Subdivision and G-spline hybrid constructions for high-quality geometric and analysis-suitable surfaces

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Abstract. This survey of piecewise polynomial surface constructions for filling multi-sided holes in a smooth spline complex focusses on a class of hybrid constructions that, while heterogeneous, combines all the practical advantages of state-of-the-art for modelling and analysis: good shape, easy implementation and simple refinability up a pre-defined level.
After reviewing the three ingredients – subdivision, G-spline and guided surfaces – the hybrid is defined to consist essentially of one macro-patch for each of n sectors, leaving just a tiny n-sided central hole to be filled by a G-spline construction. Here tiny means both geometrically small, e.g. two orders of magnitude smaller than pieces of the spline complex, and small in its contribution to engineering analysis, i.e. it is unlikely to require further refinement to express additional geometric detail or resolve a function on the surface, such as the solution of a partial differential equation.
Each macro-patch has the local structure of a subdivision surface near, but excluding the central, extraordinary point: all internal transitions are the identity or scale according to contraction speed toward the extraordinary point. Both the number of pieces of the macro-patches and the speed can be chosen application-dependent and adaptively.

Keywords: free-form spline surface, subdivision surface, G-spline, guided surface, accelerated subdivision, refinability

1 Introduction

A hybrid construction for filling an n-sided hole in a spline complex consists of two parts: a main body formed by several nested surface rings and a tiny central n-sided cap, see Fig. 1, b,h. Each ring consists of n L-shaped sectors that fit together smoothly at their tips to form a curved annulus, called surface ring. The L-shape also joins smoothly with a smaller L-shaped sector of the next ring, as illustrated in Fig. 2. The surface rings form a nested sequence that contracts rapidly towards the center. Alternatively, one can think of each L-shape as a 4-sided macro-patch with one quadrant punched out to leave space for the next smaller L-shape, i.e. their union is a spline complex with a hole. This is the setup of subdivision algorithms [82] near extraordinary points, except that the process in the hybrid construction stops after a few steps, e.g. 2 or 3 nested surface rings.

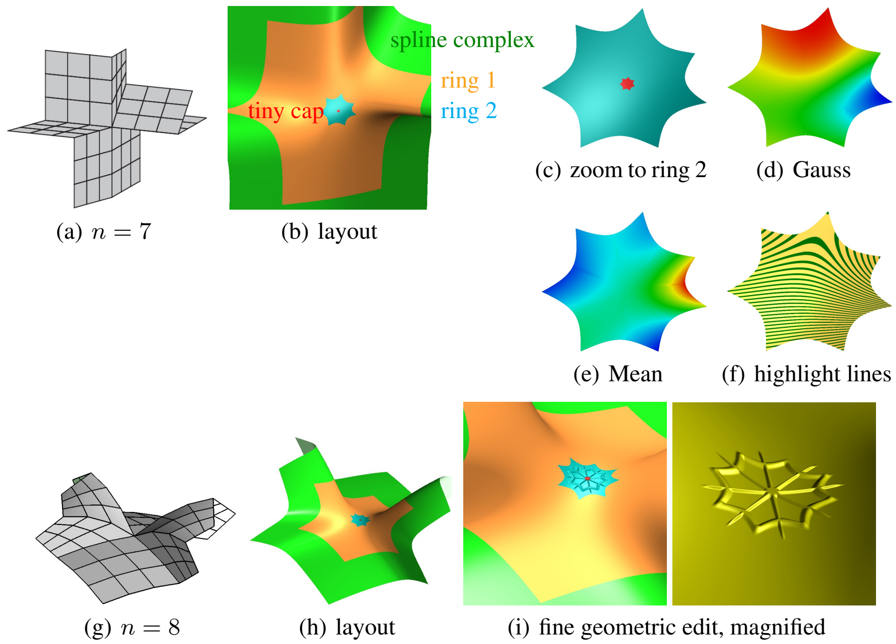


Fig. 1. Examples of high-valence hybrid surfaces. (a,g) Generalized control net (**c**-net). (b,h) layout of surface rings with almost invisible tiny cap. (c–f) Shape interrogation of ring 2 and the tiny cap: (d) Gauss curvature, (e) mean curvature. (i) Localized degrees of freedom used to emboss a pattern [46].

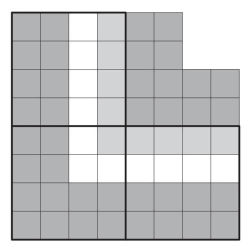


Fig. 2. L-shaped sectors forming a macro-patch (fine lines form a bi-quartic (bi-4) BB-net [20].

