https://doi.org/10.15388/vu.thesis.386
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VILNIUS UNIVERSITY
JEAN–MONNET UNIVERSITY

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# Multiscale Modelling of Viscous Flows in Domains of Complex Geometry

**DOCTORAL DISSERTATION**

Natural sciences,
Mathematics (N 001)

VILNIUS 2022

This dissertation was written between 2017 and 2021 at Vilnius university and Jean Monnet university. The research was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency.

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This doctoral dissertation will be defended in a public meeting of the Dissertation Defence Panels (Vilnius University and Jean-Monnet University).

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VILNIAUS UNIVERSITETAS
JEAN–MONNET UNIVERSITETAS

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# Daugiaskaliai skysčių tekėjimo modeliai sudėtingos geometrijos srityse

**DAKTARO DISERTACIJA**

Gamtos mokslai,
matematika (N 001)

VILNIUS 2022

Disertacija rengta 2017–2021 metais Vilniaus universitete ir Jean–Monnet universitete. Mokslinius tyrimus rėmė Liono universitetas pagal programą LABEX MILYON (ANR-10-LABX-0070) kartu su Prancūzijos nacionaline tyrimų agentūra pagal programą "Investissements d'Avenir" (ANR-11-IDEX-0007).

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Disetraciją galima peržiūrėti Vilniaus universiteto bibliotekoje ir VU interneto svetainėje adresu:
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# Contents

# Introduction

The Stokes and Navier–Stokes equations are partial differential equations used in fluid mechanics to describe the motion of an incompressible viscous fluid (for example, water, oil, blood, etc.). Besides the theoretical interest, these equations also have a huge practical impact. Modelling of some engineering or biological systems containing fluid, could not be possible without these equations. Multidisciplinary research may let us develop the Stokes and Navier–Stokes equations theory in both directions: theoretical and practical.

We are interested in mathematics applications in medicine, i.e., modelling blood circulation systems. In the thesis, we study some simplified theoretical models to characterize flow in thin cylinders, which may be used to model the blood flow in small and very small vessels and may be extended to model the blood flow in more complex systems, like a human heart, etc.

Let $\Omega$ be some domain in $\mathbb{R}^n$, $n = 2, 3$. Consider periodic in time the non-linear Navier–Stokes system in $\Omega \times (0, 2\pi)$ (without loss of generality, we may assume that the time period is $2\pi$), we get a system

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v} = \boldsymbol{\varphi}, & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega. \end{cases} \tag{1}$$

Here $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \ldots, v_n(x,t))$ and $p = p(x,t)$ are the unknown velocity and pressure, respectively, $\mathbf{f} = \mathbf{f}(x,t) = (f_1(x,t), \ldots, f_n(x,t))$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x,t) = (\varphi_1(x,t), \ldots, \varphi_n(x,t))$ denote the given external force and the boundary value, $\nu > 0$ is the viscosity coefficient, which is constant and depends on physical properties of the fluid, $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ is the periodicity condition with the period $2\pi$, $x = (x_1, \ldots, x_n) \in \Omega$.

The linearized Navier–Stokes equations are called Stokes equations. They have the following form:

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v} = \boldsymbol{\varphi}, & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega. \end{cases} \tag{2}$$

Besides the fact that these equations were formulated almost 200 years ago, they remain the source of many interesting mathematical problems. Some of their remains unsolved, such as: *Existence and smoothness of the Navier–Stokes equation* stated by the Clay Mathematics Institute on May 24, 2000, as one of the seven Millenium Prize Problems. Some long-standing problems are recently solved, such as so-called *Flux Leray's problem.* This problem was open for more than 80 years and recently was solved, but only in two-dimensional and three-dimensional axially-symmetric cases (see [42]), leaving the general three-dimensional case still open.

Since our interest is an application of the Navier–Stokes equations in medicine, i.e., possibility to create a computer model of a blood vessel network, it is very important to study the time-periodic Navier–Stokes equations in thin tube structures. Tube structure as the blood vessel network, together with the periodicity condition, as an illustration of the heart beating.

## Examples of domains

The Stokes and Navier–Stokes problems are considered in different domains. We started our research with the linearized Navier-Stokes equations in the domain with an outlet to infinity. However, for hemodynamical modelling Stokes system is insufficient, and therefore we study the non-linear Navier–Stokes system in a domain which is called a tube structure. We will introduce these domains in the following subsections.

### Domain with an outlet to infinity

Let $\Omega = \Omega_0 \cup D$ be a two-dimensional domain with an outlet to infinity (see Figure 1).

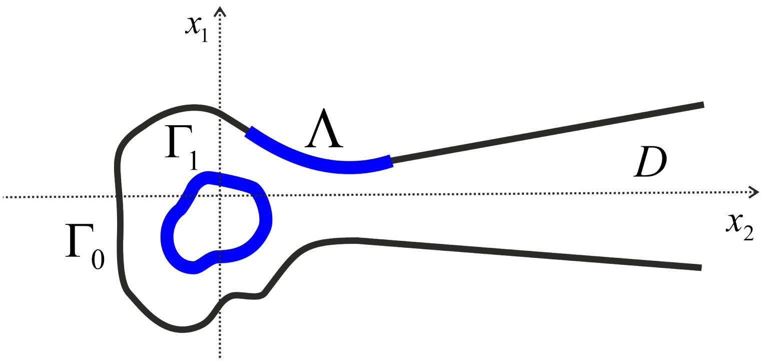


Figure 1 – Domain $\Omega$

Here $\Omega_0 = \Omega \cap B_{R_0}(0) = \Omega \cap \{x \in \mathbb{R}^2 : |x| \leqslant R_0\}$ is a bounded domain with Lipschitz boundary and an outlet $D$ is given by the formula: $D = \{x \in \mathbb{R}^2 : |x_1| < g(x_2),\ x_2 > R_0\}$. We suppose that the function $g$ satisfies the Lipschitz condition

$$|g(t_1) - g(t_2)| \leqslant L|t_1 - t_2|,\ t_1, t_2 > R_0,\ g(t) \geqslant \text{const} > 0$$

and $\partial\Omega \in C^2$. The boundary $\partial\Omega$ consists of the inner boundary $\Gamma_1$ and the outer boundary $\Gamma_0$. Besides, the inner boundary $\Gamma_1$ is compact and the origin $(0,0)$ of the coordinate system is inside the "hole" $\Gamma_1$, while the outer boundary $\Gamma_0$ is unbounded. Let us assume that boundary value $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ [1] has a compact support, and $\Lambda = \operatorname{supp} \boldsymbol{\varphi} \cap \Gamma_0 \subset \Gamma_0 \cap B_{R_0}(0)$. In this domain, we will consider the time-periodic Stokes system (2) with a non-homogeneous boundary condition.

## Tube structure

The domain called tube structure is a mathematical model aimed to describe structures that contain several tubes connected at their points called nodes. More precisely, it is a special structure of thin domains. From the physical point of view, these structures can be used as geometrical models of the blood vessel systems, hydraulic installations, and so on. The domain that contains three tubes is shown in Figure 2.

Here $\Gamma$ is a lateral boundary of the domain $B_\varepsilon$ and $\sigma^1, \sigma^2, \sigma^3$ are inflow-outflow boundaries.

1. The boundary value $\boldsymbol{\varphi}(x)$ is independent of time.

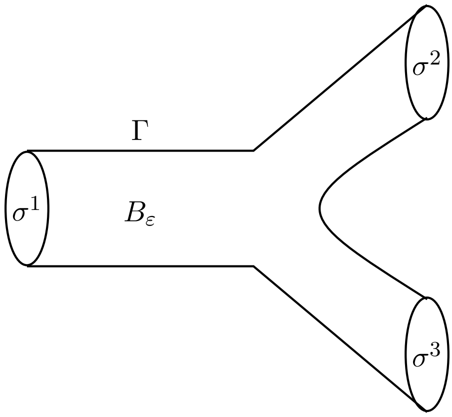


Figure 2 – Domain $B_\varepsilon$

In order to define a tube structure, let us start from the definition of the graph.

**Definition 1.** *Let $O_1, O_2, \ldots, O_N$ be $N$ different points in $\mathbb{R}^n$, $n = 2, 3$, and $e_1, e_2, \ldots, e_M$ be $M$ closed segments each connecting two of these points (i.e. each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \ldots, N\}$, $i_j \neq k_j$). All points $O_i$ are supposed to be the ends of some segments $e_j$. The segments $e_j$ are called edges of the graph. A point $O_i$ is called a node, if it is the common end of at least two edges and $O_i$ is called a vertex, if it is the end of the only one edge. Any two edges $e_j$ and $e_i$ can intersect only at the common node. The set of vertices is supposed to be non-empty.*

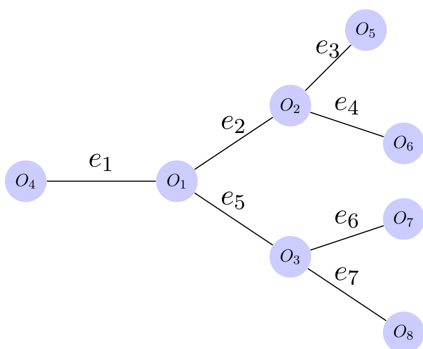


Figure 3 – Graph of the tube structure

Denote $\mathcal{B} = \bigcup\limits_{j=1}^{M} e_j$ the union of edges and assume that $\mathcal{B}$ is a connected set (see Figure 3). The union of all edges having the same end point $O_l$ is called the bundle $\mathcal{B}_l$. Figure 3 is an example of the graph, which presents a union of edges $e_1,\dots,e_7$ and points $O_1,\dots,O_8$, where points $O_1,O_2,O_3$ are called nodes and the points $O_4,O_5,O_6,O_7,O_8$ are called vertices.

Let $e$ be some edge, $e=\overline{O_iO_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in $O_i$ and the axis $O_i x_n^{(e)}$ has the direction of the ray $[O_iO_j)$; the second one has the origin in $O_j$ and the opposite direction, i.e. $O_j\tilde{x}_n^{(e)}$ is directed over the ray $[O_jO_i)$.

Below in various situations, we choose one or another coordinates system denoting the local variable in both cases by $x^{(e)}$ and pointing out which end is taken as the origin of the coordinate system.

With every edge $e_j$ we associate a bounded domain $\sigma^j\subset\mathbb{R}^{n-1}$ containing the origin $O_i$ and having $C^2$- smooth boundary $\partial\sigma^j$, $j=1,\dots,M$. For every edge $e_j=e$ and the associated $\sigma^j=\sigma^{(e)}$ we denote by $\Pi_\varepsilon^{(e)}$ the cylinder

$$\Pi_\varepsilon^{(e)}=\Big\{x^{(e)}\in\mathbb{R}^n : x_n^{(e)}\in(0,|e|),\frac{x^{(e)\prime}}{\varepsilon}\in\sigma^{(e)}\Big\},$$

where $x^{(e)\prime}=(x_1^{(e)},\dots,x_{n-1}^{(e)})$, $|e|$ is the length of the edge $e$ and $\varepsilon>0$ is a small parameter. Notice that the edges $e_j$ and the Cartesian coordinates of nodes and vertices $O_j$, as well as the domains $\sigma^j$, do not depend on $\varepsilon$.

Let $O_1,\dots,O_{N_1}$ be nodes and $O_{N_1+1},\dots,O_N$ be vertices. Let $\omega^1,\dots,\omega^{N_1}$ be bounded independent of $\varepsilon$ domains in $\mathbb{R}^n$; introduce the nodal domains $\omega_\varepsilon^j=\Big\{x\in\mathbb{R}^n:\dfrac{x-O_j}{\varepsilon}\in\omega^j\Big\}$.

Every vertex $O_j$ is the end of one and only one edge $e_k$ which will also be denoted as $e_{O_j}$; we will re-denote as well the domain $\sigma^k$ associated to this edge as $\sigma^{O_j}$. Notice that the subscript $k$ may be different from $j$.

**Definition 2.** *By a tube structure, we call the following domain*

$$B_\varepsilon=\Big(\bigcup_{j=1}^{M}\Pi_\varepsilon^{(e_j)}\Big)\bigcup\Big(\bigcup_{j=1}^{N_1}\omega_\varepsilon^j\Big).$$

*Suppose that it is a connected set and that the boundary $\partial B_\varepsilon$ of $B_\varepsilon$ is $C^2$-smooth (see Figure 4).*

For the case with given Bernoulli pressure we will define as well a semi-infinite dilated cylinder $\Pi_\infty^{(e)}=\Big\{x^{(e)}\in\mathbb{R}^n : x_n^{(e)}\in[0,\infty),\ x^{(e)\prime}\in\sigma^{(e)}\Big\}$, and slightly modified the definition of a tube structure.

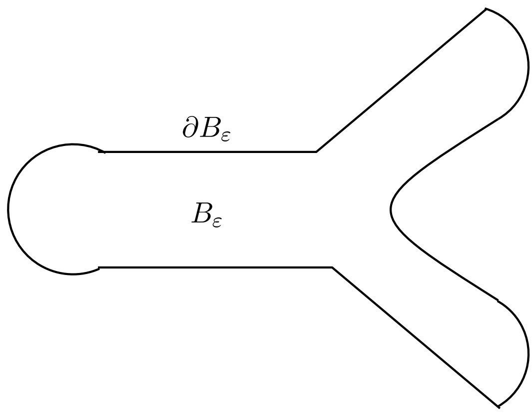


Figure 4 – Domain $B_\varepsilon$

**Definition 3.** *By a tube structure, we call the domain $B_\varepsilon$ (see Definition 2) with the boundary $\partial B_\varepsilon$ of $B_\varepsilon$ which is $C^2$-smooth except for the parts of the boundary which are the bases $\gamma_\varepsilon^j = \{x^{(e)\prime} \in \sigma^{O_j},\ x_n^{(e)} = 0\}$ of cylinders $\Pi_\varepsilon^{(e)}$ (see Figure 2).*

Let $r_1$ be the maximal diameter of the domains $\omega^i,\ i = 1,...,N$, denote $r = r_1 + 1$. Consider a node or a vertex $O_l$ and all edges $e_j$ having $O_l$ as one of their end points. We call the union of all these edges a bundle of edges and denote it $\mathcal{B}_l$, i.e., $\mathcal{B}_l = \bigcup\limits_{j:O_l \in e_j} e_j$. By a bundle of cylinders $B_{O_l}$ we call the union $\omega_\varepsilon^l \cup \Big( \bigcup\limits_{j:O_l \in e_j} \Pi_\varepsilon^{(e_j)} \Big)$. We will consider as well the half-bundle $HB_{O_l} = \omega_\varepsilon^l \cup \Big( \bigcup\limits_{j:O_l \in e_j} \{x \in \Pi_\varepsilon^{(e_j)},\ x_n^{(e_j)} \in [0, |e_j|/2]\} \Big)$. We will use also $\Omega_l = \omega^l \cup \Big( \bigcup\limits_{j:O_l \in e_j} \Pi_\infty^{(e_j)} \Big)$ as a bundle of dilated cylinders. Denote also $\Omega_l^\varepsilon = \Big\{ x \in \mathbb{R}^n : \dfrac{x}{\varepsilon} \in \Omega_l \Big\}$.

In this thesis, we consider two different cases in a thin tube structure: the time-periodic Navier–Stokes equations with Dirichlet boundary conditions and the stationary Navier–Stokes equations with given Bernoulli pressure on the inflow-outflow boundary.

In the first case, we consider the time-periodic Navier–Stokes equation with a non-homogeneous boundary condition

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, & (x,t)\in B_\varepsilon\times(0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t)\in B_\varepsilon\times(0,2\pi), \\ \mathbf{v} = \mathbf{g}, & (x,t)\in \partial B_\varepsilon\times(0,2\pi), \\ \mathbf{v}(x,t) = \mathbf{v}(x,t+2\pi), & x\in B_\varepsilon. \end{cases} \tag{3}$$

Here the velocity field $\mathbf{v}=\mathbf{v}(x,t)$ and a fluid pressure $p=p(x,t)$ are unknown, while the boundary condition $\mathbf{g}=\mathbf{g}(x,t)=(g_1(x,t),g_2(x,t),g_3(x,t))$ is zero everywhere except for some small parts of the boundary $\gamma_\varepsilon^j=\partial B_\varepsilon\cap\partial\omega_\varepsilon^j$, $j=N_1+1,\ldots,N$.

In the second case, we study the stationary Navier–Stokes equations with the given Bernoulli pressure for inflow-outflow boundary

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, & x\in B_\varepsilon, \\ \operatorname{div}\mathbf{v} = 0, & x\in B_\varepsilon, \\ \mathbf{v} = 0, & x\in \partial B_\varepsilon\setminus\cup_{j=N_1+1}^{N}\gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}} = 0, & x\in\gamma_\varepsilon^j, \\ -\nu\partial_n(\mathbf{v}\cdot\mathbf{n}) + \Big(p+\frac{1}{2}|\mathbf{v}|^2\Big) = c_j/\varepsilon^2, & x\in\gamma_\varepsilon^j,\ j=N_1+1,...,N, \end{cases} \tag{4}$$

where $\mathbf{v}_{\boldsymbol{\tau}}=\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{v}$, $\partial_n h=\nabla h\cdot\mathbf{n}$ is the normal derivative of $h$, $c_j$ are some constants.

## Actuality and literature review

The Stokes and stationary Navier–Stokes equations with homogeneous boundary conditions were intensively studied in domains with outlets to infinity during the last 50 years (see [27, 31, 50, 51, 76, 77] and the literature cited there). In the last 20 years, special attention was given to problems with non-homogeneous boundary conditions (see [9, 32, 34, 35, 58, 59, 60, 61, 63, 64]). Moreover, recently big progress was obtained in Leray's problem in the bounded and exterior domains [38, 39, 40, 41, 42, 43, 44]. On the other hand, the time-periodic problem for the Navier–Stokes equations were mainly studied only in the case of homogeneous boundary conditions (see, for example, [46, 52, 53]). The time-periodic problems with non-homogeneous boundary conditions were essentially considered by H. Morimoto [62] and

T. Kobayashi [37]. However, they consider the case only of domains with compact boundaries. A wide review and study of periodic problems could be found in the habilitation thesis of M. Kyed [47]. Also, you may find many interesting periodic problems in G. Galdi and his colleagues' works for example [6, 10, 12, 18, 19, 20, 21, 22, 23, 24, 25, 26].

We start with the linearized Navier-Stokes equations in the domain with an outlet to infinity (see Figure 1), where the boundary condition depends only on the space variable. This problem was generalized by K. Kaulakytė and K. Pileckas in [33]. The analysis of the Stokes system gives the possibility to study the time-periodic Navier–Stokes system.

However, the dissertation was motivated by the modelling of the blood circulation system. The studies were done in 2017 - 2021 Junior research fellow of the research grant "Multiscale Modeling for Viscous Flows in Domains with Complex Geometry".[2] For this research the Stokes system is insufficient, that's why we consider the time-periodic Navier–Stokes problem. Besides we have to deal with the structure which may represent the blood circulation system. For this reason, we introduce a thin tube structure (see Figure 2, 4). Each such tube structure may be schematically represented by its graph: letting the thickness of tubes tend to zero, we find out that tubes degenerate to segments, and we get a one-dimensional case. However, the existing one-dimensional models and codes cannot give the required accuracy in the neighbourhoods of the clot formation zones, stents, and bifurcation of vessels. On the other hand, the completely three-dimensional computations are currently very time-consuming and can be applied only to small parts of the blood circulation system. That is why we suggest the hybrid dimension models, combining the one-dimensional reduction in the regular zones with three-dimensional zooms in the small zones of singular behaviour. This approach for the non-stationary initial boundary value problems for the Navier–Stokes equations in thin tube structures was proposed in [70, 71]. However, for the hemodynamical modelling, more natural are time-periodic settings. In order to combine one-dimensional and three-dimensional models, we need to construct the asymptotic expansion of a solution. We consider

2. This project has received funding from European Social Fund (project No. 09.3.3-LMT-K-712-01-0012) under grant agreement with the Research Council of Lithuania (LMTLT).

the problem in two different scalings concerning the small parameter $\varepsilon$ equal to the ratio of the diameter of vessels to their length: one of them is the same as in [70], while the other generates a big coefficient of order $\varepsilon^{-2}$ of the time derivative of the velocity. And then the constructed asymptotic expansion is used for the construction of a special numerical method combining a one-dimensional description with three-dimensional zooms. This method is called a method of asymptotic partial decomposition of the domain (MAPDD). It reduces the full geometry settings to the computations in neighbourhoods of bifurcation zone of a diameter of order $\varepsilon|\ln\varepsilon|$. This method for initial data was studied in [5].

An alternative approach was developed by A. Quarteroni's team [15]. However, this method of the junction of one-dimensional and three-dimensional zones is different because it is based on multi-physics modelling: the one-dimensional hyperbolic equations and three-dimensional models are derived independently of conservation laws, and then coupling is based on the ideas of consistency of numerical schemes implementing these models. On the contrary, the MAPDD starts from the Navier–Stokes equation written everywhere in the blood flow area, it rigorously derives the one-dimensional Poiseuille type equations in the main part of the domain with three-dimensional zooms in small parts near the bifurcations of the vessels and clot formations zones. It prescribes mathematically justified size of the zoomed areas and asymptotically exact junction conditions. Numerous computational tests show that the multi-physics approach with hyperbolic one-dimensional models is more convenient for the description of thick vessels (for example arteries), while the MAPDD model works better for small vessels such as arterioles.

As was mentioned before, we use two different scalings for the Navier–Stokes equations that satisfy different types of vessels such as small and very small arterioles or capillaries. Let us describe these scalings. The experimental data depend on numerous factors: whether the human or animal blood system is considered, if it is healthy or ill, etc. So we take some averaged data for [54, 57]. The characteristic time (period) is 1 second, while the characteristic velocity is about $0.5 \times 10^{-3}$ $m/sec$. Consider two scalings for the characteristic length and diameter of vessels: (1) the length is $10^{-3}$

$m$ and the characteristic diameter $10^{-4}$ $m$, (2) the length is $10^{-2}$ $m$ and the characteristic diameter $10^{-3}$ $m$ (in both cases $\varepsilon = 0.1$). Let us make the change of the space variable $X = 10^{-3}x$ in the case (1), and $X = 10^{-2}x$ in the case (2). Consider case (1). Making the change of the velocity $\mathbf{v} = 10^{-3}\mathbf{V}$ and the change of the pressure $p = 10^3 P$ and taking into account that the dynamic viscosity of the blood is about $4 \times 10^{-3}$ $Pa\ sec$ and its density is $10^3$ $kg/m^3$, we obtain in new variables the Navier–Stokes equation with all coefficients of order one:

$$\frac{\partial \mathbf{V}}{\partial t} - 4\Delta_X \mathbf{V} + 0.5(\mathbf{V} \cdot \nabla_X)\mathbf{V} + \nabla_X P = 0, \quad \nabla \cdot \mathbf{V} = 0.$$

Consider now the case (2). Making the change of the velocity $\mathbf{v} = 10^{-4}\mathbf{V}$ and the change of the pressure $p = 10^1 P$, we obtain in new variables the Navier–Stokes equation with all coefficients, except for the time derivative term, of order one, while the coefficient of the time derivative is $10^2$, i.e., $\varepsilon^{-2}$:

$$10^2 \frac{\partial \mathbf{V}}{\partial t} - 4\Delta_X \mathbf{V} + 5(\mathbf{V} \cdot \nabla_X)\mathbf{V} + \nabla_X P = 0, \quad \nabla \cdot \mathbf{V} = 0.$$

That is why in this thesis, we consider two different settings: with the factor $\varepsilon^{-2}$ and without it.

The existence, uniqueness, and a priori estimates for the Navier–Stokes equation with the initial conditions were published in [70, 71] however, the proof of this for time-periodic settings differs significantly. For this proof, we use Stokes operator extension [79] and take the base consisting of its eigenfunctions. In this case, we obtain more precise estimates. Nevertheless, this technique requires the $C^2$-smoothness of the boundary, that is why we adding smoothing domains near the vertices of the graph of the tube structure. This modification allows as well to improve the estimates for the $J$-th partial sums of an asymptotic expansion of the solution.

The main step that may lead us to the computer simulation is the construction of the asymptotic expansion of the solution. The asymptotic behaviour of solutions of partial differential equations in thin domains is extensively studied in the vast mathematical literature. In particular, the thin tube structures, introduced in [65], are considered as a geometrical model of a blood vessel network (see other approaches to the modelling of blood vessel networks [7, 15, 17]). For the steady-state Navier–Stokes equations in a network of thin tubes, an asymptotic expansion of the solution was firstly constructed in [65, 66]. The small parameter was introduced as the ratio of the thickness to the length of tubes in the network. This asymptotic expansion was used to justify the method of asymptotic partial decomposition of the domain. This method allowed reducing the computational costs that the Navier–Stokes equations posed in thin tube structures. In particular, the full-dimensional computations are only needed in small neighbourhoods of the junction of tubes, while in the largest part of the domain, the computations are one-dimensional. The non-stationary Navier–Stokes equations in such a domain were studied in [70, 71]. However, in these papers, the inflows and outflows were described by the given velocity at the corresponding parts of the boundary. For numerical implementation, the boundary conditions involving pressure for outflow are more natural. That is why such conditions were extensively studied in mathematical literature (see [11, 55, 56]). In particular, [55] studies the stationary Navier–Stokes equations in a tube structure (a bundle of three tubes) with the given pressure at the "free" ends of the tubes. It is well known that this problem has a solution for small data only. Therefore, [55] proves a theorem of existence and uniqueness and constructs a first-order asymptotic approximation.

For the hemodynamics models, more natural, is to consider the Navier–Stokes equations with the Neumann type boundary conditions to get the appropriate asymptotic expansion. These cases are not widely studied for time-periodic data that is why we consider steady-state Navier–Stokes equations with the inflow-outflow boundary conditions for the Bernoulli pressure. These non-linear boundary conditions were studied first in [11, 28, 36] for an incompressible fluid with small data where the existence of the solution was proved. Since then, these conditions were considered in different contexts: in [45] these conditions were studied for arbitrary data in a finite pipe; in [3] a special model of the decomposition of the boundary value problem for the non-Newtonian flow with the Bernoulli boundary conditions and some special newly introduced interface conditions inside the domain was studied, and the existence of a weak solution to this problem was proved.

In this dissertation, we will construct an asymptotic expansion of a weak solution of the stationary Navier–Stokes equations in the whole thin tube structure with the Bernoulli boundary conditions for the inflows and outflows. We also prove the existence and uniqueness of the solution taking into account the domain dependence on the small parameter.

For the incompressible Navier–Stokes equation in a network of thin tubes when the flow is time-periodic, at the leading order, we obtain a well-posed [69, 72] asymptotic model for the macroscopic pressure on the graph of the network with the Poiseuille profiles in the tubes for the velocity, a Kirchhoff type at the junction of the tubes and the continuity condition for the pressure. This problem for stationary and time-dependent flow for the first time was proposed in [68] which generalizes the so-called Reynolds' equation for the case of a network of tubes. The numerical solution for the non-stationary Navier–Stokes equation with initial data in a network of thin tubes was considered in [8].

## Aims and problems

The main aim of this dissertation is to analyse the Navier–Stokes equations in a thin tube structure. We started our research with the linearized Navier-Stokes equations in the domain with an outlet to infinity. After that, we present the results obtained by studying Navier–Stokes systems in a thin tube structure. These theoretical results may be used to create and develop a numerical simulation for the blood vessels. We

- prove the existence and uniqueness of the time-periodic Stokes equation by constructing some special boundary value extension in the domain with an outlet to infinity,
- prove the existence and uniqueness of the time-periodic and steady-state Navier–Stokes equations in thin tube structures,
- construct the asymptotic expansion of the solution and justified it for the time-periodic Navier–Stokes system and a stationary Navier–Stokes system with the given Bernoulli pressure,
- develop the method of asymptotic partial decomposition of the domain (MAPDD) for the time-periodic Navier–Stokes equations.

## Methods

In the thesis, we use standard methods of functional analysis, the properties of Sobolev spaces and Stokes operator, and both partial and ordinary differential equations theory. We apply asymptotic analysis ideas and techniques to construct special asymptotic expansions. We use the method of asymptotic partial decomposition of the domain, which let us combine one-dimensional and three-dimensional models and reduce the computational cost.

## Novelty

All results of this thesis are new. To our best knowledge, the results for the time-periodic Stokes system in a domain with an outlet to infinity, the results for the time-periodic Navier–Stokes system in thin tube structures, and the results for the steady-state Navier–Stokes equations with the given Bernoulli pressure for inflow and outflow boundary are new. The asymptotic expansions of the solution to these problems in the thin tube structures were unknown.

