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# Pathwise Convergent Approximation for the Fractional SDEs

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**Abstract:** Fractional stochastic differential equation (FSDE)-based random processes are used in a wide spectrum of scientific disciplines. However, in the majority of cases, explicit solutions for these FSDEs do not exist and approximation schemes have to be applied. In this paper, we study one-dimensional stochastic differential equations (SDEs) driven by stochastic process with Hölder continuous paths of order $1/2 < \gamma < 1$. Using the Lamperti transformation, we construct a backward approximation scheme for the transformed SDE. The inverse transformation provides an approximation scheme for the original SDE which converges at the rate $h^{2\gamma}$, where $h$ is a time step size of a uniform partition of the time interval under consideration. This approximation scheme covers wider class of FSDEs and demonstrates higher convergence rate than previous schemes by other authors in the field.

**Keywords:** stochastic differential equations; fractional Brownian motion; backward approximation; Lamperti transformation

**MSC:** 60G22; 60H10; 60H05

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## 1. Introduction

Stochastic differential equations (SDEs) are used as a modeling tool in many fields of science. Currently, a lot of research is being conducted on models with fractional Brownian motion (fBm) $B^H$, $0 < H < 1$, since fBm introduces a memory element, which provides new and promising modeling possibilities. It should be noted that the definition of SDEs driven by fBm differs substantially from the definition used by standard Brownian-motion-driven SDEs. Furthermore, there is more than one way to define these SDEs (for example, in our paper, we will use the Riemann–Stieltjes integral to achieve this). The general conditions under which the fractional diffusion process has a unique solution were obtained in [1]. These conditions impose more strict restrictions on the coefficients than in the case of SDEs driven by standard Brownian motion (sBm).

Classical financial models such as Chan–Karoli–Longstaff–Sanders (CKLS), Cox–Ingersoll–Ross (CIR), and Ait–Sahalia are defined using SDEs driven by standard Brownian motion. Replacing the standard Brownian motion with a fractional Brownian motion, we obtain a fractional analogue of these classical models. In many of these models, the solution positivity is a desirable property. The positivity of fractional CIR model was studied in [2,3]. The conditions under which fractional CKLS, Ait–Sahalia and other models have positive solutions follow from [4–6]. The above-mentioned financial models have a strictly positive diffusion coefficient.

In this paper, we shall consider a one-dimensional SDE driven by an arbitrary stochastic process $Z = (Z_t)_{t \geqslant 0}$, $Z_0 = 0$, with Hölder continuous paths of order $1/2 < \gamma < 1$

$$X_t = x_0 + \int_0^t \alpha(X_s)\, ds + \int_0^t \sigma(X_s)\, dZ_s, \qquad x_0 \in \mathbb{R}, \quad t \in [0, T], \tag{1}$$

where $x_0$ is a constant, and the coefficients $\alpha, \sigma : \mathbb{R} \to \mathbb{R}$ are continuous functions. The stochastic integral in Equation (1) is a pathwise Riemann–Stieltjes integral, and thus the whole equation is understood as a pathwise Riemann–Stieltjes integral equation (special case $Z = B^H$). The conditions under which the Equation (1) has a unique solution $X$ were obtained in [7].

A lot of SDEs cannot be solved explicitly; thus, it is important to find their approximated solutions by applying some numerical methods. For SDEs driven by sBm, authors usually consider the convergence rate of strong Itô–Taylor approximation schemes of the SDEs solutions. From a strong convergence rate, one can obtain the pathwise convergence rate (see [8,9] and the references therein), which provides the error of the actual approximation.

For Equation (1) with $Z = B^H$ and rather smooth and bounded coefficients $\alpha$ and $\sigma$, different explicit approximation schemes were considered in [10–13] (see also the references therein). Particularly, in [10,12,13], Euler and other higher-order approximation schemes were considered; [11] presents a new method that converges with weaker conditions compared to the results in [12]; and in [14], less strict conditions for the coefficients were used in the explicit Euler approximation.

The purpose of this work is to obtain an approximation scheme under weaker conditions and of a higher order convergence rate than in the articles mentioned above. For this, we require the diffusion coefficient to be strictly positive, i.e., $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. Such restriction on a diffusion coefficient was used in [11,13]. Since the diffusion coefficient is strictly positive, we can use the Lamperti transformation and transform a considerable SDE into simpler SDE with constant diffusion coefficient. The transformed SDE can be approximated accordingly to the chosen scheme, and the inverse transformation provides an approximation scheme for the original SDE. This strategy has been successfully applied to classical CIR, Heston-3/2 volatility, Ait–Sahalia, and other models using a backward (also called drift-implicit) Euler–Maruyama scheme [15–17] to obtain a strong convergence rate. This idea has also been successfully applied to fractional CIR model [18], fractional CKLS, Ait–Sahalia, and other models (see [4–6]) when they have positive solutions.

The paper is organized in the following way. In Section 2, we present the main results of the paper. Section 3 contains proofs of the main theorems. In Section 4, the fractional Pearson diffusion process and the model from [11] are considered as modeling examples. Finally, in Appendix A, we propose an approximation of the integrated fBm, which was used for the simulation results. Moreover, some results on pathwise integration, fBm, and almost sure convergence are listed in Appendices B and C as well.

## 2. Main Result

Assume that for some $L > 0$ and for any $x, y \in \mathbb{R}$,

$$|\alpha(x)| + |\sigma(x)| \leqslant L(1 + |x|), \qquad |\sigma'(x)| \leqslant L, \tag{2}$$

$$|\alpha(x) - \alpha(y)| + |\sigma'(x) - \sigma'(y)| \leqslant L|x - y|, \tag{3}$$

and the process $Z$ has Hölder continuous paths of order $1/2 < \gamma < 1$, i.e., there exists a random variable $G_{\gamma,T} = G_{\gamma,T}(\omega) \in (0, \infty)$ such that

$$\left|Z_t - Z_s\right| \leqslant G_{\gamma,T}|t - s|^{\gamma}, \quad s, t \in [0, T].$$

Under these conditions given in [7] (see also [1]), Equation (1) has a unique solution $X$ such that $\|X\|_{\alpha,\infty,T} < \infty$ a.s. for any $\alpha \in (1 - \gamma, \frac{1}{2})$. The definitions of the norm $\|X\|_{\alpha,\infty,T}$ are given in Appendix B.

