* $P_{1\zeta} F^{-1}$ *as* $N \to \infty$.

Proof. Since, for all $A \in \mathcal{B}(H(D))$,

$$P_{N,F}(A) = \frac{1}{N+1}\#\{0 \le k \le N : \zeta(s+ikh,\alpha) \in F^{-1}A\} = P_N(F^{-1}A),$$

we have that $P_{N,F} = P_N F^{-1}$. Therefore, the lemma is a consequence of Lemmas 2.6 and 1.6.

We also need the explicit form of the support of the limit measure in Lemma 2.7. Since the space $H(D)$ is separable, we know that the support of a measure P on $(H(D), \mathcal{B}(H(D)))$ is a minimal closed set $S_P \subset H(D)$ such that $P(S_P) = 1$. The set S_P consists of all $g \in H(D)$ such that, for every open neighbourhood G of g, $P(G) > 0$. We will use the following assertion.

Lemma 2.8. *The supports of the measure* P_ζ *and* $P_{1\zeta}$ *both are the whole of* $H(D)$.

Proof of the lemma for P_ζ is given in [1], and, for the measure $P_{1\zeta}$, can be found in [39].

Lemma 2.9. *Let* $F : H(D) \to H(D)$ *be a continuous operator such that, for every open set* $G \subset H(D)$, *the set* $F^{-1}G$ *is non-empty. Then the supports of the measures* $P_\zeta F^{-1}$ *and* $P_{1\zeta} F^{-1}$ *are both the whole of* $H(D)$.

Proof. Let g be an arbitrary element of $H(D)$, and G be any open neighbourhood of g. Since F is continuous, the set $F^{-1}G$ is also open, and, moreover, by the assumptions of the lemma, non-empty. Thus, $F^{-1}G$ is an open neighbourhood of a certain element $g_1 \in H(D)$. Therefore, by Lemma 2.8,

$$P_\zeta(F^{-1}G) > 0 \text{ and } P_{1\zeta}(F^{-1}G) > 0.$$

Hence,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$$

and

$$P_{1\zeta} F^{-1}(G) = P_{1\zeta}(F^{-1}G) > 0.$$

Since g and G are arbitrary objects, this proves the lemma.

Lemma 2.10. *Let* $F : H(D) \to H(D)$ *be a continuous operator such that, for every polynomial* $p = p(s)$, *the set* $F^{-1}\{p\}$ *is non-empty. Then the supports of the measures* $P_\zeta F^{-1}$ *and* $P_{1\zeta} F^{-1}$ *are both the whole of* $H(D)$.

Proof. We have seen in the proof of Lemma 1.14 that the approximation in the space $H(D)$ reduces to that on compact subsets with connected complements. Therefore, we can use the Mergelyan theorem (Lemma 1.13).

Let $g \in H(D)$ be an arbitrary element, and G be any open neighbourhood of g. If $K \in \mathcal{K}$, then, in view of Lemma 1.13, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |g(s) - p(s)| < \varepsilon.$$

Thus, the polynomial $p(s) \in G$ if the number ε is small enough. Since, by the hypothesis of the lemma, the set $F^{-1}\{p\}$ is non-empty, the set $F^{-1}G$ is also non-empty. Therefore, the lemma follows from Lemma 2.9.

Lemma 2.11. *Suppose that $F : H(D) \to H(D)$ is a continuous operator. Then the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$ both are the closure of $F(H(D))$.*

Proof. Let g be an arbitrary element of $F(H(D))$, and G be any open neighbourhood of g. Then there exists $g_1 \in H(D)$ such that $F(g_1) = g$. Therefore, the open set $F^{-1}G$ is non-empty, and Lemma 2.8 gives

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$$

and

$$P_{1\zeta} F^{-1}(G) = P_{1\zeta}(F^{-1}G) > 0.$$

Moreover,

$$P_\zeta F^{-1}(F(H(D))) = P_\zeta(H(D)) = 1$$

and

$$P_{1\zeta} F^{-1}(F(H(D))) = P_{1\zeta}(H(D)) = 1.$$

Since the support is a closed set, the last four relations show that the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$ both are the closure of $F(H(D))$.

Lemma 2.12. *Let $F : H(D) \to H(D)$ be a continuous operator such that $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$. Then the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$ both contain the closure of the set $H_{a_1,\dots,a_r}(D)$.*

Proof. By Lemma 2.11, both these supports are the closure of $F(H(D))$. Since $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$, we have that the closure of $F(H(D))$ contains the closure of the set $H_{a_1,\dots,a_r}(D)$. Therefore, the supports of considered measures contain the closure of the set $H_{a_1,\dots,a_r}(D)$.

2.4. Proof of universality theorems

Proof of Theorem 2.2. By Lemma 1.13, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2.7}$$

Define

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G is an neighbourhood of the polynomial $p(s)$. In view of Lemma 2.9, the polynomial $p(s)$ is an element of the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$. Therefore,

$$P_\zeta F^{-1}(G) > 0 \text{ and } P_{1\zeta} F^{-1}(G) > 0.$$

This and Lemmas 2.7 and 1.17 imply the inequalities

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \{0 \le k \le N : F(\zeta(s + ikh, \alpha)) \in G\} \ge P_\zeta F^{-1}(G) > 0$$

for rational α, and

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \{0 \le k \le N : F(\zeta(s + ikh, \alpha)) \in G\} \ge P_{1\zeta} F^{-1}(G) > 0$$

for transcendental α. Hence, by the definition of G,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F(\zeta^N(s + ikh, \alpha)) - p(s)| < \frac{\varepsilon}{2} \right\} > 0$$

for rational and transcendental α. Combining this with (2.7) gives the assertion of the theorem.

Proof of Theorem 2.3. We repeat the proof of Theorem 2.2 with application of Lemma 2.10 instead of Lemma 2.9.

Proof of Theorem 2.4. We separate two cases, $r = 1$ and $r \ge 2$.

Let $r = 1$. Then, by Lemma 1.13, we find a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{2.8}$$

Since $f(s) \ne a_1$ on K, we have by (2.8) that $p(s) \ne a_1$ on K if ε is small enough. Therefore, there exists a continuous branch of $\log(p(s) - a_1)$ on K, which is analytic in the interior of K. Applying Lemma 1.13 once more, we find a polynomial $q(s)$ satisfying

$$\sup_{s\in K} |p(s) - a_1 - e^{q(s)}| < \frac{\varepsilon}{4}. \tag{2.9}$$

We set $g_1(s) = e^{q(s)} + a_1$. Then $g_1(s) \in H(D)$, and $g_1(s) \ne a_1$ on D. Thus, $g_1(s) \in H_{a_1}(D)$. Therefore, by the assumption of the theorem that $F(H(D)) \supset H_{a_1}(D)$ and Lemma 2.12, we have that $g_1(s)$ is an element of the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$. Define

$$G_1 = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - g_1(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G_1 is an open neighbourhood of the function $g_1(s)$, thus, by the above remark, $P_\zeta F^{-1}(G_1) > 0$ and $P_{1\zeta} F^{-1}(G_{1,}) > 0$. Therefore, Lemmas 2.7 and 1.7, and the definition of the set G_1, show that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - g_1(s)| < \frac{\varepsilon}{2} \right\} \ge P_\zeta F^{-1}(G_1) > 0 \tag{2.10}$$

for rational α, and

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - g_1(s)| < \frac{\varepsilon}{2} \right\} \ge P_{1\zeta} F^{-1}(G_1) > 0 \tag{2.11}$$

for transcendental α. Inequalities (2.8) and (2.9) imply

$$\sup_{s\in K} |f(s) - g_1(s)| < \frac{\varepsilon}{2}.$$

This shows that, for $k \in \mathbb{N}_0$ satisfying (2.10) or (2.11),

$$\sup_{s\in K} |F(\zeta(s+ikh)) - f(s)| < \varepsilon.$$

This together with (2.10) and (2.11) proves the theorem.

Now let $r \ge 2$. Define

$$G_2 = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

By the hypothesis of the theorem and Lemma 2.12, the function $f(s)$ is an element of the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$. Thus, the set G_2 is an open neighbourhood of an element of the supports of the measures $P_\zeta F^{-1}$ and $P_{1\zeta} F^{-1}$, and hence,

$$P_\zeta F^{-1}(G_2) > 0$$

and

$$P_{1\zeta} F^{-1}(G_2) > 0.$$

Therefore, using Lemmas 2.7 and 1.7 again, we obtain, by the definition of G_2, that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon \right\} > 0.$$

The theorem is proved.

Proof of Theorem 2.5. We use Lemma 2.11 and apply the same arguments as in the proof of the case $r \ge 2$ of Theorem 2.4.

Chapter 3

Zeros of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ in the critical strip

In this chapter, we present a certain information on the number of zeros of the functions $F(\zeta(s+ikh))$ and $F(\zeta(s+ikh,\alpha))$, $k \in \mathbb{N}_0$, $h > 0$, for some classes of operators $F : H(D) \to H(D)$. For this, we will apply universality theorems of Chapters 1 and 2. Moreover, we will use the classical Rouché theorem.

3.1. Zeros of functions related to $\zeta(s)$

Universality theorems for the Riemann zeta-function $\zeta(s)$, Theorems B and C, do not give any information on zeros of $\zeta(s)$ because approximated functions in these theorems belong to the class $H_0(K)$. The picture changes when composite functions $F(\zeta(s))$ are considered. Then the shifts $F(\zeta(s+i\tau))$ or $F(\zeta(s+ikh))$ approximate analytic functions from the class $H(K)$, and we can derive some information on zeros of these shifts. More precisely, this is described in the next theorem.