## Structure of the dissertation and main results

The dissertation consists of five chapters, conclusion, and bibliography. The first chapter is an introduction to the research field and the special research domains. It contains the history and actuality of the problem, examples of the studying domains as well as required information concerning the dissemination of results presented in the thesis.

Chapter 2, for the readers' convenience, introduces the notations and auxiliary results which we use in the dissertation.

In Chapter 3 we present the time-periodic Stokes system with a non-homogeneous boundary conditions in a domain with an outlet to infinity. We construct a special extension of the non-homogeneous boundary value and prove the existence and uniqueness of a weak solution in such a domain.

In Chapter 4 we consider the time-periodic Navier-Stokes equations with Dirichlet boundary conditions in a thin tube structure. The existence and uniqueness theorems for these settings are proved and the asymptotic expansion for a solution (with respect to the small parameter $\varepsilon$ equal to the ratio of the diameter of vessels to their length) was constructed. These asymptotic expansions are then used for the construction of a special numerical method combining one-dimensional description with three-dimensional zooms which is called a method of asymptotic partial decomposition of the domain (MAPDD). This method reduces the full geometry setting to the computations in neighbourhoods of bifurcation zone of a diameter of order $\varepsilon|\ln\varepsilon|$.

In Chapter 5 we consider the steady-state Navier–Stokes equations in a thin tube structure with the given Bernoulli pressure for inflow-outflow boundary. We construct an asymptotic expansion of a weak solution of the stationary Navier–Stokes equations with the non-linear boundary conditions in the whole thin tube structure, justify it, and prove the existence and uniqueness of the solution.

Finally, in Chapter 6, we formulate the conclusions.

## Dissemination

The results of this thesis were presented at the following seminar and conferences

1. Poster "Journées annuelles de l‘équipe MMCS de l‘Institut Camille Jordan", October 1-2, 2018, Lyon, France.
2. Seminar "Journées Scientifiques GDF MORPHEA, École des Mines de Saint-Etienne", June 6, 2019, Saint-Etienne, France.
3. Poster "Journée de la recherche", June 13, 2019, Saint-Etienne, France.
4. International conference "Journée EDP Auverge-Rhône-Alpes", November 7-8, 2019, Chambery, France.
5. International conference "8th European Congress of Mathematic" minisymposium "Multiscale Modeling and Methods: Application in Engineering, Biology and Medicine", June 20-26, 2020, Portorož, Slovenia.
6. International conference "Mathematical Fluid Mechanics In 2022", August 22-26, 2022, Prague, Czech Republic.

Contributing talks were given at the seminars at the Department of Differential Equations (VU) and at the Department of Mathematical Modelling (VILNIUSTECH).

## Publications

The results of this thesis are published in the following papers:

1. R. Juodagalvytė, K. Kaulakytė, Time periodic boundary value Stokes problem in a domain with an outlet to infinity, *Nonlinear Analysis: Modelling and Control* **23**(6), 866–888 (2018).
2. R. Juodagalvytė, G. Panasenko, K. Pileckas, Time periodic Navier–Stokes equations in a thin tube structure, *Boundary Value Problems* **2020**(1), 1–35 (2020).
3. R. Juodagalvytė, G. Panasenko, K. Pileckas, Steady-state Navier–Stokes equations in thin tube structure with the Bernoulli pressure inflow boundary conditions: Asymptotic analysis, *Mathematics* **9**(19), 1–20 (2021).

# Chapter 2

# Notations and preliminary results

## 2.1 Notations and inequalities

We use $c, C, C_j, c_j,\ j = 1, 2, \ldots$ notations for constants which are independent and not have significant meaning in our proof. Besides, the same symbol $c$ may be used to define different constants.

Let $V$ be a Banach space. The norm of the element $u$ in the function space $V$ is denoted by $\|u\|_V$. Vector-valued functions are denoted by bold letters and the spaces of scalar, vector-valued and tensor-valued functions are not distinguished in notation. The vector valued function $\mathbf{u} = (u_1, \ldots, u_n)$ belongs to a space $V$, if $u_i \in V,\ i = 1, \ldots, n$, and $\|\mathbf{u}\|_V = \sum\limits_{i=1}^{n} \|u_i\|_V$. The dual space of $V$ is denoted as $V^*$.

We use the standard notations for different spaces. Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. As usual, denote by $C^\infty(\Omega)$ the set of all infinitely differentiable functions defined on $\Omega$, and let $C_0^\infty(\Omega)$ be the subset of all functions from $C^\infty(\Omega)$ with compact support in $\Omega$. For the given non-negative integers $k$ and $q \geqslant 1,\ L^q(\Omega)$ and $W^{k,q}(\Omega)$ indicate as usual Lebesgue and Sobolev spaces with the norms

$$\|u\|_{L^q(\Omega)} = \Big(\int\limits_{\Omega} |u|^q \,\mathrm{d}x\Big)^{1/q} \text{ and } \|u\|_{W^{k,q}(\Omega)} = \Big(\sum_{|\alpha|=0}^{k} \int\limits_{\Omega} |D^\alpha u(x)|^q \,\mathrm{d}x\Big)^{1/q},$$

where $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},\ |\alpha| = \sum\limits_{j=1}^{n} \alpha_j$. Space $W^{k-1/q,q}(\partial\Omega),\ q > 1$, is a trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$ with the norm

$$\|u\|_{W^{k-1/q,q}}(\partial\Omega) = \inf\{\|\hat{u}\|_{W^{k,q}(\Omega)} : \hat{u} = u \text{ on } \partial\Omega\},$$

and $\mathring{W}^{k,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,q}(\Omega)$.

$L^\infty(\Omega)$ is the space of all essentially bounded function with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

The space $L^q(0,T;V)$ is the space of functions $u$ such that $u(\cdot,t)\in V$ for almost all $t\in[0,T]$ and the norm

$$\|u\|_{L^q(0,T;V)} = \Big(\int\limits_0^T \|u(\cdot,t)\|_V^q \,\mathrm{d}t\Big)^{1/q} \leqslant \infty, \quad 1\leqslant q<\infty.$$

We also use certain weighted function spaces. The corresponding definitions will be given in sections where such space appear for the first time.

Let $T$ be a positive number. The notation $V_{\mathrm{per}}$ means that elements of the space $V$ are $T$-periodic functions, i.e., $u(\,\cdot\,,t)=u(\,\cdot\,,t+T)$. Without loss of generality we may assume that $T=2\pi$. Let $C^\infty_{\mathrm{per}}(0,T;V)=\{u\in C^\infty(\mathbb{R}) : u(t)=u(t+2\pi),\ \forall t\in[0,T]\}$. We will need two more spaces of periodic functions: $L^2_{\mathrm{per}}(0,2\pi)$ and $W^{1,2}_{\mathrm{per}}(0,2\pi)$, which are supplied by the inner product of $L^2(0,2\pi)$ and $W^{1,2}(0,2\pi)$, respectively.

Let $D(\Omega)$ be the Hilbert space of vector functions formed as the closure of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\mathbf{u}\|_{D(\Omega)}=\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ generated by the scalar product

$$[\mathbf{u},\mathbf{v}]=\int\limits_\Omega \nabla\mathbf{u}:\nabla\mathbf{v}\,\mathrm{d}x,$$

where $\nabla\mathbf{u}:\nabla\mathbf{v}=\sum\limits_{j=1}^n \nabla u_j\cdot\nabla v_j=\sum\limits_{j=1}^n\sum\limits_{k=1}^n \dfrac{\partial u_j}{\partial x_k}\dfrac{\partial v_j}{\partial x_k}$. Denote the set of all solenoidal ($\operatorname{div}\mathbf{u}=0$) vector fields $\mathbf{u}$ from $C_0^\infty(\Omega)$ by $J_0^\infty(\Omega)=\{\mathbf{w}\in C_0^\infty(\Omega) : \operatorname{div}\mathbf{w}=0\}$. By $\widehat{H}(\Omega)$ we indicate the subset of $D(\Omega)$ consisting of solenoidal vector fields and by $H(\Omega)$ - the space formed as the closure of $J_0^\infty(\Omega)$ in the Dirichlet norm. Obviously, $H(\Omega)\subset\widehat{H}(\Omega)$. In general, the spaces $\widehat{H}(\Omega)$ and $H(\Omega)$ do not coincide. However, if $\Omega$ is a bounded domain with Lipschitz boundary, then $H(\Omega)=\widehat{H}(\Omega)$.

Besides, for a tube structure we introduce the function spaces $\widehat{W}^{1,2}_\gamma(B_\varepsilon)$ and $\widehat{J}^{1,2}_\gamma(B_\varepsilon)$. Let $\Gamma=\partial B_\varepsilon\setminus\cup_{j=N_1+1}^N\gamma_\varepsilon^j$ be the lateral surface of the domain $B_\varepsilon$, then

$$\widehat{W}^{1,2}_\gamma(B_\varepsilon)=\{\boldsymbol{\eta}\in W^{1,2}(B_\varepsilon) : \boldsymbol{\eta}|_\Gamma=0,\ \boldsymbol{\eta}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j}=0,\ j=N_1+1,\dots,N\},$$

$$\widehat{J}^{1,2}_\gamma(B_\varepsilon) = \{\boldsymbol{\eta} \in \widehat{W}^{1,2}_\gamma(B_\varepsilon) : \operatorname{div} \boldsymbol{\eta} = 0\}.$$

We also introduce a subspace $J^{1,2}_\gamma(B_\varepsilon)$ of $\widehat{J}^{1,2}_\gamma(B_\varepsilon)$ defined by

$$J^{1,2}_\gamma(B_\varepsilon) = \{\boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon) : \int\limits_{\gamma^j_\varepsilon} \boldsymbol{\eta}\cdot \mathbf{n}\, \mathrm{d}S = 0,\ j = N_1+1,\dots,N\}.$$

For the tube structure $B_\varepsilon$ some multiplicative inequalities hold. We shall prove it. First we construct two coverings of the domain $B_\varepsilon$. Take domains $A^{(e_j)}_{\varepsilon,k} = \{x \in \Pi^{(e_j)}_\varepsilon : x^{(e_j)}_n \in \varepsilon(k-2,k+2)\},\ j = 1,\dots,M,\ k = 2,\dots,L^j_\varepsilon,\ L^j_\varepsilon \sim |e|\varepsilon^{-1}$, and define $A^{(j)}_{\varepsilon,k} = \omega^j_\varepsilon \cup \{x \in \Pi^{(e_j)}_\varepsilon : x^{(e_j)}_n \in (0,2\varepsilon)\},\ j = N_1+1,\dots,N$ (i.e., when $O_j$ are vertices), and $A^{(j)}_{\varepsilon,k} = \omega^j_\varepsilon \cup \bigcup\limits_{k_j}\{x \in \Pi^{(e_{k_j})}_\varepsilon : x^{(e_{k_j})}_n \in (0,2\varepsilon)\}$, $j = 1,\dots,N_1$ (i.e., when $O_j$ are nodes), where the union over $k_j$ is taken over all edges of the bundle $\mathcal{B}_j$ associated with the node $O_j$.

In parallel with the covering

$$\mathfrak{A}_\varepsilon = \left(\bigcup_{j=1}^{M}\bigcup_{k=2}^{L^j_\varepsilon} A^{(e_j)}_{\varepsilon,k}\right) \bigcup \left(\bigcup_{j=1}^{N} A^{(j)}_{\varepsilon,k}\right)$$

we take the covering

$$\widetilde{\mathfrak{A}}_\varepsilon = \left(\bigcup_{j=1}^{M}\bigcup_{k=2}^{\widetilde{L}^j_\varepsilon} \widetilde{A}^{(e_j)}_{\varepsilon,k}\right) \bigcup \left(\bigcup_{j=1}^{N} \widetilde{A}^{(j)}_{\varepsilon,k}\right)$$

of $B_\varepsilon$ containing larger domain: $\widetilde{A}^{(e_j)}_{\varepsilon,k} = \{x \in \Pi^{(e_j)}_\varepsilon : x^{(e_j)}_n \in \varepsilon(k-3,k+3)\}$, $j = 1,\dots,M,\ k = 3,\dots,\widetilde{L}^j_\varepsilon,\ \widetilde{L}^j_\varepsilon \sim |e|\varepsilon^{-1}$. Then we define $A^{(j)}_{\varepsilon,k} = \omega^j_\varepsilon \cup \{x \in \Pi^{(e_j)}_\varepsilon : x^{(e_j)}_n \in (0,3\varepsilon)\}$, $j = N_1+1,\dots,N$, and $A^{(j)}_{\varepsilon,k} = \omega^j_\varepsilon \cup \bigcup\limits_{k_j}\{x \in \Pi^{(e_{k_j})}_\varepsilon : x^{(e_{k_j})}_n \in (0,3\varepsilon)\}$, $j = 1,\dots,N_1$. Obviously,

$$A^{(e_j)}_{\varepsilon,k} \subset \widetilde{A}^{(e_j)}_{\varepsilon,k}, \quad A^{(j)}_{\varepsilon,k} \subset \widetilde{A}^{(j)}_{\varepsilon,k}.$$

The constructed covering has a finite multiplicity $\kappa_0$ which is independent of $\varepsilon$.

**Lemma 2.1.1.** *(Poincaré–Friedrich's inequality) The following inequality*

$$\|u\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon \|\nabla u\|_{L^2(B_\varepsilon)}, \quad \forall u \in \mathring{W}^{1,2}(B_\varepsilon) \tag{2.1}$$

*holds with the constant $c$ independent of $\varepsilon$.*

*Proof.* In any bounded domain $\Omega$ for $u \in \mathring{W}^{1,2}(\Omega)$ holds the Poincaré inequality (see [14])

$$\|u\|^2_{L^2(\Omega)} \leqslant c\|\nabla_y u\|^2_{L^2(\Omega)},$$

where $c$ is an absolute constant.

By scaling $y = \dfrac{x}{\varepsilon}$, in any domain $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, we get the estimate

$$\|u\|^2_{L^2(A_\varepsilon)} \leqslant c\varepsilon^2\|\nabla_x u\|^2_{L^2(A_\varepsilon)}.$$

Summing the above inequalities over all domains $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, we obtain

$$\|u\|^2_{L^2(B_\varepsilon)} \leqslant \sum_{A_\varepsilon}\|u\|^2_{L^2(A_\varepsilon)} \leqslant c\varepsilon^2\sum_{A_\varepsilon}\|\nabla_x u\|^2_{L^2(A_\varepsilon)} \leqslant c\varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}.$$

□

**Lemma 2.1.2.** *(Ladyzhenskaya inequalities)*

$$\|u\|_{L^4(B_\varepsilon)} \leqslant 2^{1/4}\|u\|^{1/2}_{L^2(B_\varepsilon)}\|\nabla u\|^{1/2}_{L^2(B_\varepsilon)} \quad \forall u \in \mathring{W}^{1,2}(B_\varepsilon),\ B_\varepsilon \subset \mathbb{R}^2, \tag{2.2}$$

$$\|u\|_{L^4(B_\varepsilon)} \leqslant (4/3)^{3/8}\|u\|^{1/4}_{L^2(B_\varepsilon)}\|\nabla u\|^{3/4}_{L^2(B_\varepsilon)} \ \forall u \in \mathring{W}^{1,2}(B_\varepsilon),\ B_\varepsilon \subset \mathbb{R}^3, \tag{2.3}$$

$$\|u\|_{L^6(B_\varepsilon)} \leqslant 48^{1/6}\|\nabla u\|_{L^2(B_\varepsilon)} \qquad \forall u \in \mathring{W}^{1,2}(B_\varepsilon),\ B_\varepsilon \subset \mathbb{R}^3. \tag{2.4}$$

*The constants in* (2.2)-(2.4) *are independent of* $\varepsilon$. *In particular,*

$$\|u\|_{L^4(B_\varepsilon)} \leqslant c\varepsilon^{1/2}\|\nabla u\|_{L^2(B_\varepsilon)} \quad \forall u \in \mathring{W}^{1,2}(B_\varepsilon),\ B_\varepsilon \subset \mathbb{R}^2, \tag{2.5}$$

$$\|u\|_{L^4(B_\varepsilon)} \leqslant c\varepsilon^{1/4}\|\nabla u\|_{L^2(B_\varepsilon)} \quad \forall u \in \mathring{W}^{1,2}(B_\varepsilon),\ B_\varepsilon \subset \mathbb{R}^3. \tag{2.6}$$

The proofs of (2.2)-(2.4) can be found in [49].

**Lemma 2.1.3.** *Let* $B_\varepsilon \subset \mathbb{R}^2$, $u \in W^{1,2}(B_\varepsilon)$. *Then*

$$\|u\|^4_{L^4(B_\varepsilon)} \leqslant c\varepsilon^{-2}\|u\|^2_{L^2(B_\varepsilon)}\left(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}\right) \tag{2.7}$$

*with the constant* $c$ *independent of* $\varepsilon$.

*Proof.* In any bounded Lipschitz domain $\Omega$ holds the inequality (see [48])

$$\|u\|^4_{L^4(\Omega)} \leqslant c(\Omega)\|u\|^2_{L^2(\Omega)}\left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right),$$

and, by Young inequality,

$$\|u\|^4_{L^4(\Omega)} \leqslant \delta^{-2}\|u\|^4_{W^{1,2}(\Omega)} + c\delta^2\|u\|^4_{L^2(\Omega)}, \quad \delta > 0.$$

By scaling, it is easy to see that in any domain $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, we get the estimate

$$\|u\|^4_{L^4(A_\varepsilon)} \leqslant \delta^{-2}\varepsilon^{-2}\left(\|u\|^2_{L^2(A_\varepsilon)}+\varepsilon^2\|\nabla u\|^2_{L^2(A_\varepsilon)}\right)^2+c\delta^2\varepsilon^{-2}\|u\|^4_{L^2(A_\varepsilon)},\quad \delta>0.$$

Summing the above inequalities terms over all domains $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, we get

$$\sum_{A_\varepsilon}\|u\|^4_{L^2(A_\varepsilon)} \leqslant \sum_{A_\varepsilon}\|u\|^2_{L^2(A_\varepsilon)}\|u\|^2_{L^2(A_\varepsilon)} \leqslant c\sum_{A_\varepsilon}\|u\|^2_{L^2(B_\varepsilon)}\|u\|^2_{L^2(A_\varepsilon)}$$
$$\leqslant c\|u\|^2_{L^2(B_\varepsilon)}\sum_{A_\varepsilon}\|u\|^2_{L^2(A_\varepsilon)} \leqslant c\|u\|^4_{L^2(B_\varepsilon)}.$$

i.e.,

$$\|u\|^4_{L^4(B_\varepsilon)} \leqslant \delta^{-2}\varepsilon^{-2}\left(\|u\|^2_{L^2(B_\varepsilon)}+\varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}\right)^2+c\delta^2\varepsilon^{-2}\|u\|^4_{L^2(B_\varepsilon)},\quad \delta>0.$$

Putting now $\delta^2=\left(\|u\|^2_{L^2(B_\varepsilon)}+\varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}\right)\|u\|^{-2}_{L^2(B_\varepsilon)}$ yields (2.7). □

**Lemma 2.1.4.** *Let* $B_\varepsilon\subset\mathbb{R}^n$, $n=2,3$, $u\in \mathring{W}^{1,2}(B_\varepsilon)\cap W^{2,2}(B_\varepsilon)$. *Then*

$$\|\nabla u\|^2_{L^2(B_\varepsilon)} \leqslant c\|u\|_{L^2(B_\varepsilon)}\|\nabla^2 u\|_{L^2(B_\varepsilon)}. \tag{2.8}$$

*In particular,*

$$\|\nabla u\|^2_{L^2(B_\varepsilon)} \leqslant c\varepsilon^2\|\nabla^2 u\|^2_{L^2(B_\varepsilon)} \tag{2.9}$$

*with the constant* $c$ *independent of* $\varepsilon$.

*Proof.* In any bounded Lipschitz domain $\Omega$ holds the interpolation inequality (see [48])

$$\|\nabla u\|^2_{L^2(\Omega)} \leqslant c(\delta\|\nabla^2 u\|^2_{L^2(\Omega)}+\delta^{-1}\|u\|^2_{L^2(\Omega)}),\quad \delta>0.$$

Here $\|\nabla^2 h\|^2_{L^2(\Omega)}$ means $\sum_{|\alpha|=2}\|D^\alpha h\|^2_{L^2(\Omega)}$.

By scaling in $\varepsilon$ domain $A_\varepsilon$ (i.e., scaling $A_\varepsilon$ in any direction of "size" $\varepsilon$) we derive

$$\varepsilon^2\|\nabla u\|^2_{L^2(A_\varepsilon)} \leqslant c(\delta\varepsilon^4\|\nabla^2 u\|^2_{L^2(A_\varepsilon)}+\delta^{-1}\|u\|^2_{L^2(A_\varepsilon)}),\quad \delta>0.$$

Summing the above inequalities over all domains $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, we get

$$\varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)} \leqslant c(\delta\varepsilon^4\|\nabla^2 u\|^2_{L^2(B_\varepsilon)}+\delta^{-1}\|u\|^2_{L^2(B_\varepsilon)}),\quad \delta>0.$$

Putting $\delta=\varepsilon^{-2}\|u\|_{L^2(B_\varepsilon)}\|\nabla^2 u\|^{-1}_{L^2(B_\varepsilon)}$ implies (2.8). □

**Lemma 2.1.5.** *Let $B_\varepsilon \subset \mathbb{R}^3$, $u \in W^{1,2}(B_\varepsilon)$. Then*

$$\|u\|^3_{L^3(B_\varepsilon)} \leqslant c\varepsilon^{-3/2}\|u\|^{3/2}_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)})^{3/4} \tag{2.10}$$

*with the constant $c$ independent of $\varepsilon$.*

*Proof.* By the multiplicative inequality in a bounded Lipschitz domain $\Omega$ (see [48])

$$\|u\|^3_{L^3(\Omega)} \leqslant c\|u\|^{3/2}_{L^2(\Omega)}(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{3/4}$$

and, by the Young inequality,

$$\|u\|^3_{L^3(\Omega)} \leqslant \delta^2\|u\|^2_{W^{1,2}(\Omega)} + c\delta^{-6}\|u\|^6_{L^2(\Omega)}, \quad \forall \delta > 0.$$

Then, by scaling we get, for any domains $A_\varepsilon$ from the covering $\mathfrak{A}_\varepsilon$, the estimate

$$\|u\|^3_{L^3(A_\varepsilon)} \leqslant \delta^2(\|u\|^2_{L^2(A_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(A_\varepsilon)}) + c\delta^{-6}\varepsilon^{-6}\|u\|^6_{L^2(A_\varepsilon)},$$

and summing them over all $A_\varepsilon$ from $\mathfrak{A}_\varepsilon$, we derive

$$\|u\|^3_{L^3(B_\varepsilon)} \leqslant \delta^2(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}) + c\delta^{-6}\varepsilon^{-6}\|u\|^6_{L^2(B_\varepsilon)}.$$

Taking in the last inequality

$$\delta = \varepsilon^{-3/4}\|u\|^{3/4}_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)})^{-1/8},$$

gives (2.10). □

**Lemma 2.1.6.** *Let $B_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$, $u \in W^{1,2}(B_\varepsilon)$. Then*

$$\begin{aligned}\|u\|^4_{L^4(B_\varepsilon)} &\leqslant c\varepsilon^{-2}\|u\|^2_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}) \\ &\leqslant c\varepsilon^2\|\nabla u\|^4_{L^2(B_\varepsilon)}, \quad n = 2\end{aligned} \tag{2.11}$$

*and*

$$\begin{aligned}\|u\|^4_{L^4(B_\varepsilon)} &\leqslant c\varepsilon^{-3}\|u\|_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)})^{3/2} \\ &\leqslant c\varepsilon\|\nabla u\|^4_{L^2(B_\varepsilon)}, \quad n = 3\end{aligned} \tag{2.12}$$

*with the constant $c$ independent of $\varepsilon$.*

This lemma is proved by the same way as the previous one.

**Lemma 2.1.7.** *Let $B_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$, $u \in W^{1,2}(B_\varepsilon)$. Then*

$$\|u\|^2_{L^2(\gamma^j_\varepsilon)} \leqslant c\varepsilon^{-1}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)}) \leqslant c\varepsilon\|\nabla u\|^2_{L^2(B_\varepsilon)} \tag{2.13}$$

*with the constant $c$ independent of $\varepsilon$.*

*Proof.* The inequality (2.13) follows immediately from well known trace estimate

$$\|v\|_{L^2(\partial\Omega)} \leqslant c\|v\|_{W^{1,2}(\Omega)}$$

and scaling argument. □

**Lemma 2.1.8.** *(Agmon's inequality, n=3) Let $B_\varepsilon \subset \mathbb{R}^3$, $u \in \mathring{W}^{1,2}(B_\varepsilon) \cap W^{2,2}(B_\varepsilon)$. Then*

$$\begin{aligned} &\|u\|^4_{L^\infty(B_\varepsilon)} \\ &\leqslant c\varepsilon^{-6}\|u\|_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(B_\varepsilon)})^{3/2}, \end{aligned} \tag{2.14}$$

*with the constant $c$ independent of $\varepsilon$. In particular,*

$$\|u\|_{L^\infty(B_\varepsilon)} \leqslant c\varepsilon^{1/4}\|\nabla^2 u\|_{L^2(B_\varepsilon)}. \tag{2.15}$$

*Proof.* In any bounded Lipschitz domains $\Omega$ holds the multiplicative inequality (see Lemma 13.2 in [1])

$$\|u\|^4_{L^\infty(\Omega)} \leqslant c\|u\|_{L^2(\Omega)}\|u\|^3_{W^{2,2}(\Omega)}.$$

By scaling it is easy to see that in $A_\varepsilon$ we have

$$\begin{aligned} &\|u\|^4_{L^\infty(A_\varepsilon)} \\ &\leqslant c(\varepsilon^{-3/2}\|u\|_{L^2(A_\varepsilon)})(\varepsilon^{-3}\|u\|^2_{L^2(A_\varepsilon)} + \varepsilon^{-1}\|\nabla u\|^2_{L^2(A_\varepsilon)} + \varepsilon\|\nabla^2 u\|^2_{L^2(A_\varepsilon)})^{3/2} \\ &\leqslant c\varepsilon^{-6}\|u\|_{L^2(A_\varepsilon)}(\|u\|^2_{L^2(A_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(A_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(A_\varepsilon)})^{3/2} \\ &\leqslant c\varepsilon^{-6}\|u\|_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(B_\varepsilon)})^{3/2}. \end{aligned}$$

Taking the supremum over all $A_\varepsilon \in \mathfrak{A}_\varepsilon$, we obtain (2.14). The inequality (2.15) follows from (2.14), (2.1) and (2.9). □

**Lemma 2.1.9.** *(Agmon's inequality, n=2) Let $B_\varepsilon \subset \mathbb{R}^2$, $u \in \mathring{W}^{1,2}(B_\varepsilon) \cap W^{2,2}(B_\varepsilon)$. Then*

$$\|u\|^4_{L^\infty(B_\varepsilon)} \leqslant c\varepsilon^{-4}\|u\|^2_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(B_\varepsilon)}), \tag{2.16}$$

*with the constant $c$ independent of $\varepsilon$. In particular,*

$$\|u\|_{L^\infty(B_\varepsilon)} \leqslant c\varepsilon^{1/2}\|\nabla^2 u\|_{L^2(B_\varepsilon)}. \tag{2.17}$$

*Proof.* In any bounded Lipschitz two-dimensional domain $\Omega$ holds the interpolation inequality (see Lemma 13.2 in [1])

$$\|u\|^4_{L^\infty(\Omega)} \leqslant c\|u\|^2_{L^2(\Omega)}\|u\|^2_{W^{2,2}(\Omega)}.$$

By scaling,

$$\begin{aligned}\|u\|^4_{L^\infty(A_\varepsilon)} &\leqslant c\varepsilon^{-4}\|u\|^2_{L^2(A_\varepsilon)}(\|u\|^2_{L^2(A_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(A_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(A_\varepsilon)})\\ &\leqslant c\varepsilon^{-4}\|u\|^2_{L^2(B_\varepsilon)}(\|u\|^2_{L^2(B_\varepsilon)} + \varepsilon^2\|\nabla u\|^2_{L^2(B_\varepsilon)} + \varepsilon^4\|\nabla^2 u\|^2_{L^2(B_\varepsilon)}).\end{aligned}$$

Taking the supremum over all $A_\varepsilon \in \mathfrak{A}_\varepsilon$ we get (2.16). □

## 2.2 Stokes operator

Consider in $B_\varepsilon$ the Dirichlet problem for the Stokes system

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & x \in B_\varepsilon,\\ \operatorname{div}\mathbf{v} = 0, & x \in B_\varepsilon,\\ \mathbf{v} = 0, & x \in \partial B_\varepsilon. \end{cases} \tag{2.18}$$

The weak solution $\mathbf{v} \in H(B_\varepsilon) = \{\mathbf{v} \in \mathring{W}^{1,2}(B_\varepsilon) : \operatorname{div}\mathbf{v} = 0\}$ to (2.18) satisfies the integral identity

$$\nu\int\limits_{B_\varepsilon} \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\,\mathrm{d}x = \int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x \quad \forall\boldsymbol{\eta} \in H(B_\varepsilon),$$

and, hence, the estimate

$$\|\nabla\mathbf{v}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)} \tag{2.19}$$

holds.