In addition to the conditions formulated in (2)–(3), we will require that the diffusion term be such that

$$(\mathbf{H}) \quad \inf_{x\in\mathbb{R}} \sigma(x) > 0.$$

Note that condition (3) implies that the function $1/\sigma(x)$ is continuously differentiable on $\mathbb{R}$. Thus, under condition $(\mathbf{H})$, the Lamperti transform

$$F(x) = \int_0^x \frac{1}{\sigma(y)}\, dy, \qquad x \in \mathbb{R}$$

has the inverse function $F^{-1} : \mathbb{R} \to \mathbb{R}$, which is strictly monotone and differentiable

$$(F^{-1})'(x) = \sigma(F^{-1}(x)), \qquad x \in \mathbb{R}. \tag{4}$$

Set $Y_t = F(X_t)$. By chain rule, we obtain

$$Y_t = Y_0 + \int_0^t F'(X_s)\, dX_s = Y_0 + \int_0^t \frac{\alpha(X_s)}{\sigma(X_s)}\, ds + Z_t = y_0 + \int_0^t f(Y_s)\, ds + Z_t,$$

where $y_0 = F(x_0)$,

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

Since under conditions (2)–(3) there exists a unique solution of (1), the equation

$$Y_t = y_0 + \int_0^t f(Y_s)\, ds + Z_t \tag{5}$$

has a unique solution under these same conditions.

To state our main results, we use the following requirements on function $f$:

$(\mathbf{C}_0)$ $f$ is continuously differentiable on $\mathbb{R}$;

$(\mathbf{C}_1)$ Assume that there exists a constant $K \in \mathbb{R}$ such that $f'(x) \leqslant K$ for all $x \in \mathbb{R}$;

$(\mathbf{C}_2)$ Assume that there exists a constant $M \geqslant 0$ such that $g'(x) \geqslant -M$ for all $x \in \mathbb{R}$, where $g(x) = f(x)f'(x)$;

$(\mathbf{C}_3)$ Assume that the function $f$ on $\mathbb{R}$ is twice continuous differentiable and there exists a constant $N \geqslant 0$ such that $|f''(x)| \leqslant N$ for all $x \in \mathbb{R}$.

Let $\pi = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ be a sequence of uniform partitions of the interval $[0, T]$, and let $h = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$. For the solution $Y$ of the SDE (5), define the following backward approximation scheme:

$$\begin{aligned} &Y_{n,k+1} - f(Y_{n,k+1})h + f'(Y_{n,k+1})f(Y_{n,k+1})\,\frac{h^2}{2} \\ &\qquad = Y_{n,k} + \left(Z_{t_{k+1}^n} - Z_{t_k^n}\right) - f'(Y_{n,k}) \int_{t_k^n}^{t_{k+1}^n} \left(Z_{t_{k+1}^n} - Z_s\right) ds, \\ &Y_{n,0} = y_0,\ 0 \leqslant k \leqslant n-1. \end{aligned} \tag{6}$$

For the simplicity of notation, we introduce the symbol $O_\omega$. Let $(\xi_n)$ be a sequence of r.v.s, let $\varsigma$ be an a.s. nonnegative r.v., and let $(a_n) \subset (0, \infty)$ be a vanishing sequence. Then $\xi_n = O_\omega(a_n)$ means that $|\xi_n| \leq \varsigma \cdot a_n$ for all $n$. In particular, $\xi_n = O_\omega(1)$ means that the sequence $(\xi_n)$ is a.s. bounded.

Define

$$h_0 := \frac{\sqrt{2M + (K^+)^2} - K^+}{M} \tag{7}$$

where constants $K$ and $M$ are given in $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$, $K^+ = \max\{0, K\}$.

**Theorem 1.** *Suppose that the function $f$ in (5) satisfies conditions $(\mathbf{C}_0)$–$(\mathbf{C}_3)$. Assume that the sequence of uniform partitions $\pi$ of the interval $[0, T]$ is such that $h < h_0$. Then for $\gamma \in (\frac{1}{2}, 1)$, it follows that*

$$\max_{1 \leqslant k \leqslant n} \left|Y_{t_k^n} - Y_{n,k}\right| = O_\omega(h^{2\gamma}).$$

**Remark 1.** *Note that this result is not applicable for CKLS, Heston-3/2 volatility, or Ait–Sahalia models, since condition* $(\boldsymbol{C}_3)$ *is not satisfied.*

**Theorem 2.** *Assume that SDE* (1) *has unique solution and conditions of Theorem 1 are satisfied. Then for* $\gamma \in (\frac{1}{2}, 1)$*, it follows that*

$$\max_{1 \leqslant k \leqslant n} \left| X_{t_k^n} - F^{-1}(Y_{n,k}) \right| = O_\omega(h^{2\gamma}).$$

## 3. Proofs of Theorems

Firstly, we prove that the backward approximation (6) is well defined.

**Lemma 1.** *Let the conditions* $(\boldsymbol{C}_1)$ *and* $(\boldsymbol{C}_2)$ *be satisfied. The function*

$$H(x) = x - f(x)h + f'(x)f(x)\frac{h^2}{2}, \qquad x \in \mathbb{R},$$

*is strictly monotone and* $\lim_{x \to +\infty} H(x) = +\infty$, $\lim_{x \to -\infty} H(x) = -\infty$ *for any* $h < h_0$.

**Remark 2.** *Note that for* $K \leqslant 0$ *and* $g'(x) \geqslant 0$*, there is no restriction on* $h$.

**Proof.** Under the assumptions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$, the function $H(x)$ is strictly monotone. Indeed,

$$\begin{aligned}(x-y)\big(H(x) - H(y)\big) =& (x-y)^2 - (x-y)\big(f(x) - f(y)\big)h \\ &+ (x-y)\big(f'(x)f(x) - f'(y)f(y)\big)\frac{h^2}{2} \\ \geqslant& \Big(1 - K^+h - M\frac{h^2}{2}\Big)(x-y)^2 > 0 \end{aligned} \tag{8}$$

for $h < h_0$.

Now we find the limits of the function $H(x)$ as $x \to \pm\infty$. The proof is similar as in [17]. From (8), it follows that

$$H(x) \leqslant H(x_0) + \Big(1 - K^+h - M\frac{h^2}{2}\Big)(x - x_0)$$

for $x < x_0$, which provides $\lim_{x \to -\infty} H(x) = -\infty$. Inequality (8) implies

$$H(x) \geqslant H(x_0) + \Big(1 - K^+h - M\frac{h^2}{2}\Big)(x - x_0)$$

for $x > x_0$ and thus $\lim_{x \to \infty} H(x) = \infty$. $\square$

From Lemma 1, it follows that for each $b \in \mathbb{R}$, the equation $F(x) = b$ has a unique solution for $0 < h < h_0$. Consequently, the backward approximation scheme is well defined if conditions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$ are satisfied and $h < h_0$.