Theorem 3.1. *Suppose that the operator F is as in one of Theorems 1.1-1.3. Then, for arbitrary σ_1 and σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, F, h)$ such that, for sufficiently large $N \in \mathbb{N}$, the function $F(\zeta(s+ikh))$ has a zero in the disc*

$$|s - \widehat{\sigma}| \le \frac{\sigma_2 - \sigma_1}{2}, \quad , \widehat{\sigma} = \frac{\sigma_1 + \sigma_2}{2},$$

for more than cN numbers k, $0 \le k \le N$.

Before the proof of Theorem 3.1, we remind the Rouché theorem.

Lemma 3.2. *Let G be a region on the complex plane, K be a compact subset of G, and $f(s)$ and $g(s)$ be analytic functions in G such that*

$$|f(s) - g(s)| < |f(s)|$$

for every point s in the boundary of K. Then the functions $f(s)$ and $g(s)$ have the same number of zeros in the interior of the set K, taking into account multiplicities.

Proof of the lemma can be found, for example, in [55].

Proof of Theorem 3.1. Define

$$\sigma_0 = \max\left(\left|\sigma_1 - \frac{3}{4}\right|, \left|\sigma_2 - \frac{3}{4}\right|\right),$$

$f(s) = s - \widehat{\sigma}$ and $0 < \varepsilon < \frac{\sigma_2 - \sigma_1}{20}$. Moreover, let $K = \left\{s \in \mathbb{C} : \left|s - \frac{3}{4}\right| \leq \sigma_0\right\}$. Then, clearly, $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, by Theorems 1.1-1.3 we have that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leq k \leq N : \sup_{\left|s-\frac{3}{4}\right|\leq\sigma_0} |F(\zeta(s+ikh)) - f(s)| < \varepsilon\right\} > 0.$$

From this, it follows that there exists a constant $c = c(\sigma_1, \sigma_2, F, h) > 0$ such that, for sufficiently large N,

$$\frac{1}{N+1} \# \left\{0 \leq k \leq N : \sup_{\left|s-\frac{3}{4}\right|\leq\sigma_0} |F(\zeta(s+ikh)) - f(s)| < \varepsilon\right\} > c. \tag{3.1}$$

The circle

$$|s - \widehat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2} \tag{3.2}$$

lies in the disc

$$\left|s - \frac{3}{4}\right| \leq \sigma_0. \tag{3.3}$$

Indeed, suppose that s lies on the circle (3.2). Then

$$\left|s - \frac{3}{4}\right| = \left|(s-\widehat{\sigma}) + \left(\widehat{\sigma} - \frac{3}{4}\right)\right| \leq |s - \widehat{\sigma}| + \left|\widehat{\sigma} - \frac{3}{4}\right| = \frac{\sigma_2 - \sigma_1}{2} + \left|\widehat{\sigma} - \frac{3}{4}\right|. \tag{3.4}$$

If $\widehat{\sigma} \geq \frac{3}{4}$, then, in view of (3.4), we have that

$$\left|s - \frac{3}{4}\right| \leq \frac{\sigma_2 - \sigma_1}{2} + \frac{\sigma_1 + \sigma_2}{2} - \frac{3}{4} = \sigma_2 - \frac{3}{4} \leq \sigma_0.$$

If $\widehat{\sigma} < \frac{3}{4}$, then $\sigma_1 < \frac{3}{4}$, and

$$\left|s - \frac{3}{4}\right| \leq \frac{\sigma_2 - \sigma_1}{2} + \frac{3}{4} - \frac{\sigma_1 + \sigma_2}{2} = \frac{3}{4} - \sigma_1 \leq \sigma_0.$$

Thus, we have that in both the cases, s lies in the disc (3.3). Therefore, for $k \in \mathbb{N}_0$ satisfying (3.1), we obtain that

$$\max_{|s-\widehat{\sigma}|=\frac{\sigma_2-\sigma_1}{2}} |F(\zeta(s+ikh)) - (s-\widehat{\sigma})| < \frac{\sigma_2-\sigma_1}{20}. \tag{3.5}$$

This shows that, in the disc

$$|s-\widehat{\sigma}| \le \frac{\sigma_2-\sigma_1}{2}, \tag{3.6}$$

the functions $f(s) = s-\widehat{\sigma}$ and $g(s) = F(\zeta(s+ikh))$ satisfy the hypotheses of Lemma 3.2. However, the function $s-\widehat{\sigma}$ has precisely one zero $s=\widehat{\sigma}$ in that disc. Therefore, the function $F(\zeta(s+ikh))$ also has one zero in the disc (3.5). Since, in view of (3.1), the number of such k, $0 \le k \le N$, satisfying (3.5), is larger than cN, this proves the theorem.

3.2. Zeros of the Hurwitz zeta-function

In this section, we obtain an analogue of Theorem 3.1 for the Hurwitz zeta-function $\zeta(s,\alpha)$. For this, we apply a discrete universality theorem (Theorem F) for $\zeta(s,\alpha)$, and we have the following result. We preserve the notation of Section 3.1.

Theorem 3.3. *Suppose that α is transcendental or rational number $\neq 1, \frac{1}{2}$. In the case of rational α, let $h>0$ be arbitrary fixed number, while in the case of transcendental α, let $h>0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Then, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, h) > 0$ such that, for sufficiently large $N \in \mathbb{N}$, the function $\zeta(s+ikh,\alpha)$ has a zero in the disc*

$$|s-\widehat{\sigma}| \le \frac{\sigma_2-\sigma_1}{2}$$

for more than cN numbers k, $0 \le k \le N$.

Proof. We take $f(s) = s-\widehat{\sigma}$, $K = \left\{s \in \mathbb{C} : \left|s-\frac{3}{4}\right| \le \sigma_0\right\}$ and $0 < \varepsilon < \frac{\sigma_2-\sigma_1}{20}$ in Theorem F. Then, by Theorem F, we have that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{\left|s-\frac{3}{4}\right| \le \sigma_0} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Therefore, there exists a positive constant $c = c(\sigma_1, \sigma_2, \alpha, h)$ such that, for sufficiently large N,

$$\# \left\{ 0 \le k \le N : \sup_{\left|s-\frac{3}{4}\right| \le \sigma_0} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > cN. \tag{3.7}$$

Since the circle $|s - \widehat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2}$ lies in the disc $\left|s - \frac{3}{4}\right| \leq \sigma_0$, we have that, for $k \in \mathbb{N}$ satisfying (3.7), the inequality

$$\max_{|s-\widehat{\sigma}|=\frac{\sigma_2-\sigma_1}{2}} |\zeta(s + ikh, \alpha) - (s - \widehat{\sigma})| < \frac{\sigma_2 - \sigma_1}{20}.$$

is satisfied. Thus, the functions $s - \widehat{\sigma}$ and $\zeta(s + ikh, \alpha)$ on the disc K satisfy the hypotheses of Lemma 3.2. Hence, repeating the arguments of the proof of Theorem 3.1, we obtain the assertion of the theorem.

3.3. Zeros of functions related to $\zeta(s, \alpha)$

This section is devoted for analogues of Theorem 3.3 for composite functions $F(\zeta(s, \alpha))$ with some operators $F : H(D) \to H(D)$.

Theorem 3.4. *Suppose that the numbers α and h are as in Theorem 3.3, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty, or $s - a \in F(H(D))$ for all $a \in \left(\frac{1}{2}, 1\right)$. Then, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, h, F) > 0$ such that, for sufficiently large $N \in \mathbb{N}$, the function $F(\zeta(s + ikh, \alpha))$ has a zero in the disc*

$$|s - \widehat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}$$

for more than cN numbers k, $0 \leq k \leq N$.

We give an example of an operator $F : H(D) \to H(D)$ such that $s - a \in F(H(D))$ for all $a \in \left(\frac{1}{2}, 1\right)$. Let

$$F(g) = gg', \; g \in H(D).$$

We solve the equation

$$gg' = s - a$$

with respect to g. We have that

$$\frac{1}{2}(g^2)' = s - a.$$

Hence, we find that

$$g^2 = s^2 - 2as + C$$

with arbitrary $C \in \mathbb{C}$, and

$$g = \pm(s^2 - 2as + C)^{\frac{1}{2}}.$$

If $s^2 - 2as + C = 0$, then

$$s = a \pm \sqrt{a^2 - C}.$$

Therefore, there exists $C_0 \in \mathbb{C}$ such that $s^2 - 2as + C_0 \neq 0$ on D. Thus, there exists a function $g \in H(D)$, we can take, for example, $g(s) = (s^2 - 2as + C_0)^{\frac{1}{2}}$, such that $F(g) = s - a$. Hence, $s - a \in F(H(D))$.

Proof of Theorem 3.4. First we observe that if $s - a \in F(H(D))$, where $F : H(D) \to H(D)$ is a continuous operator, then, for every compact subset $K \subset D$ and every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh, \alpha)) - (s - a)| < \varepsilon \right\} > 0. \tag{3.8}$$

Really, this is partial case of Theorem 2.5 with $f(s) = s - a$.