**Lemma 2.2.1.** *Let $\partial B_\varepsilon \in C^2$. Then*

$$\|\nabla^2\mathbf{v}\|_{L^2(B_\varepsilon)} \leqslant c\|\mathbf{f}\|_{L^2(B_\varepsilon)} \tag{2.20}$$

*with the constant $c$ independent of $\varepsilon$.*

*Proof.* Let $A_\varepsilon \subset \widetilde{A}_\varepsilon$ be domains from the covering $\mathfrak{A}_\varepsilon$ and $\widetilde{\mathfrak{A}}_\varepsilon$ of $B_\varepsilon$. Consider (2.18) in $\widetilde{A}_\varepsilon$. Making the change of variables $x = \varepsilon^{-1}y$ we transform $A_\varepsilon$ and $\widetilde{A}_\varepsilon$ into the fixed (independent of $\varepsilon$) domains $A$ and $\widetilde{A}$. The Stokes problem in coordinates $y$ takes the form

$$\begin{cases} -\nu\Delta_y \mathbf{v} + \nabla_y(\varepsilon p) = \varepsilon^2 \mathbf{f}, & x \in \widetilde{A}, \\ \operatorname{div}_y \mathbf{v} = 0, & x \in \widetilde{A}, \\ \mathbf{v} = 0, & x \in \partial B_\varepsilon \cap \partial \widetilde{A}. \end{cases} \tag{2.21}$$

ADN local estimates for elliptic problems (see [2]) yield the inequality

$$\begin{aligned} &\|\mathbf{v}\|^2_{L^2(A)} + \|\nabla_y \mathbf{v}\|^2_{L^2(A)} + \|\nabla_y^2 \mathbf{v}\|^2_{L^2(A)} \\ &\leqslant c\Big(\varepsilon^4 \|\mathbf{f}\|^2_{L^2(\widetilde{A})} + \|\mathbf{v}\|^2_{L^2(\widetilde{A})} + \|q - \overline{q}\|^2_{L^2(\widetilde{A})}\Big), \end{aligned} \tag{2.22}$$

where $q = \varepsilon p$, $\overline{q} = \dfrac{1}{|\widetilde{A}|}\displaystyle\int_{\widetilde{A}} q(y)\,\mathrm{d}y$. Since $\displaystyle\int_{\widetilde{A}} (q(y) - \overline{q})\,\mathrm{d}y = 0$, there exists $\mathbf{w} \in \mathring{W}^{1,2}(\widetilde{A})$ such that $\operatorname{div}\mathbf{w} = q(y) - \overline{q}$ in $\widetilde{A}$ and

$$\|\nabla \mathbf{w}\|_{L^2(\widetilde{A})} \leqslant c\|q - \overline{q}\|_{L^2(\widetilde{A})}$$

(see [50]). Multiplying (2.21) by $\mathbf{w}$ and integrating by parts yields

$$\begin{aligned} \|q - \overline{q}\|^2_{L^2(\widetilde{A})} &= \int_{\widetilde{A}} q(y)(q(y) - \overline{q})\,\mathrm{d}y = \int_{\widetilde{A}} q(y)\operatorname{div}\mathbf{w}\,\mathrm{d}y \\ &= \nu\int_{\widetilde{A}} \nabla\mathbf{v} : \nabla\mathbf{w}\,\mathrm{d}y - \varepsilon^2 \int_{\widetilde{A}} \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}y \\ &\leqslant \nu\|\nabla\mathbf{v}\|_{L^2(\widetilde{A})}\|\nabla\mathbf{w}\|_{L^2(\widetilde{A})} + \varepsilon^2\|\mathbf{f}\|_{L^2(\widetilde{A})}\|\mathbf{w}\|_{L^2(\widetilde{A})} \\ &\leqslant c\|\nabla\mathbf{v}\|_{L^2(\widetilde{A})}\|q - \overline{q}\|_{L^2(\widetilde{A})} + c\varepsilon^2\|\mathbf{f}\|_{L^2(\widetilde{A})}\|q - \overline{q}\|_{L^2(\widetilde{A})}. \end{aligned}$$

Therefore,

$$\|q - \overline{q}\|_{L^2(\widetilde{A})} \leqslant c\Big(\|\nabla\mathbf{v}\|_{L^2(\widetilde{A})} + \varepsilon^2\|\mathbf{f}\|_{L^2(\widetilde{A})}\Big). \tag{2.23}$$

From (2.22), using (2.23) and Poincaré–Friedrich's inequality, we derive

$$\|\mathbf{v}\|^2_{L^2(A)} + \|\nabla_y \mathbf{v}\|^2_{L^2(A)} + \|\nabla_y^2 \mathbf{v}\|^2_{L^2(A)} \leqslant c\Big(\varepsilon^4\|\mathbf{f}\|^2_{L^2(\widetilde{A})} + \|\nabla_y\mathbf{v}\|^2_{L^2(\widetilde{A})}\Big).$$

Returning to coordinates $x$, we obtain

$$\varepsilon^4\|\nabla^2\mathbf{v}\|^2_{L^2(A_\varepsilon)} \leqslant c\Big(\varepsilon^4\|\mathbf{f}\|^2_{L^2(\widetilde{A}_\varepsilon)} + \varepsilon^2\|\nabla\mathbf{v}\|^2_{L^2(\widetilde{A}_\varepsilon)}\Big). \tag{2.24}$$

Summing (2.24) by all domains $A_\varepsilon \subset \widetilde{A}_\varepsilon$ we get

$$\|\nabla^2 \mathbf{v}\|^2_{L^2(B_\varepsilon)} \leqslant c\Big(\|\mathbf{f}\|^2_{L^2(B_\varepsilon)} + \varepsilon^{-2}\|\nabla \mathbf{v}\|^2_{L^2(B_\varepsilon)}\Big). \tag{2.25}$$

Estimating the last term in the right hand side of (2.25) by (2.19) we derive (2.20). □

Problem (2.18) can be rewritten in the operator form (without loss of generality we suppose that $\mathbf{f} \in J_0(B_\varepsilon)$, where $J_0(B_\varepsilon)$ is the closure of the set $\{\mathbf{v} \in C_0^\infty(B_\varepsilon) : \operatorname{div} \mathbf{v} = 0\}$ in $L^2(B_\varepsilon)$-norm, by Helmholtz-Weyl theorem (see [49]) $L^2(\Omega) = J_0(\Omega) \bigoplus G(\Omega)$, where $G(\Omega) = \{\mathbf{v} \in L^2(\Omega) : \mathbf{v} = \nabla p$ for some $p \in L^2(\Omega)\}$, we understand $\nabla p$ in the sense of distribution)

$$\widetilde{\Delta} \mathbf{v} = \mathbf{f}, \tag{2.26}$$

where $\widetilde{\Delta} = P\Delta : H(B_\varepsilon) \cap W^{2,2}(B_\varepsilon) \mapsto J_0(B_\varepsilon)$ is an unbounded operator with the domain $H(B_\varepsilon) \cap W^{2,2}(B_\varepsilon)$, and $P$ is the Leray projection onto divergence free vector fields. By the same notation we denote the Friedrich's extension of this operator to the whole space $H(B_\varepsilon)$. $\widetilde{\Delta}$ is called the Stokes operator. It is known that (see [49], [79]):

(i) The Stokes operator has a discrete spectrum:

$$\widetilde{\Delta} \mathbf{w} = \lambda \mathbf{w}, \quad \mathbf{w} \in H(B_\varepsilon),\ \mathbf{w} \neq 0;$$

$\lambda_i > 0$, $\lim\limits_{i\to\infty} \lambda_i \to +\infty$.

(ii) The eigenfunctions $\{\mathbf{w}_k\}_{k=1}^\infty$ of $\widetilde{\Delta}$ constitute an orthogonal basis in $J_0(B_\varepsilon)$ and $H(B_\varepsilon)$, $\|\nabla \mathbf{w}_k\|_{L^2(B_\varepsilon)} = \sqrt{\lambda_k}$, $\|\mathbf{w}_k\|_{L^2(B_\varepsilon)} = 1$. If $\partial B_\varepsilon \in C^2$, then $\mathbf{w}_k \in H(B_\varepsilon) \cap W^{2,2}(B_\varepsilon)$.

For given $\mathbf{w} \in H(B_\varepsilon) \cap W^{2,2}(B_\varepsilon)$ we have

$$\int\limits_{B_\varepsilon} \widetilde{\Delta} \mathbf{w} \cdot \mathbf{v}\, dx = \int\limits_{B_\varepsilon} (-\nu \Delta \mathbf{w} + \nabla p) \cdot \mathbf{v}\, dx = -\nu \int\limits_{B_\varepsilon} \Delta \mathbf{w} \cdot \mathbf{v}\, dx = \nu \int\limits_{B_\varepsilon} \nabla \mathbf{w} \cdot \nabla \mathbf{v}\, dx$$

for any divergence free $\mathbf{v} \in C_0^\infty(B_\varepsilon)$. Then, by density arguments, it follows that

$$\int\limits_{B_\varepsilon} |\widetilde{\Delta} \mathbf{w}|^2\, dx = -\nu \int\limits_{B_\varepsilon} \Delta \mathbf{w} \cdot \widetilde{\Delta} \mathbf{w}\, dx.$$

Thus,

$$\|\widetilde{\Delta} \mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c\|\Delta \mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c\|\nabla^2 \mathbf{w}\|_{L^2(B_\varepsilon)}. \tag{2.27}$$

Moreover, from the equality $\int\limits_{B_\varepsilon} \widetilde{\Delta}\mathbf{w}\cdot\mathbf{v}\,\mathrm{d}x = \nu\int\limits_{B_\varepsilon} \nabla\mathbf{w}\cdot\nabla\mathbf{v}\,\mathrm{d}x$, we obtain

$$\|\widetilde{\Delta}\mathbf{w}\|_{H(B_\varepsilon)^*} \leqslant \nu\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)}.$$

Here $V^*$ means the dual space to $V$.

From (2.20) we get the estimate

$$\|\nabla^2\mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c\|\widetilde{\Delta}\mathbf{w}\|_{L^2(B_\varepsilon)},$$

which together with (2.27) gives

$$c_1\|\nabla^2\mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant \|\widetilde{\Delta}\mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c_2\|\nabla^2\mathbf{w}\|_{L^2(B_\varepsilon)}. \tag{2.28}$$

**Lemma 2.2.2.** *Let* $B_\varepsilon\subset\mathbb{R}^n,\ n=2,3,\ \mathbf{w}\in H(B_\varepsilon)\cap W^{2,2}(B_\varepsilon)$*. Then*

$$\|\mathbf{w}\|_{L^\infty(B_\varepsilon)} \leqslant c\varepsilon^{1/4}\|\nabla^2\mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{1/4}\|\widetilde{\Delta}\mathbf{w}\|_{L^2(B_\varepsilon)}, \quad \text{for } n=3, \tag{2.29}$$

*and*

$$\|\mathbf{w}\|_{L^\infty(B_\varepsilon)} \leqslant c\varepsilon^{1/2}\|\nabla^2\mathbf{w}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{1/2}\|\widetilde{\Delta}\mathbf{w}\|_{L^2(B_\varepsilon)}, \quad \text{for } n=2, \tag{2.30}$$

*with the constant $c$ independent of $\varepsilon$.*

*Proof.* The inequality (2.29) follows from (2.15) and (2.28), while the inequality (2.30) from (2.17) and (2.28). □

## 2.3 Stokes equation in a half-cylinder with Neumann's condition on the base and no-slip condition on the lateral boundary

Let $\Omega$ be a half-cylinder $\omega\times(0,+\infty)$, where $\omega$ is a bounded domain in $\mathbb{R}^{n-1}$ with Lipschitz boundary. $\Gamma$ denotes the lateral boundary $\partial\omega\times(0,+\infty)$, and $\gamma$ the base $\omega\times\{0\}$. Consider the stationary Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{v}(x)+\nabla p(x) = \mathbf{f}(x)+\sum\limits_{m=1}^{n}\dfrac{\partial\mathbf{f}_m(x)}{\partial x_m}, & x\in\Omega,\\ \operatorname{div}\mathbf{v}(x)=0, & x\in\Omega,\\ \mathbf{v}(x)=0, & x\in\Gamma,\\ p(x)=\psi(x'), & x\in\gamma,\\ \mathbf{v}_\tau(x)=0, & x\in\gamma. \end{cases} \tag{2.31}$$

Define $\mathbf{J}_{\Gamma,0} = \{\boldsymbol{\eta} \in W^{1,2}(\Omega) : \operatorname{div} \boldsymbol{\eta} = 0,\ \boldsymbol{\eta}|_\Gamma = 0,\ \boldsymbol{\eta}_\tau|_\gamma = 0\}$. Assume that $\mathbf{f}, \mathbf{f}_m \in L^2(\Omega)$ and $\psi \in L^2(\gamma)$. By a weak solution of problem (2.31) we understand a vector field $\mathbf{v} \in \mathbf{J}_{\Gamma,0}$ satisfying the integral identity

$$\nu \int\limits_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, \mathrm{d}x = \int\limits_\gamma \psi(x') \cdot \boldsymbol{\eta}(x') \cdot \mathbf{n} \, \mathrm{d}x' + \int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x - \sum_{m=1}^n \int\limits_\Omega \mathbf{f}_m \cdot \frac{\partial \boldsymbol{\eta}}{\partial x_m} \, \mathrm{d}x \quad (2.32)$$

for every vector field $\boldsymbol{\eta} \in \mathbf{J}_{\Gamma,0}$.

**Theorem 2.3.1.** *Assume that* $\mathbf{f}$, $\mathbf{f}_m \in L^2(\Omega)$ *and* $\psi \in L^2(\gamma)$. *Then there exists a unique weak solution* $\mathbf{v}$ *of problem* (2.31). *It satisfies the estimate*

$$\|\mathbf{v}\|_{W^{1,2}(\Omega)} \leqslant C\Big(\|\mathbf{f}\|_{L^2(\Omega)} + \sum_{m=1}^n \|\mathbf{f}_m\|_{L^2(\Omega)} + \|\psi\|_{L^2(\gamma)}\Big) \qquad (2.33)$$

*with a constant* $C$ *independent of* $\mathbf{f}$, $\mathbf{f}_m$ *and* $\psi$.

*Proof.* Define in $\mathbf{J}_{\Gamma,0}$ the inner product $[\mathbf{v}, \boldsymbol{\eta}] = \nu \int\limits_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \ \mathrm{d}x$. Using Cauchy-Schwarz inequality and the trace theorem, we get

$$\Big| \int\limits_\gamma \psi(x') \cdot \boldsymbol{\eta}(x') \cdot \mathbf{n} \ \mathrm{d}x' \Big| \leqslant \|\psi\|_{L^2(\gamma)} \|\boldsymbol{\eta} \cdot \mathbf{n}\|_{L^2(\gamma)} \leqslant C \|\psi\|_{L^2(\gamma)} \|\nabla \boldsymbol{\eta}\|_{L^2(\Omega)}.$$

This is linear continuous functional for all $\boldsymbol{\eta} \in \mathbf{J}_{\Gamma,0}$. Then by Riesz representation theorem, we obtain that there exist a unique element $\boldsymbol{\Psi} \in \mathbf{J}_{\Gamma,0}$, such that

$$\int\limits_\gamma \psi(x') \cdot \boldsymbol{\eta}(x') \cdot \mathbf{n} \ \mathrm{d}x' = [\boldsymbol{\Psi}, \boldsymbol{\eta}].$$

Again using Cauchy-Schwarz and Poincaré–Friedrich's inequalities, we prove that the right-hand side of the equation (2.32) is a linear continuous functional, i.e.,

$$\Big| \int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta} \ \mathrm{d}x \Big| \leqslant \|\mathbf{f}\|_{L^2(\Omega)} \|\boldsymbol{\eta}\|_{L^2(\Omega)} \leqslant C \|\mathbf{f}\|_{L^2(\Omega)} \|\nabla \boldsymbol{\eta}\|_{L^2(\Omega)};$$

$$\Big| \sum_{m=1}^n \int\limits_\Omega \mathbf{f}_m \cdot \frac{\partial \boldsymbol{\eta}}{\partial x_m} \ \mathrm{d}x \Big| \leqslant \sum_{m=1}^n \|\mathbf{f}_m\|_{L^2(\Omega)} \Big\| \frac{\partial \boldsymbol{\eta}}{\partial x_m} \Big\|_{L^2(\Omega)} \leqslant \sum_{m=1}^n \|\mathbf{f}_m\|_{L^2(\Omega)} \|\nabla \boldsymbol{\eta}\|_{L^2(\Omega)}.$$

By Riesz theorem there exist a unique $\mathbf{F}, \mathbf{F}_m \in \mathbf{J}_{\Gamma,0}$. Hence

$$\int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta} \ \mathrm{d}x = [\mathbf{F}, \boldsymbol{\eta}]; \quad \sum_{m=1}^n \int\limits_\Omega \mathbf{f}_m \cdot \frac{\partial \boldsymbol{\eta}}{\partial x_m} \ \mathrm{d}x = \sum_{m=1}^n [\mathbf{F}_m, \boldsymbol{\eta}],$$

for every $\boldsymbol{\eta} \in \mathbf{J}_{\Gamma,0}$. So, we can rewrite (2.32) integral identity and we obtain

$$\mathbf{v} = \frac{1}{\nu}\boldsymbol{\Psi} + \frac{1}{\nu}\mathbf{F} - \sum_{m=1}^{n} \mathbf{F}_m,$$

i.e., the solution exist and it is unique.

To get estimate (2.33) in the (2.32) we take $\boldsymbol{\eta} = \mathbf{v}$, then using trace theorem and Poincaré–Friedrich's inequality, we get

$$\|\nabla \mathbf{v}\|^2_{L^2(\Omega)} \leqslant C\|\psi(x')\|_{L^2(\gamma)}\|\nabla \mathbf{v}\|_{L^2(\Omega)} + C\|\mathbf{f}\|_{L^2(\Omega)}\|\nabla \mathbf{v}\|_{L^2(\Omega)} + \sum_{m=1}^{n} \|\mathbf{f}_m\|_{L^2(\Omega)}\|\nabla \mathbf{v}\|_{L^2(\Omega)},$$

i.e.,

$$\|\mathbf{v}\|_{W^{1,2}(\Omega)} \leqslant C\Big(\|\psi(x')\|_{L^2(\gamma)} + \|\mathbf{f}\|_{L^2(\Omega)} + \sum_{m=1}^{n} \|\mathbf{f}_m\|_{L^2(\Omega)}\Big).$$

□

Let us define in a half-cylinder $\Omega$ weighted function spaces. Denote

$$E_\beta(x) = \exp(2\beta x_n). \tag{2.34}$$

Denote by $\mathcal{W}^{l,2}_\beta(\Omega)$, $l \geqslant 0$, the space of functions obtained as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|v\|_{\mathcal{W}^{l,2}_\beta(\Omega)} = \Big(\sum_{|\alpha|=0}^{l} \int_\Omega E_\beta(x)|D^\alpha v(x)|^2 \, \mathrm{d}x\Big)^{1/2}$$

and set $\mathcal{W}^{0,2}_\beta(\Omega) = \mathcal{L}^2_\beta(\Omega)$. Notice that for $\beta > 0$ elements of the space $\mathcal{W}^{l,2}_\beta(\Omega)$ exponentially vanish as $x_n \to \infty$.

Denote $\Omega_\delta = \{x \in \Omega : x_n > \delta\}$. There holds the following theorem.

**Theorem 2.3.2.** *Assume that* $\mathbf{f}$, $\mathbf{f}_m \in \mathcal{L}^2_\beta(\Omega)$, $\beta > 0$. *If* $\beta$ *is sufficiently small, then the weak solution* $\mathbf{v}$ *of problem* (2.31) *belongs to the space* $\mathcal{W}^{1,2}_\beta(\Omega)$.

*Moreover, if* $\partial\omega \in C^2$ *and* $\mathbf{f}_m = 0$, *then for any* $\delta > 0$, $\mathbf{v} \in \mathcal{W}^{2,2}_\beta(\Omega_\delta)$ *and there exists a function* $p \in L^2_{loc}(\Omega)$ *with* $\nabla p \in \mathcal{L}^2_\beta(\Omega_\delta)$ *such that the pair* $(\mathbf{v}, p)$ *satisfies equations* (2.31) *almost everywhere in* $\Omega_\delta$. *There exists a constant* $\widehat{a}$ *such that* $\lim\limits_{x \in \Omega, |x| \to \infty} p(x) = \widehat{a}$ *in the sense*

$$\int_{\Omega_\delta} \exp\{2\beta_1 x_n\}|p(x) - \widehat{a}|^2 \, \mathrm{d}x < \infty \quad \forall \beta_1 \in (0, \beta). \tag{2.35}$$

This assertion is a corollary of Theorem A.1, Theorem A.2 and Proposition A.1 of [70], see also [73], [74]. The regularity of the solution in $\Omega_\delta$, needed for the proof, follows from ADN estimates (see [2]).

# Chapter 3

# Time-periodic Stokes equations

## 3.1 Formulation of the problem

Let us consider the time-periodic Stokes system with non-homogeneous boundary condition

$$\begin{cases} \mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{v}(x,t) = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v}(x,t) = \boldsymbol{\varphi}(x), & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega, \end{cases} \tag{3.1}$$

in two-dimensional multiply connected unbounded domain $\Omega$ with an outlet to infinity (see Figure 3.1).

We will prove the existence and uniqueness of a weak solution to problem (3.1) in such domain.

By a weak solution of problem (3.1) we understand a solenoidal vector field $\mathbf{v}$ with $\nabla\mathbf{v}, \mathbf{v}_t \in L^2(0,2\pi; L^2(\Omega))$ satisfying the boundary condition $\mathbf{v}|_{\partial\Omega} = \boldsymbol{\varphi}$, the time periodicity condition $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ and the integral identity

$$\int_0^{2\pi}\int_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t + \nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t = \int_0^{2\pi}\int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t$$

for all time-periodic $\boldsymbol{\eta} \in L^2(0,2\pi; H(\Omega))$.

Since $\mathbf{v}$ is divergence free, integrating $\operatorname{div}\mathbf{v} = 0$ over the domain $\Omega \cap \{x \in$

$D : x_2 = R\}$ with sufficiently large $R$, we get

$$0 = \int\limits_{\Omega\cap\{x\in\mathbb{R}:x_2=R\}} \operatorname{div}\mathbf{v}\, \mathrm{d}x = \int\limits_{\partial(\Omega\cap\{x\in\mathbb{R}:x_2=R\})} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}x$$
$$= \int\limits_{\Gamma_1} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S + \int\limits_{\Lambda} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S + \int\limits_{\sigma(R)} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S$$
$$= \int\limits_{\Gamma_1} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S + \int\limits_{\Lambda} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S + \int\limits_{\sigma(R)} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S,$$

where $\sigma(R) = (-g(R), g(R))$ is a cross-section of an outlet to infinity $D$ by the line $x_2 = R$. $\Gamma_1$ is the inner boundary and $\Lambda = \operatorname{supp}\boldsymbol{\varphi}\cap\Gamma_0 \subset \Gamma_0\cap B_{R_0}(0)$ (see Figure 3.1).

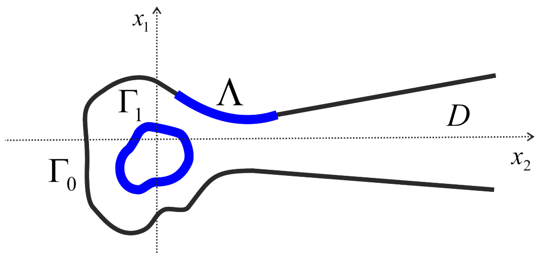


Figure 3.1 – Domain $\Omega$

So, the flow rate of $\mathbf{v}$ over the cross-section $\sigma(R)$ do not depend on $R$ and the following condition has to be fulfilled. Let $\mathcal{F}^{(\mathrm{inn})} = \int\limits_{\Gamma_1} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S$ and $\mathcal{F}^{(\mathrm{out})} = \int\limits_{\Lambda} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S$ be the fluxes of the boundary value $\boldsymbol{\varphi}$ over the inner and the outer boundary, respectively. Then

$$\int\limits_{\sigma(R)} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S = -(\mathcal{F}^{(\mathrm{inn})} + \mathcal{F}^{(\mathrm{out})}).$$

## 3.2 Construction of the extension of the boundary value

In order to reduce a non-homogeneous boundary value condition to the homogeneous one, we shall construct an extension $\mathbf{A}$ of the boundary value $\boldsymbol{\varphi}$. Since $\boldsymbol{\varphi}$ is independent of time, the extension of the boundary value could be constructed using similar ideas as in [32]. Additionally, we need to estimate the term $\|\nabla \mathbf{A}\|$. We construct the extension $\mathbf{A}$ in the following form:

$$\mathbf{A}(x) = \mathbf{B}^{(\mathrm{inn})}(x) + \mathbf{B}^{(\mathrm{out})}(x),$$

where $\mathbf{B}^{(\mathrm{inn})}$ extends the boundary value $\boldsymbol{\varphi}$ from the inner boundary $\Gamma_1$, and $\mathbf{B}^{(\mathrm{out})}$ extends $\boldsymbol{\varphi}$ from the outer boundary $\Gamma_0$.

### 3.2.1 Construction of the extension $\mathbf{B}^{(\mathrm{inn})}$

First, we construct a vector field $\mathbf{b}^{(\mathrm{inn})}$ such that

$$\operatorname{div} \mathbf{b}^{(\mathrm{inn})} = 0, \quad \mathbf{b}^{(\mathrm{inn})}|_{\partial D \cap \partial \Omega} = 0, \quad \int\limits_{\sigma(R)} \mathbf{b}^{(\mathrm{inn})} \cdot \mathbf{n} \, \mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}.$$

Let $\Delta_{\gamma_+}$, $\Delta_{\partial D \cap \partial \Omega}$ and $\Delta_{\partial \Omega \setminus \Lambda}$ be the regularized distances from the point $x \in D$ to the line $\gamma_+ = \overline{\{x : x_1 = 0, \ x_2 \leqslant 0\}}$ and to the boundary $\partial D \cap \partial \Omega$ and to the set $\overline{\partial \Omega \setminus \Lambda}$, respectively. The regularized distance satisfies the following properties.

**Lemma 3.2.1.** *(See [78]) Let $\mathcal{M}$ be a closed set in $\mathbb{R}^2$. Denote by $\Delta_{\mathcal{M}}(x)$ the regularized distance from the point $x$ to the set $\mathcal{M}$. Function $\Delta_{\mathcal{M}}(x)$ is infinitely differentiable in $\mathbb{R}^2 \setminus \mathcal{M}$, and the following estimates*

$$a_1 d_{\mathcal{M}}(x) \leqslant \Delta_{\mathcal{M}}(x) \leqslant a_2 d_{\mathcal{M}}(x), \quad |D^{\alpha} \Delta_{\mathcal{M}}(x)| \leqslant a_3 d_{\mathcal{M}}^{1-|\alpha|}(x), \tag{3.2}$$

*hold, where $d_G(x) = dist(x, G)$ is the distance from $x$ to $\mathcal{M}$, positive constants $a_1, a_2$ and $a_3$ are independent of $\mathcal{M}$, and $|\alpha|$ is an order of differentiation.*

Define in $D$ a Hopf's-type cut-off function

$$\xi(x) = \Psi\left( \ln \frac{\varrho(\Delta_{\gamma_+}(x))}{\Delta_{\partial D \cap \partial \Omega}(x)} \right),$$

where $\Psi$ and $\varrho(\tau)$ are smooth monotone functions

$$\Psi(t) = \begin{cases} 0, & t \leqslant 0, \\ 1, & t \geqslant 1, \end{cases} \quad \text{and} \quad \varrho(\tau) = \begin{cases} \frac{a_1}{2} d_0, & \tau \leqslant \frac{a_2}{2} d_0, \\ t, & \tau \geqslant a_2 d_0, \end{cases} \tag{3.3}$$

where $d_0$ is a positive number such that $\operatorname{dist}(\gamma_+, \partial D \cap \partial \Omega) \geqslant d_0$, and $a_1, a_2$ are positive constants from the estimates of the regularized distance (see Lemma 3.2.1).