### 3.1. Proof of Theorem 1

Applying the chain rule, we obtain that

$$\begin{aligned}
&f(Y_{t^n_{k+1}}) - f(Y_s) \\
&\quad = \int_s^{t^n_{k+1}} f'(Y_u)\, dY_u = \int_s^{t^n_{k+1}} f'(Y_u) f(Y_u)\, du + \int_s^{t^n_{k+1}} f'(Y_u)\, dZ_u \\
&\quad = -\int_s^{t^n_{k+1}} [f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}}) - f'(Y_u) f(Y_u)] du + f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}})(t^n_{k+1} - s) \\
&\qquad + \int_s^{t^n_{k+1}} [f'(Y_u) - f'(Y_s)]\, dZ_u + [f'(Y_s) - f'(Y_{t^n_k})]\big(Z_{t^n_{k+1}} - Z_s\big) \\
&\qquad + f'(Y_{t^n_k})\big(Z_{t^n_{k+1}} - Z_s\big).
\end{aligned} \tag{9}$$

From Equation (5), we obtain that

$$\begin{aligned}
Y_{t^n_{k+1}} =& Y_{t^n_k} + f(Y_{t^n_{k+1}})h - \int_{t^n_k}^{t^n_{k+1}} [f(Y_{t^n_{k+1}}) - f(Y_s)]\, ds + \big(Z_{t^n_{k+1}} - Z_{t^n_k}\big) \\
=& Y_{t^n_k} + f(Y_{t^n_{k+1}})h + \big(Z_{t^n_{k+1}} - Z_{t^n_k}\big) - f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}}) \int_{t^n_k}^{t^n_{k+1}} (t^n_{k+1} - s)\, ds \\
&- f'(Y_{t^n_k}) \int_{t^n_k}^{t^n_{k+1}} \big[Z_{t^n_{k+1}} - Z_s\big] ds + R_{n,k+1} \\
=& Y_{t^n_k} + f(Y_{t^n_{k+1}})h + \big(Z_{t^n_{k+1}} - Z_{t^n_k}\big) - f'(Y_{t^n_k+1}) f(Y_{t^n_k+1}) \frac{h^2}{2} \\
&- f'(Y_{t^n_k}) \int_{t^n_k}^{t^n_{k+1}} \big[Z_{t^n_{k+1}} - Z_s\big] ds + R_{n,k+1},
\end{aligned}$$

where

$$\begin{aligned}
R_{n,k+1} =& \int_{t^n_k}^{t^n_{k+1}} \int_s^{t^n_{k+1}} [f'(Y_{t^n_{k+1}}) f(Y_{t^n_{k+1}}) - f'(Y_u) f(Y_u)]\, du ds \\
&- \int_{t^n_k}^{t^n_{k+1}} \int_s^{t^n_{k+1}} [f'(Y_u) - f'(Y_s)]\, dZ_u ds \\
&- \int_{t^n_k}^{t^n_{k+1}} [f'(Y_s) - f'(Y_{t^n_k})]\big(Z_{t^n_{k+1}} - Z_s\big)\, ds
\end{aligned} \tag{10}$$

is the remainder term.

For simplicity of notation, we introduce the following:

$$\begin{aligned}
\zeta_{n,k+1} =& f'(Y_{t^n_{k+1}} + \theta(Y_{n,k+1} - Y_{t^n_{k+1}})), \quad \text{where } \theta = \theta_{n,k+1} \in (0,1), \\
\eta_{n,k} =& f''(Y_{t^n_k} + \vartheta(Y_{n,k} - Y_{t^n_k})), \quad \text{where } \vartheta = \vartheta_{n,k} \in (0,1), \\
\rho_{n,k+1} =& g'(Y_{t^n_{k+1}} + \kappa(Y_{n,k+1} - Y_{t^n_{k+1}})), \quad \text{where } \kappa = \kappa_{n,k+1} \in (0,1),\ g(x) = f'(x) f(x).
\end{aligned}$$

Then the difference of Equation (9) and approximation (6) is

$$
\begin{aligned}
&Y_{t^n_{k+1}} - Y_{n,k+1} \\
&\quad = Y_{t^n_k} - Y_{n,k} + h\big(f(Y_{t^n_{k+1}} - f(Y_{n,k+1})\big) - \big(f'(Y_{t^n_{k+1}})f(Y_{t^n_{k+1}}) \\
&\qquad - f'(Y_{n,k+1})f(Y_{n,k+1})\big)\frac{h^2}{2} - \big(f'(Y_{t^n_k}) - f'(Y_{n,k})\big)\int_{t^n_k}^{t^n_{k+1}} \big[Z_{t^n_{k+1}} - Z_s\big]ds + R_{n,k+1} \\
&\quad = Y_{t^n_k} - Y_{n,k} + \zeta_{n,k+1}\big(Y_{t^n_{k+1}} - Y_{n,k+1}\big)h - \rho_{n,k+1}\big(Y_{t^n_{k+1}} - Y_{n,k+1}\big)\frac{h^2}{2} \\
&\qquad - \eta_{n,k}\big(Y_{t^n_k} - Y_{n,k}\big)\int_{t^n_k}^{t^n_{k+1}} \big[Z_{t^n_{k+1}} - Z_s\big]ds + R_{n,k+1}
\end{aligned}
$$

and

$$
(Y_{t^n_{k+1}} - Y_{n,k+1})\left[1 - \zeta_{n,k+1}h + \rho_{n,k+1}\frac{h^2}{2}\right] = (Y_{t^n_k} - Y_{n,k})[1 - \lambda_{n,k}] + R_{n,k+1}, \tag{11}
$$

where

$$
\lambda_{n,k+1} = \eta_{n,k}\int_{t^n_k}^{t^n_{k+1}} \big(Z_{t^n_{k+1}} - Z_s\big)ds.
$$

Applying assumptions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$, we transform equality (11) into a recursive inequality. Indeed,

$$
1 - \zeta_{n,k+1}h + \rho_{n,k+1}\frac{h^2}{2} \geqslant 1 - K^+h - M\frac{h^2}{2} > 0
$$

for $h < h_0$. Thus,

$$
(Y_{t^n_{k+1}} - Y_{n,k+1})\Big(1 - K^+h - M\frac{h^2}{2}\Big) \leqslant (Y_{t^n_k} - Y_{n,k})(1 - \lambda_{n,k+1}) + R_{n,k+1}.
$$

Further, by applying inequality $1 + x \leqslant e^x$, $\forall x \geqslant 0$, we obtain that

$$
|y_{n,k+1}| \leqslant |y_{n,k}|\exp\{|\lambda_{n,k+1}|\}\varepsilon_h^{-1} + |R_{n,k+1}|\varepsilon_h^{-1}, \tag{12}
$$

where $y_{n,k} = Y_{t^n_k} - Y_{n,k}$ and $\varepsilon_h = 1 - K^+h - M\frac{h^2}{2}$.