Since $a \in \left(\frac{1}{2}, 1\right)$, we can take $a = \widehat{\sigma} = \frac{\sigma_1+\sigma_2}{2}$. Then an application of Theorem 2.2 and (3.8) shows that there exists a positive constant $c = c(\sigma_1, \sigma_2, \alpha, h, F)$ such that, for sufficiently large N,

$$\# \left\{ 0 \leq k \leq N : \sup_{s\in K} |F(\zeta(s + ikh, \alpha)) - (s - \widehat{\sigma})| < \frac{\sigma_2 - \sigma_1}{20} \right\} > cN. \tag{3.9}$$

Moreover, we have that the functions $s-\widehat{\sigma}$ and $F(\zeta(s+ikh, \alpha))$, in the disc $K = \left\{ s \in \mathbb{C} : \left|s - \frac{3}{4}\right| \leq \sigma_0 \right\}$, satisfy the hypotheses of Lemma 3.2. Hence, the theorem follows.

Theorem 3.5. *Suppose that the numbers α and h are as in Theorem 3.3, and that $F : H(D) \to H(D)$ is a continuous operator such that $F(H(D)) \supset H_{a_1,\ldots,a_r}(D)$, where $\mathrm{Re} a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \ldots, r$. Then the assertion of Theorem 3.4 is true.*

Proof. Let $f(s)$ and K be as in the proof of previous theorems of this chapter. Since $\mathrm{Re} a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, we have that $f(s) = s - \widehat{\sigma} \neq a_j$ on D, $j = 1, \ldots, r$. Indeed, for $s \in D$,

$$-\frac{1}{2} < \mathrm{Re}(s - \widehat{\sigma}) < \frac{1}{2},$$

since $\frac{1}{2} < \sigma < 1$ and $\frac{1}{2} < \widehat{\sigma} < 1$. Therefore, we have that the function $f(s)$ on the disc K satisfies the hypotheses of Theorem 2.4, and the further proof runs in the same way as that of Theorem 3.3.

We consider the example $F(g) = \sin g$. Suppose that $f(s) \in H_{-1,1}(D)$ and solve the equation

$$\sin g = \frac{e^{ig} - e^{-ig}}{2i} = f.$$

Using the notation $y = e^{ig}$, we arrive to the equation

$$y^2 - 2iyf - 1 = 0.$$

Thus,

$$y = if \pm \sqrt{-f^2 + 1},$$

and

$$g = \log(if \pm \sqrt{-f^2 + 1}).$$

Since $f(s) \in H_{-1,1}(D)$, the function $\sqrt{-f^2+1} \in H(D)$, and

$$g = \log(if + \sqrt{-f^2 + 1}) \in H(D).$$

Therefore, $F(H(D)) \supset H_{-1,1}(D)$. Moreover, since $a_1 = -1$ and $a_2 = 1$, we have that $\mathrm{Re} a_j \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, 2$. Thus, all hypotheses of Theorem 3.5 are satisfied for the operator $F(g) = \sin g$, $g \in H(D)$, and, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a positive constant $c = c(\sigma_1, \sigma_2, \alpha, h)$ such that, for sufficiently large $N \in \mathbb{N}$, the function $\sin(\zeta(s + ikh, \alpha))$ has a zero in the disc

$$|s - \widehat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}$$

for more than cN numbers k, $0 \leq k \leq N$.

Chapter 4

Discrete universality theorems for the Hurwitz zeta-function

The discrete universality of the Hurwitz zeta-function $\zeta(s, \alpha)$ is described by Theorem F. However, in that theorem, the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. The case of rational α is completely solved, in this case the number $h > 0$ is arbitrary. In the case of transcendental α, it is required that $h > 0$ must be such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ would be rational. It remains the case of algebraic irrational α which is an open problem. In this chapter, we extend the case of transcendental parameter α.

4.1. Results

Let $\mathbb{Q}$ be the set of all rational numbers. Denote by $\mathbb{Q}_1^+$ the subset of $\mathbb{Q}$ of positive rational numbers $\neq 1$, and define, for $q \in \mathbb{Q}_1^+$, the set

$$L(\alpha, q) = \{(\log(m + \alpha) : m \in \mathbb{N}_0, \log q)\}.$$

The main result of this chapter is the following theorem.

Theorem 4.1. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over the field $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, and every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

It is not difficult to see that, for transcendental α, the set $L(\alpha, q)$ is linearly independent over $\mathbb{Q}$. Indeed, suppose that there exists the numbers $k_1, \dots, k_r,\ k \in \mathbb{Z} \setminus \{0\}$ such that

$$k_1 \log(m_1 + \alpha) + \cdots + k_r \log(m_r + \alpha) + k \log q = 0.$$

Then

$$(m_1 + \alpha)^{k_1} \cdots (m_r + \alpha)^{k_r} q^k - 1 = 0.$$

This shows that the number α is a root of the polynomial

$$(m_1 + s)^{k_1} \cdots (m_r + s)^{k_r} q^k - 1$$

with integer coefficients, and this contradicts the transcendence of α. The equality

$$k_1 \log(m_1 + \alpha) + \cdots + k_r \log(m_r + \alpha) = 0$$

with $k_1, \dots, k_r,\ k \in \mathbb{Z} \setminus \{0\}$ also leads to the contradiction.

On the other hand, it can happen that the set $L(\alpha, q)$ is linearly independent over $\mathbb{Q}$ with some algebraic irrational α. This conjecture is supported by the Cassels theorem [9] that at least 51 percent of elements of the set

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$$

with algebraic irrational α in the sense of density are linearly independent over $\mathbb{Q}$. Therefore, one can conjecture that there exists an algebrais irrational α such that the set $L(\alpha)$ in linearly independent over $\mathbb{Q}$. Hence, the set $L(\alpha, q)$ also can be linearly independent over $\mathbb{Q}$. Thus, Theorem 4.1 extend the case of transcendental α in Theorem F. However, Theorem 4.1 is non-effective because we do not know any algebraic irrational α with linearly independent over $\mathbb{Q}$ the set $L(\alpha, q)$.

Obviously, the set $L(\alpha, q)$ with rational α is linearly dependent over $\mathbb{Q}$. Indeed, let $\alpha = \frac{a}{b},\ a, b \in \mathbb{N},\ b > 1,\ (a, b) = 1$. Then the set

$$\{\log \frac{a}{b}, \log q\}$$

with $q = \frac{a}{b}$ is linearly dependent over $\mathbb{Q}$ because it contains two equal elements.

Theorem 4.1 can be generalized for composite functions in the same way as Theorem F was generalized in Chapter 2. We present only an analogue of Theorem 2.3.

Theorem 4.2. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$, $F : H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the preimage $F^{-1}\{p\}$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s \in K} |F(\zeta(s + ikh, \alpha)) - f(s)| < \varepsilon \right\} > 0.$$

As in the case of Theorem 2.3, we have the universality of the function $F(\zeta(s+ikh,\alpha))$ with an operator

$$F(g) = c_1 g' + \cdots + c_r g^{(r)},\ c_1,\dots,c_r \in \mathbb{C}\setminus\{0\},\ g \in H(D).$$

For proving of Theorems 4.1 and 4.2, we use limit theorems on weakly convergent probability measures.

4.2. Main lemma

In section 2.3, we have defined the torus Ω_1 which is the product over the set $\mathbb{N}_0$ of unit circles. For convenience of the notation, in this section, we will denote Ω_1 by Ω, i.e.,

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. For $m \in \mathbb{N}_0$, we will denote the projection of $\omega \in \Omega$ to the circle γ_m. In this section, we will prove a discrete limit theorems for probability measures on $(\Omega, \mathcal{B}(\Omega))$.

The torus Ω is a compact topological Abelian group. Therefore, for proving of a limit theorem, with will apply the method of Fourier transforms. We recall that a continuous function $\chi : \Omega \to \gamma$ is a character of Ω if $\chi(\omega_1\omega_2) = \chi(\omega_1)\chi(\omega_2)$. All character of Ω form a group $\mathcal{G}$ which is called the dual group or character group. The Fourier transform of the measure μ on $(\Omega, \mathcal{B}(\Omega))$ is defined by

$$\int_{\Omega} \chi(\omega) d\mu,\ \chi \in \mathcal{G}.$$

It is known [33] that the dual group of Ω is isomorphic to

$$\mathcal{D} \stackrel{def}{=} \bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\bigoplus$ denotes the direct sum, and $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$. An element $\underline{k} = (k_0, k_1, \dots) \in \mathcal{D}$ acts on Ω by the formula

$$\underline{k} \to \omega^{\underline{k}} = \prod_{m=0}^{\infty} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero. Therefore, the Fourier transform of the measure μ on $(\Omega, \mathcal{B}(\Omega))$ is

$$\int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) d\mu. \tag{4.1}$$

For probability measures on compact groups, a continuity theorems in terms of Fourier transforms is valid [18]. We state a theorem of such a type in the form of the following lemma.

Lemma 4.3. *Let P_n, $n \in \mathbb{N}$, be a probability measure on $(\Omega, \mathcal{B}(\Omega))$, and $g_n(\underline{k})$ be the corresponding Fourier transform. Suppose that $g_n(\underline{k})$, for all $\underline{k} \in \mathcal{D}$, converges to a continuity function $g(\underline{k})$ as $n \to \infty$. Then on $(\Omega, \mathcal{B}(\Omega))$, there exists a probability measure P such that P_n converges weakly to P as $n \to \infty$. In this case, $g(\underline{k})$ is the Fourier transform of the measure P.*

The lemma is a partial case of Theorem 1.4.2 from [18].

Let m_H be the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Thus, we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Then $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent complex-valued random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega, m_H))$.

Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_N(A) \stackrel{def}{=} \frac{1}{N+1} \# \left\{ 0 \le k \le N : ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in A \right\}.$$

In this section, we consider the weak convergence of Q_N as $N \to \infty$. The next lemma is the main result of the section.

Lemma 4.4. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for any $h > 0$ such that $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measure Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. In view of (4.1), we have that the Fourier transform $g_N(\underline{k})$ of Q_N is

$$\int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) dQ_N,$$

where only a finite number of integers k_m are distinct from zero. Thus, by the definition of Q_N,

$$g_N(\underline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m=0}^{\infty} (m+\alpha)^{-ikk_m h} = \frac{1}{N+1} \sum_{k=0}^{\infty} \exp\left\{ -ikh \sum_{m=0}^{\infty} k_m \log(m+\alpha) \right\}, \tag{4.2}$$

where only a finite number of integers k_m are distinct from zero. The linear independence over $\mathbb{Q}$ of the set $L(\alpha, q)$ implies, obviously, that of the set $L(\alpha)$. Therefore,

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$. Here and in the sequel, we have in mind that the above sum is finite. Moreover, we observe that

$$\exp\left\{ -ih \sum_{m=0}^{\infty} k_m \log(m+\alpha) \right\} \neq 1 \tag{4.3}$$

for $\underline{k} \neq \underline{0}$. Indeed, if the above inequality is not true, then

$$\exp\left\{-ih\sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\} = e^{2\pi i a}$$

with some $a \in \mathbb{Z}$. Hence,

$$-ih\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 2\pi i b$$

for some $b \in \mathbb{Z} \setminus \{0\}$, and

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) = \frac{2\pi c}{h} \tag{4.4}$$

with some $c \in \mathbb{Z} \setminus \{0\}$. Since the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the number $\exp\left\{\frac{2\pi c}{h}\right\}$ is also rational number, say, $q \neq 1$. If $q > 1$, then by (4.4),

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) - \log q = 0,$$

where only a finite number of integers k_m are not zero. However, this contradicts the linear independence of the set $L(\alpha, q)$. If $q < 1$, similar arguments hold with $\log \frac{1}{q}$.

Taking into account (4.3) and using the formula for the sum of geometric progression with denominator

$$\exp\left\{-ih\sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\},$$

we find from (4.2) that, for $\underline{k} = \underline{0}$,

$$g_N(\underline{k}) = \frac{1 - \exp\left\{-i(N+1)h\sum\limits_{m=0}^{\infty} k_m \log(m+\alpha)\right\}}{(N+1)\left(1 - \exp\left\{-ih\sum\limits_{m=0}^{\infty} k_m \log(m+\alpha)\right\}\right)}. \tag{4.5}$$

Clearly, $g_N(\underline{0}) = 1$. Therefore, by (4.5),

$$g_N(\underline{k}) = \begin{cases} 1 \ \text{ if } \ \underline{k} = \underline{0}, \\ \frac{1 - \exp\left\{-i(N+1)h\sum\limits_{m=0}^{\infty} k_m \log(m+\alpha)\right\}}{(N+1)\left(1 - \exp\left\{-ih\sum\limits_{m=0}^{\infty} k_m \log(m+\alpha)\right\}\right)} \ \text{ if } \ \underline{k} \neq \underline{0}. \end{cases}$$

Hence,

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 \ \text{ if } \ \underline{k} = \underline{0}, \\ 0 \ \text{ if } \ \underline{k} \neq \underline{0}. \end{cases}$$

The function

$$g(\underline{k}) = \begin{cases} 1 \ \text{ if } \ \underline{k} = \underline{0}, \\ 0 \ \text{ if } \ \underline{k} \neq \underline{0}, \end{cases}$$

is continuous in the discrete topology. Moreover, $g(\underline{k})$ is the Fourier transform of the Haar measure m_H. Indeed, the measure m_H is the product of the Haar measures on $(\gamma_m, \mathcal{B}(\gamma_m))$, $m \in \mathbb{N}_0$. Thus, the Fourier transform $\widehat{g}(\underline{k})$ of m_H is

$$\widehat{g}(\underline{k}) = \prod_{m=0}^{\infty} \int_{\gamma_m} \omega^{k_m}(m) d\mu, \tag{4.6}$$

where μ is the Haar measure on $(\gamma_m, \mathcal{B}(\gamma_m))$. Clearly, if $\underline{k} = 0$, then $\widehat{g}(\underline{k}) = 1$. If $k_m \neq 0$ for some $k_m \in \mathbb{Z}$, then

$$\int_{\gamma_m} \omega^{k_m}(m) d\mu = \int_0^{2\pi} e^{ik_m x} dx = \frac{e^{2\pi i k_m} - 1}{k_m} = 0.$$

Thus, in view of (4.6), $g(\underline{k}) = 0$ for $\underline{k} \neq \underline{0}$. This shows that $\widehat{g}(\underline{k}) = g(\underline{k})$. Now Lemma 4.3 proves that Q_N converges weakly to m_H as $N \to \infty$. The lemma is proved.

4.3. Limit theorems for absolutely convergent series

In this section, using Lemma 4.4, we will obtain limit theorems on the weak convergence of probability measures on $(H(D), \mathcal{B}(H(D)))$ defined by terms of absolutely convergent Dirichlet series.

For a fixed number $\sigma_1 > \frac{1}{2}$, and $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let

$$v_n(m, \alpha) = \exp\left\{ - \left(\frac{m+\alpha}{n+\alpha} \right)^{\sigma_1} \right\},$$

and define

$$\zeta_n(s, \alpha) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m) v_n(m, \alpha)}{(m+\alpha)^s}, \ \omega \in \Omega.$$

In [33], it was proved that the Dirichlet series for the functions $\zeta_n(s, \alpha)$ and $\zeta_n(s, \alpha, \omega)$ converge absolutely for $\sigma > \frac{1}{2}$. For a fixed $\widehat{\omega} \in \Omega$, define

$$P_{N,n}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta_n(s + ikh, \alpha) \in A\}, \ A \in \mathcal{B}(H(D)),$$

and

$$P_{N,n,\widehat{\omega}}(A) = \frac{1}{N+1} \#\{0 \leq k \leq N : \zeta_n(s + ikh, \alpha, \widehat{\omega}) \in A\}, \ A \in \mathcal{B}(H(D)).$$

Lemma 4.5. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, $P_{N,n}$ and $P_{N,n,\widehat{\omega}}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. For proving of the theorem, we apply standard arguments used in the continuous case [33]. Define the function $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = \zeta_n(s, \alpha, \omega).$$

The absolute convergence of the series for $\zeta_n(s, \alpha, \omega)$ implies the continuity of the function u_n. Moreover, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} P_{N,n}(A) &= \frac{1}{N+1}\#\{0 \le k \le N : \zeta_n(s, \alpha, ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0)) \in A\} \\ &= \frac{1}{N+1}\#\{0 \le k \le N : u_n(((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in A\} \\ &= \frac{1}{N+1}\#\{0 \le k \le N : ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in u_n^{-1}A\} = Q_N(u_n^{-1}A). \end{aligned}$$

Therefore, we have that $P_{N,n} = Q_N u_n^{-1}$. The Lemmas 4.4 and 1.6 together with the continuity of u_n show that $P_{N,n}$ converges weakly to $P_n = m_H u_n^{-1}$ as $N \to \infty$.

It remains to consider the measure $P_{N,n,\widehat{\omega}}$. Let the function $u_{n,\widehat{\omega}} : \Omega \to H(D)$ be given by the formula

$$u_{n,\widehat{\omega}}(\omega) = \zeta_n(s, \alpha, \widehat{\omega}\omega).$$

Then, similarly as above, we find that the measure $P_{N,n,\widehat{\omega}}$ converges weakly to $m_H u_{n,\widehat{\omega}}^{-1}$ as $N \to \infty$. Let $u : \Omega \to \Omega$ be given by $u(\omega) = \widehat{\omega}\omega$. Then we have that $u_{n,\widehat{\omega}} = u_n(u)$. Now we use the invariance of the Haar measure m_H, i.e., for all $A \in \mathcal{B}(\Omega)$ and $\omega \in \Omega$,

$$m_H(A) = m_H(\omega A) = m_H(A\omega).$$

Therefore, in view of the definition of u,

$$m_H u_{n,\widehat{\omega}}^{-1} = m_H(u_n u)^{-1} = (m_H u^{-1})u_n^{-1} = m_H u_n^{-1} = P_n.$$

Thus, the measures $P_{N,n}$ and $P_{N,n,\widehat{\omega}}$ both converge weakly to $P_n = m_H u_n^{-1}$ as $N \to \infty$.

4.4. Approximation in the mean

For $s \in D$ and $\omega \in \Omega$, define

$$\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}.$$

Then in [33] it is proved that the latter series converges uniformly on compact subset of the strip D for almost all $\omega \in \Omega$. In other words, $\zeta(s, \alpha, \omega)$ is a $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

To pass from the function $\zeta_n(s,\alpha)$ to $\zeta(s,\alpha)$, and from the function $\zeta_n(s,\alpha,\omega)$ to $\zeta(s,\alpha,\omega)$, we need certain approximation results. In this section, we approximate the above functions in the mean.

We start with auxiliary results.

Lemma 4.6. *Suppose that $\frac{1}{2} < \sigma < 1$. Then, for $T \to \infty$,*

$$\int_0^T |\zeta(\sigma+it,\alpha)|^2 dt \ll T.$$

The lemma is Theorem 3.3.1 of [33].