**Lemma 3.2.2.** *The function $\xi(x)=0$ at those points of $D$ where $\varrho(\Delta_{\gamma_+}(x))\leqslant \Delta_{\partial D\cap\partial\Omega}(x)$, while the $\mathrm{d}_0/2$-neighbourhood of the line $\gamma_+$ is contained in this set; $\xi(x)=1$ at those points of $D$ where $\Delta_{\partial D\cap\partial\Omega}\leqslant e^{-1}\varrho(\Delta_{\gamma_+}(x))$. The following estimates hold:*

$$\left|\frac{\partial\xi(x)}{\partial x_k}\right|\leqslant\frac{c}{\Delta_{\partial D\cap\partial\Omega}(x)},\qquad \left|\frac{\partial^2\xi(x)}{\partial x_k\partial x_l}\right|\leqslant\frac{c}{\Delta^2_{\partial D\cap\partial\Omega}(x)},$$

$$\left|\frac{\partial^3\xi(x)}{\partial x_k^2\partial x_l}\right|\leqslant\frac{c}{\Delta^3_{\partial D\cap\partial\Omega}(x)}.$$

*Proof.* The proof of the lemma follows directly from the definition of the cut-off function $\xi(x)$, properties of the regularized distance and the fact that $\operatorname{supp}\nabla\xi(x)$ is contained in the set where $\Delta_{\partial D\cap\partial\Omega}(x)\leqslant\varrho(\Delta_{\gamma_+}(x))$. For more details see [76]. □

Let us define the vector field

$$\mathbf{b}_1^{(\mathrm{inn})}(x)=-\mathcal{F}^{(\mathrm{inn})}\left(\frac{\partial\tilde{\xi}(x)}{\partial x_2};-\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right),\qquad x\in D^+=\overline{\{x\in D:x_1>0\}},\quad (3.4)$$

where

$$\tilde{\xi}(x)=\begin{cases}\xi(x), & x\in D^+,\\ 0, & x\in D\setminus D^+.\end{cases}$$

**Lemma 3.2.3.** *The solenoidal vector field $\mathbf{b}_1^{(\mathrm{inn})}(x)$ is infinitely differentiable, vanishes near the boundary $\partial D\cap\partial\Omega$ and the contour $\gamma_+$, the support of $\mathbf{b}_1^{(\mathrm{inn})}(x)$ is contained in the set of points $x\in D^+$ satisfying the inequalities*

$$\varrho(\Delta_{\gamma_+}(x))e^{-1}\leqslant\Delta_{\partial D\cap\partial\Omega}(x)\leqslant\varrho(\Delta_{\gamma_+}(x)).\quad (3.5)$$

*Moreover,*

$$\int\limits_{\sigma(R)}\mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\;\mathrm{d}S=\mathcal{F}^{(\mathrm{inn})},\quad (3.6)$$

*and the following estimates*

$$|\mathbf{b}_1^{(\mathrm{inn})}(x)|\leqslant\frac{c|\mathcal{F}^{(\mathrm{inn})}|}{d(x)},\quad x\in D^+,\ d(x)=\operatorname{dist}(x,\partial D\cap\partial\Omega),\quad (3.7)$$

$$|\mathbf{b}_1^{(\mathrm{inn})}(x)|\leqslant\frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g(x_2)},\quad x\in D,\quad (3.8)$$

$$|\nabla\mathbf{b}_1^{(\mathrm{inn})}(x)|\leqslant\frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^2(x_2)},\quad |\Delta\mathbf{b}_1^{(\mathrm{inn})}(x)|\leqslant\frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^3(x_2)},\quad x\in D,\quad (3.9)$$

*hold.*

*Proof.* Relation (3.5) follows directly from Lemma 3.2.2.

By the construction of $\mathbf{b}_1^{(\mathrm{inn})}$ we easily show (3.7):

$$\int\limits_{\sigma(R)} \mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{-g(R)}^{g(R)} \mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = -\mathcal{F}^{(\mathrm{inn})}\int\limits_{-g(R)}^{g(R)} \left(\frac{\partial\tilde{\xi}(x)}{\partial x_2}; -\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right)\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}x_1$$

$$= -\mathcal{F}^{(\mathrm{inn})}\int\limits_{-g(R)}^{g(R)}\left(-\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right)\mathrm{d}x_1 = \mathcal{F}^{(\mathrm{inn})}\int\limits_{-g(R)}^{g(R)}\frac{\partial\tilde{\xi}(x)}{\partial x_1}\,\mathrm{d}x_1$$

$$= \mathcal{F}^{(\mathrm{inn})}\big(\tilde{\xi}(g(R),R) - \tilde{\xi}(-g(R),R)\big) = \mathcal{F}^{(\mathrm{inn})}.$$

This proves that the flux is independent of $R$.

According to the definition of $\mathbf{b}_1^{(\mathrm{inn})}(x)$ and Lemma 3.2.2, we obtain the following estimates:

$$|\mathbf{b}_1^{(\mathrm{inn})}(x)| \leqslant |\mathcal{F}^{(\mathrm{inn})}|\sqrt{\left(\frac{\partial\tilde{\xi}(x)}{\partial x_2}\right)^2 + \left(\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta_{\partial D\cap\partial\Omega}(x)}; \qquad (3.10)$$

$$|\nabla\mathbf{b}_1^{(\mathrm{inn})}(x)| \leqslant |\mathcal{F}^{(\mathrm{inn})}|\sqrt{\left(\frac{\partial^2\tilde{\xi}(x)}{\partial x_1\partial x_2}\right)^2 + \left(\frac{\partial^2\tilde{\xi}(x)}{\partial x_2\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta^2_{\partial D\cap\partial\Omega}(x)}; \quad (3.11)$$

$$|\Delta\mathbf{b}_1^{(\mathrm{inn})}(x)| \leqslant |\mathcal{F}^{(\mathrm{inn})}|\sqrt{\left(\frac{\partial^3\tilde{\xi}(x)}{\partial x_1^2\partial x_2}\right)^2 + \left(\frac{\partial^3\tilde{\xi}(x)}{\partial x_2^2\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta^3_{\partial D\cap\partial\Omega}(x)}. \quad (3.12)$$

Due to estimates for the regularized distance (3.2), inequality (3.7) follows from (3.10). Notice that for points $x\in\operatorname{supp}\mathbf{b}_1^{(\mathrm{inn})}$ the inequalities

$$c_1 g(x_2) \leqslant d(x) \leqslant c_2 g(x_2)$$

hold, where $c_1, c_2$ are positive constants (see [76] for details). Then estimates (3.8), (3.9) follow from inequalities (3.10)-(3.12). $\square$

Let us define on $\partial\Omega_0$ a vector field

$$\mathbf{h}_1(x) = \begin{cases} 0, & x\in\Gamma_1,\\ \mathbf{b}_1^{(\mathrm{inn})} + \mathbf{b}_\#^{(\mathrm{inn})}, & x\in\partial\Omega_0\cap\partial D,\\ \mathbf{b}_\#^{(\mathrm{inn})}, & x\in\partial\Omega_0\setminus(\Gamma_1\cup(\partial\Omega_0\cap\partial D)),\end{cases}$$

with $\mathbf{b}_1^{(\mathrm{inn})}$ given by (3.4) and $\mathbf{b}_\#^{(\mathrm{inn})}$ defined as following:

$$\mathbf{b}_\#^{(\mathrm{inn})}(x) = \mathcal{F}^{(\mathrm{inn})}\nabla q(x),$$

where $q(x) = -\dfrac{1}{2\pi}\ln|x|$ is a fundamental solution of the Laplace operator in $\mathbb{R}^2$.

Notice that $\mathbf{b}_{\#}^{(\mathrm{inn})}(x)$ is a solenoidal vector field:

$$\operatorname{div}\mathbf{b}_{\#}^{(\mathrm{inn})} = \operatorname{div}\mathcal{F}^{(\mathrm{inn})}\nabla q(x) = \mathcal{F}^{(\mathrm{inn})}\operatorname{div}\nabla q(x) = \mathcal{F}^{(\mathrm{inn})}\Delta q(x) = 0.$$

Since

$$\int\limits_{\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = 1, \qquad \int\limits_{\partial\Omega_0\setminus\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = -1,$$

we have that

$$\int\limits_{\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\Gamma_1} \mathcal{F}^{(\mathrm{inn})}\nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}\int\limits_{\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})},$$

$$\int\limits_{\partial\Omega_0\setminus\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\partial\Omega_0\setminus\Gamma_1} \mathcal{F}^{(\mathrm{inn})}\nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}\int\limits_{\partial\Omega_0\setminus\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = -\mathcal{F}^{(\mathrm{inn})}.$$

Then according to the properties of the vector fields $\mathbf{b}_1^{(\mathrm{inn})}$ and $\mathbf{b}_{\#}^{(\mathrm{inn})}$, we get

$$\begin{aligned}\int\limits_{\partial\Omega_0} \mathbf{h}_1\cdot\mathbf{n}\,\mathrm{d}S &= \int\limits_{\partial\Omega_0\cap\partial D} \mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S + \int\limits_{\partial\Omega_0\setminus\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S \\ &= \mathcal{F}^{(\mathrm{inn})} - \mathcal{F}^{(\mathrm{inn})} = 0.\end{aligned}$$

In order to extend $\mathbf{h}_1$ into $\Omega_0$, first, we define the solenoidal vector field

$$\widetilde{\mathbf{b}}_{01}^{(\mathrm{inn})} = \left(\frac{\partial\mathbf{H}(x)}{\partial x_2}; -\frac{\partial\mathbf{H}(x)}{\partial x_1}\right),$$

where $\mathbf{H}\in W^{2,2}(\Omega_0)$ satisfies the following boundary conditions:

$$\left.\frac{\partial\mathbf{H}(x)}{\partial x_2}\right|_{\partial\Omega_0\cap\partial D} = \left.(b_{11}^{(\mathrm{inn})} + b_{\#1}^{(\mathrm{inn})})\right|_{\partial\Omega_0\cap\partial D},$$

$$\left.-\frac{\partial\mathbf{H}(x)}{\partial x_1}\right|_{\partial\Omega_0\cap\partial D} = \left.(b_{12}^{(\mathrm{inn})} + b_{\#2}^{(\mathrm{inn})})\right|_{\partial\Omega_0\cap\partial D},$$

$$\left.\frac{\partial^2\mathbf{H}(x)}{\partial x_2^2}\right|_{\partial\Omega_0\cap\partial D} = \left.\left(\frac{\partial b_{11}^{(\mathrm{inn})}}{\partial x_2} + \frac{\partial b_{\#1}^{(\mathrm{inn})}}{\partial x_2}\right)\right|_{\partial\Omega_0\cap\partial D},$$

$$\left.\left(\frac{\partial\mathbf{H}(x)}{\partial x_2}; -\frac{\partial\mathbf{H}(x)}{\partial x_1}\right)\right|_{\partial\Omega_0\setminus\Gamma_1\cup(\partial\Omega_0\cap\partial D)} = \mathbf{b}_{\#}^{(\mathrm{inn})}|_{\partial\Omega_0\setminus\Gamma_1\cup(\partial\Omega_0\cap\partial D)}.$$

For the proof of the existence of $\mathbf{H}$, see [49]. Then we extend $\mathbf{h}_1$ into $\Omega_0$ in the form

$$\mathbf{b}_{01}^{(\mathrm{inn})}(x) = \left(\frac{\partial(\kappa(x)\mathbf{H}(x))}{\partial x_2}; -\frac{\partial(\kappa(x)\mathbf{H}(x))}{\partial x_1}\right),$$

where the support of Hopf's-type smooth cut-off function $\kappa$ is contained in the neighbourhood of $\Omega_0 \setminus \Gamma_1$. We get $\mathbf{b}_{01}^{(\mathrm{inn})} \in W^{2,2}(\Omega_0)$ and $\mathbf{b}_{01}^{(\mathrm{inn})}$ satisfies the following estimate:

$$\|\mathbf{b}_{01}^{(\mathrm{inn})}\|_{W^{2,2}(\Omega_0)} \leqslant c\|\mathbf{h}_1\|_{W^{3/2,2}(\partial\Omega_0)}$$
$$\leqslant c(\|\mathbf{b}_{\#}^{(\mathrm{inn})}\|_{W^{3/2,2}(\partial\Omega_0\setminus\Gamma_1)} + \|\mathbf{b}_1^{(\mathrm{inn})}\|_{W^{3/2,2}(\partial\Omega_0\cap\partial D)}) \leqslant c|\mathcal{F}^{(\mathrm{inn})}|,$$

where the constant $c$ depends only on the domain $\Omega_0$ (see [49]).

Next, we define the vector field, which "removes" the non-zero flux from the inner boundary $\Gamma_1$:

$$\mathbf{b}^{(\mathrm{inn})} = \begin{cases} \mathbf{b}_{\#}^{(\mathrm{inn})} - \mathbf{b}_{01}^{(\mathrm{inn})}, & x \in \Omega_0, \\ \mathbf{b}_1^{(\mathrm{inn})}, & x \in D. \end{cases}$$

Notice that by the construction the function $\mathbf{b}^{(\mathrm{inn})}$ and its derivatives $\frac{\partial \mathbf{b}^{(\mathrm{inn})}}{\partial x_1}$, $\frac{\partial \mathbf{b}^{(\mathrm{inn})}}{\partial x_2}$ have no jump discontinuity passing from $\Omega_0$ to $D$. Therefore, $\mathbf{b}^{(\mathrm{inn})} \in W^{2,2}(\Omega)$. Then we define a vector field

$$\mathbf{h}_0 = \begin{cases} \boldsymbol{\varphi} - \mathbf{b}_{\#}^{(\mathrm{inn})}, & x \in \Gamma_1, \\ 0, & x \in \partial\Omega_0 \setminus \Gamma_1, \end{cases}$$

which satisfies the following condition:

$$\int\limits_{\Gamma_1} \mathbf{h}_0 \cdot \mathbf{n}\, \mathrm{d}S = \int\limits_{\Gamma_1} \boldsymbol{\varphi} \cdot \mathbf{n}\, \mathrm{d}S - \int\limits_{\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})} \cdot \mathbf{n}\, \mathrm{d}S = \mathcal{F}^{(\mathrm{inn})} - \mathcal{F}^{(\mathrm{inn})} = 0.$$

Therefore, the function $\mathbf{h}_0$ can be extended inside $\Omega$ in the form

$$\mathbf{b}_0^{(\mathrm{inn})}(x) = \left(\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_2}; -\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_1}\right),$$

where $\mathbf{E}(x) \in W^{2,2}(\Omega_0)$, $\left(\frac{\partial \mathbf{E}(x)}{\partial x_2}, -\frac{\partial \mathbf{E}(x)}{\partial x_1}\right) = \mathbf{h}_0$, the support of Hopf's-type smooth cut-off function $\chi$ is contained in the neighbourhood of $\Gamma_1$ (see [49]).

Finally, we put

$$\mathbf{B}^{(\mathrm{inn})}(x) = \mathbf{b}^{(\mathrm{inn})}(x) + \mathbf{b}_0^{(\mathrm{inn})}(x).$$

The properties of the extension $\mathbf{B}^{(\mathrm{inn})}$ are formulated in the following lemma.

**Lemma 3.2.4.** *The vector field* $\mathbf{B}^{(\mathrm{inn})}$ *is solenoidal,* $\mathbf{B}^{(\mathrm{inn})}|_{\Gamma_1} = \boldsymbol{\varphi}|_{\Gamma_1}$, $\mathbf{B}^{(\mathrm{inn})}|_{\partial\Omega\setminus\Gamma_1} = 0$, $\mathbf{B}^{(\mathrm{inn})} \in W^{2,2}(\bar{\Omega})$ *and satisfies the following estimates:*

$$|\mathbf{B}^{(\mathrm{inn})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g(x_2)}, \quad x \in D,$$

$$|\nabla \mathbf{B}^{(\mathrm{inn})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^2(x_2)}, \quad |\Delta \mathbf{B}^{(\mathrm{inn})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^3(x_2)}, \quad x \in D,$$

$$|\mathbf{B}^{(\mathrm{inn})}(x)| + |\nabla \mathbf{B}^{(\mathrm{inn})}(x)| + |\Delta \mathbf{B}^{(\mathrm{inn})}(x)| \leqslant c|\mathcal{F}^{(\mathrm{inn})}|, \quad x \in \Omega \setminus D.$$

### 3.2.2 Construction of the extension $\mathbf{B}^{(\mathrm{out})}$

Take any point $x^{(1)} \in \Lambda \subset \Gamma_0$. Let $\gamma$ be a smooth simple curve, which intersects $\partial\Omega$ at the point $x^{(1)}$, and

$$\gamma = \hat{\gamma} \cup \gamma_0,$$

where $\hat{\gamma}$ is a semi-infinite line lying in $D$, $\gamma_0$ is a finite simple curve connecting $\hat{\gamma}$ and the point $x^{(1)}$. Assume that $\inf\limits_{x\in\gamma, y\in\partial\Omega\setminus\Lambda} |x-y| \geqslant d_0$.

Define a Hopf's-type cut-off function

$$\zeta(x) = \Psi\left( \ln \frac{\varrho(\Delta_\gamma(x))}{\Delta_{\partial\Omega\setminus\Lambda}(x)} \right),$$

where functions $\Psi$ and $\varrho$ are defined by (3.3).

**Lemma 3.2.5.** *Function* $\zeta(x) = 0$ *if* $\varrho(\Delta_\gamma(x)) \leqslant \Delta_{\partial\Omega\setminus\Lambda}(x)$, *while the* $d_0/2$*-neighbourhood of the curve is contained in this set. Function* $\zeta(x) = 1$ *at those points where* $\Delta_{\partial\Omega\setminus\Lambda}(x) \leqslant e^{-1}\varrho(\Delta_\gamma(x))$. *Moreover, the following estimates hold:*

$$\left|\frac{\partial\zeta(x)}{\partial x_k}\right| \leqslant \frac{c}{\Delta_{\partial\Omega\setminus\Lambda}(x)}, \quad \left|\frac{\partial^2\zeta(x)}{\partial x_k \partial x_l}\right| \leqslant \frac{c}{\Delta^2_{\partial\Omega\setminus\Lambda}(x)},$$

$$\left|\frac{\partial^3\zeta(x)}{\partial x_k^2 \partial x_l}\right| \leqslant \frac{c}{\Delta^3_{\partial\Omega\setminus\Lambda}(x)}.$$

*Proof.* The proof follows directly from the definition of $\zeta(x)$, properties of the regularized distance and the fact that $\operatorname{supp} \nabla\zeta(x)$ is contained in the set where $\Delta_{\partial\Omega\setminus\Lambda}(x) \leqslant \varrho(\Delta_\gamma(x))$. □

Let us introduce the vector field

$$\mathbf{b}^{(\mathrm{out})}(x) = \mathcal{F}^{(\mathrm{out})}\left(\frac{\partial\tilde{\zeta}(x)}{\partial x_2}; -\frac{\partial\tilde{\zeta}(x)}{\partial x_1}\right),$$

where $\tilde{\zeta}(x) = \zeta(x)$ above the curve $\gamma$, and $\tilde{\zeta}(x) = 0$ under the curve $\gamma$.

**Lemma 3.2.6.** *The vector field $\mathbf{b}^{(\mathrm{out})}$ is infinitely differentiable and solenoidal, vanishes near the set $\partial\Omega \setminus \Lambda$ and in a small neighbourhood of the curve $\gamma$. The following estimates hold:*

$$|\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c}{d_{\partial\Omega\setminus\Lambda}(x)}, \quad x \in D, \tag{3.13}$$

$$|\nabla\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c}{d^2_{\partial\Omega\setminus\Lambda}(x)}, \quad |\Delta\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c}{d^3_{\partial\Omega\setminus\Lambda}(x)}, \quad x \in D, \tag{3.14}$$

$$|\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g(x_2)}, \quad x \in D, \tag{3.15}$$

$$|\nabla\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^2(x_2)}, \quad |\Delta\mathbf{b}^{(\mathrm{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^3(x_2)}, \quad x \in D, \tag{3.16}$$

$$\int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\ \mathrm{d}S = \mathcal{F}^{(\mathrm{out})}. \tag{3.17}$$

*Proof.* Estimates (3.13)-(3.16) could be proven in the same way as in Lemma 3.2.3. Due to the construction of $\mathbf{b}^{(\mathrm{out})}$, we get (3.17):

$$\int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\,\mathrm{d}S = -\int\limits_{\sigma(R)} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\,\mathrm{d}S = -\int\limits_{-g(R)}^{g(R)} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\,\mathrm{d}S$$

$$= -\mathcal{F}^{(\mathrm{out})}\int\limits_{-g(R)}^{g(R)}\left(\frac{\partial\tilde{\zeta}(x)}{\partial x_2}; -\frac{\partial\tilde{\zeta}(x)}{\partial x_1}\right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,\mathrm{d}x_1 = -\mathcal{F}^{(\mathrm{out})}\int\limits_{-g(R)}^{g(R)}\left(-\frac{\partial\tilde{\zeta}(x)}{\partial x_1}\right)\mathrm{d}x_1$$

$$= \mathcal{F}^{(\mathrm{out})}\int\limits_{-g(R)}^{g(R)} \frac{\partial\tilde{\zeta}(x)}{\partial x_1}\,\mathrm{d}x_1 = \mathcal{F}^{(\mathrm{out})}\Big(\tilde{\zeta}(g(R),R) - \tilde{\zeta}(-g(R),R)\Big) = \mathcal{F}^{(\mathrm{out})}.$$

□

Let us take

$$\mathbf{h}(x) = \boldsymbol{\varphi}(x)|_{\Lambda} - \mathbf{b}^{(\mathrm{out})}(x)|_{\Lambda}.$$

Then

$$\int\limits_{\Lambda} \mathbf{h}(x)\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\Lambda} \boldsymbol{\varphi}(x)\cdot\mathbf{n}\,\mathrm{d}S - \int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})}(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{out})} - \mathcal{F}^{(\mathrm{out})} = 0,$$

and $\mathbf{h}$ can be extended (see [49]) inside $\Omega$ in the form

$$\mathbf{b}_0^{(\text{out})}(x) = \left(\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_2}; -\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_1}\right),$$

where $\mathbf{E}(x) \in W^{2,2}(\Omega_0)$, $\left(\frac{\partial \mathbf{E}(x)}{\partial x_2}; -\frac{\partial \mathbf{E}(x)}{\partial x_1}\right)\Big|_\Lambda = \mathbf{h}$ and $\chi$ is a Hopf's-type cut-off function such that $\chi = 1$ on $\Lambda$, $\operatorname{supp}\chi$ is contained in a small neighbourhood of $\Lambda$.

Finally, we put

$$\mathbf{B}^{(\text{out})}(x) = \mathbf{b}^{(\text{out})}(x) + \mathbf{b}_0^{(\text{out})}(x).$$

The properties of the extension $\mathbf{B}^{(\text{out})}$ are formulated in the following lemma.

**Lemma 3.2.7.** *The vector field* $\mathbf{B}^{(\text{out})}$ *is solenoidal,* $\mathbf{B}^{(\text{out})}|_\Lambda = \boldsymbol{\varphi}|_\Lambda$, $\mathbf{B}^{(\text{out})}|_{\partial\Omega\setminus\Lambda} = 0$, $\mathbf{B}^{(\text{out})} \in W^{2,2}(\bar{\Omega})$ *and satisfies the following estimates:*

$$|\mathbf{B}^{(\text{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\text{out})}|}{g(x_2)}, \quad x \in D,$$

$$|\nabla \mathbf{B}^{(\text{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\text{out})}|}{g^2(x_2)}, \quad |\Delta \mathbf{B}^{(\text{out})}(x)| \leqslant \frac{c|\mathcal{F}^{(\text{out})}|}{g^3(x_2)}, \quad x \in D,$$

$$|\mathbf{B}^{(\text{out})}(x)| + |\nabla \mathbf{B}^{(\text{out})}(x)| + |\Delta \mathbf{B}^{(\text{out})}(x)| \leqslant c|\mathcal{F}^{(\text{out})}|, \quad x \in \Omega \setminus D.$$

The sum we have constructed is sought extension $\mathbf{A} = \mathbf{B}^{(\text{inn})} + \mathbf{B}^{(\text{out})}$ of the boundary value $\boldsymbol{\varphi}$. The properties of $\mathbf{A}$ are given in the following theorem.

**Theorem 3.2.8.** *The constructed extension* $\mathbf{A} \in W^{2,2}(\Omega)$ *is solenoidal, satisfies the boundary condition* $\mathbf{A}|_{\partial\Omega} = \boldsymbol{\varphi}$ *and the following estimates:*

$$|\mathbf{A}(x)| \leqslant \frac{c(|\mathcal{F}^{(\text{inn})}| + |\mathcal{F}^{(\text{out})}|)}{g(x_2)}, \quad x \in D \tag{3.18}$$

$$|\nabla \mathbf{A}(x)| \leqslant \frac{c(|\mathcal{F}^{(\text{inn})}| + |\mathcal{F}^{(\text{out})}|)}{g^2(x_2)}, \quad x \in D, \tag{3.19}$$

$$|\Delta \mathbf{A}(x)| \leqslant \frac{c(|\mathcal{F}^{(\text{inn})}| + |\mathcal{F}^{(\text{out})}|)}{g^3(x_2)}, \quad x \in D, \tag{3.20}$$

$$|\mathbf{A}(x)| + |\nabla \mathbf{A}(x)| + |\Delta \mathbf{A}(x)| \leqslant c(|\mathcal{F}^{(\text{inn})}| + |\mathcal{F}^{(\text{out})}|), \quad x \in \Omega \setminus D. \tag{3.21}$$

*Proof.* Since $\mathbf{A}(x) = \mathbf{B}^{(\text{inn})}(x) + \mathbf{B}^{(\text{out})}(x)$, estimates (3.18)-(3.21) follows form Lemma 3.2.4 and 3.2.7. □

## 3.3 Solvability of the problem

We look for the solution of problem (3.1) in the form

$$\mathbf{v}(x,t) = \mathbf{A}(x) + \mathbf{u}(x,t),$$

where $\mathbf{A}$ is the suitable extension of the boundary value $\boldsymbol{\varphi}$ constructed in the previous section. Then problem (3.1) is reduced to the homogeneous boundary condition problem for

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \nu\Delta\mathbf{A} + \mathbf{f}, & (x,t) \in \Omega\times(0,2\pi), \\ \operatorname{div}\mathbf{u} = 0, & (x,t) \in \Omega\times(0,2\pi), \\ \mathbf{u} = \mathbf{0}, & (x,t) \in \partial\Omega\times(0,2\pi), \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), & x \in \Omega, \end{cases} \tag{3.22}$$

and now we look for the new unknown velocity field $\mathbf{u}$.

Let us denote the following space:

$$L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega)) := \overline{C^\infty_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))}^{L^2(0,2\pi)},$$

$L^2_1(\Omega)$ is weighted space with the norm

$$\|w\|_{L^2_1(\Omega)} = \sqrt{\int\limits_D |w|^2 g^2 \,\mathrm{d}x + \int\limits_{\Omega_0} |w|^2 \,\mathrm{d}x}.$$

**Definition 3.1.** *Suppose* $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))$. *By the weak solution of problem* (3.22) *we understand a solenoidal vector field* $\mathbf{u}$ *with* $\nabla\mathbf{u}, \mathbf{u}_t \in L^2(0,2\pi;L^2(\Omega))$ *satisfying the homogeneous boundary condition* $\mathbf{u}|_{\partial\Omega} = 0$, *the time periodicity condition* $\mathbf{u}(x,0) = \mathbf{u}(x,2\pi)$ *and the integral identity:*

$$\begin{aligned} &\int\limits_0^{2\pi}\int\limits_\Omega \mathbf{u}_t\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu\int\limits_0^{2\pi}\int\limits_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\nu\int\limits_0^{2\pi}\int\limits_\Omega \nabla\mathbf{A} : \nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.23}$$

*for all time-periodic* $\boldsymbol{\eta} \in L^2(0,2\pi;H(\Omega))$.

**Theorem 3.3.1.** *Assume that the domain* $\Omega \subset \mathbb{R}^2$ *has one outlet to infinity, boundary value* $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ *has a compact support,* $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))$.

