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Article

A Generalized Bohr–Jessen Type Theorem for the Epstein Zeta-Function

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Abstract: Let Q be a positive defined $n \times n$ matrix and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$. The Epstein zeta-function $\zeta(s;Q)$, $s = \sigma + it$, is defined, for $\sigma > \frac{n}{2}$, by the series $\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s}$, and is meromorphically continued on the whole complex plane. Suppose that $n \geqslant 4$ is even and $\varphi(t)$ is a differentiable function with a monotonic derivative. In the paper, it is proved that $\frac{1}{T}\mathrm{meas}\{t \in [0,T] : \zeta(\sigma + i\varphi(t);Q) \in A\}$, $A \in \mathcal{B}(\mathbb{C})$, converges weakly to an explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.

Keywords: Epstein zeta-function; limit theorem; weak convergence; Haar measure

MSC: 11M46; 11M06

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1. Introduction

It is well known that the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad s = \sigma + it, \quad \sigma > 1,$$

shows analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$, and satisfies functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

where $\Gamma(s)$ denotes the Euler gamma-function. The majority of other zeta-functions also have similar equations, which are referred to as the Riemann type. Epstein in [1] raised a question to find the most general zeta-function with a functional equation of the Riemann type and introduced the following zeta-function. Let Q be a positive defined quadratic $n \times n$ matrix, and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$ for $\underline{x} \in \mathbb{Z}^n$. Epstein defined, for $\sigma > \frac{n}{2}$, the function

$$\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s},$$

continued analytically it to the whole complex plane, except for a simple pole at the point $s = \frac{n}{2}$ with residue $\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\sqrt{\det Q}}$, and proved the functional equation

$$\pi^{-s} \Gamma(s) \zeta(s;Q) = (\det Q)^{-\frac{1}{2}} \pi^{s - \frac{n}{2}} \Gamma\left(\frac{n}{2} - s\right) \zeta\left(\frac{n}{2} - s; Q^{-1}\right).$$

In [2], Bohr and Jessen proved a probabilistic limit theorem for the function $\zeta(s)$. We recall its modern version. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the topological space $\mathbb{X}$, and by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that, for $\sigma > \frac{1}{2}$,

$$\frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}), \tag{1}$$

converges weakly to P_σ as $T \to \infty$ (see, for example, [3] (Theorem 1.1, p. 149). In [4], the latter limit theorem was generalized for the Epstein zeta-function $\zeta(s;Q)$ with even $n \geq 4$ and integers $Q[\underline{x}]$. Namely, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists an explicitly given probability measure $P_{Q,\sigma}$ such that, for $\sigma > \frac{n-1}{2}$,

$$\frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma + it; Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{Q,\sigma}$ as $T \to \infty$.

For the function $\zeta(s)$, more general limit theorems are also considered. In place of (1), the weak convergence for

$$\frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma + i\varphi(t)) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

with certain measurable function $\varphi(t)$ is studied. For example, theorems of such a kind follow from limit theorems in the space of analytic functions proved in [5].

Suppose that the function $\varphi(t)$ is defined for $t \geq T_0 > 0$, is increasing to $+\infty$, and has a monotonic derivative $\varphi'(t)$ satisfying the estimate

$$\varphi(2t)\frac{1}{\varphi'(t)} \ll t, \quad t \to \infty.$$

Denote the class of the above functions by $W(T_0)$.

The aim of this paper is to prove a limit theorem for

$$\hat{P}_{T,Q,\sigma}(A) \overset{def}{=} \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma + i\varphi(t); Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

when $\varphi(t) \in W(T_0)$. In place of $\hat{P}_{T,Q,\sigma}$ one can consider

$$P_{T,Q,\sigma}(A) \overset{def}{=} \frac{1}{T}\text{meas}\{t \in [T,2T] : \zeta(\sigma + i\varphi(t); Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