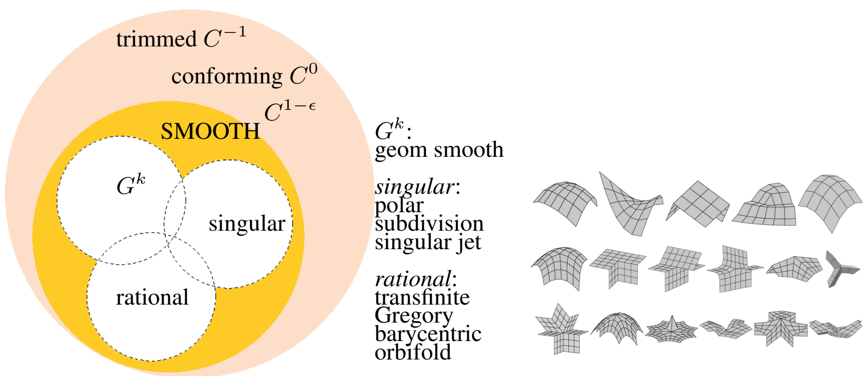
The pieces join to form the main body of the surface without change of variables, except possibly scaled derivatives as we adjust the contraction 'speed' of the rings. The main body of the surface is therefore amenable to refinement by knot insertion to increase of degrees of freedom. The same holds for functions on the surface, such as textures.

Only the tiny central n-sided cap is assembled by joining pieces so that derivatives agree after a change of variables (reparameterization), i.e. the cap is G^k continuous. The cap can be one of many in the literature, see Section 4. Typically the cap is so small that it needs not be refined. If it needs to be refined, [55] provides a refinable G^2 tiny cap construction.

The hybrid structure seems to just delay the multi-sided hole-filling challenge. The motivation for this seemingly more complex approach is that, if the cap is chosen to be smaller than the maximally anticipated refinement – for geometric modeling or computing on surfaces – then all surface pieces share a simple parameterization for refinement. On the other hand, capping the rings avoids the infinite recursion and central singularity of subdivision surfaces that necessitate smart data structures and special algorithms for proper integration over infinitely many polynomial pieces. By construction a hybrid surface is therefore 'analysis-suitable', in the sense that it offers a uniform increase in degrees of freedom by knot insertion – up to the refinement level of the tiny cap (and further, at some cost in complexity, if the cap is chosen to be refinable).

To ensure that the hybrid constructions are 'geometry-suitable', i.e. yield good shape, a third technique is applied: both the main body and the cap follow a guide surface. A guide surface need not match the surrounding spline complex but is sampled ever closer to the central point to guarantee that subsequent rings and the tiny cap do not freely oscillate but follow a consistent, structurally simpler shape.

Overview. Section 2 classifies constructions for filling n-sided holes and explains requirements for geometric design and engineering analysis. Section 3 reviews subdivision algorithms, Section 4 reviews G-splines, and Section 5 reviews guided constructions, the three ingredients from which subdivision-and-G-spline-hybrids are built in Section 6.



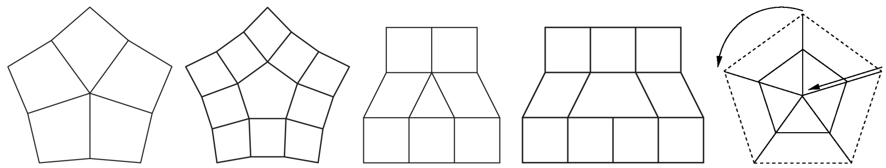
(a) Classification of constructions

(b) quad-net obstacle course

Fig. 3. (a) Categories of surface parameterizations for irregular configurations from [80]. (b) Examples of meshes in the quad-net obstacle course [37] that serve as test control nets for the geometric suitability of constructions analogous to the finite element obstacle course for engineering analysis.

2 Categories of and requirements for surfaces with irregular configurations

Fig. 3(a) proposes a partition of the universe of spline constructions for irregular configurations. The main irregular configurations are illustrated in Fig. 4: star-configurations, multi-sided facets, T-junctions and polar configurations. The partition groups by smoothness, polynomiality and singularity of the surface parameterization. The topmost entry, *trimmed NURBS* surfaces, is the de facto industry standard. Trimming means restricting the domain of surface pieces by curves in the domain. In styling software, polynomial



(a) star-net (b) n-gon-net (c) τ_0-net (d) τ_1-net (e) polar-net

Fig. 4. Locally quad-dominant mesh patterns (see [54]). The valence in (a,b,e) is $n = 5$. The pentagon in (d) is a T_1-gon, the triangle in (c) is a T_0-gon.

pieces are typically laid down to capture primary shape, cut back where they overlap and blended by fillet transition surfaces. This approach preserves simple, elegant shape in the large but ultimately forces stylists to devote ever more time to ever smaller blends between the primary surfaces [105].