Lemma 4.7. *Suppose that $\frac{1}{2} < \sigma < 1$. Then, for $T \to \infty$,*

$$\int_0^T |\zeta'(\sigma+it,\alpha)|^2 dt \ll T.$$

Proof. By the integral Cauchy formula, we have that

$$\zeta'(s,\alpha) = \frac{1}{2\pi i} \int_{|z-s|=\delta} \frac{\zeta(z,\alpha)}{(s-z)^2} dz,$$

where the circle $|z-s|=\delta$ is lying in D. Then, for some σ_1, $\frac{1}{2} < \sigma_1 < 1$, by Lemma 4.6,

$$\int_0^T |\zeta'(\sigma+it,\alpha)|^2 dt = \int_0^T \left| \frac{1}{2\pi} \int_{|z-\sigma|=\delta} \frac{\zeta(z+it)}{(z-\sigma)^2} dz \right|^2 dt \ll \int_0^{2T} |\zeta(\sigma_1+it)|^2 dt \ll T.$$

The Gallagher lemma connects continuous and discrete mean-values. We state it as the following lemma.

Lemma 4.8. *Let T_0 and $T \geq \delta > 0$ be real numbers, and $\mathcal{T}$ be a finite set in the interval $[T_0+\frac{\delta}{2}, T_0+T-\frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t\in\mathcal{T} \\ |t-x|<\delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T+T_0]$ having a continuous derivative on $(T_0, T+T_0)$. Then

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx + \left(\int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{\frac{1}{2}}.$$

The lemmas is Lemma 1.4 of [44], where its proof is given.

Let ϱ be the metric of the space $H(D)$ defined in Chapter 1.

Lemma 4.9. *The relation*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varrho(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)) = 0$$

holds for all $h > 0$ *and* α, $0 \le \alpha < 1$.

Proof. An application of Lemma 4.8 gives, for fixed σ, $\frac{1}{2} < \sigma < 1$,

$$\sum_{k=0}^{N} |\zeta(\sigma+ikh+it,\alpha)|^2 \le \frac{1}{h} \int_0^{Nh} |\zeta(\sigma+i\tau+it,\alpha)|^2 d\tau$$
$$+ \left(\int_0^{Nh} |\zeta(\sigma+i\tau+it,\alpha)|^2 d\tau \int_0^{Nh} |\zeta'(\sigma+i\tau+it,\alpha)|^2 d\tau \right)^{\frac{1}{2}}.$$

Therefore, Lemmas 4.6 and 4.7 imply the estimate

$$\sum_{k=0}^{N} |\zeta(\sigma+ikh+it,\alpha)|^2 \ll N(1+|t|). \tag{4.7}$$

Let $\sigma_1 > \frac{1}{2}$ be the same as in Section 4.3. Define

$$l_n(s,\alpha) = \frac{s}{\sigma_1} \Gamma\left(\frac{s}{\sigma_1}\right)(n+\alpha)^s.$$

Then a straightforward application of the Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} ds = e^{-a}, \ a, b > 0,$$

leads to a formula

$$\zeta_n(s,\alpha) = \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha) \frac{dz}{z}. \tag{4.8}$$

Let $\frac{1}{2} < \sigma_2 < 1$ and $\sigma_2 < \sigma$. Then, moving the line of integration in (4.8) to the left, we obtain, by the residue theorem, that

$$\zeta_n(s,\alpha) - \zeta(s,\alpha) = \frac{1}{2\pi i} \int_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha) \frac{dz}{z} + R_n(s,\alpha), \tag{4.9}$$

where

$$R_n(s,\alpha) = \mathrm{Res}_{z=1-s} \zeta(s+z,\alpha) l_n(z,\alpha) z^{-1}.$$

Let K be an arbitrary compact subset of the strip D, and let L be a simple closed contour of length $|L|$ lying in same compact set of D and enclosing the set K such that the distance δ of L from the set K in strongly positive. Then, applying once more the integral Cauchy formula, we find that

$$\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)| \ll \frac{1}{2\pi\delta}\int\limits_L |\zeta(z+ikh,\alpha)-\zeta_n(z+ikh,\alpha)||dz|.$$

Hence,

$$\begin{aligned}
&\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)| \\
&\ll \frac{1}{N\delta}\int\limits_L |dz| \left(\sum_{k=0}^{N}|\zeta(\mathrm{Re}z+ikh+i\mathrm{Im}z,\alpha)-\zeta_n(\mathrm{Re}z+ikh+i\mathrm{Im}z,\alpha)|\right) \\
&\ll \frac{|L|}{N\delta}\sup_{s\in L}\sum_{k=0}^{N}|\zeta(\sigma+it+ikh,\alpha)-\zeta_n(\sigma+it+ikh,\alpha)|. \qquad (4.10)
\end{aligned}$$

Moreover, in view of (4.9),

$$\begin{aligned}
&\zeta(\sigma+it+ikh,\alpha)-\zeta_n(\sigma+it+ikh,\alpha) \\
&\ll \int\limits_{-\infty}^{\infty}|\zeta(\sigma_2+it+ikh+i\tau,\alpha)||l_n(\sigma_2-\sigma+i\tau,\alpha)|d\tau+|R_n(\sigma+it+ikh,\alpha)|
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma+it+ikh,\alpha)-\zeta_n(\sigma+it+ikh,\alpha)| \\
&\ll \int\limits_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau,\alpha)|\left(\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma_2+it+ikh+i\tau,\alpha)|\right)d\tau \\
&+\frac{1}{N}\sum_{k=0}^{N}|R_n(\sigma+it+ikh,\alpha)|. \qquad (4.11)
\end{aligned}$$

We observe that t is bounded for $s\in L$. Thus, using (4.7) and the Cauchy type inequality, we find

$$\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma_2+it+ikh+i\tau,\alpha)| \ll \left(\frac{1}{N}\sum_{k=0}^{N}|\zeta(\sigma_2+it+ikh+i\tau,\alpha)|^2\right)^{\frac{1}{2}} \ll 1+|\tau|. \qquad (4.12)$$

Taking account the properties of the gamma-function, see, for example, Chapter 1 of [33], and applying Lemma 4.8 once more, we obtain that

$$\frac{1}{N}\sum_{k=0}^{N}|R_n(\sigma+it+ikh,\alpha)| = o(1) \qquad (4.13)$$

as $N\to\infty$. Now, from (4.10)-(4.13), we conclude that

$$\frac{1}{N}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)| \ll \sup_{s\in L}\int\limits_{-\infty}^{\infty}|l_n(\sigma_2-\sigma+i\tau,\alpha)|(1+|\tau|)d\tau+o(1) \qquad (4.14)$$

as $N \to \infty$. By the choice of σ_2, we have that $\sigma_2 - \sigma < 0$. This implies, in view of the definition of $l_n(s, \alpha)$, that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |l_n(\sigma_2 - \sigma + i\tau, \alpha)|(1 + |\tau|)d\tau = 0,$$

and (4.14) implies the equality

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |\zeta(s + ikh, \alpha) - \zeta_n(s + ikh, \alpha)| = 0.$$

This and the definition of the metric ϱ prove the lemma.

The case of the approximation of $\zeta(s, \alpha, \omega)$ by $\zeta_n(s, \alpha, \omega)$ is more complicated, and we need a result of the ergodic theory.

Let, for $\tau \in \mathbb{R}$,

$$a_{\tau,\alpha} = \{(m + \alpha)^{-i\tau} : m \in \mathbb{N}_0\},$$

and let the one-parameter family $\{\varphi_\tau : \tau \in \mathbb{R}\}$ of transformations on Ω be defined by

$$\varphi_{\tau,\alpha}(\omega) = a_{\tau,\alpha}\omega, \ \omega \in \Omega.$$

Then we have that $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable transformations on Ω. Indeed, $a_{\tau,\alpha}$ is an element of Ω. Therefore, $a_{\tau,\alpha}^{-1} A \in \mathcal{B}(\Omega)$ for $A \in \mathcal{B}(\Omega)$. Hence,

$$\varphi_{\tau,\alpha}^{-1}(A) = \{\omega \in \Omega : \varphi_{\tau,\alpha}(\omega) \in A\} = \{\omega \in \Omega : \omega \in a_{\tau,\alpha}^{-1} A\} \in \mathcal{B}(\Omega),$$

and $\varphi_{\tau,\alpha}$ is measurable for every $\tau \in \mathbb{R}$. Obviously, the set $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ forms a group. We have that $\varphi_{\tau_1,\alpha} \cdot \varphi_{\tau_2,\alpha} = \varphi_{\tau_1+\tau_2,\alpha}$, $\varphi_{0,\alpha}$ is the unit element, $\varphi_{-\tau,\alpha}$ is the inverse element of $\varphi_{\tau,\alpha}$, and, clearly, all axioms of a group are satisfied.

Moreover, the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is measure preserving. From the invariance of the Haar measure m_H, we find that, for every $A \in \mathcal{B}(\Omega)$ and $\tau \in \mathbb{R}$,

$$m_H(\omega \in \Omega : \varphi_{\tau,\alpha}(\omega) \in A) = m_H(\omega \in \Omega : \omega \in a_{\tau,\alpha}^{-1} A) = m_H(A).$$

Now we recall some notions of the ergodic theory. A set $A \in \mathcal{B}(\Omega)$ is called invariant with respect to $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ if the sets A and $A_\tau = \varphi_{\tau,\alpha}(A)$ differ one from another at most by a set m_H-measure zero. All invariants sets form a sub-σ field of the field $\mathcal{B}(\Omega)$. The group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is called ergodic if all invariant sets have m_H-measure 1 or 0.