It is easily seen that the weak convergence of $\hat{P}_{T,Q,\sigma}$ to $P_{Q,\sigma}$ as $T \to \infty$ is equivalent to that of $P_{T,Q,\sigma}$. Actually, if $\hat{P}_{T,Q,\sigma}$ converges weakly to $P_{Q,\sigma}$ as $T \to \infty$, then

$$\lim_{T\to\infty} \hat{P}_{T,Q,\sigma}(A) = P_{Q,\sigma}(A)$$

for every continuity set A of the measure $P_{Q,\sigma}$. Since

$$P_{T,Q,\sigma}(A) = 2\hat{P}_{2T,Q,\sigma}(A) - \hat{P}_{T,Q,\sigma}(A),$$

we obtain that

$$\lim_{T\to\infty} P_{T,Q,\sigma}(A) = P_{Q,\sigma}(A), \tag{2}$$

i.e., $P_{T,Q,\sigma}$ converges weakly to $P_{Q,\sigma}$ as $T \to \infty$.

Now, suppose that (2) is true. Then

$$XP_{X,Q,\sigma}(A) = XP_{Q,\sigma}(A) + g_A(X)X,$$

where $g_A(X) \to 0$ as $X \to \infty$. Taking $X = T2^{-j}$ and summing the above equality over $j \in \mathbb{N}$, we obtain, ue of σ-additivity of the Lebesgue measure,

$$\hat{P}_{T,Q,\sigma}(A) = P_{Q,\sigma}(A) + \sum_{j=1}^{\infty} g_A(T2^{-j})2^{-j}. \tag{3}$$

Let $\epsilon > 0$. We fix j_0 such that

$$\sum_{j>j_0} 2^{-j} < \epsilon.$$

Then

$$\sum_{j=1}^{\infty} g_A(T2^{-j})2^{-j} \ll_A \sum_{j\leq j_0} g_A(T2^{-j}) + \epsilon.$$

Thus, taking $T \to \infty$ and then $\epsilon \to 0$, we find

$$\lim_{T\to\infty} \sum_{j=1}^{\infty} g_A(T2^{-j})2^{-j} = 0.$$

This together with (3) shows that

$$\hat{P}_{T,Q,\sigma}(A) = P_{Q,\sigma}(A) + o(1), \quad T \to \infty,$$

i.e., $\hat{P}_{T,Q,\sigma}$ converges weakly to $P_{Q,\sigma}$ as $T \to \infty$.

Since, in the case of $P_{T,Q,\sigma}$ the function $\varphi(t)$ occurs for large values of t, the study of $P_{T,Q,\sigma}$ sometimes is more convenient than that of $\hat{P}_{T,Q,\sigma}$. Therefore, we will prove a limit theorem for $P_{T,Q,\sigma}$.

As in [3], we use the decomposition [6]

$$\zeta(s;Q) = \zeta(s;E_Q) + \zeta(s;F_Q),$$

where $\zeta(s;E_Q)$ and $\zeta(s;F_Q)$ are zeta-functions of certain Eisenstein series and of a certain cusp form, respectively. The latter decomposition and the results of [7], [8]—see also [9]—imply that, for $\sigma > \frac{n-1}{2}$,

$$\zeta(s;Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L\Big(s - \frac{n}{2} + 1, \psi_l\Big) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \tag{4}$$

where $a_{kl} \in \mathbb{C}$, $K, L \in \mathbb{N}$, $L(s,\chi_k)$ and $L(s,\psi_l)$ are Dirichlet L-functions, and the series is absolutely convergent for $\sigma > \frac{n-1}{2}$. Equality (4) is the main relation for investigation of the function $\zeta(s;Q)$. Before the statement of a limit theorem, we construct a $\mathbb{C}$-valued random element connected to $\zeta(s;Q)$.