Conforming C^0 surfaces, including curved triangulations, are ubiquitous and sufficient for basic engineering analysis (volume, homogeneous material moments, linear elasticity, etc.) and in computer graphics, where visual appearance trumps accurate geometry, [101, 64, 28]. Generalized barycentric patches, e.g. [65, 94], and non-4-sided transfinite constructions, e.g. [15, 86, 99, 100, 88], cover multi-sided holes with single, typically C^∞ patches. Due to an underlying very high rational degree, exactly computing their higher-order derivatives is costly. Due to its focus on free-form surfaces, the classification leaves out analysis-suitable functions and their graphs, e.g. [95, 33]. Subdivision surfaces and G^k constructions will be discussed individually in the Section 3 and Section 4.

2.1 Requirements on the geometry

The quality of a surface is a subjective term. Two tools are commonly used to assess geometric quality: 'curvature profiles' superimpose the curvature function of planar cuts orthogonal to the surface (Fig. 1d,e show the less common shading of the whole surface by Gauss and Mean curvature); and 'highlight lines' [6] approximate the effect of parallel arrangement of tube lights in a car show room. Unless explicitly intended as a surface feature, artifacts such as abrupt changes in the distribution of curvature or highlight lines are not wanted because they distort reflections and make the product appear less well designed. When the surfaces generated from the obstacle course quad-meshes (see Fig. 3(b)) avoid artifacts, the construction will be deemed *geometry-suitable*. The automotive design industry also uses the term 'Class A surface' [104] to describe spline surfaces with aesthetic, non-oscillating highlight lines. Class A surfaces satisfy, depending on the application area and contractual agreement, certain hard geometric constraints [2]. Remarkably, they allow a mismatch of normals across curves between two surface pieces up to one tenth of a degree, justifying the surface category $C^{1-\epsilon}$ in Fig. 3(a).

2.2 The isogeometric approach

Higher orders of continuity of the solution space are required for correctly solving higher-order differential equations such as the biharmonic equation, Kirchhoff-Love shell formulations [9, 8, 58], the Cahn-Hilliard phase-field model [23] or simplified Navier-Stokes-Korteweg equations. C^1-continuous finite element spaces are needed even when these problems are expressed in the weak formulation [103], e.g. via Galerkin's discretization for solving a partial differential equation [102, 75]. Where the partitions deviate from regular lattices, classical finite elements join typically only C^0. Ignoring the required differentiability between elements and computing piecemeal in the larger C^0 space can give rise to extraneous solutions that do not correspond to physical solutions. For other classes of differential equations, smoothness, while not formally necessary, has been found to improve accuracy, stability or convergence (see e.g. for contact problems [68, 97]).

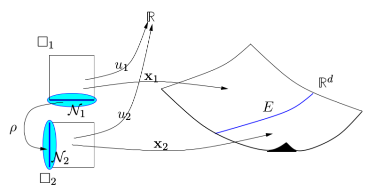


Fig. 5. Atlas $\mathbf{x}$ and analysis functions u_α in the formulation of the fundamental theorem: elements on manifolds $b_i(\mathbf{x}_\alpha) := u_\alpha \circ \mathbf{x}_\alpha^{-1}$ built with geometric continuity across an interface E with the corresponding edges of the domains $\square_1$ and $\square_2$ related by the change of variable ρ.

For a grid-like layout of quadrilateral surface pieces, the practice of using bi-variate splines both for modeling the geometry of the domain and for computing functions on this domain goes back at least to the 1980s [13]. To emphasize the use of B-splines for both geometry and engineering analysis, the term iso-geometric analysis (IGA) was coined in the 2005 publication [29]. Publications that developed the isogeometric approach in the 1980s and 1990s used the less specific and less memorable terms 'higher-order', 'isoparametric' or 'finite elements using NURBS' [93, 3, 4, 91].

A fundamental theorem, [26], asserts that generalized splines whose irregular layout captures free-form shape of surfaces, e.g. G-splines, can directly serve to define isogeometric finite elements. The composition of maps (where $\mathbf{x}_\alpha^{-1}$ is the pull back of $\mathbf{x}_\alpha$) that define the finite elements is illustrated in Fig. 5. G-spline implementations in [72, 35, 73, 74, 89] have confirmed this fact by solving problems of the classical 'finite element obstacle course' [7].