Now, recursively from (12), we obtain that

$$
|y_{n,k+1}| \leqslant \sum_{j=0}^{k} \varepsilon_h^{-(k+1-j)} \exp\left\{\sum_{i=j+1}^{k} |\lambda_{n,i+1}|\right\}|R_{n,j+1}|.
$$

We have $\ln\frac{1}{1-x} \leqslant \frac{x}{1-x}$, $0 \leqslant x < 1$; thus, we obtain that

$$
\begin{aligned}
\varepsilon_h^{-(k+1-j)} &\leqslant \exp\left(n\ln\frac{1}{\varepsilon_h}\right) \leqslant \exp\left(n\,\frac{K^+h + M\frac{h^2}{2}}{\varepsilon_h}\right) \\
&\leqslant \exp\left(\frac{K^+T + MT}{\varepsilon_h}\right),
\end{aligned}
$$

if $h \leqslant 2$. Consequently,

$$
|y_{n,k+1}| \leqslant \exp\left(\frac{K^+T + MT}{\varepsilon_h}\right)\sum_{j=0}^{k}\exp\left\{\sum_{i=j+1}^{k}|\lambda_{n,i+1}|\right\}|R_{n,j+1}| \leqslant \psi_n\sum_{j=0}^{k}|R_{n,j+1}|,
$$

where

$$\psi_n = \exp\left(\frac{K^+T + MT}{\varepsilon_h}\right)\exp\left\{n\max_{1\leqslant i\leqslant n}|\lambda_{n,i}|\right\}.$$

To finish the proof of Theorem 1, it remains to estimate $\max_{1\leqslant j\leqslant n}|R_{n,j}|$ and $\max_{1\leqslant i\leqslant n}|\lambda_{n,i}|$. From Lemmas 2 and 3 below, it follows that

$$\psi_n = \exp\left(\frac{K^+T + MT}{\varepsilon_h}\right)\exp\left\{nO_\omega\left(h^{1+\gamma}\right)\right\} = O_\omega(1).$$

Thus,

$$\max_{1\leqslant k\leqslant n}|y_{n,k}| = O_\omega\left(h^{2\gamma}\right).$$

**Lemma 2.** *Under the assumptions of Theorem 1*

$$\max_{1\leqslant j\leqslant n}|R_{n,j}| = O_\omega(h^{1+2\gamma}).$$

**Proof.** We start by estimating the first term $R_{n,j}$ in (10). Since the function $g'(x)$ is continuous and the process $Y$ is continuous, then $\sup_{0\leqslant t\leqslant T}|g'(Y_s)| < \infty$. From the Hölder continuity of $Z$, we obtain

$$\begin{aligned}\left|Y_t - Y_s\right| &\leqslant \int_s^t |f(Y_u)|\,du + \left|Z_t - Z_s\right| \\ &\leqslant (t-s)\sup_{0\leqslant u\leqslant T}|f(Y_u)| + G_{\gamma,T}|t-s|^\gamma = O_\omega\left(|t-s|^\gamma\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{t_k^n}^{t_{k+1}^n}\int_s^{t_{k+1}^n}\left|g(Y_u) - g(Y_s)\right|du\,ds \\ &\quad\leqslant \sup_{0\leqslant t\leqslant T}|g'(Y_s)|\sup_{t_k^n\leqslant s\leqslant t_{k+1}^n}\sup_{s\leqslant u\leqslant t_{k+1}^n}\left|Y_u - Y_s\right|h^2 = O_\omega\left(h^{2+\gamma}\right).\end{aligned}$$

From the Love–Young inequality, it follows that

$$\left|\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u\right| \leqslant C_{\gamma,\gamma}K_Y K_Z\sup_{0\leqslant t\leqslant T}|f''(Y_t)|h^{2\gamma} = O_\omega(h^{2\gamma}). \tag{13}$$

Therefore, the second term $R_{n,j}$ in (10) has the following estimate:

$$\begin{aligned}&\left|\int_{t_k^n}^{t_{k+1}^n}\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u ds\right| \\ &\quad\leqslant \int_{t_k^n}^{t_{k+1}^n}\left|\int_s^{t_{k+1}^n}[f'(Y_u) - f'(Y_s)]\,dZ_u\right|ds = O_\omega(h^{1+2\gamma}).\end{aligned}$$

Finally, for the third term $R_{n,j}$ in (10), we obtain the estimate

$$\begin{aligned}&\int_{t_k^n}^{t_{k+1}^n}\left|f'(Y_s) - f'(Y_{t_k^n})\right|\cdot\left|Z_{t_{k+1}^n} - Z_s\right|ds \\ &\quad\leqslant \sup_{0\leqslant t\leqslant T}|f''(Y_t)|\int_{t_k^n}^{t_{k+1}^n}\left|Y_s - Y_{t_k^n}\right|\cdot\left|Z_{t_{k+1}^n} - Z_s\right|ds = O_\omega\left(h^{1+2\gamma}\right).\end{aligned}$$

The proof is complete. □

**Lemma 3.** *Assume that condition* $(\mathbf{C}_3)$ *is satisfied. Then*

$$\max_{1\leqslant k\leqslant n} |\lambda_{n,k}| = O_\omega\big(h^{1+\gamma}\big).$$

**Proof.** Applying condition $(\mathbf{C}_3)$, we obtain

$$|\eta_{n,k}| \leqslant N.$$

Since

$$\int_{t_k^n}^{t_{k+1}^n} \big|Z_{t_{k+1}^n} - Z_s\big|\, ds = O_\omega\big(h^{1+\gamma}\big),$$

then the proof is complete. □

*3.2. Proof of Theorem 2*

Note that

$$\begin{aligned}\big|X_{t_k^n} - F^{-1}(Y_{n,k})\big| =& \big|F^{-1}(Y_{t_k^n}) - F^{-1}(Y_{n,k})\big| \\ =& \big|\big(F^{-1}\big(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})\big)\big)'\big| \cdot \big|Y_{t_k^n} - Y_{n,k}\big| \\ =& \big|\sigma\big(F^{-1}\big(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})\big)\big)\big| \cdot \big|Y_{t_k^n} - Y_{n,k}\big|\end{aligned} \tag{14}$$

where $\theta = \theta_{n,k} \in (0,1)$, and

$$\max_{1\leqslant k\leqslant n} \big|Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})\big| \leqslant \sup_{0\leqslant t\leqslant T} |Y_t| + O_\omega(h^{2\gamma}) = O_\omega(1). \tag{15}$$

Since the function $\sigma(x)$ satisfies condition (2), then

$$\big|\sigma\big(F^{-1}(x)\big)\big| \leqslant L\big(1 + \big|F^{-1}(x)\big|\big).$$

In addition, as the function $F^{-1}(x)$ increases and (15) is satisfied, then there exists a random variable $0 < \zeta < \infty$ such that

$$\max_{1\leqslant k\leqslant n} \big|F^{-1}\big(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})\big)\big| \leqslant \max\{\big|F^{-1}(-\zeta)\big|, \big|F^{-1}(\zeta)\big|\} = O_\omega(1).$$

Thus, the required result follows from Theorem 1.