We remind that $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$.

Lemma 4.10. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. We have seen in the proof of Lemma 4.4 that a character χ of Ω is of the form

$$\chi(\omega) = \prod_{m=0}^{\infty} \omega^{k_m}(m),\ \omega \in \Omega,$$

where only a finite number of integers k_m are distinct from zero. Let χ be a non-trivial character of Ω, i.e., $\chi(\omega) \not\equiv 1$. Then we have that

$$\chi(a_{\tau,\alpha}) = \prod_{m=0}^{\infty} (m+\alpha)^{-i\tau k_m} = \exp\left\{-i\tau \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}.$$

Since the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, then

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \{k_m\} = \underline{0}$. Therefore, there exists $\tau_0 \in \mathbb{R}$ such that

$$\chi(a_{\tau_0,\alpha}) \neq 1. \tag{4.15}$$

Let $A \in \mathcal{B}(\Omega)$ be an invariant set of the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$, and I_A be its indicator function, i.e.,

$$I_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ 0 \text{ if } \omega \notin A. \end{cases}$$

Then

$$I_A(a_{\tau_0,\alpha}, \omega) = I_A(\omega)$$

for almost all $\omega \in \Omega$ because the sets A and $a_{\tau_0}A$ differ one from on other at most by a set of m_H-measure zero. Let $\widehat{f}(\chi)$ denote the Fourier transform of f. Then we have that

$$\begin{aligned}\widehat{I}_A(\chi) &= \int\limits_{\Omega} \chi(\omega) I_A(\omega) dm_H = \int\limits_{\Omega} \chi(a_{\tau_0,\alpha}\omega) I_A(a_{\tau_0,\alpha}\omega) dm_H \\ &= \chi(a_{\tau_0,\alpha}) \int\limits_{\Omega} \chi(\omega) I_A(\omega) dm_H = \chi(a_{\tau_0,\alpha}) \widehat{I}_A(\chi).\end{aligned}$$

Therefore, in view of (4.15), for a non-trivial character χ,

$$\widehat{I}_A(\chi) = 0. \tag{4.16}$$

Now let χ_0 be the trivial character of Ω, i.e., $\chi_0(\omega) \equiv 1$. Suppose that $\widehat{I}_A(\chi_0) = u$. Then, using the orthogonality of characters

$$\int\limits_{\Omega} \chi(\omega) dm_H = \begin{cases} 0 \text{ if } \chi \neq \chi_0, \\ 1 \text{ if } \chi = \chi_0, \end{cases}$$

and (4.16), we find that, for any character χ of Ω,

$$\widehat{I}_A(\chi) = u \int\limits_{\Omega} \chi(\omega) dm_H = u\widehat{1}(\chi) = \widehat{u}(\chi). \tag{4.17}$$

The function $I_A(\omega)$ is uniquely determined by its Fourier transform $\widehat{I}_A(\chi)$. Thus, by (4.17), we see that $I_A(\omega) = u$ for almost all $\omega \in \Omega$. Thence, $I_A(\omega) = 1$ or $I_A(\omega) = 0$ for almost all $\omega \in \Omega$. Therefore, $m_H(A) = 1$ or $m_H(A) = 0$. This proves the ergodicity of the group $\{\varphi_{\tau,\alpha} : \tau \in \mathbb{R}\}$.

For the sequel, we need the classical Birkhoff-Khintchine ergodicity theorem. First we remind some definitions. Let $\xi(\tau, \omega)$, $\tau \in \mathcal{T}$, be a random process defined on the probability space $(\widehat{\Omega}, \mathcal{F}, \mu)$, and X be the space of all finite real functions $x(\tau)$, $\tau \in \mathcal{T}$. Then the family of finite-dimensional distributions of the process $\xi(\tau, \omega)$ determines a probability measure Q on $(X, \mathcal{B}(X))$. Then on the probability space $(X, \mathcal{B}(X), Q)$, a translation transformation g_u mapping each function $x(\tau) \in X$ to $x(\tau + u)$, can be defined.

A random process $\xi(\tau, \omega)$ is called strongly stationary if all its finite-dimensional distributions are invariant with respect to the transformation g_u.

Let $A_u = g_u(A)$, $A \in \mathcal{B}(X)$. A set $A \in \mathcal{B}(X)$ is called an invariant set of the process $\xi(\tau, \omega)$ if, for every $u \in \mathbb{R}$, the sets A and A_u can differ one from another at most by a set of Q-measure zero. A strongly stationary process is called ergodic if its σ-field of invariant sets consists only of sets of Q-measure 1 or 0.

Now we state the Birkhoff-Khintchine theorem as the following lemma. Denote by $\mathbb{E}\xi$ the expectation of a random element ξ.

Lemma 4.11. *Suppose that the random process $\xi(\tau, \omega)$ is ergodic, $\mathbb{E}|\xi(\tau, \omega)| < \infty$, and that its sample paths are integrable almost surely in the Riemann sense over every finite interval. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int\limits_0^T \xi(\tau, \omega) d\tau = \mathbb{E}\xi(0, \omega).$$

The above remarks and the proof the lemma can be found, for example, in [12].

Lemma 4.12. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $\frac{1}{2} < \sigma < 1$. Then, for $T \to \infty$,*

$$\int\limits_0^T |\zeta(\sigma + it, \alpha, \omega)|^2 dt \ll T$$

for almost all $\omega \in \Omega$.

Proof. Let, for brevity,

$$\widehat{\zeta}(\sigma,\alpha,\omega)=|\zeta(\sigma,\alpha,\omega)|^2=\left|\sum_{m=0}^{\infty}\frac{\omega(m)}{(m+\alpha)^{\sigma}}\right|^2.$$

Using the pairwise orthogonality of the random variables $\omega(m)$,

$$\int\limits_{\Omega}\omega(m)\overline{\omega(n)}dm_H=\begin{cases}1 \text{ if } m=n,\\ 0 \text{ if } m\neq n,\end{cases}$$

we find that

$$\mathbb{E}\widehat{\zeta}(\sigma,\alpha,\omega)=\int\limits_{\Omega}\widehat{\zeta}(\sigma,\alpha,\omega)dm_H=\int\limits_{\Omega}\sum_{m=0}^{\infty}\frac{\omega(m)}{(m+\alpha)^{\sigma}}\sum_{m=0}^{\infty}\frac{\overline{\omega(n)}}{(n+\alpha)^{\sigma}}=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^{2\sigma}}<\infty. \quad (4.18)$$

Moreover, by the definition of the transformation $\varphi_{\tau,\alpha}$, we have that

$$\widehat{\zeta}(\sigma,\alpha,\varphi_{\tau,\alpha}(\omega))=|\zeta(\sigma,\alpha,a_{\tau,\alpha}\omega)|^2=|\zeta(\sigma+i\tau,\alpha,\omega)|^2,$$

and, in view of Lemma 4.10, the random process $|\zeta(\sigma+i\tau,\alpha,\omega)|$ is ergodic. Therefore, by Lemma 4.11 and (4.18), we obtain that

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T|\zeta(\sigma+i\tau,\alpha,\omega)|^2d\tau=\mathbb{E}\widehat{\zeta}(\sigma,\alpha,\omega)<\infty$$

for almost all $\omega\in\Omega$. This proves the lemma.

Now we are ready to prove an analogue of Lemma 4.9.

Lemma 4.13. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for almost all $\omega\in\Omega$,*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho(\zeta(s+ikh,\alpha,\omega),\zeta_n(s+ikh,\alpha,\omega))=0.$$

Proof. We apply arguments similar to those used in proof of Lemma 4.9.

From Lemma 4.12 and the integral Cauchy formula, it follows that, for $\frac{1}{2}<\sigma<1$ and $T\to\infty$,

$$\int\limits_0^T|\zeta'(\sigma+it,\alpha,\omega)|^2dt\ll T$$

for almost all $\omega\in\Omega$. This, Lemmas 4.12 and 4.8 lead, for almost all $\omega\in\Omega$, to the estimate

$$\sum_{k=0}^{N}|\zeta(s+ikh+it,\alpha)|^2\ll N(1+|t|).$$

Therefore, the remained part of the proof completely coincides with the corresponding part of the proof of Lemma 4.9 with one difference that all estimates are valid for almost all $\omega\in\Omega$.

4.5. Discrete limit theorems for $\zeta(s, \alpha)$ and $\zeta(s, \alpha, \omega)$

For $A \in \mathcal{B}(H(D))$ and $\omega \in \Omega$, define

$$P_N(A) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha) \in A\}$$

and

$$P_{N,\omega}(A) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\}.$$

In this section, we will consider the weak convergence of P_N and $P_{N,\omega}$ as $N \to \infty$. For this aim, we need some results from the theory of weak convergence of probability measures.

Let $\{P\}$ be a family of probability measures on $(X, \mathcal{B}(X))$. This family is called tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset X$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$. The family $\{P\}$ is called relatively compact if each sequence $\{P_n\} \subset \{P\}$ contains a weakly convergent subsequence to a certain probability measure on $(X, \mathcal{B}(X))$.

The Prokhorov theorem connects the notions of tightness and relative compactness. We need only the direct Prokhorov theorem which is contained in the next lemma.

Lemma 4.14. *Suppose that the family $\{P\}$ is tight. Then it is relative compact.*

Proof of the lemma can be found in [2], Theorem 6.1.