2.3 Requirements for analysis: flexibility-increasing refinability

Examples of functions on surfaces are textures in graphics and deformation stress in engineering analysis. Both may need to have increased resolution, even when the surface remains the same. [16] shows that C^2-connected bi-cubics have a sub-optimal approximation order in the presence of extraordinary points and, more generally, that refinement of the domain (h-refinement) in the presence of non-trivial reparameterizations fuses the refined polynomial surface pieces of degree p when the continuity is C^{p-1}. Similarly, in the context of geometric modeling, [79] proved that for bi-3 (bicubic) spline patches the interdependence of partial derivatives forces a minimum separation of the extraordinary points when polynomial pieces are joined G^1 or else they fuse. ([16] uses the term 'locking' for the artificial algebraic stiffness; while evocative this term already has a fixed meaning in the thin-shell community). An alternative term for flexibility-increasing refinability is *analysis-suitability*, hence the title of this paper.

To formalize flexibility-increasing refinability, we follow the exposition of [55] and define smoothness between non-overlapping pieces of manifolds as follows.

Definition 1 **(G^κ constraints).** *Two regular and sufficiently smooth surface pieces* $\tilde{\mathbf{f}}, \mathbf{f} : (u,v) \in \mathbb{R}^2 \to \mathbb{R}^d$ *that share a boundary curve* $\mathbf{e}$ join G^κ along $\mathbf{e}$ *if there exists a suitably oriented and non-singular reparameterization* $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ *so that the partial derivatives* $\partial^k \tilde{\mathbf{f}}$ *and* $\partial^k(\mathbf{f} \circ \rho)$, $k = 0, 1, \ldots, \kappa$, *agree along* $\mathbf{e}$.

Let, according to the focus of this survey, the surface pieces be sufficiently smooth piecewise polynomials, e.g. tensor-product splines, and fix the reparameterization $\rho_{\mathbf{e}}$ for every edge $\mathbf{e}$. Then the splines joined with these reparameterizations form a space G_ρ linear in the unconstrained polynomial coefficients. That is, any linear combination of elements in G_ρ associated with these free coefficients (set to 1 and all others to 0) is again in G_ρ.

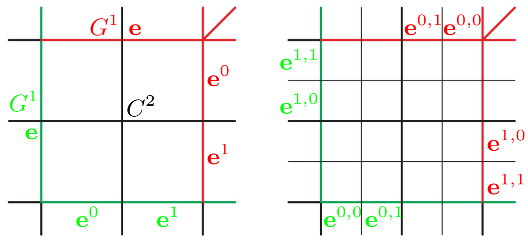
The space G_ρ is (binarily) *refinable* to a space $\dot{G}_{\dot{\rho}}$ in the following sense, see Fig. 6. For each piece $\mathbf{f} \in G_\rho$ restricted to $\square := [0..1]^2$, the space $\dot{G}_{\dot{\rho}}$ has four polynomial pieces $\mathbf{f}^r$, $r = 1, 2, 3, 4$ (wlog. of the same degree as $\mathbf{f}$) defined on the four quarters of $\square$ and joined by the following reparameterizations $\dot{\rho}$:

$$\dot{\rho}_{\mathbf{e}^0}(u,v) = \rho_{\mathbf{e}}(\frac{u}{2}, \frac{v}{2}), \quad \dot{\rho}_{\mathbf{e}^1}(u,v) = \rho_{\mathbf{e}}(\frac{1}{2} + \frac{u}{2}, \frac{v}{2}).$$

Then there is a choice of $\mathbf{f}^r$, namely applying de Casteljau's algorithm to $\mathbf{f}$ at $u = v = 1/2$, so that any element $\mathbf{f} \in G_\rho$ can be represented in $\dot{G}_{\dot{\rho}}$. That is, $\dot{G}_{\dot{\rho}}$ refines G_ρ. However, $\dot{G}_{\dot{\rho}}$ is a larger space than G_ρ since many other choices of macro-patches $\mathbf{f}^r$ are allowable. $\dot{G}_{\dot{\rho}}$ can therefore be expected to provide more flexibility than G_ρ. The additional degrees of freedom are new coefficients not constrained by enforcing smoothness.

Definition 2 **(flexibility-increasing refinable).** *A construction is* flexibility-increasing refinable *if, for each domain piece* $\square$, $\dot{G}_{\dot{\rho}} \supsetneq G_\rho$ *has more degrees of freedom than* G_ρ, *both along map boundaries and in the interior.*

Since all new internal transitions arising from refinement must be parametrically C^2 to reproduce the original polynomial pieces by the finer construction, macro-patches are internally C^k rather than G^k, see the black transitions in Fig. 6 are C^k. Having covered the requirements, we now survey in more detail the three ingredients of hybrid constructions: subdivision, G-spline and guide surfaces.