*If* $\int\limits_{R_0}^{+\infty} \frac{\mathrm{d}x_2}{g^3(x_2)} < +\infty$, *then problem* (3.1) *has a unique weak solution* $\mathbf{v} = \mathbf{A} + \mathbf{u}$ *satisfying the following estimate:*

$$\begin{aligned} &\|\mathbf{v}_t\|_{L^2(0,2\pi;L^2(\Omega))} + \|\nabla \mathbf{v}\|_{L^2(0,2\pi;L^2(\Omega))} \\ &\leqslant c\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1+\int\limits_{R_0}^{+\infty}\frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right)\right)^{1/2} + \|\mathbf{f}\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))}\right). \end{aligned} \tag{3.24}$$

*Proof.* Let us choose $\Omega_k$, i.e.,

$$\Omega_k = \Omega_0 \cup D_k,$$

where $\Omega_0 = \Omega \cap B_{R_0}$ and $D_k = \{x \in D : x_2 < R_k\}$ with $R_1 = R_0 + \frac{g(R_0)}{2L}$, $R_{k+1} = R_k + \frac{g(R_k)}{2L}$, $k \geqslant 1$.

The existence of a unique solution $\mathbf{u}$ satisfying the integral identity (3.23) could be proved by three following steps. First, we prove the existence of the approximate solution $\mathbf{u}^{(k,N)}$ to the problem in the bounded domain $\Omega_k$

$$\begin{cases} \mathbf{u}_t^{(k,N)} - \nu\Delta\mathbf{u}^{(k,N)} + \nabla p^{(k,N)} = \nu\Delta\mathbf{A} + \mathbf{f}^{(N)}, & (x,t) \in \Omega_k \times (0,2\pi), \\ \operatorname{div}\mathbf{u}^{(k,N)} = 0, & (x,t) \in \Omega_k \times (0,2\pi), \\ \mathbf{u}^{(k,N)} = \mathbf{0}, & (x,t) \in \partial\Omega_k \times (0,2\pi), \\ \mathbf{u}^{(k,N)}(x,0) = \mathbf{u}^{(k,N)}(x,2\pi), & x \in \Omega_k. \end{cases} \tag{3.25}$$

Second, we show the convergence of the approximate solution $\mathbf{u}^{(k,N)}$ to the solution $\mathbf{u}^{(k)}$, which satisfies

$$\begin{cases} \mathbf{u}_t^{(k)} - \nu\Delta\mathbf{u}^{(k)} + \nabla p^{(k)} = \nu\Delta\mathbf{A} + \mathbf{f}, & (x,t) \in \Omega_k \times (0,2\pi), \\ \operatorname{div}\mathbf{u}^{(k)} = 0, & (x,t) \in \Omega_k \times (0,2\pi), \\ \mathbf{u}^{(k)} = \mathbf{0}, & (x,t) \in \partial\Omega_k \times (0,2\pi), \\ \mathbf{u}^{(k)}(x,0) = \mathbf{u}^{(k)}(x,2\pi), & x \in \Omega_k. \end{cases} \tag{3.26}$$

Finally, passing to the limit as $k \to +\infty$, we prove that the limit function $\mathbf{u}$ is a weak solution to problem (3.22).

Consider problem (3.25). It is well known that every $2\pi$-periodic function in $L^2(0,2\pi)$ could be represented as Fourier series:

$$\mathbf{f}(x,t) = \frac{\mathbf{f}_0^{(c)}(x)}{2} + \sum_{n=1}^{\infty}(\mathbf{f}_n^{(s)}(x)\sin(nt) + \mathbf{f}_n^{(c)}(x)\cos(nt)). \tag{3.27}$$

Let $\mathbf{f}^{(N)}$ be a partial sum of (3.27).

We look for the approximate solution $(\mathbf{u}^{(k,N)}, p^{(k,N)})$ in the form

$$\mathbf{u}^{(k,N)}(x,t) = \frac{\mathbf{b}_0^{(k,c)}(x)}{2} + \sum_{n=1}^{N} (\mathbf{a}_n^{(k,s)}(x)\sin(nt) + \mathbf{b}_n^{(k,c)}(x)\cos(nt)), \quad (3.28)$$

$$p^{(k,N)}(x,t) = \frac{p_0^{(k,c)}(x)}{2} + \sum_{n=1}^{N} (p_n^{(k,s)}(x)\sin(nt) + p_n^{(k,c)}(x)\cos(nt)). \quad (3.29)$$

In order to prove the existence of the approximate solution, we need to prove the existence of Fourier coefficients $\mathbf{a}_n^{(k,s)}$ and $\mathbf{b}_n^{(k,c)}$, $n = 0, 1, \dots, N$. To do this, we substitute (3.27)-(3.29) into the problem (3.25), and by collecting the coefficients of sin and cos functions we obtain the following stationary problems:

$$\begin{cases} -\nu\Delta \mathbf{b}_0^{(k,c)}(x) + \nabla p_0^{(k,c)}(x) = 2\nu\Delta\mathbf{A}(x) + \mathbf{f}_0^{(c)}(x), \\ \operatorname{div} \mathbf{b}_0^{(k,c)}(x) = 0, \quad \mathbf{b}_0^{(k,c)}(x)|_{\partial\Omega_k} = \mathbf{0}, \end{cases} \quad (3.30)$$

$$\begin{cases} n\mathbf{a}_n^{(k,s)}(x) - \nu\Delta \mathbf{b}_n^{(k,c)}(x) + \nabla p_0^{(k,c)}(x) = \mathbf{f}_n^{(c)}(x), \\ -n\mathbf{b}_n^{(k,c)}(x) - \nu\Delta \mathbf{a}_n^{(k,s)}(x) + \nabla p_0^{(k,s)}(x) = \mathbf{f}_n^{(s)}(x), \\ \operatorname{div} \mathbf{a}_n^{(k,s)}(x) = 0, \quad \operatorname{div} \mathbf{b}_n^{(k,c)}(x) = 0, \\ \mathbf{a}_n^{(k,s)}(x)|_{\partial\Omega_k} = \mathbf{0}, \quad \mathbf{b}_n^{(k,c)}(x)|_{\partial\Omega_k} = \mathbf{0}, \quad n = 1, 2, \dots, N. \end{cases} \quad (3.31)$$

Notice that (3.30) is the Stokes system with homogeneous boundary condition and the existence of a weak solution of (3.30) is well known (see [49]).

In order to prove the existence of a unique solution to problem (3.31), we multiply $(3.31)_1$ by $\boldsymbol{\eta} \in H(\Omega_k)$ and $(3.31)_2$ by $\boldsymbol{\xi} \in H(\Omega_k)$. Then integrating by parts over $\Omega_k$ we obtain the following system:

$$\begin{cases} n\int\limits_{\Omega_k} \mathbf{a}_n^{(k,s)} \cdot \boldsymbol{\eta} \, \mathrm{d}x + \nu \int\limits_{\Omega_k} \nabla \mathbf{b}_n^{(k,c)} : \nabla \boldsymbol{\eta} \, \mathrm{d}x = \int\limits_{\Omega_k} \mathbf{f}_n^{(c)} \cdot \boldsymbol{\eta} \, \mathrm{d}x, \\ -n\int\limits_{\Omega_k} \mathbf{b}_n^{(k,s)} \cdot \boldsymbol{\xi} \, \mathrm{d}x + \nu \int\limits_{\Omega_k} \nabla \mathbf{a}_n^{(k,c)} : \nabla \boldsymbol{\xi} \, \mathrm{d}x = \int\limits_{\Omega_k} \mathbf{f}_n^{(s)} \cdot \boldsymbol{\xi} \, \mathrm{d}x. \end{cases} \quad (3.32)$$

To prove the existence of the unique solution of (3.32), we use Fredholm alternative by reducing (3.32) to the system of operator equations

$$\begin{cases} \mathcal{B}\mathbf{a}_n^{(k,s)} + \nu \mathbf{b}_n^{(k,c)} = \mathbf{F}^{(c)}, \quad \forall \boldsymbol{\eta} \in H(\Omega_k), \\ -\mathcal{B}\mathbf{b}_n^{(k,c)} + \nu \mathbf{a}_n^{(k,s)} = \mathbf{F}^{(s)}, \quad \forall \boldsymbol{\xi} \in H(\Omega_k), \end{cases}$$

where $\mathcal{B} = n \int\limits_{\Omega_k} \mathbf{s}_n \cdot \boldsymbol{\psi} \, \mathrm{d}x$ is linear completely continuous operator, then $\mathbf{s}_n$ is equal $\mathbf{a}_n^{(k,s)}$ or $\mathbf{b}_n^{(k,c)}$ and $\boldsymbol{\psi}$ is equal $\boldsymbol{\eta}$ or $\boldsymbol{\xi}$, respectively.

Then we consider homogeneous operator equations

$$\begin{cases} \mathcal{B}\mathbf{a}_n^{(k,s)} + \nu \mathbf{b}_n^{(k,c)} = 0 & \forall \boldsymbol{\eta} \in H(\Omega_k), \\ -\mathcal{B}\mathbf{b}_n^{(k,c)} + \nu \mathbf{a}_n^{(k,s)} = 0 & \forall \boldsymbol{\xi} \in H(\Omega_k), \end{cases}$$

i.e.,

$$\begin{cases} n \int\limits_{\Omega_k} \mathbf{a}_n^{(k,s)} \cdot \boldsymbol{\eta} \, \mathrm{d}x + \nu \int\limits_{\Omega_k} \nabla \mathbf{b}_n^{(k,c)} : \nabla \boldsymbol{\eta} \, \mathrm{d}x = 0, \\ -n \int\limits_{\Omega_k} \mathbf{b}_n^{(k,c)} \cdot \boldsymbol{\xi} \, \mathrm{d}x + \nu \int\limits_{\Omega_k} \nabla \mathbf{a}_n^{(k,s)} : \nabla \boldsymbol{\xi} \, \mathrm{d}x = 0. \end{cases}$$

After substituting $\boldsymbol{\eta}(x) = \mathbf{b}_n^{(k,c)}(x)$ and $\boldsymbol{\xi}(x) = \mathbf{a}_n^{(k,s)}(x)$ and summing up the equations, we obtain

$$\nu \int\limits_{\Omega_k} |\nabla \mathbf{b}_n^{(k,c)}(x)|^2 \, \mathrm{d}x + \nu \int\limits_{\Omega_k} |\nabla \mathbf{a}_n^{(k,s)}(x)|^2 \, \mathrm{d}x = 0.$$

Then it follows that

$$\mathbf{b}_n^{(k,c)}(x) = 0, \quad \mathbf{a}_n^{(k,s)}(x) = 0.$$

According to Fredholm alternative, we obtained that (3.31) has a unique solution. Therefore, the existence and uniqueness of the approximate solution $\mathbf{u}^{(k,N)}$ to problem (3.25) is proved.

In order to prove the convergence of an approximate solution $\mathbf{u}^{(k,N)}(x,t)$ to $\mathbf{u}^{(k)}(x,t)$ in bounded domains $\Omega_k$, we need to obtain the estimates for the norms of $\mathbf{u}^{(k,N)}(x,t)$. To do this, we multiply equation $(3.31)_1$ by $\mathbf{u}^{(k,N)}(x,t)$, and integrating by parts over $\Omega_k$, we get

$$\begin{aligned} &\int\limits_{\Omega_k} \mathbf{u}_t^{(k,N)} \cdot \mathbf{u}^{(k,N)} \, \mathrm{d}x + \nu \int\limits_{\Omega_k} |\nabla \mathbf{u}^{(k,N)}|^2 \, \mathrm{d}x \\ &= -\nu \int\limits_{\Omega_k} \nabla \mathbf{A} : \nabla \mathbf{u}^{(k,N)} \, \mathrm{d}x + \int\limits_{\Omega_k} \mathbf{f}^{(N)} \cdot \mathbf{u}^{(k,N)} \, \mathrm{d}x. \end{aligned} \tag{3.33}$$

Since

$$\mathbf{u}_t^{(k,N)} \cdot \mathbf{u}^{(k,N)} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{u}^{(k,N)}|,$$

from (3.33) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_k}\left|\mathbf{u}^{(k,N)}\right|^2\,\mathrm{d}x+\nu\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}\right|^2\,\mathrm{d}x$$
$$=-\nu\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\mathbf{u}^{(k,N)}\,\mathrm{d}x+\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}^{(k,N)}\,\mathrm{d}x.$$

Integration with respect to time variable $t$ from 0 till $2\pi$ yields

$$\frac{1}{2}\int\limits_{\Omega_k}\left|\mathbf{u}^{(k,N)}(x,2\pi)\right|^2\mathrm{d}x-\frac{1}{2}\int\limits_{\Omega_k}\left|\mathbf{u}^{(k,N)}(x,0)\right|^2\mathrm{d}x+\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}\right|^2\mathrm{d}x\,\mathrm{d}t$$
$$=-\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t+\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.$$

Using the periodicity condition $\mathbf{u}^{(k,N)}(x,0)=\mathbf{u}^{(k,N)}(x,2\pi)$, we derive

$$\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=-\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t+\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.34}$$

Notice that we need to get estimates with the constant independent of the domain $\Omega_k$. To do this, we rewrite equation (3.34) as follows:

$$\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=-\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t+\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot g\cdot g^{-1}\cdot\mathbf{u}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.$$

By Cauchy-Schwarz inequality,

$$\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant\nu\Bigg(\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{A}(x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{1/2}\Bigg(\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{1/2}$$
$$+\|\mathbf{f}^{(\mathbf{N})}(x,t)\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega_k))} \tag{3.35}$$
$$\times\Bigg(\int\limits_0^{2\pi}\int\limits_{\Omega_0}\left|\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_0^{2\pi}\int\limits_{D_k}\frac{\left|\mathbf{u}^{(k,N)}(x,t)\right|^2}{\left|g(x_2)\right|^2}\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{1/2}.$$

Since, due to Poincaré–Friedrich's inequality, we have that

$$\int_0^{2\pi}\int_{D_k}\frac{\left|\mathbf{u}^{(k,N)}(x,t)\right|^2}{\left|g(x_2)\right|^2}\,\mathrm{d}x\,\mathrm{d}t=\int_0^{2\pi}\int_{R_0}^{R_k}\frac{1}{|g(x_2)|^2}\,\mathrm{d}x_2\int_{-g(x_2)}^{g(x_2)}|\mathbf{u}^{(k,N)}(x,t)|^2\mathrm{d}x_1\mathrm{d}t$$

$$\leqslant c\int_0^{2\pi}\int_{R_0}^{R_k}\mathrm{d}x_2\int_{-g(x_2)}^{g(x_2)}\left|\frac{\partial}{\partial x_1}\mathbf{u}^{(k,N)}(x,t)\right|^2\mathrm{d}x_1\mathrm{d}t\leqslant c\int_0^{2\pi}\int_{D_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t,$$

from (3.35) we obtain

$$\nu\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant\nu\Big(\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{A}(x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}$$

$$+c\|\mathbf{f}^{(N)}(x,t)\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega_k))}\Big(\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}$$

$$\leqslant\left(\nu\sqrt{2\pi}\Big(\int_{\Omega_k}\left|\nabla\mathbf{A}(x)\right|^2\,\mathrm{d}x\Big)^{1/2}+c\|\mathbf{f}^{(N)}(x,t)\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega_k))}\right)$$

$$\times\Big(\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.$$

Dividing both sides by $\nu\Big(\int_0^{2\pi}\int_{\Omega_k}\left|\nabla\mathbf{u}^{(k,N)}(x,t)\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}$, we rewrite the last estimate as follows:

$$\|\nabla\mathbf{u}^{(k,N)}\|_{L^2(0,2\pi;L^2(\Omega_k))}\leqslant C(\|\nabla\mathbf{A}\|_{L^2(\Omega_k)}+\|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2_1(\Omega_k))}),\qquad(3.36)$$

where the constant $C$ is independent of the domain $\Omega_k$.

Due to Theorem 3.2.8 the norm $\|\nabla\mathbf{A}\|^2_{L^2(\Omega_k)}$ admits the estimate:

$$\|\nabla\mathbf{A}\|^2_{L^2(\Omega_k)}=\int_{\Omega_k}|\nabla\mathbf{A}|^2\,\mathrm{d}x$$

$$\leqslant\int_{\Omega_0}\Big(c(|\mathcal{F}^{(\mathrm{inn})}|+|\mathcal{F}^{(\mathrm{out})}|)\Big)^2\,\mathrm{d}x+\int_{D_k}\left(\frac{c(|\mathcal{F}^{(\mathrm{inn})}|+|\mathcal{F}^{(\mathrm{out})}|)}{g^2(x_2)}\right)^2\,\mathrm{d}x$$

$$\leqslant c(|\mathcal{F}^{(\text{inn})}|^2+|\mathcal{F}^{(\text{out})}|^2)\left(1+\int\limits_{R_0}^{R_k}\int\limits_{-g(x_2)}^{g(x_2)}\frac{1}{g^4(x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)$$
$$\leqslant c(|\mathcal{F}^{(\text{inn})}|^2+|\mathcal{F}^{(\text{out})}|^2)\left(1+\int\limits_{R_0}^{R_k}\frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right). \tag{3.37}$$

According to the fact that

$$|\mathcal{F}^{(\text{inn})}|^2+|\mathcal{F}^{(\text{out})}|^2\leqslant c\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)},$$

from (3.36), using (3.37), we get

$$\|\nabla\mathbf{u}^{(k,N)}\|_{L^2(0,2\pi;L^2(\Omega_k))}$$
$$\leqslant C\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1+\int\limits_{R_0}^{R_k}\frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right)\right)^{1/2}+\|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2_1(\Omega_k))}\right), \tag{3.38}$$

where $C$ is independent of $\Omega_k$.

Let us estimate the norm of the term $\mathbf{u}_t^{(k,N)}$. Multiplying equation $(3.25)_1$ by $\mathbf{u}_t^{(k,N)}(x,t)$ and integrating by parts over $\Omega_k$, we arrive at

$$\int\limits_{\Omega_k}|\mathbf{u}_t^{(k,N)}|^2\,\mathrm{d}x+\nu\int\limits_{\Omega_k}\nabla\mathbf{u}^{(k,N)}:\nabla\mathbf{u}_t^{(k,N)}\,\mathrm{d}x$$
$$=\nu\int\limits_{\Omega_k}\Delta\mathbf{A}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x+\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x. \tag{3.39}$$

Since

$$\nabla\mathbf{u}^{(k,N)}:\nabla\mathbf{u}_t^{(k,N)}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla\mathbf{u}^{(k,N)}|^2),$$

from (3.39) it follows that

$$\int\limits_{\Omega_k}|\mathbf{u}_t^{(k,N)}|^2\,\mathrm{d}x+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_k}|\nabla\mathbf{u}^{(k,N)}|^2\,\mathrm{d}x$$
$$=\nu\int\limits_{\Omega_k}\Delta\mathbf{A}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x+\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x.$$

Then integrating with respect to time variable $t$ from 0 till $2\pi$, we obtain

$$\int\limits_0^{2\pi}\int\limits_{\Omega_k}|\mathbf{u}_t^{(k,N)}|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\nu}{2}\int\limits_{\Omega_k}|\nabla\mathbf{u}^{(k,N)}(x,2\pi)|^2\,\mathrm{d}x-\frac{\nu}{2}\int\limits_{\Omega_k}|\nabla\mathbf{u}^{(k,N)}(x,0)|^2\,\mathrm{d}x$$
$$=\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\Delta\mathbf{A}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t+\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{u}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.$$

Using the periodicity condition $\nabla \mathbf{u}^{(k,N)}(x,0)=\nabla \mathbf{u}^{(k,N)}(x,2\pi)$, the last equality reduces to

$$\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}|^2 \,\mathrm{d}x\,\mathrm{d}t = \nu \int_0^{2\pi}\int_{\Omega_k} \Delta\mathbf{A}\cdot\mathbf{u}_t^{(k,N)} \,\mathrm{d}x\,\mathrm{d}t + \int_0^{2\pi}\int_{\Omega_k} \mathbf{f}^{(N)}\cdot\mathbf{u}_t^{(k,N)} \,\mathrm{d}x\,\mathrm{d}t.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}|^2 \,\mathrm{d}x\,\mathrm{d}t &\leqslant \nu\Big(\int_0^{2\pi}\int_{\Omega_k} |\Delta\mathbf{A}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\quad + \Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{f}^{(N)}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \\
&\leqslant \Big(\nu\sqrt{2\pi}\Big(\int_{\Omega_k} |\nabla\mathbf{A}|^2 \,\mathrm{d}x\Big)^{1/2} + \Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{f}^{(N)}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big)\Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.
\end{aligned}$$

Then dividing both sides by $\Big(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{u}_t^{(k,N)}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}$, we obtain the last estimate as follows:

$$\|\mathbf{u}_t^{(k,N)}\|_{L^2(0,2\pi;L^2(\Omega_k))} \leqslant C_1\big(\|\Delta\mathbf{A}\|_{L^2(\Omega_k)} + \|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2(\Omega_k))}\big), \tag{3.40}$$

where $C_1$ is independent of the domain $\Omega_k$.

Due to Theorem 3.2.8

$$\begin{aligned}
\|\Delta\mathbf{A}\|^2_{L^2(\Omega_k)} &= \int_{\Omega_k} |\Delta\mathbf{A}|^2 \,\mathrm{d}x \\
&\leqslant \int_{\Omega_0} \Big(c(|\mathcal{F}^{(\mathrm{inn})}|+|\mathcal{F}^{(\mathrm{out})}|)\Big)^2 \,\mathrm{d}x + \int_{D_k} \Big(\frac{c(|\mathcal{F}^{(\mathrm{inn})}|+|\mathcal{F}^{(\mathrm{out})}|)}{g^3(x_2)}\Big)^2 \,\mathrm{d}x \\
&\leqslant c(|\mathcal{F}^{(\mathrm{inn})}|^2+|\mathcal{F}^{(\mathrm{out})}|^2)\Big(1+\int_{R_0}^{R_k}\int_{-g(x_2)}^{g(x_2)} \frac{1}{g^6(x_2)} \,\mathrm{d}x_1\,\mathrm{d}x_2\Big) \\
&\leqslant c(|\mathcal{F}^{(\mathrm{inn})}|^2+|\mathcal{F}^{(\mathrm{out})}|^2)\Big(1+\int_{R_0}^{R_k} \frac{\mathrm{d}x_2}{g^5(x_2)}\Big).
\end{aligned} \tag{3.41}$$

According to the fact that

$$|\mathcal{F}^{(\mathrm{inn})}|^2+|\mathcal{F}^{(\mathrm{out})}|^2 \leqslant c\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)},$$

it follows form (3.40) using (3.41) the following estimate:

$$\begin{aligned}
&\|\mathbf{u}_t^{(k,N)}\|_{L^2(0,2\pi;L^2(\Omega_k))} \leqslant C_1(\|\Delta \mathbf{A}\|_{L^2(\Omega_k)} + \|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2(\Omega_k))})\\
&\leqslant C_1\Bigg(\Bigg(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\Bigg(1+\int\limits_{R_0}^{R_k}\frac{\mathrm{d}x_2}{g^5(x_2)}\Bigg)\Bigg)^{1/2} + \|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2(\Omega_k))}\Bigg)\\
&\leqslant C_1\Bigg(\Bigg(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\Bigg(1+\int\limits_{R_0}^{R_k}\frac{\mathrm{d}x_2}{g^3(x_2)}\Bigg)\Bigg)^{1/2} + \|\mathbf{f}^{(N)}\|_{L^2(0,2\pi;L^2_1(\Omega_k))}\Bigg),
\end{aligned} \tag{3.42}$$

where $C_1$ is independent of $\Omega_k$.

For the fixed $k$, from estimates (3.38), (3.42) we conclude that $\{\nabla \mathbf{u}^{(k,N)}\}$ and $\{\mathbf{u}_t^{(k,N)}\}$ are bounded sequences in the space $L^2(0,2\pi;L^2(\Omega_k))$. Hence there exists a subsequence $\{\mathbf{u}^{(k,N_m)}\}$ such that $\{\nabla \mathbf{u}^{(k,N_m)}\}$ and $\{\mathbf{u}_t^{(k,N_m)}\}$ converge weakly to $\{\nabla \mathbf{u}^{(k)}\}$ and $\{\mathbf{u}_t^{(k)}\}$ in $L^2(0,2\pi;L^2(\Omega_k))$. Moreover, $\{\mathbf{f}^{(N)}\}$ converges to $\{\mathbf{f}\}$ in the space $L^2(0,2\pi;L^2(\Omega_k))$. For the approximate solution, the following integral identity holds:

$$\begin{aligned}
&\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{u}_t^{(k,N_m)}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{u}^{(k,N_m)}:\nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = -\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N_m)}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for all time-periodic $\boldsymbol{\eta}\in L^2(0,2\pi;H(\Omega_k))$. Passing to the limit as $N_m\to+\infty$, we get

$$\begin{aligned}
&\int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{u}_t^{(k)}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{u}^{(k)}:\nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = -\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A}:\nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.43}$$

Thus, $\mathbf{u}^{(k)}$ are weak solutions of problem (3.26) in bounded domains $\Omega_k$.

Finally, we will get the solution in the whole domain $\Omega$. Since the estimates for the approximate solution $\mathbf{u}^{(k,N)}$ remain valid for the limit solution $\mathbf{u}^{(k)}$, from (3.38) and (3.42), it follows that

$$\begin{aligned}
&\|\mathbf{u}_t^{(k)}\|_{L^2(0,2\pi;L^2(\Omega_k))} + \|\nabla\mathbf{u}^{(k)}\|_{L^2(0,2\pi;L^2(\Omega_k))}\\
&\leqslant c\Bigg(\Bigg(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\Bigg(1+\int\limits_{R_0}^{R_k}\frac{\mathrm{d}x_2}{g^3(x_2)}\Bigg)\Bigg)^{1/2} + \|\mathbf{f}\|_{L^2(0,2\pi;L^2(\Omega_k))}\Bigg),
\end{aligned} \tag{3.44}$$

where constant $c$ is independent of domain $\Omega_k$.

Since $\int\limits_{R_0}^{+\infty} \frac{1}{g^3(x_2)}\,\mathrm{d}x_2 < +\infty$, the right-hand side of (3.44) is bounded by a constant independent of $\Omega_k$. So $\{\nabla \mathbf{u}^{(k)}\}$ and $\{\mathbf{u}_t^{(k)}\}$ are bounded sequences in the space $L^2(0,2\pi;L^2(\Omega_k))$. Therefore, there exists a subsequence $\{\mathbf{u}^{(k_m)}\}$ such that $\{\nabla \mathbf{u}^{(k_m)}\}$ and $\{\mathbf{u}_t^{(k_m)}\}$ converge weakly to $\{\nabla \mathbf{u}\}$ and $\{\mathbf{u}_t\}$ as $k_m \to +\infty$ in the space $L^2(0,2\pi;L^2(\Omega))$. Taking in integral identity (3.43) an arbitrary time-periodic test function $\boldsymbol{\eta} \in L^2(0,2\pi;H(\Omega))$ with a compact support, we can pass to a limit as $k \to +\infty$. As a result, we get for the limit function $\mathbf{u}$ integral identity (3.23).

The uniqueness is obtained by standard way assuming that (3.22) has two weak solutions $\mathbf{w}_1$ and $\mathbf{w}_2$, which satisfy the integral identity

$$\int\limits_0^{2\pi}\int\limits_\Omega \frac{\partial}{\partial t}\mathbf{w}_i \cdot \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu \int\limits_0^{2\pi}\int\limits_\Omega \nabla \mathbf{w}_i : \nabla \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\nu \int\limits_0^{2\pi}\int\limits_\Omega \nabla \mathbf{A} : \nabla \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_\Omega \mathbf{f}\cdot \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t, \quad i = 1,2.$$

Subtracting the identities, we get

$$\int\limits_0^{2\pi}\int\limits_\Omega \frac{\partial}{\partial t}(\mathbf{w}_1 - \mathbf{w}_2)\cdot \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu \int\limits_0^{2\pi}\int\limits_\Omega \nabla(\mathbf{w}_1 - \mathbf{w}_2) : \nabla \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Taking $\boldsymbol{\eta} = \mathbf{w}_1 - \mathbf{w}_2$, we have

$$\int\limits_0^{2\pi}\int\limits_\Omega \frac{\partial}{\partial t}(\mathbf{w}_1 - \mathbf{w}_2)\cdot(\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\nu \int\limits_0^{2\pi}\int\limits_\Omega \nabla(\mathbf{w}_1 - \mathbf{w}_2) : \nabla(\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since $\frac{\partial(\mathbf{w}_1 - \mathbf{w}_2)\cdot(\mathbf{w}_1 - \mathbf{w}_2)}{\partial t} = \frac{1}{2}\frac{\partial |\mathbf{w}_1 - \mathbf{w}_2|^2}{\partial t}$, integrating by interval $[0,2\pi]$, it follows that

$$\nu \int\limits_0^{2\pi}\int\limits_\Omega |\nabla(\mathbf{w}_1 - \mathbf{w}_2)|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

and hence $\mathbf{w}_1 = \mathbf{w}_2$.