## 4. Modeling

In this section, we will apply obtained theoretical results for concrete SDEs. In particular, the Pearson fractional diffusion process and the model from [11] based on trigonometric functions satisfy the conditions of Theorem 1. We chose to investigate the fractional Pearson diffusion process as it is better known and more widely used. However, the same simulation and investigation methodology can be applied to the second model as well.

*4.1. Pearson Diffusion*

Consider the Pearson diffusion process

$$dX_t = \alpha(X_t)\, dt + \sigma(X_t)\, dB_t^H \tag{16}$$

with

$$\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0 + \sigma_1 x + \sigma_2 x^2}.$$

Assume that $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. Then the diffusion coefficient $\sigma(x)$ has bounded first- and second-order derivatives, and Equation (16) has a unique solution since the drift and diffusion coefficients satisfy conditions (2)–(3).

Then Equation (16) after Lamperti transform has the form

$$Y_t = Y_0 + \int_0^t f(Y_s)\, ds + B_t^H, \tag{17}$$

where

$$f(x) = \widehat{f}(F^{-1}(x)), \qquad \widehat{f}(x) = \frac{\alpha(x)}{\sigma(x)}.$$

To check the conditions $(\mathbf{C}_0)$–$(\mathbf{C}_3)$, we have to find the expressions of the functions $f'(x)$, $(ff')'(x)$, $f''(x)$.

Note that

$$\begin{aligned} f'(x) =& \big(\widehat{f}(F^{-1}(x))\big)' = \widehat{f}'(F^{-1}(x))\big(F^{-1}\big)'(x) = \widehat{f}'(F^{-1}(x))\sigma(F^{-1}(x)) \\ =& \Big(\alpha' - \frac{\alpha\sigma'}{\sigma}\Big)(F^{-1}(x)), \end{aligned} \tag{18}$$

where

$$\widehat{f}'(x) = \Big(\frac{\alpha'\sigma - \alpha\sigma'}{\sigma^2}\Big)(x).$$

Since

$$(\widehat{f}\widehat{f}')(x) = \Big(\frac{\alpha\alpha'\sigma - \alpha^2\sigma'}{\sigma^3}\Big)(x) = \Big(\frac{\alpha\alpha'}{\sigma^2} - \frac{\alpha^2\sigma'}{\sigma^3}\Big)(x)$$

and

$$\begin{aligned} (\widehat{f}\widehat{f}')'(x) =& \frac{[(\alpha'(x))^2 + \alpha(x)\alpha''(x)]\sigma(x) - 2\alpha(x)\alpha'(x)\sigma'(x)}{\sigma^3(x)} \\ &- \frac{[2\alpha(x)\alpha'(x)\sigma'(x) + \alpha^2(x)\sigma''(x)]\sigma(x) - 3\alpha^2(x)(\sigma'(x))^2}{\sigma^4(x)}, \end{aligned}$$

then

$$\begin{aligned} (ff')'(x) =& (\widehat{f}\,\widehat{f}')'(F^{-1}(x))\sigma(F^{-1}(x)) \\ =& \Big(\frac{[(\alpha')^2 + \alpha\alpha'']\sigma - 2\alpha\alpha'\sigma'}{\sigma^2}\Big)(F^{-1}(x)) \\ &- \Big(\frac{[2\alpha\alpha'\sigma' + \alpha^2\sigma'']\sigma - 3\alpha^2(\sigma')^2}{\sigma^3}\Big)(F^{-1}(x)). \end{aligned}$$

Further,

$$\begin{aligned} (\widehat{f}')'(x) =& \frac{\alpha''(x)\sigma(x) - \alpha'(x)\sigma'(x)}{\sigma^2(x)} \\ &- \frac{[\alpha'(x)\sigma'(x) + \alpha(x)\sigma''(x)]\sigma^2(x) - 2\alpha(x)\sigma(x)(\sigma'(x))^2}{\sigma^4(x)}. \end{aligned}$$

Thus,

$$\begin{aligned} f''(x) =& \widehat{f}''(F^{-1}(x))\sigma(F^{-1}(x)) \\ =& \Big(\frac{\alpha''\sigma - \alpha'\sigma'}{\sigma}\Big)(F^{-1}(x)) - \Big(\frac{[\alpha'\sigma' + \alpha\sigma'']\sigma^2 - 2\alpha\sigma(\sigma')^2}{\sigma^3}\Big)(F^{-1}(x)). \end{aligned}$$

To simplify the analysis of derivatives and to allow the simulation itself, we take specific expressions of coefficients. Let

$$\sigma(x) = \sqrt{x^2 + 2x + 2}, \qquad a = 1, \qquad b = 2. \tag{19}$$

The Lamperti transformation of $\sigma$ and its inverse are defined as follows:

$$F(x) = \int_0^x \frac{dy}{\sqrt{y^2+2y+2}} = \ln\left(x + \sqrt{(x+1)^2+1} + 1\right) = \sinh^{-1}(x+1),$$
$$F^{-1}(x) = \frac{1}{2}\left(-e^{-x} + e^x - 2\right) = \sinh x - 1.$$

Note that

$$(F^{-1})'(x) = \frac{1}{2}\left(e^{-x} + e^x\right) \leqslant e^{|x|}.$$

Thus, from (15), it follows that

$$\left(F^{-1}\left(Y_{t_k^n} + \theta_{n,k}(Y_{t_k^n} - Y_{n,k})\right)\right)' = O_\omega(1).$$

This is enough for Theorem 2 to be true (see equality (14)) if the conditions of Theorem 1 are satisfied.