Let $\mathbb{P}$ is the set of all prime numbers, $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. The infinite-dimensional torus Ω is a compact topological Abelian group; therefore, the probability Haar measure can be defined on $(\Omega, \mathcal{B}(\Omega))$. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the pth, $p \in \mathbb{P}$, component of an element $\omega \in \Omega$, and extend the function $\omega(p)$ to the whole set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, for $\sigma > \frac{n-1}{2}$, define the $\mathbb{C}$-valued random element by

$$\begin{aligned} \zeta(\sigma,\omega;Q) &= \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}\omega(k)\omega(l)}{k^\sigma l^\sigma} L(\sigma,\omega,\chi_k) L\Big(\sigma - \frac{n}{2} + 1, \omega, \psi_l\Big) \\ &+ \sum_{m=1}^{\infty} \frac{f_Q(m)\omega(m)}{m^\sigma}, \end{aligned}$$

where

$$L(\sigma,\omega,\chi_k) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\chi_k(p)\omega(p)}{p^\sigma}\right)^{-1},$$

and

$$L\Big(\sigma - \frac{n}{2} + 1, \omega, \psi_l\Big) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\psi_l(p)\omega(p)}{p^{\sigma - \frac{n}{2}+1}}\right)^{-1}.$$

Now, denote by $P_{\zeta,Q,\sigma}$ the distribution of $\zeta(\sigma,\omega;Q)$, i.e.,

$$P_{\zeta,Q,\sigma}(A) = m_H\{\omega \in \Omega : \zeta(\sigma,\omega;Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

Because $n \geq 4$, $\sigma - \frac{n}{2} + 1 > \frac{1}{2}$ for $\sigma > \frac{n-1}{2}$. Therefore, the second Euler product for Dirichlet L-function is convergent for almost all ω and defines a random variable.

The main the result of the paper is the following theorem.

Theorem 1. *Suppose that $\varphi(t) \in W(T_0)$, $n \geq 4$ and $\sigma > \frac{n-1}{2}$ is fixed. Then $P_{T,Q,\sigma}$ converges weakly to the measure $P_{\zeta,Q,\sigma}$ as $T \to \infty$.*

Since the representation (4) depends on Q, the random element $\zeta(\sigma,\omega;Q)$ depends on Q. Thus, the limit measure $P_{\zeta,Q,\sigma}$ also depends on Q.

2. Some Estimates

We will consider the measure $P_{T,Q,\sigma}$; therefore, we suppose that $t \in [T, 2T]$ with large T. Let χ be a Dirichlet character modulo q, and $L(s,\chi)$ be a corresponding Dirichlet L-function.

Lemma 1. *Suppose that $\varphi(t) \in W(T_0)$ and $\sigma > \frac{1}{2}$ is fixed. Then, for $\tau \in \mathbb{R}$,*

$$\int_T^{2T} |L(\sigma + i\varphi(t) + i\tau, \chi)|^2 \mathrm{d}\tau \ll_{\sigma,\chi,\varphi} T(1 + |\tau|).$$

Proof. It is well known that, for fixed $\sigma > \frac{n-1}{2}$,

$$\int_{-T}^{T} |L(\sigma + it, \chi)|^2 \mathrm{d}t \ll_{\sigma,\chi} T. \tag{5}$$

An application of the mean value theorem, in view of (5), gives

$$\begin{aligned}
I(T,\chi,\sigma) &\overset{def}{=} \int_T^{2T} |L(\sigma+i\varphi(t)+i\tau,\chi)|^2 \mathrm{d}t = \int_T^{2T} \frac{1}{\varphi'(t)} |L(\sigma+i\varphi(t)+i\tau,\chi)|^2 \mathrm{d}\varphi(t) \\
&= \int_T^{2T} \frac{1}{\varphi'(t)} \mathrm{d}\left(\int_T^{\varphi(t)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \right) = \frac{1}{\varphi'(T)} \int_T^{\xi} \mathrm{d}\left(\int_T^{\varphi(t)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \right) \\
&= \frac{1}{\varphi'(T)} \int_{\varphi(T)+\tau}^{\varphi(\xi)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \le \frac{1}{\varphi'(T)} \int_{\varphi(T)-|\tau|}^{\varphi(2T)+|\tau|} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \\
&\le \frac{1}{\varphi'(T)} \int_{-\varphi(2T)-|\tau|}^{\varphi(2T)+|\tau|} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \ll_{\sigma,\chi} \frac{1}{\varphi'(T)} (\varphi(2T)+|\tau|),
\end{aligned}$$

where $T \le \xi \le 2T$ and $\varphi'(t)$ is increasing. Thus, by the definition of the class $W(T_0)$,

$$I(T,\chi,\sigma) \ll_{\sigma,\chi} \frac{\varphi(2T)}{\varphi'(T)} \left(1 + \frac{|\tau|}{\varphi(2T)}\right) \ll_{\sigma,\chi,\varphi} T(1+|\tau|).$$