Let $\{\xi_n\}$ be a sequence of X-valued random elements defined on a certain probability space $(\widehat{\Omega}, \mathcal{F}, \mu)$. This sequence converges to a random element ξ in distribution as $n \to \infty$ $\left(\xi_n \xrightarrow[n \to]{\mathcal{D}} \xi\right)$ if the distribution of ξ_n

$$\mu(\widehat{\omega} \in \widehat{\Omega} : \xi_n(\widehat{\omega}) \in A), \ A \in \mathcal{B}(X),$$

converges weakly to the distribution of ξ

$$\mu(\widehat{\omega} \in \widehat{\Omega} : \xi(\widehat{\omega}) \in A), \ A \in \mathcal{B}(X),$$

as $n \to \infty$.

We will use the following assertion.

Lemma 4.15. *Let the metric space (X, d) is separable, and let $\eta_n, \xi_{1n}, \xi_{2n}, \dots$ be X-valued random elements on the probability space $(\widehat{\Omega}, \mathcal{F}, \mu)$. Suppose that, for each k, $\xi_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} \xi_k$, $\xi_k \xrightarrow[k\to\infty]{\mathcal{D}} \xi$, and that, for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(\xi_{kn}, \eta_n) \geq \varepsilon\} = 0.$$

Then $\eta_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi$.

The lemma is Theorem 4.2 from [2], where its proof is given.

Lemma 4.16. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measures P_N and $P_{N,\omega}$ both converge to the same probability measure P as $N \to \infty$.*

Proof. By Lemma 4.5, the measure $P_{N,n}$ converges weakly to the measure P_n as $N \to \infty$. Let $X_n(s)$ be the $H(D)$-valued random element with the distribution P_n. Moreover, let θ_N be a discrete random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mu)$ and having the distribution

$$\mu(\theta_N = kh) = \frac{1}{N+1}, \ l = 0, \dots, N.$$

Define an $H(D)$-valued random element $X_{N,n}(s)$ by

$$X_{N,n}(s) = \zeta_n(s + i\theta_N, \alpha).$$

Then we can write the assertion of Lemma 4.5 for the measure $P_{N,n}$ in the form

$$X_{N,n}(s) \xrightarrow[N\to\infty]{\mathcal{D}} X_n(s). \tag{4.19}$$

Let K_l be a compact set in the definition of the metric ϱ. Then, using (4.7) with $t = 0$,

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh, \alpha)|^2 \ll N, \ \frac{1}{2} < \sigma < 1,$$

we deduce by the integral Cauchy formula that

$$\sum_{k=0}^{N} \sup_{s\in K_l} |\zeta(\sigma + ikh, \alpha)|^2 \ll_l N.$$

For this and from the relation

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_l} |\zeta(s + ikh, \alpha) - \zeta_n(s + ikh, \alpha)| = 0$$

obtained in the proof of Lemma 4.9, we find that

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_l} |\zeta_n(s + ikh, \alpha)| \leq R_l < \infty. \tag{4.20}$$

Now let $\varepsilon > 0$ be an arbitrary number, and $M = M_l(\varepsilon) = 2^l R_l \varepsilon^{-1}$. Then (4.20) and the Chebyshev type inequality yield

$$\mu\left(\sup_{s\in K_l} |X_{N,n}(s)| > M\right) =$$
$$\frac{1}{N+1}\#\left\{0 \le k \le N : \sup_{s\in K_l} |\zeta_n(s+ikh,\alpha)| > M\right\}$$
$$\le \frac{1}{(N+1)M}\sum_{k=0}^{N}\sup_{s\in K_l}|\zeta_n(s+ikh,\alpha)| \le \frac{\varepsilon}{2^l}. \tag{4.21}$$

Clearly, the relation (4.19) implies

$$\sup_{s\in K_l} |X_{N,n}(s)| \xrightarrow[N\to\infty]{\mathcal{D}} |X_n(s)|.$$

Combining this with (4.21) gives

$$\mu\left(\sup_{s\in K_l} |X_n(s)| > M\right) \le \frac{\varepsilon}{2^l}. \tag{4.22}$$

Let

$$K = K(\varepsilon) = \{g \in H(D) : \sup_{s\in K_l} |g(s)| \le M_l,\ l \in \mathbb{N}\}.$$

Then the set K is uniformly bounded on compact subsets, therefore, it is a compact subset of $H(D)$. Moreover, in view of (4.22),

$$\mu(X_n(s) \in K) \ge 1 - \varepsilon\sum_{l=1}^{\infty}\frac{1}{2^l} = 1 - \varepsilon,$$

or

$$P_n(K) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 4.14, this family is relatively compact. Hence, there exists a subsequence $\{P_{n_r}\} \subset \{P_n\}$ such that P_{n_r} converges weakly to a certain probability measure P on $(H(D)), \mathcal{B}(H(D))$ as $r \to \infty$. In other words, we have that

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{4.23}$$

Define one more $H(D)$-valued random element

$$X_N(s) = \zeta(s + i\theta_N, \alpha).$$

Then, in view of Lemma 4.9, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mu\left(\varrho\left(X_N(s), X_{N,n}(s)\right) \ge \varepsilon\right)$$
$$= \lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\#\{0 \le k \le N : \varrho(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)) \ge \varepsilon\}$$

$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \varrho(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)) = 0.$$

This, (4.19), (4.23) and Lemma 4.15 show that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{4.24}$$

and the later relation is equivalent to the weak convergence of P_N to P as $N \to \infty$. Moreover, relation (4.24) shows that the measure P does not depend on the sequence $\{P_{n_r}\}$. Thus, the relative compactness of the family $\{P_n : n \in \mathbb{N}\}$ implies the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{4.25}$$

It remains to consider the weak convergence of the measure $P_{N,\omega}$. Define the $H(D)$-valued random elements

$$X_{N,n,\omega}(s) = \zeta_n(s + i\theta_N, \alpha, \omega)$$

and

$$X_{N,\omega}(s) = \zeta(s + i\theta_N, \alpha, \omega).$$

Then, repeating the above arguments, and using (4.25) and Lemma 4.13, we find that $P_{N,\omega}$ also converges weakly to P as $N \to \infty$.

4.6. Main limit theorem

In this section, we will prove a limit theorem on the weak convergence of the measure P_N with explicitly given limit measure. Denote by P_ζ the distribution of the random element $\zeta(s,\alpha,\omega)$,i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s,\alpha,\omega) \in A), \ A \in \mathcal{B}(H(D)).$$

Theorem 4.17. *Suppose that the set $L(\alpha,q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measure P_N converges weakly to the measure P_ζ as $N \to \infty$.*

In view of Lemma 4.16, it remains to identify the limit measure in that lemma. For this, we will apply some elements of ergodic theory.

Let

$$a_{h,\alpha} = \{(m+\alpha)^{-ih} : m \in \mathbb{N}_0\},$$

and, for $\omega \in \Omega$, let

$$\varphi_{h,\alpha}(\omega) = a_{h,\alpha}\omega.$$

Then, similarly to the case of $\varphi_{\tau,\alpha}$, we have that $\varphi_{h,\alpha}$ is a measurable measure preserving transformation on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. A set $A \in \mathcal{B}(\Omega)$ is invariant with respect to the transformation $\varphi_{h,\alpha}$ if the sets A and $A = \varphi_{h,\alpha}(A)$ can differ one from another at most by a set of zero m_H-measure. The transformation $\varphi_{h,\alpha}$ is ergodic if its σ-field of invariant sets consist only of sets hawing m_H-measure equal to 1 or 0.

Lemma 4.18. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then the transformation $\varphi_{h,\alpha}$ is ergodic.*

Proof. Let χ be a non-trivial character of Ω. As in the proof of Lemma 4.4, we have that

$$\chi(\omega) = \prod_{m=0}^{\infty} \omega^{k_m}(m),\ \omega \in \Omega,$$

where only a finite number of integers k_m are distinct from zero. Clearly, $a_{h,\alpha} \in \Omega$. Therefore,

$$\chi(a_{h,\alpha}) = \prod_{m=0}^{\infty} (\omega + \alpha)^{-ik_m h} = \exp\left\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}.$$

Thus, in view of (4.3),

$$\chi(a_{h,\alpha}) \neq 1. \tag{4.26}$$

Let $A \in \mathcal{B}(\Omega)$ be an invariant set of the transformation $\varphi_{h,\alpha}$, and let I_A be its indicator function. Then

$$I_A(a_{h,\alpha}\omega) = I_A(\omega)$$

for almost all $\omega \in \Omega$. Hence, denoting by $\widehat{f}$ the Fourier transform of a function f, we have that

$$\begin{aligned}
\widehat{I}_A(\chi) &= \int_{\Omega} \chi(\omega) I_A(\omega) dm_H \\
&= \int_{\Omega} \chi(a_{h,\alpha}\omega) I_A(a_{h,\alpha}\omega) dm_H \\
&= \chi(a_{h,\alpha}) \int_{\Omega} \chi(\omega) I_A(\omega) dm_H = \chi(a_{h,\alpha}) \widehat{I}_A(\chi)
\end{aligned}$$

in virtue of the invariance of the Haar measure m_H, and multiplicativity of χ. Therefore, by inequality (4.26), for a non-trivial character χ,

$$\widehat{I}_A(\chi) = 0. \tag{4.27}$$

Now let χ_0 be the trivial character of Ω, and $\widehat{I}_A(\chi_0) = u$. Then, using the relations

$$\int\limits_{\Omega} \chi(\omega) dm_H = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ 1 & \text{if } \chi = \chi_0, \end{cases}$$

and (4.27), we find that, for any character χ of Ω,

$$\widehat{I}_A(\chi) = u \int\limits_{\Omega} \chi(\omega) dm_H = u\widehat{1}(\chi) = \widehat{u}(\chi).$$

Hence, as in the case of Lemma 4.10, we obtain that $m_H(A) = 0$ or $m_H(A) = 1$, i.e., the transformation $\varphi_{h,\alpha}$ is ergodic.