Thus, we have proved that $\mathbf{v} = \mathbf{A} + \mathbf{u}$ is a unique weak solution of problem (3.1). Estimate (3.24) for $\mathbf{u}$ follows from (3.44). Since, for $\mathbf{A}$, the analogues to (3.24) estimate is also valid, we obtain (3.24) for the sum $\mathbf{v} = \mathbf{A} + \mathbf{u}$.

$\square$

# Chapter 4

# Time-periodic Navier–Stokes equations in a thin tube structure

## 4.1 Formulation of the problem

Consider in the tube structure $B_\varepsilon$ (see Definition 2) the time-periodic boundary value problem for the Navier–Stokes equations

$$\begin{cases} \frac{1}{\varepsilon^\beta}\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, \ \beta = 0,2, \ (x,t)\in B_\varepsilon\times(0,2\pi), \\ \qquad\qquad\qquad \operatorname{div}\mathbf{v} = 0, \qquad\qquad (x,t)\in B_\varepsilon\times(0,2\pi), \\ \qquad\qquad\qquad \mathbf{v} = \mathbf{g}, \qquad\qquad (x,t)\in \partial B_\varepsilon\times(0,2\pi), \\ \qquad \mathbf{v}(x,t) = \mathbf{v}(x,t+2\pi), \qquad x\in B_\varepsilon. \end{cases} \tag{4.1}$$

Assume that the fluid velocity $\mathbf{g}$ at the boundary $\partial B_\varepsilon$ has the following structure: $\mathbf{g} = 0$ everywhere on $\partial B_\varepsilon$ except for the set $\gamma_\varepsilon^{N_1+1},\dots,\gamma_\varepsilon^N$, where $\gamma_\varepsilon^j = \partial B_\varepsilon \cap \partial\omega_\varepsilon^j$, $j = N_1+1,\dots,N$, i.e.,

$$\mathbf{g}(x,t)|_{\gamma_\varepsilon^j} = \mathbf{g}^j\left(\frac{x-O_j}{\varepsilon},t\right)\Big|_{\gamma_\varepsilon^j}, \quad j = N_1+1,\dots,N,$$
$$\mathbf{g}(x,t)\Big|_{\partial B_\varepsilon\setminus\bigcup\limits_{j=N_1+1}^{N}\gamma_\varepsilon^j} = 0,$$

where $\mathbf{g}^j \in C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{3/2,2}(\gamma^j))$, $[\alpha]$- is the integer part of $\alpha$, $\gamma^j = \varepsilon^{-1}(\gamma_\varepsilon^j - O_j)$ denote the corresponding dilated part of the boundary, and $\mathbf{g}\in C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{3/2,2}(\partial B_\varepsilon))$.

Denote $e = e_{O_j}$ (the edge with the end $O_j$) and $x^{(e)}$ the Cartesian coordinates corresponding to the origin $O_j$ and the edge $e$, i.e., $x^{(e)} =$

$\mathcal{P}^{(e)}(x-O_j)$, $\mathcal{P}^{(e)}$ is the orthogonal matrix relating the global coordinates $x$ with the local ones $x^{(e)}$, $\sigma_\varepsilon^j = \{x : \dfrac{x^{(e)\prime}}{\varepsilon} \in \sigma,\ x_n^{(e)} = 0\}$. Denote $\mathbf{g}^{(e)} = \mathcal{P}^{(e)}\mathbf{g}^j$.

Let

$$
\begin{aligned}
\widetilde{F}^j(t) &= \int\limits_{\gamma_\varepsilon^j} \mathbf{g}(x,t)\cdot\mathbf{n}(x)\ \mathrm{d}S = \int\limits_{\gamma_\varepsilon^j} \mathbf{g}^j\left(\frac{x-O_j}{\varepsilon},\ t\right)\cdot\mathbf{n}(x)\ \mathrm{d}S \\
&= \varepsilon^{n-1}\int\limits_{\gamma^j} \hat{g}_n^j(y^{(e)\prime},t)\ \mathrm{d}y^{(e)\prime} \equiv \varepsilon^{n-1}F^j(t), \quad j = N_1+1,\dots,N,
\end{aligned}
\tag{4.2}
$$

where $\mathbf{n}$ is the unit outward (with respect to $B_\varepsilon$) normal vector to $\gamma_\varepsilon^j$, $y^{(e)\prime} = \dfrac{x^{(e)\prime}}{\varepsilon}$, $\hat{\mathbf{g}}^j(y^{(e)\prime},t) = \mathbf{g}^j((\mathcal{P}^{(e)})^* y^{(e)\prime},t)$. Since $\mathbf{g}(x,t)$ is time periodic, $F^j(t)$ also must be time periodic. Moreover, since we will need the divergence free extension of $\mathbf{g}$, we assume the compatibility condition for the flow rates $F^j(t)$:

$$
\sum_{j=N_1+1}^{N} F^j(t) = 0 \quad \forall t \in [0,2\pi]. \tag{4.3}
$$

Let $\mathbf{g}$ be the divergence free time periodic extension of the boundary function $\mathbf{g}$ (here we use the same symbol $\mathbf{g}$, $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi; W^{2,2}(B_\varepsilon))$) satisfying for all $t \in [0,2\pi]$ the following asymptotic estimates

$$
\begin{aligned}
&\sup_{x\in B_\varepsilon} |\mathbf{g}(x,t)| \leqslant c, \qquad \|\nabla \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-3}{2}} \quad \forall t \in [0,2\pi], \\
&\|\mathbf{g}_t\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-1}{2}}, \quad \|\nabla^2 \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-5}{2}} \quad \forall t \in [0,2\pi],
\end{aligned}
\tag{4.4}
$$

where the constant $c$ is independent of $\varepsilon$.

Below, we construct the special extension $\mathbf{g}$ in the form of asymptotic representation of the solution $\mathbf{g}$, such that the discrepancy, let us denote it by $\mathbf{f}$, of this extension in the equations (4.1), is small. But first, we consider the following variational problem: find a vector-field $\mathbf{v} = \mathbf{u}+\mathbf{g}$ with $\operatorname{div}\mathbf{u} = 0$, $\mathbf{u} \in L^\infty_{\mathrm{per}}(0,2\pi; \mathring{W}^{1,2}(B_\varepsilon)\cap W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t \in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))$, satisfying the integral identity

$$
\begin{aligned}
&\int\limits_{B_\varepsilon} \left(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\eta} - ((\mathbf{u}+\mathbf{g})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{g}\right)\mathrm{d}x \\
&\qquad = \int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\ \mathrm{d}x,
\end{aligned}
\tag{4.5}
$$

for every divergence free vector field $\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)$. Here $\mathbf{g}$ is an arbitrary extension satisfying (4.4) and $\mathbf{f}$ is an arbitrary function such that $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))$.

Denote

$$A_1(t) = \|\mathbf{f}(\cdot,t)\|^2_{L^2(B_\varepsilon)}. \tag{4.6}$$

## 4.2 Solvability of the problem

In this section we prove the existence and uniqueness of the problem (4.1) in two and three-dimensional cases.

### 4.2.1 Two-dimensional case

In this subsection we prove the existence of the solution to problem (4.5) when $n = 2$.

**Theorem 4.2.1.** *Let $B_\varepsilon \subset \mathbb{R}^2$, $\partial B_\varepsilon \in C^2$. Suppose that the extended function $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi; W^{2,2}(B_\varepsilon))$ satisfies the conditions* (4.2), (4.3), (4.4), *and $\mathbf{f} \in L^2_{\text{per}}(0,2\pi; L^2(B_\varepsilon))$. Then for sufficiently small $\varepsilon$, the variational problem* (4.5) *admits a solution $\mathbf{u}$, satisfying the estimates*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2+\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t, \tag{4.7}$$

$$\begin{aligned}\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ +\varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla^2 \mathbf{u}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t\end{aligned} \tag{4.8}$$

*with constants $c$ independent of $\varepsilon$.*

*Proof.* We prove the solvability of problem (4.5) by Galerkin method (see [49], [79]). The main purpose is to obtain suitable a priori estimates. The remaining part is standard.

If $\mathbf{u}$ is a weak solution, then taking in (4.5) $\boldsymbol{\eta} = \mathbf{u}$, we obtain

$$\frac{1}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_\varepsilon} |\mathbf{u}|^2 \,\mathrm{d}x + \nu\int_{B_\varepsilon} |\nabla\mathbf{u}|^2 \,\mathrm{d}x = \int_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}x + \int_{B_\varepsilon} \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x.$$

Using (4.4) and the Poincaré-Friedrich's inequality (2.1), we derive the estimate

$$\left|\int_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{g}\,\mathrm{d}x\right| \leqslant \|\mathbf{u}\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\mathbf{g}\|_{L^\infty(B_\varepsilon)} \leqslant c_1\varepsilon\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}.$$

If $c_1\varepsilon \leqslant \dfrac{\nu}{4}$, this gives

$$\frac{1}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon}|\mathbf{u}|^2\,\mathrm{d}x+\nu\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x \leqslant c_1\varepsilon\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+c\varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}$$
$$\leqslant \frac{\nu}{2}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+c\varepsilon^2\|\mathbf{f}\|^2_{L^2(B_\varepsilon)}.$$

Then

$$\frac{1}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}+\frac{\nu}{2}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\leqslant c\varepsilon^2\|\mathbf{f}\|^2_{L^2(B_\varepsilon)}\equiv c\varepsilon^2A_1(t) \quad (4.9)$$

and, hence, multiplying this relation by $2\varepsilon^\beta$ and using the Poincaré-Friedrich's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}+c_*\varepsilon^{\beta-2}\|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\leqslant c_2\varepsilon^{\beta+2}A_1(t).$$

Multiplying this inequality with omitted second term in the left-hand side by $e^{c_*\varepsilon^{\beta-2}t}$ and integrating over $t$ we obtain

$$\begin{aligned}&\|\mathbf{u}(\cdot,2\pi)\|^2_{L^2(B_\varepsilon)}\\ &\leqslant \|\mathbf{u}(\cdot,0)\|^2_{L^2(B_\varepsilon)}\cdot e^{-c_*2\pi\varepsilon^{\beta-2}}+c_2\varepsilon^{\beta+2}\int\limits_0^{2\pi}A_1(t)\cdot e^{c_*\varepsilon^{\beta-2}(t-2\pi)}\,\mathrm{d}t\\ &\leqslant \|\mathbf{u}(\cdot,0)\|^2_{L^2(B_\varepsilon)}\cdot e^{-c_*2\pi\varepsilon^{\beta-2}}+c_2\varepsilon^{\beta+2}\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t.\end{aligned} \quad (4.10)$$

Consider Galerkin approximations of the solution to problem (4.5) defined by the following system of ordinary differential equations

$$\int\limits_{B_\varepsilon}\left(\frac{1}{\varepsilon^\beta}\mathbf{u}_t^{(N)}\cdot\boldsymbol{\psi}_l+\nu\nabla\mathbf{u}^{(N)}:\nabla\boldsymbol{\psi}_l-\Big((\mathbf{u}^{(N)}+\mathbf{g})\cdot\nabla\Big)\boldsymbol{\psi}_l\cdot\mathbf{u}^{(N)}-(\mathbf{u}^{(N)}\cdot\nabla)\boldsymbol{\psi}_l\cdot\mathbf{g}\right)\mathrm{d}x$$
$$=\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\psi}_l\,\mathrm{d}x, \quad (4.11)$$

where $l=1,\dots,N$, $\mathbf{u}^{(N)}(x,t)=\sum\limits_{k=1}^N\gamma_k^{(N)}(t)\psi_k(x)$, and $\{\psi_k\}_{k=1}^\infty$ is a basis in the space $H(B_\varepsilon)$.

From estimate (4.10), which remains valid for Galerkin approximations, it follows that for every $N$, the map $M:\mathbf{u}^{(N)}(0)\mapsto\mathbf{u}^{(N)}(2\pi)$ brings the ball in $L^2(B_\varepsilon)$ of radius $r_0=\sqrt{\dfrac{c_2\varepsilon^{\beta+2}}{1-e^{-c_*2\pi\varepsilon^{\beta-2}}}\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t}$ into itself. The operator

$M$ is continuous (see [75]). By Brouwer's fixed-point theorem there exist a fixed point of the map $M$. This insure the existence of a $2\pi$-periodic solution to the Galerkin approximations (4.11) for each fixed $N$.

Now we derive a set of a priori estimates for Galerkin approximations $\mathbf{u}^{(N)}$. Integrate (4.9) with respect to $t$. Using the periodicity condition $\mathbf{u}^{(N)}(x,0) = \mathbf{u}^{(N)}(x,2\pi)$, we obtain

$$\int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}^{(N)}(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c\varepsilon^2 \int_0^{2\pi} A_1(t)\, \mathrm{d}t. \tag{4.12}$$

Because of the Poincaré-Friedrich's inequality and the mean value theorem for Lebesgue integrals there exist a point $t_* \in [0,2\pi]$ such that

$$\begin{aligned} \frac{1}{\varepsilon^2 c}\|\mathbf{u}^{(N)}(\cdot,t_*)\|^2_{L^2(B_\varepsilon)} &\leqslant \|\nabla \mathbf{u}^{(N)}(\cdot,t_*)\|^2_{L^2(B_\varepsilon)} \\ \leqslant \frac{1}{2\pi}\int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}^{(N)}(x,t)|^2 \, \mathrm{d}x\, \mathrm{d}t &\leqslant c\varepsilon^2 \int_0^{2\pi} A_1(t)\, \mathrm{d}t. \end{aligned} \tag{4.13}$$

Without loss of generality we may assume that $t_* = 0$ (if not, we can consider problem (4.5) on the interval $[t_*, t_* + 2\pi]$ and reduce it, by change of variable $t \to t - t_*$, to the interval $[0,2\pi]$). Integrating (4.9) from 0 to $t$ and using (4.13), we get

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}^{(N)}(\cdot,t)\|^2_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{2+\beta} \int_0^{2\pi} A_1(t)\, \mathrm{d}t. \tag{4.14}$$

Estimates (4.13) and (4.14) are valid for Galerkin approximations constructed using an arbitrary basis and for arbitrary bounded Lipschitz domains. In order to estimate the higher derivatives of $\mathbf{u}$, we have to assume that $\partial B_\varepsilon \in C^2$, and as a basis we shall use the eigenfunctions of the Stokes operator.

Multiplying (4.11) by $\lambda_k \gamma_k^{(N)}(t)$ and summing from $k=1$ to $k=N$, we obtain

$$\begin{aligned} &\int_{B_\varepsilon} \Bigg( \frac{1}{\varepsilon^\beta} \sum_{k=1}^N \mathbf{u}_t^{(N)} \cdot \psi_l \lambda_k \gamma_k^{(N)} - \nu \sum_{k=1}^N \Delta \mathbf{u}^{(N)} \cdot \psi_l \lambda_k \gamma_k^{(N)} \\ &- \sum_{k=1}^N ((\mathbf{u}^{(N)} + \mathbf{g}) \cdot \nabla) \lambda_k \gamma_k^{(N)} \psi_l \cdot \mathbf{u}^{(N)} - \sum_{k=1}^N (\mathbf{u}^{(N)} \cdot \nabla) \lambda_k \gamma_k^{(N)} \psi_l \cdot \mathbf{g} \Bigg) \mathrm{d}x \\ &= \int_{B_\varepsilon} \sum_{k=1}^N \mathbf{f} \cdot \psi_l \lambda_k \gamma_k^{(N)} \, \mathrm{d}x, \quad l = 1, \dots, N. \end{aligned}$$

Using the properties of the Stokes operator (see Chapter 2), and omitting the subscript $N$ (below $\mathbf{u}$ means $\mathbf{u}^{(N)}$), we rewrite the last equality as

$$\int\limits_{B_\varepsilon}\left(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\widetilde{\Delta}\mathbf{u}-\nu\Delta\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}+((\mathbf{u}+\mathbf{g})\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\widetilde{\Delta}\mathbf{u}\right)\mathrm{d}x$$
$$=\int\limits_{B_\varepsilon}\mathbf{f}\cdot\widetilde{\Delta}\mathbf{u}\,\mathrm{d}x.$$

This is equivalent to

$$\begin{gathered}\frac{\nu}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x+\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x=-\int\limits_{B_\varepsilon}((\mathbf{u}+\mathbf{g})\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\,\mathrm{d}x\\ -\int\limits_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\widetilde{\Delta}\mathbf{u}\,\mathrm{d}x+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\widetilde{\Delta}\mathbf{u}\,\mathrm{d}x=\sum_{i=1}^{3}J_i.\end{gathered}\tag{4.15}$$

Let us estimate the right hand side of (4.15). Using the inequality (2.5), we obtain

$$|J_3|\leqslant\delta\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\ \mathrm{d}x+c_\delta A_1(t),\tag{4.16}$$

$$\begin{aligned}|J_2|&\leqslant\|\mathbf{u}\|_{L^4(B_\varepsilon)}\|\nabla\mathbf{g}\|_{L^4(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)}\\ &\leqslant c\varepsilon^{\frac12}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{g}\|_{L^4(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)}\\ &\leqslant c_\delta\varepsilon\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\|\nabla\mathbf{g}\|^2_{L^4(B_\varepsilon)}+\delta\|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)}.\end{aligned}$$

By (2.7) and (4.4),

$$\|\nabla\mathbf{g}\|^2_{L^4(B_\varepsilon)}\leqslant c\varepsilon^{-1}\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\Big(\|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)}+\varepsilon^2\|\nabla^2\mathbf{g}\|^2_{L^2(B_\varepsilon)}\Big)^{1/2}\leqslant c\varepsilon^{-2}.$$

Therefore,

$$|J_2|\leqslant c_\delta\varepsilon^{-1}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+\delta\|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)}.\tag{4.17}$$

Further, by (4.4),

$$\begin{aligned}\left|\int\limits_{B_\varepsilon}(\mathbf{g}\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\ \mathrm{d}x\right|&\leqslant\|\mathbf{g}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)}\\ &\leqslant c_\delta\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+\delta\|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)}.\end{aligned}\tag{4.18}$$

Finally, applying (2.2) and (2.7), we get

$$\left|\int\limits_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\ \mathrm{d}x\right|\leqslant\|\mathbf{u}\|_{L^4(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^4(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)}$$

$$
\begin{aligned}
&\leqslant c_\delta \varepsilon^{-1} \|\mathbf{u}\|_{L^2(B_\varepsilon)} \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} \big(\|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} + \varepsilon^2 \|\nabla^2 \mathbf{u}\|^2_{L^2(B_\varepsilon)}\big)^{\frac{1}{2}} \\
&\quad + \frac{\delta}{2} \|\widetilde{\Delta} \mathbf{u}\|^2_{L^2(B_\varepsilon)} \leqslant c_\delta \varepsilon^{-1} \|\mathbf{u}\|_{L^2(B_\varepsilon)} \|\nabla \mathbf{u}\|^3_{L^2(B_\varepsilon)} \\
&\quad + c_\delta \|\mathbf{u}\|_{L^2(B_\varepsilon)} \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} \|\nabla^2 \mathbf{u}\|_{L^2(B_\varepsilon)} + \frac{\delta}{2} \|\widetilde{\Delta} \mathbf{u}\|^2_{L^2(B_\varepsilon)} \\
&\quad \leqslant c_\delta (1 + \|\mathbf{u}\|^2_{L^2(B_\varepsilon)}) \|\nabla \mathbf{u}\|^4_{L^2(B_\varepsilon)} + \delta \|\widetilde{\Delta} \mathbf{u}\|^2_{L^2(B_\varepsilon)} \\
&\overset{(4.14)}{\leqslant} c_\delta \Big(1 + \varepsilon^{2+\beta} \int_0^{2\pi} A_1(t)\,\mathrm{d}t\Big) \|\nabla \mathbf{u}\|^4_{L^2(B_\varepsilon)} + \delta \|\widetilde{\Delta} \mathbf{u}\|^2_{L^2(B_\varepsilon)}.
\end{aligned} \tag{4.19}
$$

Substituting (4.16)-(4.19) into (4.15) and taking $\delta = \dfrac{1}{8}$ we obtain

$$
\begin{aligned}
&\frac{\nu}{\varepsilon^\beta} \frac{\mathrm{d}}{\mathrm{d}t} \int_{B_\varepsilon} |\nabla \mathbf{u}|^2 \,\mathrm{d}x + \int_{B_\varepsilon} |\widetilde{\Delta} \mathbf{u}|^2 \,\mathrm{d}x \\
&\leqslant C_1 \Big(1 + \varepsilon^{2+\beta} \int_0^{2\pi} A_1(t)\,\mathrm{d}t\Big) \|\nabla \mathbf{u}\|^4_{L^2(B_\varepsilon)} \\
&\quad + C_3 (1 + \varepsilon^{-1}) \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} + C_2 A_1(t).
\end{aligned} \tag{4.20}
$$

Denote $Y(t) = \int_{B_\varepsilon} |\nabla \mathbf{u}(x,t)|^2 \,\mathrm{d}x$, $d_1 = C_1 \Big(1 + \varepsilon^{2+\beta} \int_0^{2\pi} A_1(t)\,\mathrm{d}t\Big)$. Then, we can rewrite (4.20) as

$$
Y'(t) \leqslant \frac{\varepsilon^\beta}{\nu} d_1 \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} Y(t) + C_3 \frac{\varepsilon^\beta}{\nu} (1 + \varepsilon^{-1}) \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} + \frac{\varepsilon^\beta}{\nu} C_2 A_1(t).
$$

Hence,

$$
\begin{aligned}
&\bigg(Y(t) e^{-\frac{\varepsilon^\beta}{\nu} \int_0^t d_1 \|\nabla \mathbf{u}(\cdot,\tau)\|^2_{L^2(B_\varepsilon)} \,\mathrm{d}\tau}\bigg)' \\
&\leqslant C_4 \varepsilon^\beta \Big((1 + \varepsilon^{-1}) \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} + A_1(t)\Big) e^{-\frac{\varepsilon^\beta}{\nu} \int_0^t d_1 \|\nabla \mathbf{u}(\cdot,\tau)\|^2_{L^2(B_\varepsilon)} \,\mathrm{d}\tau} \\
&\quad \leqslant C_4 \varepsilon^\beta \Big((1 + \varepsilon^{-1}) \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} + A_1(t)\Big).
\end{aligned}
$$

Integrating the last inequality and using (4.12) and (4.13) with $t_* = 0$, we obtain

$$
\begin{aligned}
&Y(t) \leqslant c Y(0) + c \varepsilon^\beta \int_0^{2\pi} A_1(s) \,\mathrm{d}s \\
&\leqslant c(\varepsilon^2 + \varepsilon^\beta) \int_0^{2\pi} A_1(t)\,\mathrm{d}t \leqslant c \varepsilon^\beta \int_0^{2\pi} A_1(t)\,\mathrm{d}t.
\end{aligned} \tag{4.21}
$$

From (4.21) we have

$$\sup_{t\in[0,2\pi]} \|\nabla \mathbf{u}^{(N)}(\cdot,t)\|^2_{L^2(B_\varepsilon)} \leqslant c\varepsilon^\beta \int_0^{2\pi} A_1(t)\,\mathrm{d}t. \tag{4.22}$$

Substituting (4.22) into (4.20) and integrating by $t$ from 0 to $2\pi$ imply

$$\begin{gathered}\int_0^{2\pi}\int_{B_\varepsilon} |\widetilde{\Delta}\mathbf{u}^{(N)}|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c\int_0^{2\pi} A_1(t)\,\mathrm{d}t \\ +c\varepsilon^\beta\Big(1+\varepsilon^{2+\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t\Big)\int_0^{2\pi} A_1(t)\,\mathrm{d}t\int_0^{2\pi} \|\nabla\mathbf{u}^{(N)}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t \\ +c(1+\varepsilon^{-1})\int_0^{2\pi} \|\nabla\mathbf{u}^{(N)}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t \leqslant C\int_0^{2\pi} A_1(t)\,\mathrm{d}t.\end{gathered} \tag{4.23}$$

Let us estimate the norm of $\mathbf{u}_t^{(N)}$. Multiply (4.11) by $\dfrac{\mathrm{d}}{\mathrm{d}t}\gamma_k^{(N)}$ and sum up the obtained equalities over $k$ from $k=1$ to $k=N$. Again omitting the subscript $N$, we obtain

$$\begin{gathered}\frac{1}{\varepsilon^\beta}\int_{B_\varepsilon} |\mathbf{u}_t|^2\,\mathrm{d}x + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_\varepsilon} |\nabla\mathbf{u}|^2\,\mathrm{d}x \\ = -\int_{B_\varepsilon} ((\mathbf{u}+\mathbf{g})\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x - \int_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\mathbf{u}_t\,\mathrm{d}x + \int_{B_\varepsilon}\mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x \\ \leqslant \big(\|\mathbf{u}\|_{L^\infty(B_\varepsilon)}+\|\mathbf{g}\|_{L^\infty(B_\varepsilon)}\big)\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)} \\ +\|\mathbf{u}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)} + \|\mathbf{f}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)} \\ \leqslant \delta\|\mathbf{u}_t\|^2 + c_\delta\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)} + c_\delta\|\mathbf{g}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)} \\ +c_\delta\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)} + c_\delta A_1(t).\end{gathered}$$

Taking sufficiently small $\delta$ and integrating over $[0,2\pi]$, we obtain the inequality

$$\begin{gathered}\left(\frac{1}{\varepsilon^\beta}-\frac{1}{2}\right)\int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\int_0^{2\pi} A_1(t)\,\mathrm{d}t + c\int_0^{2\pi} \|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t \\ +c\int_0^{2\pi} \|\mathbf{g}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t + c\int_0^{2\pi} \|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)}\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\,\mathrm{d}t \\ \overset{(2.30)}{\leqslant} c\int_0^{2\pi} A_1(t)\,\mathrm{d}t + c\varepsilon\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\int_0^{2\pi}\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\end{gathered}$$

$$+c \sup_{t\in[0,2\pi]} \|\mathbf{g}\|^2_{L^\infty(B_\varepsilon)} \int\limits_0^{2\pi} \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)}\, \mathrm{d}t$$

$$+c\varepsilon \sup_{t\in[0,2\pi]} \|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)} \int\limits_0^{2\pi} \|\nabla^2 \mathbf{u}\|^2_{L^2(B_\varepsilon)}\, \mathrm{d}t$$

$$\overset{(2.28)}{\leqslant} c\int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t + c\varepsilon \sup_{t\in[0,2\pi]} \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)} \int\limits_0^{2\pi} \|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)}\, \mathrm{d}t$$

$$+c \sup_{t\in[0,2\pi]} \|\mathbf{g}\|^2_{L^\infty(B_\varepsilon)} \int\limits_0^{2\pi} \|\nabla \mathbf{u}\|^2_{L^2(B_\varepsilon)}\, \mathrm{d}t$$

$$+c\varepsilon \sup_{t\in[0,2\pi]} \|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)} \int\limits_0^{2\pi} \|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)}\, \mathrm{d}t$$

$$\overset{(4.4),(4.12),(4.22),(4.23)}{\leqslant} C\int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t.$$

Thus,

$$\int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |\mathbf{u}_t^{(N)}|^2\, \mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^\beta \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t. \tag{4.24}$$

Estimates (4.13), (4.23) and (4.24) ensure (in standard way, see [49], [79]) the convergence of a subsequence of the Galerkin approximation and guaranty the existence of the solution $\mathbf{u}$. □

**Theorem 4.2.2.** *For sufficiently small* $\varepsilon$ *the solution of problem* (4.5), $n=2$, *is unique.*

*Proof.* Suppose that there are two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of problem (4.5) satisfying the conditions of Theorem 4.2.1. Subtracting the identity (4.5) for $\mathbf{u}_2$ from the one for $\mathbf{u}_1$, we obtain for the difference $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ the following identity

$$\int\limits_{B_\varepsilon} \Big(\frac{1}{\varepsilon^\beta}\mathbf{w}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{w} : \nabla\boldsymbol{\eta} - (\mathbf{u}_1\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{w} - (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}_2$$
$$-(\mathbf{g}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{w} - (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{g}\Big)\,\mathrm{d}x = 0,$$

for every divergence free vector-field $\boldsymbol{\eta}\in \mathring{W}^{1,2}(B_\varepsilon)$. Taking $\boldsymbol{\eta} = \mathbf{w}$ we obtain

$$\frac{1}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon} |\mathbf{w}|^2\,\mathrm{d}x + \nu\int\limits_{B_\varepsilon} |\nabla\mathbf{w}|^2\,\mathrm{d}x = \int\limits_{B_\varepsilon} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}_2\,\mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{g}\,\mathrm{d}x$$

$$\leqslant \|\mathbf{w}\|_{L^4(B_\varepsilon)}\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)}\Big(\|\mathbf{u}_2\|_{L^4(B_\varepsilon)}+\|\mathbf{g}\|_{L^4(B_\varepsilon)}\Big)$$

$$\overset{(2.7),(2.1)}{\leqslant} c\varepsilon\|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}\Big(\|\nabla\mathbf{u}_2\|_{L^2(B_\varepsilon)}+\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\Big).$$

Integrating the last inequality with respect to $t$ we derive

$$\nu\int_0^{2\pi}\int_{B_\varepsilon}|\nabla\mathbf{w}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant c\varepsilon\Big(\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}_2\|_{L^2(B_\varepsilon)}+\sup_{t\in[0,2\pi]}\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\Big)\int_0^{2\pi}\|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t$$

$$\overset{(4.4),(4.8)}{\leqslant} c\varepsilon\left(\Big(\varepsilon^\beta\int_0^{2\pi}A_1(t)\,\mathrm{d}t\Big)^{1/2}+\varepsilon^{-1/2}\right)\int_0^{2\pi}\|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t.$$

For sufficiently small $\varepsilon$ (i.e., if $c\varepsilon\left(\Big(\varepsilon^\beta\int_0^{2\pi}A_1(t)\,\mathrm{d}t\Big)^{1/2}+\varepsilon^{-1/2}\right)<\nu$) this implies

$$\int_0^{2\pi}\int_{B_\varepsilon}|\nabla\mathbf{w}|^2\,\mathrm{d}x\,\mathrm{d}t=0,$$

and, hence, $\mathbf{u}_1=\mathbf{u}_2$. □

### 4.2.2 Three-dimensional case

In this subsection we prove the existence of the unique weak solution of problem (4.5) when $n=3$.