(a) initial (b) refined

Fig. 6. Illustrating flexibility-increasing refinement of a macro-patch. To be flexibility-increasing, new degrees of freedom must appear both in the interior and along the red sector-separating curve boundaries and the darkgreen boundaries that connect the cap to the remaining surface. Second superscripts in (b) enumerate the (second) splitting of the original edges into parts 0 and 1.

3 Subdivision surfaces

More than 40 years ago Catmull and Clark [14] suggested a simple way to smoothly approximate quad meshes by an infinite sequence of nested rings of bi-3 splines; and Doo and Sabin [19] independently generalized bi-2 splines while deriving important mathematical machinery to analyze the resulting surfaces. Following the landmark papers [18, 96] that demonstrated industry support, Catmull-Clark surfaces are nowadays the tool of choice for computer animation. More recently, an efficient set of open source libraries for subdivision on massively parallel CPU and GPU architectures has been agreed upon by Pixar and Microsoft [83] and used by Pixar's proprietary animation system.

Closer mathematical analysis shows that Catmull-Clark surfaces have systemic shape deficiencies that preclude their use in high-end modeling for manufacturing [56, 49]: limits of convex meshes become hyperbolic surfaces, transitions at T-junctions are unduly flat and saddle-like configurations result in undesirable pinched highlight lines. Indeed, distortion of highlight lines for higher valences is a challenge for two members of the category 'singular' in Fig. 3(a): subdivision surfaces and singular jet surfaces. (Only polar surfaces, where one edge is collapsed into a pole, thrive on high valence. Subdivision surfaces can be thought of as collapsing patch size and singular jet surfaces as collapsing the Taylor expansion at a vertex[77, 84, 85, 74, 98]) Indeed, the monograph on the mathematics of subdivision surfaces characterizes subdivision surfaces near irregularities as spline surfaces with singularities [82].

Optimizations of parameters and prescription of the expansion at the extraordinary central point [34, 5, 61, 69] have improved shape outcomes by making the subdivision matrix less sparse. (The subdivision matrix maps control nets surrounding an irregular node to a contracted control net, each of which defines a surface ring, see Fig. 7.) Guided subdivision [38, 50] stabilizes the shape at the cost of a yet denser subdivision matrix. This enables accelerated contraction towards the extraordinary point without noticeable harm to the shape [48]. For example, one high speed contraction step can shrink the remaining hole by more than two steps of Catmull-Clark subdivision as illustrated in Fig. 11. Accelerated, guided subdivision is a key ingredient of hybrid constructions.

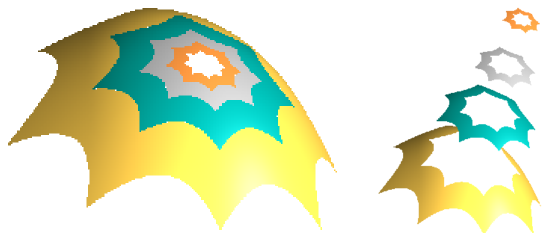


Fig. 7. (*left*) Subdivision surface (from [38]) built from (*right*) a sequence of contracting surface rings.

4 G-splines

While early publications [10, 87, 30, 17, 62, 11, 21] hint at the potential of polynomial pieces joined by a change of variables to fill multi-sided holes in a spline complex, it is arguably Hahn's use of geometric continuity [27] that set the standard for the first generation of G-spline constructions in the 1990s. The focus of the early constructions was on varying the parameterization ρ of Eq. (1) to allow different relations between patch derivatives at irregular non-4-valent points versus regular ones. Expansion by the chain rule of differentiation yields for example G^1 and G^2 constraints in terms of univariate scalar maps $a, b, d, e : u \in \mathbb{R} \to \mathbb{R}$ (partial derivatives of the two coordinates of ρ evaluated on the edge parameterized by $(u, 0)$) and the vector-valued functions $\mathbf{f}$, $\tilde{\mathbf{f}}$ evaluated at $(u, 0)$:

$$\partial_v \tilde{\mathbf{f}} = a\, \partial_v \mathbf{f} + b\, \partial_u \mathbf{f}, \tag{1}$$

$$\partial^2_{vv} \tilde{\mathbf{f}} = a^2\, \partial^2_{vv} \mathbf{f} + 2a\, b\, \partial_u \partial_v \mathbf{f} + b^2\, \partial^2_{uu} \mathbf{f} + d\, \partial_v \mathbf{f} + e\, \partial_u \mathbf{f}. \tag{2}$$

Since neither $\mathbf{f}$ (the rules for constructing the polynomial pieces) nor ρ (the reparameterization) are known a priori, the constraints define a large non-linear space to explore. Considering ρ free to choose increases the space of possible construction. However all sufficiently smooth constructions have to obey a constraint that arises from the circular arrangement of patches surrounding a point, the vertex enclosure constraint [76]. Moreover, the degree of ρ is bounded by the degree of the surface pieces [76] due to what is now understood to be a syzygy relation, see e.g. [70]. Moreover, to create a manifold, the composition of n copies of ρ (across each of the sector-separating curves around an interior point) has to form the identity map, up to the degree of smoothness [78].