If $\varphi'(t)$ is decreasing, then similarly we have

$$\begin{aligned}
I(T,\chi,\sigma) &= \frac{1}{\varphi'(2T)} \int_{\xi}^{2T} d\left(\int_T^{\varphi(t)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \right) = \frac{1}{\varphi'(2T)} \int_{\varphi(\xi)+\tau}^{\varphi(2T)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \\
&\le \frac{1}{\varphi'(2T)} \int_{\varphi(2T)+\tau}^{\varphi(2T)+\tau} |L(\sigma+iu,\chi)|^2 \mathrm{d}u \ll_{\sigma,\chi} \frac{1}{\varphi'(2T)} (\varphi(2T)+|\tau|) \\
&\ll_{\sigma,\chi} \frac{\varphi(4T)}{\varphi'(2T)} (1+|\tau|) \ll_{\sigma,\chi,\varphi} T(1+|\tau|).
\end{aligned}$$

□

Let $\theta > 0$ be a fixed number, and

$$v_N(m) = \exp\left\{ -\left(\frac{m}{N}\right)^{\theta} \right\}, \quad m, N \in \mathbb{N},$$

where $\exp\{a\} = \mathrm{e}^a$. Put

$$L_N(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m) v_N(m)}{m^s}.$$

Then, by the exponential decreasing of $v_N(m)$, the latter series is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. Define

$$\zeta_N(s;Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L_N\left(s - \frac{n}{2} + 1, \psi_l\right) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}.$$

Then the series for $\zeta_N(s;Q)$ is absolutely convergent for $\sigma > \frac{n-1}{2}$. It turns out that $\zeta_N(s;Q)$ approximates well in the mean the function $\zeta(s;Q)$. More precisely, we have the following result.

Lemma 2. *Suppose that $\varphi(t) \in W(T_0)$ and $\sigma > \frac{n-1}{2}$ is fixed. Then*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} |\zeta(\sigma + i\varphi(t); Q) - \zeta_N(\sigma + i\varphi(t); Q)| \, \mathrm{d}t = 0.$$

Proof. Let θ be from the definition of $v_N(m)$; $\Gamma(s)$ denotes the Euler gamma-function, and

$$l_N(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)N^s.$$

Then, the Mellin formula

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} \, \mathrm{d}s = \mathrm{e}^{-a}, \quad a, b > 0,$$

leads, for $\theta_1 > \frac{1}{2}$, to

$$L_N(s, \chi) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} L(s + z, \chi) l_N(z) \frac{\mathrm{d}z}{z}. \tag{6}$$

Denote by χ_0 the principal Dirichlet character modulo q. Since the function $L(s, \chi)$ is entire for $\chi \neq \chi_0$, and $L(s, \chi_0)$ has a simple pole at the point $s = 1$ with residue

$$a_q \stackrel{def}{=} \prod_{p|q} \left(1 - \frac{1}{p}\right),$$

the residue theorem and (6) give

$$L_N(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\theta_2 - i\infty}^{-\theta_2 + i\infty} L(s + z, \chi) l_N(z) \frac{\mathrm{d}z}{z} + R_N(s, \chi),$$

where $0 < \theta_2 < 1$ and

$$R_N(s, \chi) = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ a_q \frac{l_N(1-s)}{1-s} & \text{if } \chi = \chi_0. \end{cases}$$

Therefore,

$$\begin{aligned} & |L(\sigma + i\varphi(t), \chi) - L_N(\sigma + i\varphi(t), \chi)| \\ \ll & \int_{-\infty}^{\infty} |L(\sigma - \theta_2 + i\varphi(t) + i\tau, \chi)| \frac{|l_N(-\theta_2 + i\tau)|}{|-\theta_2 + i\tau|} \, \mathrm{d}\tau + |R_N(\sigma + i\varphi(t), \chi)|. \end{aligned}$$