Now we verify conditions $(\mathbf{C}_0)$–$(\mathbf{C}_3)$. Condition $(\mathbf{C}_0)$ is satisfied (see (18)). By inserting the expressions of coefficients $a$ and $b$, we obtain

$$\begin{aligned}
|\widehat{f}'(x)\sigma(x)| &= \left|-\frac{3x+4}{x^2+2x+2}\right| < 2.09 \\
|\widehat{f}''(x)\sigma(x)| &= \left|\frac{3(2x^2+5x+2)}{(x^2+2x+2)^2}\right| < 1.56 \\
\left|(\widehat{f}\widehat{f}')'(x)\sigma(x)\right| &= \left|\frac{-6x^3+6x^2+48x+28}{(x^2+2x+2)^{5/2}}\right| < 9.63.
\end{aligned}$$

Since the function $F^{-1}(x)$ is continuous and increasing then from above, it follows that the functions $f'(x)$, $f''(x)$, and $(ff')'(x)$ are bounded. Thus, conditions $(\mathbf{C}_1)$–$(\mathbf{C}_3)$ are satisfied.

### 4.2. Numerical Simulation

Notice that the approximation Scheme (6) is not final. In order to obtain the original process $X_t$ approximation, the inverse Lamperti transform has to be applied. However, Scheme (6) is implicit and both the number of calculations needed and the general complexity of the approximation can be reduced by combining the Scheme (6) and Lamperti transform simultaneously:

$$\begin{aligned}
&F(X_{n,k+1}) - \widehat{f}(X_{n,k+1})h + \widehat{f}'(X_{n,k+1})\widehat{f}(X_{n,k+1})\frac{h^2}{2} \\
&\quad = F(X_{n,k}) + \left(B^H_{t^n_{k+1}} - B^H_{t^n_k}\right) - \widehat{f}'(X_{n,k})\int_{t^n_k}^{t^n_{k+1}}\left(B^H_{t^n_{k+1}} - B^H_s\right)ds, \\
&X_{n,0} = x_0,\ 0 \leqslant k \leqslant n-1.
\end{aligned} \tag{20}$$

Now, using this finalized Scheme (20) we can generate trajectories of any process satisfying conditions of Theorem 1 (see Figure 1).

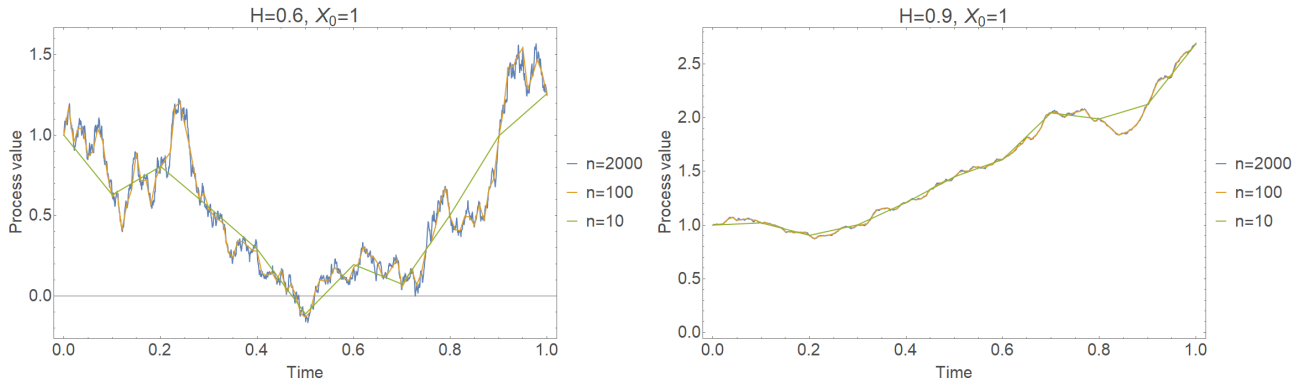


**Figure 1.** Approximation trajectories of Pearson process for conditions (19).

To compare the theoretical and empirical convergence rates of the Pearson process (19), we simulate the "exact" solution trajectories by using an approximation scheme (20) for a comparatively much smaller step size $h = 2 \times 10^{-3}$. Since the object of the numerical experiment is investigation of the convergence rate, which is not dependent on the values of constants, the most basic set of values for (16) is sufficient (i.e., $X_0 = a = \sigma_2 = 1, b = \sigma_0 = \sigma_1 = 2$). We see from Figure 2 that the empirical maximum error coincides with the theoretical result in Theorem 2 (reference slope).

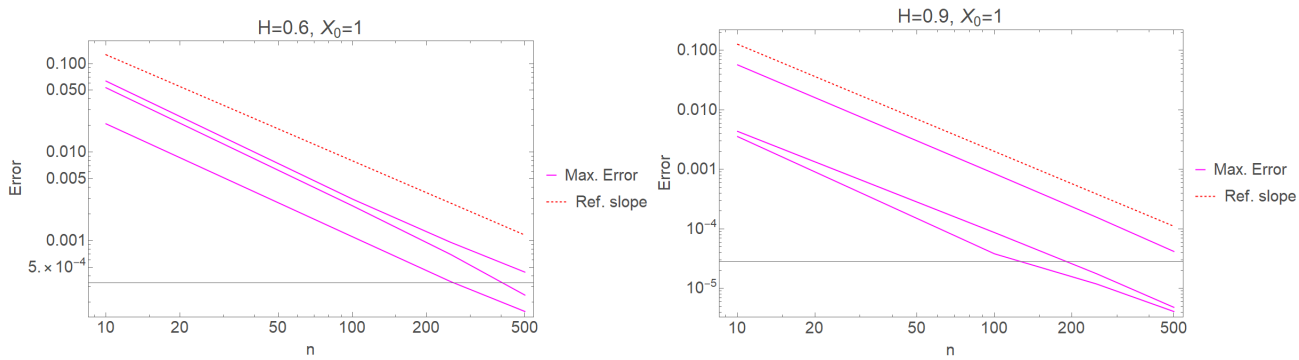


**Figure 2.** Maximum error of several approximation trajectories of Pearson process for conditions (19) in comparison to reference slope.

Of course, it should be noted that we assume fBm values $B_t^H$ provided by *Wolfram Mathematica* programming language function *FractionalBrownianMotionProcess* to be not approximate but "true" values of fractional Brownian motion and that the same assumption is being made about the approximation of integrated fBm too.