Choosing ρ to Hermite interpolate and so separate the computation at either end of an edge between two patches, simplifies solving the system of equations in the polynomial coefficients of Eqs. (1) and (2) but results in many unconstrained coefficients: Hahn and Gregory's early G^2 construction used pieces of degree 18 in u and in v (bi-18) [24] and [106, 57] of degree bi-9. Modern G-splines include G^2 constructions of degree as low as bi-5. Throughout the 1990s work focused on optimizing ρ to minimize the polynomial degree of the surface. Although compatible with the NURBS standard adopted by the manufacturing industries, the shape of early G-spline constructions was often worse than that of subdivision surfaces.

The first decade of this millennium saw improved shape, e.g. [63, 66] still of degree bi-7. Also the distinction between smoothness in the large, assessed via highlight lines, and infinitesimal smoothness, measured as matching derivatives came into focus: a formally only G^1 construction of degree bi-5 or bi-4 [42, 49] showed better curvature distribution than higher degree curvature continuous constructions; and, remarkably, $C^{1-\epsilon}$ construction in [41] trades a slight mismatch in the normal (still within the bounds of class A surfaces) for good highlight line distributions over the multi-sided patch constructed from a finite number of bi-3 patches; the quality of the highlight line distributions could not be matched by the subdivision surfaces of the time (except of the guided variety, see e.g. [47] and Section 5). Additionally, the degree of formally G^2 surfaces with a single patch per sector was reduced to bi-6 [44]. and to degree bi-5 for 2×2 macro-patches [40]. Special scaffold- and sphere-like configurations even allow for curvature continuous bi-4 constructions [43].

A next step forward for G-splines was to allow not only irregular points but also T-junctions as part of a generalized B-spline control net for free-form modeling [36]. T-junctions occur where two quads on one side meet one facet on the other and serve in polyhedral modeling to start or stop quad-strips and so increase or decrease the number of free points to be set. T-junctions prominently feature in quad-dominant remeshing (see e.g. [1, 60, 31, 90]) where they allow to side-step the otherwise stringent global quad-meshing constraints (see e.g. [32, 12, 71]). Also popular in this context are (isolated) triangles that merge and so reduce the number of quad-strips, see Fig. 8. The novelty is that all mesh nodes act as coefficients of linear combinations of piecewise polynomials, i.e. as B-spline-like control points.

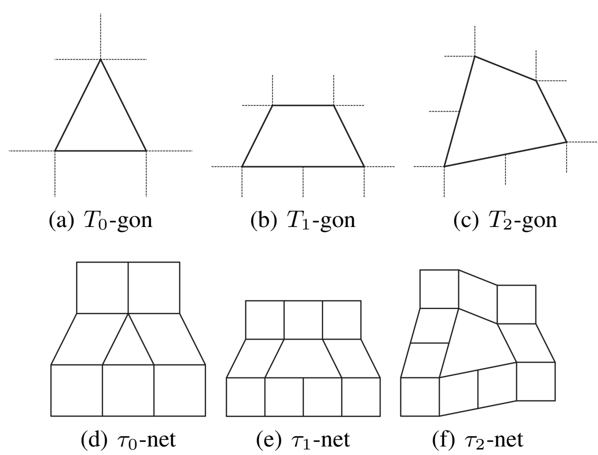
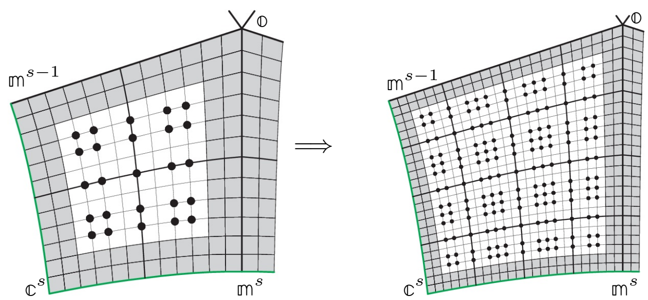


Fig. 8. T-gons and τ-configurations. The subscript counts the number of T-junctions.

While G-splines for multi-sided holes or generalized subdivision can, in principle, convert quad-dominant meshes with T-junctions into smooth surfaces, they do not preserve the two preferred directions and so cause visible shape artifacts. Hierarchical and T-splines such as [59, 92, 22] need to carefully coordinate knot intervals to admit meshes with T-gons as control nets. For many meshes a globally consistent choice of intervals is impossible [36]. That is, these approaches excel at refining tensor-product patches, but may not be able to produce a smooth surface from a given polyhedral mesh including T-junctions.