Hence, we have that

$$\frac{1}{T} \int_T^{2T} |L(\sigma + i\varphi(t), \chi) - L_N(\sigma + i\varphi(t), \chi)| \, \mathrm{d}t \ll I_1 + I_2, \tag{7}$$

where

$$I_1 = \int_{-\infty}^{\infty} \left(\left(\frac{1}{T} \int_T^{2T} |L(\sigma - \theta_2 + i\varphi(t) + i\tau, \chi)| \mathrm{d}t \right) \frac{l_N(-\theta_2 + i\tau)}{|-\theta_2 + i\tau|} \right) \mathrm{d}\tau$$

and

$$I_2 = \frac{1}{T}\int_T^{2T} |R_N(\sigma + i\varphi(t), \chi)|\, \mathrm{d}t.$$

It is well known that, uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad |t| \geq t_0 > 0, \quad c > 0. \tag{8}$$

Therefore,

$$\frac{l_N(1-\sigma - i\varphi(t))}{1-\sigma - i\varphi(t)} \ll_\theta N^{1-\sigma} \exp\Big\{-\frac{c}{\theta}\varphi(t)\Big\},$$

and

$$I_2 \ll_{\theta,q} N^{1-\sigma}\frac{1}{T}\int_T^{2T} \exp\Big\{-\frac{c}{\theta}\varphi(t)\Big\}\mathrm{d}t \ll_{\theta,q} N^{1-\sigma} \exp\Big\{-\frac{c}{\theta}\varphi(T)\Big\}. \tag{9}$$

Suppose that $\sigma > \frac{1}{2}$ and θ_2 is such that $\sigma - \theta_2 > \frac{1}{2}$. Then, in view of (8) again,

$$\frac{l_N(-\theta_2 + i\tau)}{-\theta_2 + i\tau} \ll_\theta N^{-\theta_2} \exp\Big\{-\frac{c}{\theta}|\tau|\Big\},$$

Therefore, Lemma 1 implies

$$I_1 \ll_{\theta,\sigma,\theta_2,\chi} N^{-\theta_2} \int_{-\infty}^{\infty} (1+|\tau|) \exp\Big\{-\frac{c}{\theta}|\tau|\Big\}\mathrm{d}\tau \ll_{\theta,\sigma,\theta_2,\chi} N^{-\theta_2}.$$

This, (9) and (7) show that, for fixed $\sigma > \frac{1}{2}$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} |L(\sigma + i\varphi(t), \chi) - L_N(\sigma + i\varphi(t), \chi)|\, \mathrm{d}t = 0.$$

Since $\sigma > \frac{n-1}{2}$, we have $\sigma - \frac{n}{2} + 1 > \frac{1}{2}$. Therefore, for $\sigma > \frac{n-1}{2}$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} \Big|L\Big(\sigma - \frac{n}{2} + 1 + i\varphi(t), \psi_l\Big) - L_N\Big(\sigma - \frac{n}{2} + 1 + i\varphi(t), \psi_l\Big)\Big|\, \mathrm{d}t = 0.$$

Hence,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} |\zeta(\sigma + i\varphi(t); Q) - \zeta_N(\sigma + i\varphi(t); Q)|\, \mathrm{d}t$$

$$\ll_Q \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\sum_{l=1}^{L}\int_T^{2T} \Big|L\Big(\sigma - \frac{n}{2} + 1 + i\varphi(t), \psi_l\Big) - L_N\Big(\sigma - \frac{n}{2} + 1 + i\varphi(t), \psi_l\Big)\Big|\, \mathrm{d}t = 0.$$

□

3. Limit Theorems

We divide the proof of Theorem 1 into lemmas that are limit theorems in some spaces. We start with a lemma on the torus Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_T(A) = \frac{1}{T}\mathrm{meas}\Big\{t \in [T, 2T] : \Big(p^{-i\varphi(t)} : p \in \mathbb{P}\Big) \in A\Big\}.$$