## 5. Conclusions

We introduced an improved type of approximation scheme for certain SDEs. In comparison with the previous research in the field, due to less strict conditions, this approximation covers a wider class of stochastic processes and is proven to have a higher convergence rate. These theoretical results were supported by numerical experiments. Furthermore, we proposed a simple and direct approximation scheme with estimated convergence rate for integrated fractional Brownian motion, which can be applied by other authors modelling their own approximation schemes.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| CEV | Constant elasticity of variance model |
| CIR | Cox–Ingersoll–Ross model |
| CKLS | Chan–Karolyi–Longstaff–Sanders model |
| FSDE | Fractional stochastic differential equation |
| fBm | Fractional Brownian motion |
| sBm | Standard Brownian motion |
| SDE | Stochastic differential equation |

## Appendix A. Approximation of Integrated fBm

For the modelling of process $Y$ using approximation (20), we need to calculate the integral

$$\int_{t_m^n}^{t_{m+1}^n} B_s^H \, ds. \tag{A1}$$

Analogously to the approximation of standard Riemann integral, we propose that the integral (A1) can be replaced by the sum $\frac{T}{n^2}\sum_{k=1}^n B_{s_{k,m}}^H$, where $t_m^n = \frac{mT}{n}$, $0 \leqslant m \leqslant n$ and $s_{k,m}^n = t_m^n + \frac{kT}{n^2}$, $0 \leqslant k \leqslant n$. For simplicity of notation, $t_m = t_m^n$. This method enables simple and direct use of fractional Brownian motion simulation packages provided in the most mathematical programming languages (*Wolfram Mathematica* was used for our simulations).

Now, to prove the correctness of this approximation, we estimate

$$\mathbf{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2}\sum_{k=1}^n B_{s_{k,m}}^H\right)^2.$$

After that, applying Lemma A1 (see Appendix C), we will obtain the rate of a. s. convergence of the difference

$$\int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2}\sum_{k=1}^n B_{s_{k,m}}^H.$$

First, we shall introduce the concept of generalized harmonic numbers (GHN). We define

$$H_n^{(r)} := \sum_{k=1}^n \frac{1}{k^r},$$

where $r = \sigma + it$ is a complex variable as generalized harmonic numbers.

Here are some important properties of GHN sums [19].

**Proposition A1.** *The following identity is true*

$$\sum_{i=1}^n i^{-a} H_i^{(b)} + \sum_{i=1}^n i^{-b} H_i^{(a)} = H_n^{(a)} H_n^{(b)} + H_n^{(a+b)}, \tag{A2}$$

*where* $a, b \in \mathbb{R}$.

**Proposition A2.** *The following identity is true*

$$\sum_{i=1}^{n-1} H_i^{(a)} = nH_n^{(a)} - H_n^{(a-1)}, \tag{A3}$$

*where* $a \in \mathbb{R}$.

Using these properties, the following result can be proven.

**Theorem A1.** *These equalities are true:*

1. *Integral expectation*

$$\mathbf{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds\right)^2 = \frac{T^{2H+2}}{(2H+1)n^{2H+2}}\left[(m+1)^{2H+1} - m^{2H+1} - \frac{1}{2H+2}\right]$$

2. *Product sum expectation*

$$\mathbf{E}\sum_{k=1}^{n}\sum_{l=1}^{n} B_{s_{k,m}}^H B_{s_{l,m}}^H = \frac{T^{2H}}{n^{4H}} H_n^{(-2H-1)} + \frac{T^{2H}}{n^{4H-1}}\left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)} - H_n^{(-2H)}\right].$$

3. *Mixed products*

$$\begin{aligned} 2\mathbf{E}\sum_{k=1}^{n} B_{s_{k,m}}^H \int_{t_m}^{t_{m+1}} B_s^H ds =& \frac{T^{2H+1}}{(2H+1)n^{2H}}\left[(m+1)^{2H+1} - m^{2H+1} + n^{-1}\right] \\ &+ \frac{T^{2H+1}}{n^{4H+1}}\left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)}\right] - \frac{2T^{2H+1}}{(2H+1)n^{4H+2}} H_n^{(-2H-1)}. \end{aligned}$$

**Proof.** Using the covariance properties fBm as in [20], we obtain for integrals

$$\mathbf{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds \int_{t_m}^{t_{m+1}} B_u^H du\right) = (t_{m+1} - t_m)\int_{t_m}^{t_{m+1}} u^{2H} du - \frac{1}{2H+1}\int_0^{t_{m+1}-t_m} u^{2H+1} du.$$

and for sums

$$\begin{aligned} \mathbf{E}\sum_{k=1}^{n}\sum_{l=1}^{n} B_{s_{k,m}}^H B_{s_{l,m}}^H =& \frac{T^{2H}}{2n^{4H}}\sum_{k=1}^{n}\sum_{l=1}^{n}\left[(mn+k)^{2H} + (mn+l)^{2H} - |k-l|^{2H}\right] \\ =& \frac{T^{2H}}{n^{4H-1}}\left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)}\right] - \frac{T^{2H}}{n^{4H}}\sum_{k=1}^{n-1} H_k^{(-2H)}. \end{aligned}$$

Now, applying Proposition A2 provides us the result.
The mixed product of integral and sum is estimated the following way:

$$\begin{aligned} 2\mathbf{E}\sum_{k=1}^{n} B_{s_{k,m}}^H \int_{t_m}^{t_{m+1}} B_s^H ds =& \sum_{k=1}^{n}\int_{t_m}^{t_{m+1}}\left(s_{k,m}^{2H} + s^{2H} - |s - s_{k,m}|^{2H}\right) ds \\ =& \frac{T^{2H+1}}{n^{4H+1}}\left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)}\right] + \frac{T^{2H+1}}{(2H+1)n^{2H}}\left[(m+1)^{2H+1} - m^{2H}\right] \\ &- \frac{T^{2H+1}}{(2H+1)n^{4H+2}}\sum_{k=1}^{n}\left[(n-k)^{2H+1} + k^{2H+1}\right] \\ =& \frac{T^{2H+1}}{n^{4H+1}}\left[H_{n(m+1)}^{(-2H)} - H_{nm}^{(-2H)}\right] + \frac{T^{2H+1}}{(2H+1)n^{2H}}\left[(m+1)^{2H+1} - m^{2H}\right] \\ &- \frac{T^{2H+1}}{(2H+1)n^{4H+2}}\left[2H_n^{(-2H-1)} - n^{2H+1}\right]. \end{aligned}$$

□

For further reasoning, we are going to need one elementary theorem from mathematical analysis.