The G^1 constructions [51, 53, 52] differ in their polynomial degree, their flexibility-increasing refinability and how close their τ-nets can be placed to each other and to irregular nodes. Built from pieces of degree 3×5, [51] is suitable for modeling class A surfaces. And by complementing the existing G-spline constructions for multi-sided facets, splines for the τ-configurations of Fig. 8 allow interpreting locally quad-dominant meshes as spline control nets.



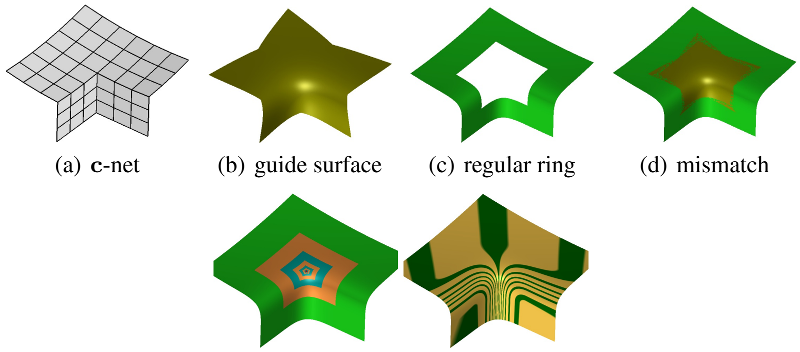
(a) initial 2×2 bi-6 G^2 (b) once-refined bi-6 G^2

Fig. 9. Internal degrees of freedom of refinement of [55] marked as •.

Since the majority of high-end surface constructions now are G-splines, their flexibility-increasing refinability has come under scrutiny. The detailed analysis of flexibility-increasing-refinement of [42] showed that, unlike for tensor-product splines, the resulting unconstrained new coefficients (degrees of freedom) are not convenient geometric handles due to their irregular distribution and support [45]. Indeed the full characterization of G^2 flexibility-increasing refinability in [55] proves that multi-sided refinement of G-splines can not be flexibility-increasing when the construction uses bi-5 macro-patches, regardless of the number of $N \times N$ pieces. Conversely, [55] exhibits a bi-6 construction with 2×2 pieces per sector that is G^2 flexibility-increasing refinable, see Fig. 9. (A bi-4 publication, currently under review, presents parallel results for the construction of multi-sided G^1 surfaces.)

5 Guided surfaces

Guided Subdivision [38] is an effective tool to overcome the defects of standard subdivision algorithms. Moreover, unlike standard subdivision, guided subdivision can easily define curvature continuous subdivision surfaces with controllable polynomial reproduction at the limit point. Fig. 10 illustrates the underlying principle: the control mesh

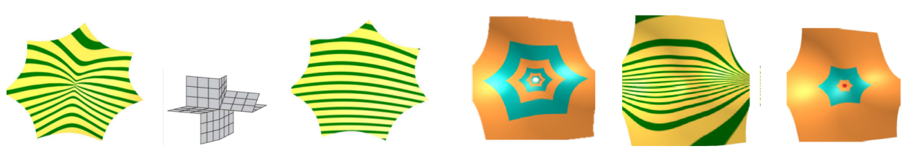


(a) **c**-net (b) guide surface (c) regular ring (d) mismatch

(e) guided rings

Fig. 10. Guided Subdivision. At each step, the preceding ring provides a Hermite prolongation to set the outer coefficients of the next, nested ring while sampling the guide shape yields the inner coefficients.

(a) defines a guide surface (b) and the Hermite prolongation of the surrounding regular ring (c). Since the guide surface and the surrounding surface frame do not fit together as illustrated in (d), the subdivision step retains the outer part of the ring that fits its predecessor ring and determines its new inner part by sampling the guide. The result is shown in (e). We note that the guide surface can have a different structure, smoothness and polynomial degree than the final surface. For example, it can consist of 3-sided patches, whereas the final surface consists of 4-sided patches. Remarkably the overall process is linear and stationary and can so be interpreted as subdivision with large stencil, i.e. with a denser subdivision matrix than Catmull-Clark subdivision. While the rules become more complex, the mathematical analysis of the limit becomes much simpler – and the shape is far better [49].



(a) CC (b) $n = 7$ (c) Guided (d) 6 rings (e) shared (f) 3 rings

Fig. 11. Guided Subdivision and Accelerated Guided Subdivision. The highlight line distributions for input **c**-net (b) for (a) Catmull Clark subdivision and (c) Guided Subdivision. For a different input $n = 6$, surfaces (d) and (f) have visually the same highlight line distribution shown in (e).