Lemma 3. *Suppose that $\varphi(t) \in W(T_0)$. Then Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We will apply the Fourier transform method. Let $g_T(\underline{k})$, $\underline{k} = \big(k_p : k_p \in \mathbb{Z}, p \in \mathbb{P}\big)$ be the Fourier transform of Q_T, i.e.,

$$g_T(\underline{k}) = \int\limits_{\Omega} \left({\prod_{p\in\mathbb{P}}}^{*} \omega^{k_p}(p) \right) \mathrm{d}Q_T,$$

where "*" indicates that only a finite number of integers k_p are distinct from zero. Thus, by the definition of Q_T,

$$g_T(\underline{k}) = \frac{1}{T} \int\limits_{T}^{2T} {\prod_{p\in\mathbb{P}}}^{*} \Big(p^{-ik_p\varphi(t)} \Big) \mathrm{d}t = \frac{1}{T} \int\limits_{T}^{2T} \exp\left\{ -i\varphi(t) {\sum_{p\in\mathbb{P}}}^{*} k_p \log p \right\} \mathrm{d}t. \tag{10}$$

Obviously,

$$g_T(\underline{0}) = 1. \tag{11}$$

Now, suppose that $\underline{k} \neq \underline{0}$. Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers, we have

$$A_{\underline{k}} \overset{def}{=} {\sum_{p\in\mathbb{P}}}^{*} k_p \log p \neq 0.$$

Then

$$\begin{aligned}
\int\limits_{T}^{2T} \cos\big(A_{\underline{k}}\varphi(t)\big)\mathrm{d}t &= \frac{1}{A_{\underline{k}}} \int\limits_{T}^{2T} \frac{1}{\varphi'(t)} \mathrm{d}\sin\big(A_{\underline{k}}\varphi(t)\big) \\
&= \frac{1}{A_{\underline{k}}} \begin{cases} (\varphi'(T))^{-1} \int\limits_{T}^{\xi} \mathrm{d}\sin\big(A_{\underline{k}}\varphi(t)\big) & \text{if } \varphi'(t) \text{ is increasing,} \\ (\varphi'(2T))^{-1} \int\limits_{\xi}^{2T} \mathrm{d}\sin\big(A_{\underline{k}}\varphi(t)\big) & \text{if } \varphi'(t) \text{ is decreasing} \end{cases} \\
&\ll \frac{1}{|A_{\underline{k}}|} \begin{cases} (\varphi'(T))^{-1} & \text{if } \varphi'(t) \text{ is increasing,} \\ (\varphi'(2T))^{-1} & \text{if } \varphi'(t) \text{ is decreasing,} \end{cases}
\end{aligned}$$

where $T \leq \xi \leq 2T$. Since $\varphi(t) \in W(T_0)$,

$$\begin{cases} (\varphi'(T))^{-1} & \text{if } \varphi'(t) \text{ is increasing,} \\ (\varphi'(2T))^{-1} & \text{if } \varphi'(t) \text{ is decreasing} \end{cases} = o(T)$$

as $T \to \infty$. Therefore,

$$\int\limits_{T}^{2T} \cos\big(A_{\underline{k}}\varphi(t)\big)\mathrm{d}t = o(T), \quad T \to \infty. \tag{12}$$

Similarly, we find that

$$\int\limits_{T}^{2T} \sin\big(A_{\underline{k}}\varphi(t)\big)\mathrm{d}t = o(T), \quad T \to \infty.$$

Thus, (10)–(12) show that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H, the lemma is proved. □

For $A \in \mathcal{B}(\mathbb{C})$, define

$$P_{T,N,Q,\sigma}(A) = \frac{1}{T}\text{meas}\{t \in [T, 2T] : \zeta_N(\sigma + i\varphi(t); Q) \in A\}.$$