**Theorem A2.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing continuous function and let*

$$S = \sum_{i=1}^{n} f(i), \qquad I = \int_1^n f(x)dx.$$

*Then*

$$I + f(1) \leq S \leq I + f(n).$$

**Proposition A3.** *The following asymptotics occur*

$$\mathbf{E}(Y_{m,k,T})^2 = O(n^{-2H-3}),$$

*where*

$$Y_{m,k,T} = \int_{t_m}^{t_{m+1}} B_s^H ds - \frac{T}{n^2} \sum_{k=1}^{n} B_{s_{k,m}}^H.$$

**Proof.** First, note that from Theorem A1, we obtain

$$\begin{aligned}
\mathbf{E}Y_{m,k,T}^2 =& \mathbf{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds\right)^2 + \mathbf{E}\left(\frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H\right)^2 - 2\mathbf{E}\left(\int_{t_m}^{t_{m+1}} B_s^H ds \frac{T}{n^2}\sum_{k=1}^{n} B_{s_{k,m}}^H\right) \\
=& -\frac{T^{2H+2}}{(2H+1)(2H+2)n^{2H+2}} - \frac{T^{2H+2}}{(2H+1)n^{2H+3}} \\
& - \frac{T^{2H+2}}{n^{4H+3}} H_n^{(-2H)} + \frac{(2H+3)T^{2H+2}}{(2H+1)n^{4H+4}} H_n^{(-2H-1)} \\
\leqslant& -\frac{T^{2H+2}}{(2H+1)(2H+2)n^{2H+2}} - \frac{T^{2H+2}}{(2H+1)n^{2H+3}} - \frac{T^{2H+2}}{n^{4H+3}}\left[\frac{n^{2H+1}}{2H+1} + \frac{2H}{2H+1}\right] \\
& + \frac{(2H+3)T^{2H+2}}{(2H+1)n^{4H+4}}\left[\frac{n^{2H+2}}{2H+2} - \frac{1}{2H+2} + n^{2H+1}\right] \\
\leq& Cn^{-2H-3}.
\end{aligned}$$

□

## Appendix B. Pathwise Integration and fBm

For any $0 < \lambda \leqslant 1$, denote $C^\lambda([0,T])$ the space of $\lambda$-Hölder continuous functions $f\colon [0,T] \to \mathbb{R}$ equipped with the norm

$$\|f\|_\lambda := \|f\|_\infty + \sup_{\substack{s,t\in[0,T] \\ s\neq t}} \frac{|f(t)-f(s)|}{|s-t|^\lambda}, \qquad \|f\|_\infty = \sup_{t\in[0,T]} |f(t)|.$$

Let $1/2 < \gamma < 1$, $\alpha \in (1-\gamma, 1/2)$. Denote $W_\infty^\alpha([0,T])$, the space of real-valued measurable functions $f\colon [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\infty,\alpha;T} = \sup_{s\in[0,T]} \left(|f(s)| + \int_0^s |f(s)-f(u)|(s-u)^{-1-\alpha} du\right) < \infty.$$

**Theorem A3** (Love–Young inequality ([21], p. 10)). *Let $f \in C^{\lambda}([0,T])$ and $g \in C^{\mu}([0,T])$ with $\lambda + \mu > 1$ and*

$$K_f = \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{|f(t)-f(s)|}{|s-t|^{\lambda}}, \qquad K_g = \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{|g(t)-g(s)|}{|s-t|^{\mu}},$$

*Love–Young inequality has the form, for any $y \in [0,T]$,*

$$\left| \int_0^T f\,dg - f(y)\big[g(T)-g(0)\big] \right| \leqslant C_{\lambda,\mu} K_f K_g T^{\lambda+\mu}. \tag{A4}$$

*where $C_{\lambda,\mu} = \zeta(\lambda+\mu)$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.*

**Corollary 3.** *Let $F$ be a continuous differentiable function, $y \in C^{\lambda}([0,T]$, $h \in C^{\mu}([0,T]$, $\lambda+\mu > 1$. Then*

$$\left| \int_0^T [F(y_s) - F(y_a)]\,dh_s \right| \leqslant \sup_{0\leqslant s\leqslant T} F'(y_s) C_{\lambda,\mu} K_y K_h T^{\lambda+\mu}.$$

**Proof.** Note that $F(y) \in C^{\lambda}([0,T])$. In fact, note that

$$|F(y_t) - F(y_s)| \leqslant \sup_{0\leqslant u\leqslant T} F'(y_u)|y_t - y_s| \leqslant \sup_{0\leqslant u\leqslant T} F'(y_u) K_y |t-s|^{\lambda}.$$

**Theorem A4** (Chain rule (see [21], p. 10)). *Let $f = (f_1,\ldots,f_d) : [0,T] \to \mathbb{R}^d$ be a function such that for each $k = 1,\ldots,d$, $f_k \in C^{\lambda}([0,T])$, $\lambda \in (1/2,1]$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function with locally Lipschitz partial derivatives $g'_k$, $k = 1,\ldots,d$. Then each $g'_l \circ f$ is Riemann–Stieltjes integrable with respect to $f_k$ and*

$$(g\circ f)(T) - (g\circ f)(0) = \sum_{k=1}^{d} \int_0^T (g'_k \circ f)\,df_k.$$

**Theorem A5** (Hölder continuity of $B^H$ (see [21], p. 4)). *It is known that almost all sample paths of an fBm $B^H$ are locally Hölder of order strictly less than $H \in (0,1)$. To be more precise, for all $T > 0$, there exists a nonnegative random variable $G_{\gamma,T}$ such that $\mathbb{E}(|G_{\gamma,T}|^p) < \infty$ for all $p \geqslant 1$, and*

$$|B^H_t - B^H_s| \leqslant G_{\gamma,T}|t-s|^{\gamma} \quad a.s.$$

*for all $s,t \in [0,T]$, where $\gamma \in (0,H)$.*

## Appendix C. Almost sure convergence

**Lemma A1** ([8]). *Let $\alpha > 0$ and $K(p) \in (0,\infty)$ for $p \geqslant 1$. In addition, let $Z_n$, $n \in \mathbb{N}$, be a sequence of random variables such that*

$$(\mathbf{E}|Z_n|^p)^{1/p} \leqslant K(p)\cdot n^{-\alpha}$$

*for all $p \geqslant 1$ and all $n \in \mathbb{N}$. Then for all $\varepsilon > 0$, there exists a random variable $\eta_\varepsilon$ such that*

$$|Z_n| \leqslant \eta_\varepsilon \cdot n^{-\alpha+\varepsilon} \qquad \textit{almost surely}$$

*for all $n \in \mathbb{N}$. Moreover, $\mathbf{E}|\eta_\varepsilon|^p < \infty$ for all $p \geqslant 1$.*

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