Moreover, since guided subdivision stabilizes the shape, an additional technique can be leveraged: *accelerated* guided subdivision [81, Sect 5], [48] widens the surface rings and so achieves a more rapid contraction of the remaining gap, see Fig. 11. When the surface rings and final cap follow a guiding shape, good highlight line distributions are obtained for the shape obstacle course Fig. 3(a), already when the accelerated sequence of C^k-joined surface rings is G^1 completed after 2 or 3 rings, see Fig. 11d, e.

We note that guides can equally well be applied to finite constructions in order to harness excess free parameters. In hybrid surfaces, the same guide surface is applied both to accelerated subdivision rings and the final tiny cap.

6 Hybrid surfaces

While guided subdivision improves shape, it does not address the problem of infinite recursion. Stopping the recursion after a few steps and filling the remaining hole with a triangulated multi-sided facet leads to noticeable flaws in the highlight line rendering. A better solution is to fit a multi-sided G^1-cap. Since the resulting surface consists of a fixed number of surface pieces, it is industry compatible. To combine the best features of subdivision and geometrically continuous surface constructions, we observe that in practice, design and analysis can often predict a maximal level of refinement. When the maximal anticipated refinement level at the irregularity suffices, the cap need not be refined and refinement in the surrounding finite coarser surface requires only standard spline knot insertion [46]. If the anticipated level does not suffice, it is easy to reconstruct with additional surface rings – or once can use a more complex geometrically smooth flexibility-increasing-refinable construction such as Fig. 9.

The earliest hybrid construction, [41], addressed whether bi-cubic surfaces can be class A. The construction, of a main body and cap of degree bi-3, is formally only C^0, with empirically less than a $0.1°$ normal mismatch between the surrounding surface and the main body. The default uses just three pieces per L-shape and one piece for the limit. The results, summarized visually in [41, Fig. 1] show the hybrid to noticeably improve on methods that yield a formally smoother result [14, 25, 67]. The focus of [47] is on improving the shape of subdivision surfaces by constructing both a C^2 subdivision algorithm generating surfaces of polynomial degree bi-6. An appendix adds G^1 parameterized tiny cap. The full concept of a hybrid construction, including acceleration to obtain few pieces, is realized in [46] (see Fig. 1). In [48] the degree of the C^2 main body is reduced to bi-5. In [49], see Fig. 2, the degree is bi-4 using macro-patches. The approaches use guide surfaces built from n three-sided patches with high smoothness at the central point. Finally [55] proves that the central caps of [46, 48, 49] are not flexibility-increasing refinable and presents a bi-6 tiny cap that both completes the surface G^2 to be is flexibility-increasing refinable. For practical purposes the latter may be more than needed: G^1 suffices up to fourth order differential operators and good geometric shape can be guaranteed by tiny G^1 caps whose refinement is much simpler, pointing to [48] as the method of choice.

The key ingredient of hybrid constructions is the guide surface and the sampling, via characteristic maps Fig. 12, of Hermite data along the sequences of boundary curves of the patches. Presented in BB-form these Hermite data are called tensor-borders. For details on the characteristic maps see [48]. The rapid contraction of the guided rings means that the resulting macro-patches consist of few pieces per sector, e.g. of seven pieces: three for each of two polynomial L-shapes and one for the central G-spline patch. This yields smooth surfaces consisting of a finite number of pieces whose for-

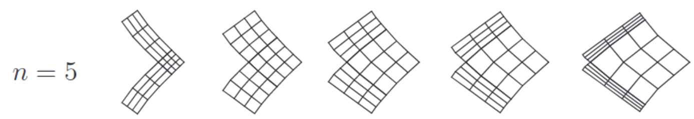


Fig. 12. A sector of the characteristic map for $n = 5$ and speeds $\frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{7}{8}$ from [39].

mulas derive linearly from the input control net, hence can be implemented, for fixed contraction speed and valence, as a single matrix multiplication applied to the local control net. In practice, for [48], one splits the matrix into a 21×31 matrix for creating a sector of the guide, a 21×21 matrix for the de Casteljau steps that re-represent the guide for a contracted domain, a 27×21 matrix defining one sector of the prolongation of the L-shape and 36×21 matrix for one sector of the tiny cap.

7 Conclusion

In order to advertise a useful class of piecewise polynomial surface constructions for filling multi-sided holes in a smooth spline complex, we surveyed subdivision, G-spline and guided surfaces. Combining accelerated guided subdivision with a tiny G-spline cap yields a number of advantages for modelling and analysis: good shape, easy implementation and simple refinability up to a pre-defined level.

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