To prove the weak convergence for $P_{T,N,Q,\sigma}$ as $T \to \infty$, consider the function $u_{N,\sigma} : \Omega \to \mathbb{C}$ given by the formula

$$u_{N,\sigma}(\omega) = \zeta_N(\sigma, \omega; Q),$$

where

$$\zeta_N(\sigma, \omega; Q) = \sum_{m=1}^{\infty} \frac{w_N(m)\omega(m)}{m^\sigma},$$

and

$$\sum_{m=1}^{\infty} \frac{w_N(m)}{m^s}$$

is the Dirichlet series for $\zeta_N(s; Q)$. Clearly, the above series are absolutely convergent for $\sigma > \frac{n-1}{2}$. The absolute convergence of the series for $\zeta_N(s, \omega; Q)$ implies the continuity for the function u_N. Therefore, the function u_N is $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{C}))$-measurable, and we can define the probability measure $V_{N,\sigma} = m_H u_{N,\sigma}^{-1}$, where

$$m_H u_{N,\sigma}^{-1}(A) = m_H\left(u_{N,\sigma}^{-1} A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

Lemma 4. *Suppose that $\varphi(t) \in W(T_0)$ and $\sigma > \frac{n-1}{2}$ is fixed. Then, $P_{T,N,Q,\sigma}$ converges weakly to $V_{N,\sigma}$ as $T \to \infty$.*

Proof. By the definitions of $P_{T,N,Q,\sigma}$, Q_T and $u_{N,\sigma}$, for all $A \in \mathcal{B}(\mathbb{C})$,

$$P_{T,N,Q,\sigma}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [T, 2T] : \left(p^{-i\varphi(t)} : p \in \mathbb{P}\right) \in u_{N,\sigma}^{-1}\right\} = Q_T\left(u_{N,\sigma}^{-1}\right).$$

Thus, $P_{T,N,Q,\sigma} = Q_T u_{N,\sigma}^{-1}$. Therefore, the lemma is a consequence of Theorem 5.1 from [10], continuity of $u_{N,\sigma}$ and Lemma 3. □

The measure $V_{N,\sigma}$ is very important for the proof of Theorem 1. Since $V_{N,\sigma}$ is independent of the function $\varphi(t)$, the following limit relation is true.

Lemma 5. *Suppose that $\sigma > \frac{n-1}{2}$ is fixed. Then $V_{N,\sigma}$ converges weakly to $P_{\zeta,Q,\sigma}$ as $N \to \infty$.*

Proof. In the proof of Theorem 2 from [4], it is obtained (relation (12)) that $V_{N,\sigma}$ converges weakly to a certain measure P_σ, and, at the end of the proof, the measure P_σ is identified by showing that $P_\sigma = P_{\zeta,Q,\sigma}$. □

For convenience, we recall Theorem 4.2 of [10]. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 6. *Suppose that the space $(\mathbb{X}, \rho)$ is separable, and $\mathbb{X}$-valued random elements $Y_n, X_{1N}, X_{2N}, \ldots$ are defined on the same probability space with measure P. Let, for every k,*

$$X_{kN} \xrightarrow[N\to\infty]{\mathcal{D}} X_k,$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If, for every $\varepsilon > 0$,

$$\lim_{k\to\infty}\limsup_{N\to\infty} P(\rho(X_{kN}, Y_N) \geq \varepsilon\} = 0,$$

then $Y_N \xrightarrow[N\to\infty]{\mathcal{D}} X$.

Proof of Theorem 1. Suppose that ξ_T is a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), P)$ and distributed uniformly in $[T, 2T]$. Since the function $\varphi(t)$ is continuous, it is thus measurable, and $\varphi(\xi_T)$ is a random variable as well. Denote by $X_{N,\sigma}$ the complex valued random element having the distribution $V_{N,\sigma}$, and, on the probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), P)$, define the random element

$$X_{T,N,\sigma} = \zeta_N(\sigma + i\varphi(\xi_T); Q).$$

Then, in view of Lemma 4,

$$X_{T,N,\sigma} \xrightarrow[T\to\infty]{\mathcal{D}} X_{N,\sigma}, \tag{13}$$

and, by Lemma 5,

$$X_{N,\sigma} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\zeta,Q,\sigma}. \tag{14}$$