**Theorem 4.2.3.** *Let $B_\varepsilon\subset\mathbb{R}^3$, $\partial B_\varepsilon\in C^2$. Suppose that the extended function $\mathbf{g}$ belongs to $C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{2,2}(B_\varepsilon))$ and satisfies the conditions* (4.2), (4.3), (4.4), *$\mathbf{f}\in L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))$ and $\|\mathbf{f}\|_{L^2(B_\varepsilon)}\leqslant c_0$, the constant $c_0$ is sufficiently small and independent of $\varepsilon$. Then for sufficiently small $\varepsilon$ there exists a solution to the variational problem* (4.5)*. The following estimate*

$$\sup_{t\in[0,2\pi]}\|\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)}+\varepsilon^\beta\int_0^{2\pi}\|\nabla\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\leqslant c\varepsilon^{2+\beta}\int_0^{2\pi}A_1(t)\,\mathrm{d}t, \quad (4.25)$$

*holds. If the constant $c_0$ is sufficiently small (independently of $\varepsilon$) in the case*

*$\beta = 0$ or if $\beta = 2$, then there also holds the estimate*

$$\begin{aligned}\sup_{t\in[0,2\pi]} \|\nabla \mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t(x,t)|^2 \, dx\, dt \\ + \varepsilon^\beta \int_0^{2\pi} \|\nabla^2 \mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} dt \leqslant c\varepsilon^\beta \int_0^{2\pi} A_1(t)\, dt.\end{aligned} \tag{4.26}$$

*Proof.* As in Theorem 4.2.1, we use Galerkin approximations. First, applying the inequality (2.3) instead of (2.2), we prove, exactly in the same way as before, the existence of Galerkin approximations and the following estimate for them

$$\frac{1}{\varepsilon^\beta} \sup_{t\in[0,2\pi]} \|\mathbf{u}^{(N)}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}^{(N)}(x,t)|^2 dx\, dt \leqslant c\varepsilon^2 \int_0^{2\pi} A_1(t)\, dt. \tag{4.27}$$

In order to estimate the higher derivatives of $\mathbf{u}$, we use as a basis the eigenfunctions of the Stokes operator. Taking in (4.11) Galerkin approximations with the basis $\{\mathbf{w}_k\}_{k=1}^\infty$, multiplying it by $\lambda_k \gamma_k^{(N)}(t)$, summing up the obtained equalities from $k = 1$ to $k = N$, and using properties of the Stokes operator (see Chapter 2), we obtain (here we again omit the subscript $N$)

$$\begin{aligned}\frac{\nu}{2\varepsilon^\beta}\frac{d}{dt}\int_{B_\varepsilon} |\nabla \mathbf{u}|^2 dx + \int_{B_\varepsilon} |\widetilde{\Delta}\mathbf{u}|^2 dx = -\int_{B_\varepsilon} ((\mathbf{u}+\mathbf{g})\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\, dx \\ -\int_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\widetilde{\Delta}\mathbf{u}\, dx + \int_{B_\varepsilon} \mathbf{f}\cdot\widetilde{\Delta}\mathbf{u}\, dx = \sum_{i=1}^3 J_i.\end{aligned} \tag{4.28}$$

Let us estimate the right hand side of (4.28). Using (2.1), (2.4)–(2.10), (2.29) and (4.4) for $n = 3$, we obtain

$$|J_3| \leqslant \delta \int_{B_\varepsilon} |\widetilde{\Delta}\mathbf{u}|^2 dx + c_\delta A_1(t), \tag{4.29}$$

$$\begin{aligned}|J_2| = \left|\int_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\widetilde{\Delta}\mathbf{u}\, dx\right| \leqslant \|\mathbf{u}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)} \\ \overset{(2.29)}{\leqslant} c\ \varepsilon^{\frac{1}{4}}\|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)},\end{aligned} \tag{4.30}$$

$$\begin{aligned}|J_{11}| = \left|\int_{B_\varepsilon} (\mathbf{g}\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\, dx\right| \leqslant \|\mathbf{g}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)} \\ \overset{(2.9)}{\leqslant} c\varepsilon\|\mathbf{g}\|_{L^\infty(B_\varepsilon)}\|\nabla^2\mathbf{u}\|_{L^2(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)} \overset{(2.28)}{\leqslant} c\varepsilon\|\widetilde{\Delta}\mathbf{u}\|^2_{L^2(B_\varepsilon)},\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
|J_{12}| = \left|\int\limits_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\widetilde{\Delta}\mathbf{u}\,\mathrm{d}x\right| &\leqslant c_\delta \|\mathbf{u}\|^2_{L^6(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^3(B_\varepsilon)} \\
+\delta\int\limits_{B_\varepsilon} |\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x &\overset{(2.4),(2.10)}{\leqslant} \delta\int\limits_{B_\varepsilon} |\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x \\
+c_\delta\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\varepsilon^{-1}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}&\Big(\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+\varepsilon^2\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\Big)^{\frac{1}{2}} \\
\overset{(2.9)}{\leqslant}\delta\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\mathrm{d}x + c_\delta\varepsilon^{-1}\|\nabla\mathbf{u}\|^3_{L^2(B_\varepsilon)}&\Big(c\varepsilon^2\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}+\varepsilon^2\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\Big)^{\frac{1}{2}} \\
&\overset{(2.29)}{\leqslant} \delta\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x + c_\delta\|\nabla\mathbf{u}\|^3_{L^2(B_\varepsilon)}\|\widetilde{\Delta}\mathbf{u}\|_{L^2(B_\varepsilon)} \\
&\leqslant 2\delta\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x + c_\delta\|\nabla\mathbf{u}\|^6_{L^2(B_\varepsilon)}.
\end{aligned}
\tag{4.32}
$$

Substituting (4.29)–(4.32) into (4.28) and taking $\delta = \dfrac{1}{12}$ implies

$$
\begin{aligned}
\frac{\nu}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\mathrm{d}x + \left(\frac{3}{4}-C_1\varepsilon^{1/4}\right)\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\mathrm{d}x \\
\leqslant C_2\|\nabla\mathbf{u}\|^6_{L^2(B_\varepsilon)} + C_3A_1(t).
\end{aligned}
\tag{4.33}
$$

If $\varepsilon$ is sufficiently small $\left(\varepsilon^{\frac{1}{4}} \leqslant \dfrac{1}{4C_1}\right)$, then

$$
\frac{3}{4} - C_1\varepsilon^{\frac{1}{4}} \geqslant \frac{1}{2},
$$

and (4.33) yields

$$
\frac{\nu}{2\varepsilon^\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{B_\varepsilon}|\widetilde{\Delta}\mathbf{u}|^2\,\mathrm{d}x \leqslant C_2\|\nabla\mathbf{u}\|^6_{L^2(B_\varepsilon)} + C_3A_1(t). \tag{4.34}
$$

Denoting $z(t) = \int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x$, we rewrite (4.34) as

$$
z'(t) \leqslant \frac{2\varepsilon^\beta}{\nu}C_2 z(t)^3 + \frac{2\varepsilon^\beta}{\nu}C_3A_1(t) \leqslant (1+z(t)^2)\left(\frac{2\varepsilon^\beta}{\nu}C_2z(t) + \frac{2\varepsilon^\beta}{\nu}C_3A_1(t)\right),
$$

or, equivalently,

$$
\frac{z'(t)}{1+z(t)^2} \leqslant \frac{2\varepsilon^\beta}{\nu}C_2z(t) + \frac{2\varepsilon^\beta}{\nu}C_3A_1(t). \tag{4.35}
$$

Integrating (4.35) by $t$ and using (4.27), we obtain

$$\begin{aligned} \arctan z(t) &\leqslant \arctan z(0) + \frac{2\varepsilon^\beta}{\nu} C_2 \int\limits_0^{2\pi} z(t)\,\mathrm{d}t + \frac{2\varepsilon^\beta}{\nu} C_3 \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t \\ &\leqslant \arctan\Big(C_4\varepsilon^2 \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t\Big) + \Big(C_5\varepsilon^{\beta+2} + \frac{2\varepsilon^\beta}{\nu}C_3\Big) \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t. \end{aligned} \tag{4.36}$$

Here, as in previous subsection, we assume that $t_* = 0$. For sufficiently small $\varepsilon$ in the case $\beta = 2$ or $\|\mathbf{f}\|_{L^2(B_\varepsilon)} \leqslant c_0$ where $c_0$ is a sufficiently small constant in the case $\beta = 0$, the following inequalities

$$\Big(C_5\varepsilon^{2+\beta} + \frac{2\varepsilon^\beta}{\nu}C_3\Big) \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t < \frac{\pi}{6},$$

$$C_4\varepsilon^2 \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t + \tan\bigg[\Big(C_5\varepsilon^{2+\beta} + \frac{2\varepsilon^\beta}{\nu}C_3\Big) \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t\bigg] < \frac{1}{2}$$

hold. Then (4.36) gives

$$\begin{aligned} z(t) = \int\limits_{B_\varepsilon} |\nabla \mathbf{u}|^2\,\mathrm{d}x \leqslant C_6\bigg(\varepsilon^2 \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t + \tan\bigg[\Big(C_5\varepsilon^{2+\beta} + \frac{2\varepsilon^\beta}{\nu}C_3\Big)\int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t\bigg]\bigg) \\ \leqslant c\varepsilon^\beta \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t, \end{aligned}$$

i.e.,

$$\sup_{t\in[0,2\pi]} \|\nabla \mathbf{u}^{(N)}(\cdot, t)\|^2_{L^2(B_\varepsilon)} \leqslant c\varepsilon^\beta \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t. \tag{4.37}$$

Substituting (4.37) into (4.34) yields

$$\begin{aligned} \int\limits_0^{2\pi} \|\nabla^2 \mathbf{u}^{(N)}(\cdot, t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t &\overset{(2.28)}{\leqslant} c\int\limits_0^{2\pi} \|\widetilde{\Delta}\mathbf{u}^{(N)}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t \\ \leqslant c\bigg(\varepsilon^{3\beta}\bigg(\int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t\bigg)^3 + \int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t\bigg) &\leqslant c\int\limits_0^{2\pi} A_1(t)\,\mathrm{d}t. \end{aligned} \tag{4.38}$$

Let us estimate the norm of $\mathbf{u}_t^{(N)}$. Taking in the integral identity (4.5) $\boldsymbol{\eta} = \mathbf{u}_t^{(N)}$ (more precisely, multiplying (4.11) by $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k^{(N)}$ and summing by $k$

from 1 to $N$) we obtain (omitting the subscript $N$)

$$
\begin{aligned}
&\frac{1}{\varepsilon^\beta}\int\limits_{B_\varepsilon}|\mathbf{u}_t|^2\,\mathrm{d}x+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x\\
&=-\int\limits_{B_\varepsilon}((\mathbf{u}+\mathbf{g})\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x-\int\limits_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{g}\cdot\mathbf{u}_t\,\mathrm{d}x+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\\
&\leqslant\|\mathbf{u}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)}+\|\mathbf{g}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)}\\
&\quad+\|\mathbf{u}\|_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{g}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)}+\|\mathbf{f}\|_{L^2(B_\varepsilon)}\|\mathbf{u}_t\|_{L^2(B_\varepsilon)}\\
&\leqslant 4\delta\|\mathbf{u}_t\|^2_{L^2(B_\varepsilon)}+c_\delta\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+c_\delta\|\mathbf{g}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\\
&\quad+c_\delta\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{g}\|^2_{L^2(B_\varepsilon)}+c_\delta A_1(t).
\end{aligned}
\tag{4.39}
$$

Integrating (4.39) over $[0,2\pi]$, using the periodicity condition, (4.4) and the inequalities (2.29), (4.38), for sufficiently small $\delta$ we derive

$$
\begin{aligned}
&\left(\frac{1}{\varepsilon^\beta}-\frac{1}{2}\right)\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\int\limits_0^{2\pi}\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\\
&\quad+c\sup_{t\in[0,2\pi]}\|\mathbf{g}(\cdot,t)\|^2_{L^\infty(B_\varepsilon)}\int\limits_0^{2\pi}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\\
&\quad+c\sup_{t\in[0,2\pi]}\|\nabla\mathbf{g}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\int\limits_0^{2\pi}\|\mathbf{u}\|^2_{L^\infty(B_\varepsilon)}\,\mathrm{d}t+c\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t\\
&\overset{(2.29),(4.4)}{\leqslant} c\varepsilon^{1/2}\int\limits_0^{2\pi}\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t+c\int\limits_0^{2\pi}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\\
&\quad+c\varepsilon^{1/2}\int\limits_0^{2\pi}\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t+c\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t\leqslant\\
&\leqslant c\varepsilon^{1/2}\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\int\limits_0^{2\pi}\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t+c\int\limits_0^{2\pi}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\\
&\quad+c\varepsilon^{1/2}\int\limits_0^{2\pi}\|\nabla^2\mathbf{u}\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t+c\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t\leqslant c\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t.
\end{aligned}
\tag{4.40}
$$

Thus,

$$
\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\mathbf{u}_t^{(N)}|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\varepsilon^\beta\int\limits_0^{2\pi}A_1(t)\,\mathrm{d}t.
$$

Estimates (4.27), (4.38) and (4.40) ensure the convergence of the Galerkin approximations and guaranty the existence of the solution (see [49], [79]). □

**Theorem 4.2.4.** *For sufficiently small $\varepsilon$ the solution of problem* (4.5)*, $n=3$, is unique.*

*Proof.* The proof is absolutely identical to that in the two-dimensional case (see Theorem 4.2.2), we only have to use inequalities (2.3), (4.26) instead of (2.2), (4.8). □

## 4.3 Asymptotic expansion

Let us describe the procedure of constructing an asymptotic expansion of the solution to problem (4.1) in the case $\beta=2$. The case $\beta=0$ is completely similar to the asymptotic expansion constructed in [70, 71] with only one difference that all functions depending on time are $2\pi$-periodic (instead of being equal to zero in some neighbourhood of $t=0$).

First, we solve the time-periodic problem on the graph and find the macroscopic pressure as a periodic in time function linear on every edge with respect to the longitudinal variable $x_n^{(e)}$. At the nodes, it satisfies the Kirchhoff-type junction conditions. This problem on the graph is the time-periodic analogue of the problem considered in [68]. It defines in every cylinder $\Pi_\varepsilon^{(e)}$ the Poiseuille type velocity depending only on the transversal space variable $x^{(e)'}$ of the tube. We multiply the Poiseuille type velocity and pressure in every cylinder by cut-off functions $\zeta$ equal to one in the main middle part of the cylinder and vanishing in some $O(\varepsilon)$-neighbourhood of the nodes. This multiplication generates a residual in the right-hand side of the Navier–Stokes equations, having a support belonging to a $O(\varepsilon)$-neighbourhood of the nodes. Then we construct the boundary layer correctors, compensating this residual. These correctors are solutions to the Stokes equations in the dilated bundles of cylinders extended by outlets to infinity.

In this context the asymptotic expansion of the velocity is constructed in the form

$$\begin{aligned}\mathbf{v}^{(J)}(x,t) = &\sum_{i=1}^{M}\zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i|-x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J}\varepsilon^j\mathbf{V}_j^{(e_i)}(y^{(e_i)'},t)\\ &+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\sum_{j=-1}^{J}\varepsilon^j\mathbf{V}_j^{[BLO_l]}(y,t),\end{aligned} \tag{4.41}$$

where $y=\dfrac{x^{(e)}}{\varepsilon}$, $\zeta(\tau)$ is a smooth cut-off function independent of $\varepsilon$ with $\zeta(\tau)=0$ for $\tau\leqslant 1/3$ and $\zeta(\tau)=1$ for $\tau\geqslant 2/3$, $0\leqslant\zeta(\tau)\leqslant 1$. Here $|e|_{\min}$ is the minimal length of the edges, $r=3\max\{\text{diam } \sigma_1,\dots,\text{diam } \sigma_{\mathrm{M}}\}+1$, $\mathbf{V}_j^{(e_i)}(y^{(e_i)\prime},t)$ are the Poiseuille type velocities and the boundary layer terms $\mathbf{V}_j^{[BLO_l]}(y,t)$ exponentially decay as $|y|$ tends to infinity.

The asymptotic expansion of the pressure has the similar form:

$$\begin{aligned} p^{(J)}(x,t)=&\sum_{i=1}^{M}\zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i|-x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J}\varepsilon^{j-2}\big(-s_j^{(e_i)}(t)x_n^{(e_i)}+a_j^{(e_i)}(t)\big)\\ &+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\sum_{j=-1}^{J}\varepsilon^{j-1}P_j^{[BLO_l]}(y,t). \end{aligned} \tag{4.42}$$

The asymptotic solution is constructed by induction with respect to $j$. At the base (initial) step $j=0$, we consider the following problem on the graph: find a function $p_0\in L^2_{\mathrm{per}}(0,2\pi;W^{1,2}(\mathcal{B}))$ such that equations

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big)=0, \quad x_n^{(e)}\in(0,|e|),\forall e=e_j,\, j=1,\dots,M,\\ -\displaystyle\sum_{e:O_l\in e}\Big(L^{(e)}\frac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=0, \qquad l=1,\dots,N_1,\\ \qquad -\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=\Psi_l(t),\ l=N_1+1,\dots,N, \end{cases} \tag{4.43}$$

hold. Here $\Psi_l(t)=\displaystyle\int_{\gamma^l}\mathbf{g}_l\cdot\mathbf{n}\,\mathrm{d}S$. Operator $L^{(e)}$ relates the pressure slope $\mathcal{S}$ and the flux $\mathcal{H}$ in an infinite cylindrical pipe with section $\sigma^{(e)}$. Namely, consider the following periodic in time boundary value problem for the heat equation: for given $\mathcal{S}\in L^2_{\mathrm{per}}(0,2\pi)$ find $\mathcal{V}\in L^2_{\mathrm{per}}(0,2\pi;\mathring{W}^{1,2}(\sigma^{(e)}))$ with $\dfrac{\partial\mathcal{V}}{\partial t}\in L^2_{\mathrm{per}}(0,2\pi;L^2(\sigma^{(e)}))$ such that

$$\begin{cases} \dfrac{\partial\mathcal{V}}{\partial t}(y^{(e)\prime},t)-\nu\Delta'_{y^{(e)\prime}}\mathcal{V}(y^{(e)\prime},t)=S(t), \quad y^{(e)\prime}\in\sigma^{(e)},\ t>0,\\ \qquad \mathcal{V}(y^{(e)\prime},t)|_{\partial\sigma^{(e)}}=0, \quad \mathcal{V}(y^{(e)\prime},t)=\mathcal{V}(y^{(e)\prime},t+2\pi) \end{cases}$$

and denote

$$L^{(e)}\mathcal{S}(t)=\int_{\sigma^{(e)}}\mathcal{V}(y^{(e)\prime},t)\,\mathrm{d}y^{(e)\prime}=\mathcal{H}(t).$$

$L^{(e)}$ is bounded linear operator acting from $L^2_{\mathrm{per}}(0,2\pi)$ to $W^{1,2}_{\mathrm{per}}(0,2\pi)$ (see [4], [17]). Denote $\mathcal{M}S=\mathcal{V}$. The existence of a solution to problem (4.43) is proved in [72]. Let us represent $p_0^{(e)}$ in the form

$$p_0^{(e)}(x_n^{(e)},t) = -s_0^{(e)}(t)x_n^{(e)} + a_0^{(e)}(t).$$

For every edge $e_i$ define the Poiseuille type velocity $\mathbf{V}_0^{(e_i)}(y^{(e_i)\prime},t)$ as a vector such that in the local coordinates its last (i.e. normal) component is $\mathcal{M}s_0^{(e)}$, while the tangential components are equal to zero.

Next, we find the boundary layer correctors $(\mathbf{V}_0^{[BLO_l]}, P_0^{[BLO_l]})$ as solutions of the periodic in time Stokes equations in the dilated domain: union of semi-infinite cylinders having the common node $O_l$, and the corresponding $\omega^l$. Namely, let $O_l$ be a node which is the common end of edges $e_{i_1},\dots,e_{i_m}$ of the bundle $\mathcal{B}_l$. Define the semi-infinite cylinders

$$\Pi_{l,j_s}^{+} = \{y \in \mathbb{R}^n : \mathcal{P}^{(e_{i_s})} y \in \sigma^{i_s} \times (0,+\infty)\}$$

and the domain $\Omega_l$ with $m$ outlets to infinity corresponding to the node $O_l$:

$$\Omega_l = (\bigcup_{s=1}^{m} \Pi_{l,j_s}^{+}) \bigcup \omega^l.$$

We introduce the boundary layer pressure of the rank $-1$ as

$$P_{-1}^{[BLO_l]}(y,t) = -\Big(\sum_{e:O_l\in e} \zeta\Big(\frac{y_n^{(e)}}{3r}\Big) - 1\Big)p_0(O_l,t).$$

Here $p_0$ is a continuous function on $\mathcal{B}$ without jumps at the nodes, so that $p_0(O_l,t)$ is well defined. The boundary layer velocity of rank $-1$ is equal to zero: $\mathbf{V}_{-1}^{[BLO_l]}(y,t) = 0$.

The boundary layer terms $(\mathbf{V}_0^{[BLO_l]}, P_0^{[BLO_l]})$ are defined as a solution of the periodic in time Stokes problem in the unbounded domain $\Omega_l$:

$$\begin{cases}
\dfrac{\partial}{\partial t}\mathbf{V}_0^{[BLO_l]} - \nu\Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} \\
= \displaystyle\sum_{e:O_l\in e} \Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\frac{\partial}{\partial t}V_0^{(e)}(y^{(e)\prime},t) + \nu\frac{\partial^2}{\partial y_n^{(e)2}}\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\mathbf{V}_0^{(e)}(y^{(e)\prime},t) \\
+ \nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)y_n^{(e)}\Big)s_0^{(e)}(t) - \nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\hat{a}_1^{(e)}(t)\Big), \quad y \in \Omega_l, \\
\operatorname{div}_y \mathbf{V}_0^{[BLO_l]} = -\displaystyle\sum_{e:O_l\in e} \frac{\partial}{\partial y_n^{(e)}}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)V_{0,n}^{(e)}(y^{(e)\prime},t), \quad y \in \Omega_l, \\
\mathbf{V}_0^{[BLO_l]}|_{\partial\Omega_l} = 0, \quad \mathbf{V}_0^{[BLO_l]}(y,t) = \mathbf{V}_0^{[BLO_l]}(y,t+2\pi),
\end{cases}$$

where the local coordinates have the origin at $O_l$ and $\hat{a}_1^{(e)}(t)$ is an unknown function. This problem is decomposed to two independent ones: first we solve it without the term containing $\hat{a}_1^{(e)}(t)$ in the right-hand side for $(\mathbf{V}_0^{[BLO_l]}, \hat{P}_0^{[BLO_l]})$:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{V}_0^{[BLO_l]} - \nu\Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y \hat{P}_0^{[BLO_l]} \\ = \displaystyle\sum_{e:O_l\in e} \Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\frac{\partial}{\partial t}V_0^{(e)}(y^{(e)\prime},t) + \nu\frac{\partial^2}{\partial y_n^{(e)2}}\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\mathbf{V}_0^{(e)}(y^{(e)\prime},t) \\ + \nabla_y\Big(\zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)y_n^{(e)}\Big)s_0^{(e)}(t)\Big), \quad y\in\Omega_l, \\ \operatorname{div}_y \mathbf{V}_0^{[BLO_l]} = -\displaystyle\sum_{e:O_l\in e}\frac{\partial}{\partial y_n^{(e)}}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)V_{0,n}^{(e)}(y^{(e)\prime},t), \quad y\in\Omega_l, \\ \mathbf{V}_0^{[BLO_l]}|_{\partial\Omega_l} = 0, \quad \mathbf{V}_0^{[BLO_l]}(y,t) = \mathbf{V}_0^{[BLO_l]}(y,t+2\pi), \end{cases}$$

and find a solution $\mathbf{V}_0^{[BLO_l]}$ which tends to zero as $|y|\to\infty$, while $\hat{P}_0^{[BLO_l]}$ at each outlet $\Pi_{l,j}^+$ tends to a constant $\hat{a}_{l,j}(t)$, except for an outlet corresponding to a selected edge $e_s$ where it tends to zero (this is possible because the pressure is defined up to an additive constant). Then we solve the following problem on the graph: find a function $p_1^{(e)} \in L^2_{\text{per}}(0,2\pi;W^{1,2}(e))$ such that equations

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_1^{(e)}}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big) = 0, & x_n^{(e)}\in(0,|e|),\ \forall e = e_j,\ j=1,\dots,M, \\ -\displaystyle\sum_{e:O_l\in e}\Big(L^{(e)}\frac{\partial p_1^{(e)}}{\partial x_n^{(e)}}\Big)(0,t) = 0, & l = 1,\dots,N_1, \\ -\Big(L^{(e)}\dfrac{\partial p_1^{(e)}}{\partial x_n^{(e)}}\Big)(0,t) = 0, & l = N_1+1,\dots,N, \\ p_1^{(e)}(0,t) - p_1^{(e_s)}(0,t) = \hat{a}_1^{(e)}(t), & \forall e\subset\mathcal{B}_l,\ e\neq e_s \end{cases}$$

hold, where $e_s$ is a selected edge of the bundle. This problem has a unique (up to an additive function of $t$) solution $p_1$ and $p_1^{(e)}(x_n^{(e)},t) = -s_1^{(e)}(t)x_n^{(e)} + a_1^{(e)}(t)$. Then finally we define

$$P_0^{[BLO_l]} = \hat{P}_0^{[BLO_l]} - \Big(\sum_{e:O_l\in e, e\neq e_s}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\hat{a}_1^{(e)}(t) - \Big(\sum_{e:O_l\in e}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big) - 1\Big)p_1^{(e_s)}(0,t).$$

Analogously, if $O_l$ is a vertex, the end of the edge $e_i$, then we define the domain $\Omega_l$, corresponding to this vertex, as

$$\Omega_l = \{y\in\mathbb{R}^n : \mathcal{P}^{(e_i)}y \in \sigma^i\times(0,+\infty)\}\cup\omega^l,$$

and the boundary layer problem has the form:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{V}_0^{[BLO_l]} - \nu \Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y \hat{P}_0^{[BLO_l]} \\ = \zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)\dfrac{\partial}{\partial t}V_0^{(e)}(y^{(e)\prime},t) + \nu \dfrac{\partial^2}{\partial y_n^{(e)2}}\Big(\zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)\Big)\mathbf{V}_0^{(e)}(y^{(e)\prime},t) \\ + \nabla_y\Big(\zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)y_n^{(e)}\Big)s_0^{(e)}(t), \quad y \in \Omega_l, \\ \mathrm{div}_y\, \mathbf{V}_0^{[BLO_l]} = -\dfrac{\partial}{\partial y_n^{(e)}}\zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)V_{0,n}^{(e)}(y^{(e)\prime},t), \quad y \in \Omega_l, \\ \mathbf{V}_0^{[BLO_l]}|_{\partial\Omega_l \setminus \gamma^l} = 0, \quad \mathbf{V}_0^{[BLO_l]}|_{\gamma^l} = \mathbf{g}^l(y,t), \\ \mathbf{V}_0^{[BLO_l]}(y,t) = \mathbf{V}_0^{[BLO_l]}(y,t+2\pi). \end{cases}$$

Because of condition $(4.43)_3$, we have

$$\int\limits_{\Omega_l} \frac{\partial}{\partial y_n^{(e)}}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)V_{0,n}^{(e)}(y^{(e)\prime},t)\,\mathrm{d}y + \int\limits_{\gamma^l}\mathbf{g}^l\cdot\mathbf{n}\,\mathrm{d}S = 0.$$

This compatibility condition ensures the existence of a unique solution $(\mathbf{V}_0^{[BLO_l]}, \hat{P}_0^{[BLO_l]})$ which exponentially tends to zero at infinity (see [70, 71], [73], [74]).