Now we state the individual Birkhoff-Khintchine theorem.

Lemma 4.19. *Let φ be a measurable measure preserving ergodic transformation on the space $(\widehat{\Omega}, \mathcal{F}, \mu)$. Then, for every integrable with respect μ function g*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} f(\varphi^k \omega) = \mathbb{E} f.$$

Proof of the lemma can be found in [57].

We also will use the equivalent of weak convergence of probability measures in terms of continuity sets. We remind that a set $A \in \mathcal{B}(X)$ is called a continuity set of a measure P on $(X, \mathcal{B}(X))$ if $P(\partial A) = 0$, where ∂A is the boundary of A.

Lemma 4.20. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is a part of Theorem 2.1 from [2].

Now we are ready to prove Theorem 4.17.

Proof of Theorem 4.17. Let A be a fixed continuity set of the limit measure P in Lemma 4.16. Then, by Lemmas 4.16 and 4.20,

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\} = P(A). \tag{4.28}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable θ by

$$\theta(\omega) = \begin{cases} 1 & \text{if } \zeta(s, \alpha, \omega) \in A, \\ 0 & \text{if } \zeta(s, \alpha, \omega) \notin A. \end{cases}$$

then, clearly, the expectation of θ is

$$\mathbb{E}(\theta) = \int\limits_{\Omega} \theta dm_H = m_H(\omega \in \Omega : \zeta(s, \alpha, \omega) \in A) = P_\zeta(A). \tag{4.29}$$

Since, by Lemma 4.18, the transformation $\varphi_{h,\alpha}$ is ergodic, using of Lemma 4.19 shows that

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \theta(\varphi_{h,\alpha}^k(\omega)) = \mathbb{E}\theta \tag{4.30}$$

for almost all $\omega \in \Omega$. However, by the definitions of θ and $\varphi_{h,\alpha}$,

$$\frac{1}{N+1} \sum_{k=0}^{N} \theta(\varphi_{h,\alpha}^k(\omega)) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\}.$$

Therefore, in view of (4.29) and (4.30),

$$\lim_{N \to \infty} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\} = P_\zeta(A)$$

for almost all $\omega \in \Omega$. This and (4.28) show that $P(A) = P_\zeta(A)$ for any continuity set A of the measure P. However, it is known [2] that all continuity sets constitute a determining class. Hence, $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(H(D))$. The theorem is proved.

Theorem 4.17 implies a limit theorem for the composite function $F(\zeta(s, \alpha))$.

Let, as in Chapter 2, for $F : H(D) \to H(D)$,

$$P_{N,F}(A) \stackrel{def}{=} \frac{1}{N+1} \#\{0 \le k \le N : F(\zeta(s + ikh, \alpha)) \in A\}, \ A \in \mathcal{B}(H(D)).$$

Theorem 4.21. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is continuous operator. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measure $P_{N,F}$ converges weakly to the measure $P_\zeta F^{-1}$ as $N \to \infty$.*

Proof of the theorem uses Theorem 4.17 and completely coincides with that of Lemma 2.7.

4.7. Supports

In Chapters 1 and 2, we have seen that, for proving of universality theorems, the explicit forms of supports of the limit measures in limit theorems on weakly convergent probability measures in the space of analytic functions are needed. In this chapter, we consider limit theorems under new hypotheses, therefore, we must check the supports of the limit measures in them.

Theorem 4.22. *Under the hypotheses of Theorem 4.1, the support of measure P_ζ is the whole of $H(D)$.*

Proof. The measure P_ζ does not depend on h. Moreover, since the linear independence over $\mathbb{Q}$ of the set $L(\alpha, q)$ implies that of the set $L(\alpha)$, in [28] it was obtained that the support of P_ζ is the whole of $H(D)$.

Theorem 4.23. *Suppose that the numbers α and h, and the operator F satisfy the hypotheses of Theorem 4.2. Then the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.*

Proof. We apply the same arguments as in the proof of Lemma 2.10. Let g be an arbitrary element of $H(D)$, and G by any open neighbourhood of g. Since F is continuous, the set $F^{-1}G$ is open, too. By Lemma 1.13 and the approximation in the space $H(D)$, there exists a polynomial $p = p(s) \in G$. Since, by a hypothesis of the theorem, the preimage $F^{-1}\{p\}$ is non-empty, we have that $F^{-1}G$ is an open neighbourhood of some element of the space $H(D)$. Thus, in view of Theorem 4.22, $P_\zeta(F^{-1}G) > 0$. Therefore,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$$

for every open neighbourhood G of arbitrary element $g \in H(D)$. Thus, the support of $P_\zeta F^{-1}$ is the whole of $H(D)$.

4.8. Proof of universality theorems

Theorems 4.1 and 4.2 are consequences of Theorems 4.17, 4.22 and 4.21, 4.23, respectively.

Proof of Theorem 4.1. By Lemma 1.13, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.31}$$

Define

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G is open neighbourhood of the polynomial $p(s)$ which, by Theorem 4.22, is an element of the support of the measure P_ζ. Thus, $P_\zeta(G) > 0$. Therefore, using Lemma 1.17, we have, by Theorem 4.17, that

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha) \in G\} \ge P_\zeta(G) > 0.$$

Thus, by the definition of G,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - p(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{4.32}$$

If $k \in \mathbb{N}_0$ satisfies the inequality

$$\sup_{s\in K} |\zeta(s+ikh,\alpha) - p(s)| < \frac{\varepsilon}{2},$$

then, in view of (4.31), we find that

$$\sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon.$$

This shows that

$$\# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - p(s)| < \frac{\varepsilon}{2} \right\} \le \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\}.$$

Thus, the assertion of the theorem follows from (4.32).

Proof of Theorem 4.2. We use Theorems 4.21 and 4.23 in place of Theorems 4.17 and 4.22, and repeat the proof of Theorem 4.1.

Theorem 4.2 is only one example of universality theorems for composite functions under new hypotheses on the numbers α and h. Obviously, we can state other allied theorems of Chapter 2 under new hypotheses involving the linear independence over $\mathbb{Q}$ of the set $L(\alpha, q)$.

Conclusions

In the thesis, the following assertions of discrete type were obtained:

1. Discrete universality theorems for composite functions $F(\zeta(s))$ with some operators $F : H(D) \to H(D)$ on the approximation of a wide class of analytic functions by shifts $F(\zeta(s + ikh))$, where $\zeta(s)$ is the Riemann zeta-function, $k \in \mathbb{N}_0$, and $h > 0$ is a fixed number, are valid.

2. Discrete universality theorems for composite functions $F(\zeta(s, \alpha))$ with some operators $F : H(D) \to H(D)$ on the approximation of a wide class of analytic functions by shifts $F(\zeta(s + ikh, \alpha))$, where $\zeta(s, \alpha)$, is the Hurwitz zeta-function, $k \in \mathbb{N}_0$, and for some types of the numbers α, $0 < \alpha < 1$, and $h > 0$, are valid.

3. The composite functions $F(\zeta(s + ikh))$ and $F(\zeta(s + ikh, \alpha))$, for all σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, and sufficiently large N have a zero in the disc
$$\left| s - \frac{\sigma_1 + \sigma_2}{2} \right| \leq \frac{\sigma_2 - \sigma_1}{2}$$
for more than cN numbers k, $0 \leq k \leq N$, where $c = c(\sigma_1, \sigma_2, F, h, \alpha)$ is a positive constant, $c = c(\sigma_1, \sigma_2, F, h)$ in the case of the Riemann zeta-function $\zeta(s)$.

4. An extension of a discrete universality theorem for the Hurwitz zeta-function $\zeta(s, \alpha)$ with transcendental parameter α exists.

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Notation

$\mathbb{N}$	set of all positive integers
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{Z}$	set of all integers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
m, n, k, l	positive or non-negative integers
p	prime number
$s = \sigma + it$, $\sigma, t \in \mathbb{R}$, $i = \sqrt{-1}$	complex variable
$F^{-1}G$	preimage of a set G
$\#A$	number of elements of a set A
$\mathrm{meas}A$	Lebesque measure of a measurable set $A \subset \mathbb{R}$
$\zeta(s)$	Riemann zeta-function defined, for $\sigma > 1$,by the series $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha)$	Hurwitz zeta-function defined, for $\sigma > 1$, by the series $\zeta(s) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$\Gamma(s)$	Euler gamma-function defined, for $\sigma > 0$ by the integral $\Gamma(s) = \int\limits_0^{\infty} \mathrm{e}^{-\mathrm{x}} \mathrm{x}^{\mathrm{s}-1} \mathrm{dx}$,
$\pi(x)$	number of prime numbers not exceeding x
$\log x = \log_e x$	
$D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$	
$H(G)$	space of analytic functions on G

$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$	
$f(x) = O(g(x)),\ g(x) > 0,\ x \in X$	means that there exists a constant $C > 0$ such that, for $x \in X$, $\lvert f(x)\rvert \leq Cg(x)$
$f(x) \ll g(x),\ g(x) > 0,\ x \in X$	mean that $f(x) = O(g(x)),\ x \in X$