Define one more complex-valued random element

$$Y_{T,\sigma} = \zeta(\sigma + i\varphi(\xi_T); Q).$$

Then, an application of Lemma 2 gives, for $\varepsilon > 0$,

$$\lim_{N\to\infty}\limsup_{T\to\infty} P\{|X_{T,N,\sigma} - Y_{T,\sigma}| \geq \varepsilon\}$$
$$\leq \frac{1}{\varepsilon T}\int_T^{2T} |\zeta(s + i\varphi(t); Q) - \zeta_N(s + i\varphi(t); Q)| \mathrm{d}t = 0.$$

This, relations (13) and (14) show that all hypotheses of Lemma 6 are satisfied. Therefore,

$$Y_{T,\sigma} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\zeta,Q,\sigma},$$

and this is equivalent to the assertion of the theorem. □

4. Concluding Remarks

By Bohr and Jessen's works, it is known that the asymptotic behavior of the Dirichlet series can be characterized by probabilistic limit theorems. It turns out that Bohr–Jessen's ideas can also be applied for the Epstein zeta-function $\zeta(s; Q)$ whose definition involves a positive defined $n \times n$ matric Q. We prove that, for any fixed $\sigma > \frac{n-1}{2}$,

$$\frac{1}{T}\text{meas}\{t \in [T, 2T] : \zeta(\sigma + i\varphi(t); Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to an explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. Here $\varphi(t)$ is an increasing differentiable function such that

$$\frac{\varphi(2t)}{\varphi'(t)} \ll t, \quad t \to \infty.$$

For example, $\varphi(t)$ can be a polynomials or the Gram function. We recall that the Gram function $g(t)$ is the solution of the equation

$$\theta(\tau) = (t-1)\pi, \quad t \geq 0,$$

where $\theta(\tau)$ is the increment of the argument of the function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ along the segment connecting the points $s = \frac{1}{2}$ and $s = \frac{1}{2} + i\tau$. It is known [11] that

$$g(t) = \frac{2\pi t}{\log t}(1 + o(1))$$

and

$$g'(t) = \frac{2\pi}{\log t}(1 + o(1))$$

as $t \to \infty$.

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References

1. Epstein, P. Zur Theorie allgemeiner Zetafunktionen. *Math. Ann.* **1903**, *56*, 615–644. [CrossRef]
2. Bohr, H.; Jessen, B. Über die Wertverteilung der Riemanschen Zetafunktion, Zweite Mitteilung. *Acta Math.* **1932**, *58*, 1–55. [CrossRef]
3. Laurinčikas, A. *Limit Theorems for the Riemann Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1996.
4. Laurinčikas, A.; Macaitienė, R. A Bohr-Jessen type theorem for the Epstein zeta-function. *Results Math.* **2018**, *73*, 147–163. [CrossRef]
5. Laurinčikas, A.; Macaitienė, R.; Šiaučiūnas, D. Generalization of the Voronin Theorem. *Lith. Math. J.* **2019**, *59*, 156–168. [CrossRef]
6. Fomenko, O.M. Order of the Epstein zeta-function in the critical strip. *J. Math. Sci.* **2002**, *110*, 3150–3163. [CrossRef]
7. Hecke, E. Über Modulfunktionen und die Dirichletchen Reihen mit Eulerscher Produktentwicklung. I, II. *Math. Ann.* **1937**, *114*, 1–28. 316–351 . [CrossRef]
8. Iwaniec H. *Topics in Classical Automorphic Forms, Graduate Studies in Mathematics*; American Mathematical Society: Providence, RI, USA, 1997; Volume 17.
9. Nakamura, T.; Pańkowski, Ł. On zeros and *c*-values of Epstein zeta-functions. *Šiauliai Math. Semin.* **2013**, *8*, 181–195.
10. Billingsley, P. *Convergence of Probability Measures*; Willey: New York, NY, USA, 1968.
11. Korolev, M.A. Gram's law in the theory of the Riemann zeta-function. Part 1. *Proc. Steklov Inst. Math.* **2016**, *292*, 1–146. [CrossRef]