Suppose that all terms of asymptotic expansion corresponding to the rank less or equal $j-1$ are known and the pressure on the graph $p_j$ is known as well. Describe the passage from rank $j-1$ to the rank $j$.

**Step 1.** As pressure on the graph $p_j$ is known, define for every edge $e$ functions $s_j^{(e)}(t)$ and $a_j^{(e)}(t)$ such that

$$p_j^{(e)}(x_n^{(e)},t) = -s_j^{(e)}(t)x_n^{(e)} + a_j^{(e)}(t)$$

and define the Poiseuille type velocity $\mathbf{V}_j^{(e)}(y^{(e)\prime},t)$ as a vector field such that in the local coordinates its last (i.e. normal) component is $\mathcal{M}(s_j^{(e)})$, while the tangential components are equal to zero.

**Step 2.** The boundary layer solution is a $2\pi-$periodic in time pair $(\mathbf{V}_j^{[BLO_l]}, P_j^{[BLO_l]})$ satisfying the problem

$$\begin{cases} \dfrac{\partial \mathbf{V}_j^{[BLO_l]}}{\partial t} - \nu\Delta_y \mathbf{V}_j^{[BLO_l]} + \nabla_y P_j^{[BLO_l]} = \mathbf{f}_j^{[REGO_l]}(y^{(e)\prime},t) + \mathbf{f}_j^{[BLO_l]}(y,t), \\ \qquad \mathrm{div}_y\,\mathbf{V}_j^{[BLO_l]} = h_j^{[REGO_l]}(y^{(e)\prime},t), \quad y \in \Omega_l \\ \mathbf{V}_j^{[BLO_l]}(y,t)|_{\partial\Omega_l} = 0, \quad \mathbf{V}_j^{[BLO_l]}(y,t) = \mathbf{V}_j^{[BLO_l]}(y,t+2\pi), \end{cases}$$

where

$$h_j^{[REGO_l]}(y^{(e)'},t) = -\sum_{e:O_l\in e} \frac{\partial}{\partial y_n^{(e)}}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_{j,n}^{(e)}(y^{(e)'},t),$$

$$\mathbf{f}_j^{[REGO_l]}(y^{(e)'},t) = \sum_{e:O_l\in e}\Big[\nu\Big(\mathbf{V}_j^{(e)}(y^{(e)'},t)\frac{\partial^2}{\partial y_n^{(e)2}}\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\Big)$$

$$-\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\sum_{k=0}^{j-1}(\mathbf{V}_k^{(e)}(y^{(e)'},t)\cdot\nabla_y)\times\Big\{\zeta(\frac{y_n^{(e)}}{3r})\mathbf{V}_{j-k-1}^{(e)}(y^{(e)'},t)\Big\}$$

$$+s_j^{(e)}(t)y_n^{(e)}\cdot\nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big) - a_{j+1}^{(e)}(t)\cdot\nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\Big]$$

and

$$\mathbf{f}_j^{[BLO_l]}(y,t) = \sum_{e:O_l\in e}\Big[-\sum_{k=0}^{j-1}(\mathbf{V}_k^{[BLO_l]}(y,t)\cdot\nabla_y)\times\Big\{\ \zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_{j-k-1}^{(e)}(y^{(e)'},t)\Big\}$$

$$-\sum_{k=0}^{j-1}(\mathbf{V}_k^{[BLO_l]}(y,t)\cdot\nabla_y)\times\Big\{\ \mathbf{V}_{j-k-1}^{[BLO_l]}(y,t)\Big\}$$

$$-\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\sum_{k=0}^{j-1}(\mathbf{V}_k^{(e)}(y^{(e)'},t)\cdot\nabla_y)\times\Big\{\ \mathbf{V}_{j-k-1}^{[BLO_l]}(y,t)\Big\}\Big],$$

where $l=1,\ldots,N,\ j=-1,\ldots,J$.

This problem is solved in two steps: by the first step we find the couple $(\mathbf{V}_j^{[BLO_l]},\hat{P}_j^{[BLO_l]})$ which is the solution of the same problem without the last term in the definition of $\mathbf{f}_j^{[REGO_l]}$. It has a unique (up to an additive constant in the pressure) solution $\mathbf{V}_j^{[BLO_l]}(\cdot,t)\in\mathring{W}^{1,2}(\Omega_l)$, $\hat{P}_j^{[BLO_l]}(\cdot,t)\in L^2_{\text{loc}}(\Omega_l)$ ($t$ is a parameter) if and only if

$$\int_{\Omega_l} h_j^{[REGO_l]}(y,t)\,\mathrm{d}y = 0,\quad l=1,...,N_1.$$

This condition can be written as

$$\sum_{e:O_l\in e}\int_{\sigma^{(e)}}\mathbf{V}_j^{(e)}(y^{(e)'},t)\cdot\mathbf{n}\,\mathrm{d}y^{(e)'} = 0$$

i.e.,

$$\sum_{e:O_l\in e} L^{(e)}s_j^{(e)}(t) = 0,$$

or

$$-\sum_{e:O_l\in e}\Big(L^{(e)}\frac{\partial p_j^{(e)}}{\partial x_n^{(e)}}\Big) = 0,\quad l=1,\ldots,N_1. \tag{4.44}$$

This condition is satisfied because $p_j^{(e)}$ is a solution of the problem on the graph with junction condition (4.44) (by the inductive hypothesis). The velocity $\mathbf{V}^{[BLO_l]}$ exponentially tends to zero as $|y| \to \infty$ while the corresponding pressure function $\hat{P}_j^{[BLO_l]}$ stabilizes in outlets at infinity to some constants $\hat{a}_{l,j}^{(e)}(t)$; these constants may be different for different outlets. Since the pressure function is defined up to an additive constant, we can fix the limit constant equal to zero for the outlet corresponding the selected edge $e_s$. Define $\varphi_{l,j+1}^{(e)}(t) = \hat{a}_{l,j}^{(e)}(t)$.

Similarly, in every vertex $O_l$, $l = N_1+1,\dots,N$, we get for the pair $(\mathbf{V}_j^{[BLO_l]}, P_j^{[BLO_l]})$ the Stokes problem in $\Omega_l$ which is the same as in the case of nodes $O_l$ with only one difference: there is no summing over $e : O_i \in e$ in the right-hand sides of the equations.

**Step 3.** Solve the problem on the graph for the function $p_{j+1}^{(e)}$, $(j < J)$:

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_{j+1}^{(e)}}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big) = 0, & x_n^{(e)} \in (0,|e|),\ \forall e = e_j,\ j = 1,\dots,M, \\ -\displaystyle\sum_{e:O_l\in e}\Big(L^{(e)}\dfrac{\partial p_{j+1}^{(e)}}{\partial x_n^{(e)}}\Big)(0,t) = 0, & l = 1,\dots,N_1, \\ -\Big(L^{(e)}\dfrac{\partial p_{j+1}^{(e)}}{\partial x_n^{(e)}}\Big)(0,t) = 0, & l = N_1+1,\dots,N, \\ p_{j+1}^{(e)}(0,t) - p_{j+1}^{(e_s)}(0,t) = \varphi_{l,j+1}^{(e)}(t),\ \forall e \subset \mathcal{B}_l,\ e \neq e_s. & \end{cases}$$

The local coordinates $x^{(e)}$ are defined so that all of them have the same origin $O_l$.

**Step 4.** Finally we find the pressure $P_j^{[BLO_l]}(y,t)$ in the boundary layer problem:

$$P_0^{[BLO_l]} = \hat{P}_0^{[BLO_l]} - \Big(\sum_{e:O_l\in e, e\neq e_s}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)\hat{a}_{l,j}^{(e)}(t) - \Big(\sum_{e:O_l\in e}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big) - 1\Big)p_{j+1}^{(e_s)}(0,t).$$

For $j = J$ the last sum is absent. The last step finalizes the passage from $j$ to $j+1$.

## 4.4 Residual

Consider the asymptotic expansion $(\mathbf{v}^{(J)}, p^{(J)})$ of order J in the case $\beta = 2$ (see (4.41), (4.42)). By construction,

$$
\begin{aligned}
\mathbf{v}^{(J)} &\in L^2_{\mathrm{per}}(0,2\pi; W^{2,2}(B_\varepsilon)) \cap L^\infty_{\mathrm{per}}(0,2\pi; W^{1,2}(B_\varepsilon)),\\
\frac{\partial \mathbf{v}^{(J)}}{\partial t} &\in L^2_{\mathrm{per}}(0,2\pi; W^{1,2}(B_\varepsilon)) \cap L^\infty_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon)),\\
\nabla p^{(J)} &\in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon)).
\end{aligned}
\tag{4.45}
$$

Put $\mathcal{L}(\mathbf{v},p) = \frac{1}{\varepsilon^2}\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p$. Let us calculate $\mathcal{L}(\mathbf{v}^{(J)}, p^{(J)})$. We obtain

$$
\mathcal{L}(\mathbf{v}^{(J)}, p^{(J)}) = \mathbf{f}^{(J)}(x,t) =
$$

$$
= \sum_{l=1}^{N}\Bigg[\sum_{j=J+1}^{2J} \varepsilon^{j-2} \sum_{\substack{k,p: k+p=j-1\\ 0\leqslant k,p\leqslant J}} \sum_{e: O_l\in e} \Bigg\{\bigg(\mathbf{V}_k^{(e)}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\cdot\nabla_y\bigg)\bigg(\mathbf{V}_p^{(e)}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\bigg)
$$

$$
+\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big(\mathbf{V}_k^{(e)}(y^{(e)\prime},t)\cdot\nabla_y\Big)\mathbf{V}_p^{[BLO_l]}(y,t)
$$

$$
+\Big(\mathbf{V}_k^{[BLO_l]}(y,t)\cdot\nabla_y\Big)\bigg(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_p^{(e)}(y^{(e)\prime},t)\bigg)
$$

$$
+\Big(\mathbf{V}_k^{[BLO_l]}(y,t)\cdot\nabla_y\Big)\mathbf{V}_p^{[BLO_l]}(y,t)\Bigg\} + \varepsilon^{J-2}\sum_{e:O_l\in e} a_{J+1}^{(e)}\nabla_y\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Bigg]
$$

$$
-\sum_{l=1}^{N}\Bigg\{\mathcal{L}\bigg(\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\mathbf{V}_J^{[BLO_l]}(y,t), \zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)P_J^{[BLO_l]}(y,t)\bigg)\chi(x)\Bigg\}.
$$

Here $y = \dfrac{x-O_l}{\varepsilon}$, $y^{(e)} = \dfrac{x^{(n)}}{\varepsilon}$, $\chi = \chi_{\mathrm{supp}(1-\zeta(\frac{|x-O_l|}{|e|_{\min}}))}$ is the characteristic function of the set $\mathrm{supp}\Big(1-\zeta\big(\dfrac{|x-O_l|}{|e|_{\min}}\big)\Big)$. From the obtained formulas it follows that

$$
\begin{aligned}
\|\mathbf{f}^{(J)}\|_{L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))} = \|\mathcal{L}(\mathbf{v}^{(J)},p^{(J)})\|_{L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))} = O(\varepsilon^{J-2}),\\
\|\mathbf{f}_t^{(J)}\|_{L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))} = O(\varepsilon^{J-2}).
\end{aligned}
\tag{4.46}
$$

Let us calculate the divergence of $\mathbf{v}^{(J)}$. We have

$$
\operatorname{div}\mathbf{v}^{(J)} = -\sum_{l=1}^{N}\nabla\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\cdot\mathbf{V}_J^{[BLO_l]}(y,t) = h^{(J)}(y,t), \tag{4.47}
$$

where $h^{(J)} \in L^2_{\text{per}}(0,2\pi;\mathring{W}^{1,2}(B_\varepsilon))$. Since the support of the function $\nabla\zeta\left(\dfrac{|x-O_l|}{|e|_{\min}}\right)$ belongs to the middle third part (between the planes $x_n^{(e)} = \dfrac{1}{3}|e|_{\min}$ and $x_n^{(e)} = |e| - \dfrac{1}{3}|e|_{\min}$) of every cylinder, there hold the relations

$$\begin{aligned} \|h^{(J)}\|_{L^2_{\text{per}}(0,2\pi;W^{1,2}(B_\varepsilon))} &= O(e^{-c_1/\varepsilon}), \\ \|h_t^{(J)}\|_{L^2_{\text{per}}(0,2\pi;L^2(B_\varepsilon))} &= O(e^{-c_1/\varepsilon}). \end{aligned} \tag{4.48}$$

The boundary conditions and the periodicity conditions are satisfied exactly. It is easy to see that

$$\int\limits_{B_\varepsilon} h^{(J)}(y,t)\,\mathrm{d}y = 0.$$

Therefore, by Lemma 3.7 [see [69]], there exists a vector field $\mathbf{w}^{(J)} \in L^2_{\text{per}}(0,2\pi; W^{2,2}(B_\varepsilon) \cap \mathring{W}^{1,2}(B_\varepsilon))$ with $\mathbf{w}_t^{(J)} \in L^2_{\text{per}}(0,2\pi;\mathring{W}^{1,2}(B_\varepsilon)) \cap L^\infty_{\text{per}}(0,2\pi;L^2(B_\varepsilon))$ such that $\operatorname{div}\mathbf{w}^{(J)} = -h^{(J)}$. Moreover, there hold the estimates

$$\|\mathbf{w}^{(J)}\|_{L^2(0,2\pi;W^{2,2}(B_\varepsilon))} \leqslant \varepsilon^{-3} c \|h^{(J)}\|_{L^2(0,2\pi;W^{1,2}(B_\varepsilon))}, \tag{4.49}$$

$$\|\mathbf{w}_t^{(J)}\|_{L^2(0,2\pi;W^{1,2}(B_\varepsilon))} \leqslant \varepsilon^{-1} c \|h_t^{(J)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}. \tag{4.50}$$

Set $\mathbf{u}^{(J)} = \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$. Then $\operatorname{div}\mathbf{u}^{(J)} = 0$, $\mathbf{u}^{(J)}$ satisfies the periodicity conditions and because of (4.46), (4.48) we have

$$\|\mathbf{f}_1^{(J)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))} = O(\varepsilon^{J-2}), \tag{4.51}$$

where $\mathbf{f}_1^{(J)} = \mathcal{L}(\mathbf{u}^{(J)}, p^{(J)})$.

If $\beta = 0$ then the residual has the same form as in [70].

## 4.5 Justification of the asymptotic

Consider the Navier–Stokes problem (4.1). As an extension of the boundary value $\mathbf{g}$ we take the asymptotic approximation $\mathbf{u}^{(J)}$ constructed in the previous sections and let $p^{(J)}$ be the corresponding asymptotic approximation for the pressure $p$. By construction $\mathbf{u}^{(J)}$ satisfies the conditions (4.4). Represent $\mathbf{v}$, $p$ as the sums $\mathbf{v} = \mathbf{u} + \mathbf{u}^{(J)}$, $p = q + p^{(J)}$. Then $\mathbf{u}$, $\mathbf{u}^{(J)} \in L^2_{\text{per}}(0,2\pi;W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t$, $\mathbf{u}_t^{(J)} \in L^2_{\text{per}}(0,2\pi;L^2(B_\varepsilon))$. The difference $\mathbf{u} = \mathbf{v} - \mathbf{u}^{(J)}$ is divergence free, satisfies the periodicity condition, the boundary condition $\mathbf{u}(x,t)|_{\partial B_\varepsilon}=0$ and the integral identity

$$\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\eta}-((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x = \int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x \tag{4.52}$$

for every $\boldsymbol{\eta}\in H(B_\varepsilon)$.

The existence of the unique solution $\mathbf{u}$ of (4.52) follows from Theorems 4.2.1–4.2.4.

**Theorem 4.5.1.** *Let $n=3$. The following estimates*

$$\sup_{t\in[0,2\pi]}\|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}+\varepsilon^\beta\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\varepsilon^{2J-2+\beta}, \tag{4.53}$$

$$\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}+\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^\beta\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\nabla^2\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-4+\beta}, \tag{4.54}$$

*hold. Moreover, there exists the pressure function $q\in L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))$ such that $\int\limits_{B_\varepsilon}q(x,t)\,\mathrm{d}x=0$ and*

$$\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\eta}-((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x = \int\limits_{B_\varepsilon}q\,\mathrm{div}\,\boldsymbol{\eta}\,\mathrm{d}x+\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x\quad\forall\boldsymbol{\eta}\in\mathring{W}^{1,2}(B_\varepsilon). \tag{4.55}$$

*If $J\geqslant 2$, then the following estimate*

$$\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|q|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\varepsilon^{2J-4-\beta} \tag{4.56}$$

*holds.*

*Proof.* The estimates (4.53) and (4.54) follow from (4.25), (4.26) and (4.51). Let us prove the existence of the pressure $q$ and the estimate (4.56) for it. Consider the linear functional

$$M(\boldsymbol{\eta})=\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\eta}-((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} -(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x-\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x \tag{4.57}$$

defined on functions $\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)$. There holds the estimate

$$
\begin{aligned}
|M(\boldsymbol{\eta})| &\leqslant c\Big(\varepsilon^{1-\beta}\|\mathbf{u}_t(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\mathbf{u}(\cdot,t)\|^2_{L^4(B_\varepsilon)} \\
&+\|\mathbf{u}(\cdot,t)\|_{L^4(B_\varepsilon)}\|\mathbf{u}^{(J)}(\cdot,t)\|_{L^4(B_\varepsilon)} + \varepsilon\|\mathbf{f}_1^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)}\Big)\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)} \\
&\overset{(2.3)}{\leqslant} c\Big(\varepsilon^{1-\beta}\|\mathbf{u}_t(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon\|\mathbf{f}_1^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)}\Big)\|\nabla\boldsymbol{\eta}\|_{L^2(B_\varepsilon)}.
\end{aligned}
\tag{4.58}
$$

Thus, $M(\boldsymbol{\eta})$ is a bounded linear functional (for almost all $t \in [0,2\pi]$) defined on $\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)$. Moreover, due to (4.52), $M(\boldsymbol{\eta}) = 0$ for $\boldsymbol{\eta}$ with $\operatorname{div}\boldsymbol{\eta} = 0$. Therefore, there exists a function $q(\cdot,t) \in L^2(B_\varepsilon)$, with $\int\limits_{B_\varepsilon} q(x,t)\,\mathrm{d}x = 0$, such that

$$
M(\boldsymbol{\eta}) = \int\limits_{B_\varepsilon} q(x,t)\operatorname{div}\boldsymbol{\eta}(x)\,\mathrm{d}x \quad \forall\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)
$$

(see [50]). Since $\int\limits_{B_\varepsilon} q(x,t)\,\mathrm{d}x = 0$, there exists a function $\mathbf{w} \in \mathring{W}^{1,2}(B_\varepsilon)$ such that $\operatorname{div}\mathbf{w} = q$ in $B_\varepsilon$ and there holds the estimate

$$
\|\nabla\mathbf{w}(\cdot,t)\|_{L^2(B_\varepsilon)} \leqslant \frac{c}{\varepsilon}\|q(\cdot,t)\|_{L^2(B_\varepsilon)}
$$

with the constant $c$ independent of $\varepsilon$ (see [69]). Taking in (4.58) $\boldsymbol{\eta} = \mathbf{w}$, we get

$$
\begin{aligned}
\|q(\cdot,t)\|^2_{L^2(B_\varepsilon)} = M(\mathbf{w}) &\leqslant c\Big(\varepsilon^{1-\beta}\|\mathbf{u}_t(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon\|\mathbf{f}_1^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)}\Big)\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)} \\
&\leqslant \frac{c}{\varepsilon}\Big(\varepsilon^{1-\beta}\|\mathbf{u}_t(\cdot,t)\|_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)} \\
&+c\varepsilon^{\frac{1}{2}}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon\|\mathbf{f}_1^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)}\Big)\|q(\cdot,t)\|_{L^2(B_\varepsilon)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int\limits_0^{2\pi}\|q(\cdot,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t \leqslant \frac{c}{\varepsilon^2}\Bigg(\int\limits_0^{2\pi}\Big(\varepsilon^{2-2\beta}\|\mathbf{u}_t(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\Big)\,\mathrm{d}t \\
+\varepsilon\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\int\limits_0^{2\pi}\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t
\end{aligned}
$$

$$+\varepsilon \sup_{t\in[0,2\pi]} \|\nabla \mathbf{u}^{(J)}(\cdot,t)\|^2_{L^2(B_\varepsilon)} \int_0^{2\pi} \|\nabla \mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t$$

$$+\varepsilon^2 \int_0^{2\pi} \|\mathbf{f}_1^{(J)}(\cdot,t)\|^2_{L^2(B_\varepsilon)}\,\mathrm{d}t\Bigg) \overset{(4.53),(4.54),(4.4)}{\leqslant} c\varepsilon^{2J-4-\beta}.$$

□

The results and the proof for the two-dimensional case are absolutely the same. There holds

**Theorem 4.5.2.** *Let $n=2$. The following estimates*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-2+\beta}, \tag{4.59}$$

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla^2 \mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-4+\beta} \tag{4.60}$$

*hold. Moreover, there exists the pressure function $q \in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))$ satisfying the identity* (4.55)*. If $J \geqslant 2$, then*

$$\int_0^{2\pi}\int_{B_\varepsilon} |q|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-4-\beta}. \tag{4.61}$$

Let $n=3$ or $n=2$. In the case when the boundary value $\mathbf{g}$ is more regular, the obtained estimates can be improved. Assume that $\mathbf{g}\in C^{[\frac{J+3}{2}]+1}([0,2\pi]; W^{3/2,2}(\sigma))$. Then we can construct the asymptotic approximation $\mathbf{u}^{(J+2)}$ and the estimate (4.53) takes the following form

$$\sup_{t\in[0,2\pi]} \|\mathbf{v}(\cdot,t) - \mathbf{u}^{(J+2)}(\cdot,t)\|_{L^2(B_\varepsilon)} + \varepsilon^{\beta/2} \|\nabla \mathbf{v} - \nabla \mathbf{u}^{(J+2)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}$$
$$\leqslant c\varepsilon^{J+\beta/2+1} \leqslant c\varepsilon^{J+\beta/2}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}.$$

Comparing $\mathbf{u}^{(J)}$ and $\mathbf{u}^{(J+2)}$ we notice that

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}^{(J)}(\cdot,t) - \mathbf{u}^{(J+2)}(\cdot,t)\|_{L^2(B_\varepsilon)} + \varepsilon^{\beta/2} \|\nabla \mathbf{u}^{(J)} - \nabla \mathbf{u}^{(J+2)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}$$
$$\leqslant c\varepsilon^{J+\beta/2}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}.$$

By the triangle inequality, we get

$$\sup_{t\in[0,2\pi]} \|\mathbf{v}(\cdot,t) - \mathbf{u}^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)} + \varepsilon^{\beta/2} \|\nabla \mathbf{v} - \nabla \mathbf{u}^{(J)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}$$
$$\leqslant c\varepsilon^{J+\beta/2}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}.$$

Analogously can be obtained the improvement of estimates (4.56), (4.61).

**Theorem 4.5.3.** *If* $\mathbf{g} \in C^{[\frac{J+3}{2}]+1}([0,2\pi]; W^{3/2,2}(\sigma))$, *then*

$$\sup_{t\in[0,2\pi]} \|\mathbf{v}(\cdot,t)-\mathbf{u}^{(J)}(\cdot,t)\|_{L^2(B_\varepsilon)} + \varepsilon^{\beta/2}\|\nabla\mathbf{v}-\nabla\mathbf{u}^{(J)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}$$
$$\leqslant c\varepsilon^{J+\beta/2}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)},$$

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{v}(\cdot,t)-\nabla\mathbf{u}^{(J)}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{v}_t-\mathbf{u}_t^{(J)}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla^2\mathbf{v}-\nabla^2\mathbf{u}^{(J)}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-2+\beta}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}$$

*and*

$$\|p\|_{L^2(0,2\pi;L^2(B_\varepsilon))} \leqslant c\varepsilon^{J-2-\beta/2}\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}.$$

**Remark 4.5.4.** *The asymptotic expansion* (4.41)-(4.42) *can be slightly modified without loss of the accuracy. Namely, the argument* $\frac{|x-O_l|}{|e|_{\min}}$ *in the cut-off function* $\zeta$ *may be replaced by* $\frac{|x-O_l|}{\delta}$, *where* $\delta = C_J\varepsilon|\ln\varepsilon||e|_{\min}$ *and the constant* $C_J$ *will be chosen below.*

Denote $J' = J+2$. Consider the boundary layer functions $\mathbf{V}^{[BLO_l,J']}$ and $P^{[BLO_l,J']}$. It follows that these functions $F^{[BLO_l,J']}$ ($F$ stands for $\mathbf{V}$ or $P$) and their derivatives decay exponentially as the space variable tends to infinity in the outlets. Thus, there exist positive constants $c_1$, $c_2$ such that for all $t \in [0,2\pi]$ and for sufficiently large $R$ holds the inequality

$$\|F^{[BLO_l,J']}(\cdot,t)\|_{W^{2,2}(\Omega_l^R)} + \Big\|\frac{\partial F^{[BLO_l,J']}(\cdot,t)}{\partial t}\Big\|_{W^{2,2}(\Omega_l^R)} \leqslant c_1\exp(-c_2R),$$

where $\Omega_l^R = \Omega_l \cup \{|y| > R\}$.

Therefore, if $B_\varepsilon^l = \{x \in B_\varepsilon : |x-O_l| \geqslant C_J\varepsilon|\ln\varepsilon||e|_{\min}/3\}$, then making change of the variable $y = \frac{x-O_l}{\varepsilon}$ in the above inequality and taking $R = C_J|\ln\varepsilon||e|_{\min}/3$, we get

$$\|F^{[BLO_l,J']}(\cdot,\cdot)\|_{L^2(0,2\pi;W^{2,2}(B_\varepsilon^l))} + \Big\|\frac{\partial F^{[BLO_l,J']}(\cdot,\cdot)}{\partial t}\Big\|_{L^2(0,2\pi;W^{2,2}(B_\varepsilon^l))}$$
$$\leqslant c_1\exp\{-c_2C_J|\ln\varepsilon||e|_{\min}/3) = c_1\varepsilon^{c_2C_J|e|_{\min}/3}.$$

Choose $C_J$ such that $c_2 C_J |e|_{\min}/3 \geqslant J' = J+2$. Then for $F^{[BLO_l,J']}$ and its derivatives this upper bound is equal to $c_1\varepsilon^{J+2}$. So, for the difference

$$\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)F^{[BLO_l,J']}\Big(\frac{x-O_l}{\varepsilon},t\Big)-\zeta\Big(\frac{|x-O_l|}{\delta}\Big)F^{[BLO_l,J']}\Big(\frac{x-O_l}{\varepsilon},t\Big)$$

the following estimate

$$\Big\|\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)F^{[BLO_l,J']}(\cdot,\cdot)-\zeta\Big(\frac{|x-O_l|}{\delta}\Big)F^{[BLO_l,J']}(\cdot,\cdot)\Big\|_{L^2(0,2\pi;W^{2,2}(D))} \leqslant c_1\varepsilon^{J+2}$$

holds, where $D=\operatorname{supp}\Big\{\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)-\zeta\Big(\frac{|x-O_l|}{\delta}\Big)\Big\}$. Notice that in $D$ we have $\frac{|x-O_l|}{\delta}\geqslant\frac{1}{3}$ for sufficiently small $\varepsilon$. Because of this estimate for the approximation $\mathbf{u}^{(J+2)}$ the residual $\|\mathbf{f}_1^{(J+2)}\|_{L^2(0,2\pi;L^2(B_\varepsilon))}$ has the order $O(\varepsilon^J)$. So, the difference $\mathbf{u}^{(J+2)}-\mathbf{v}$ is of order $O(\varepsilon^J\sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)})$ in the norm of Theorem 4.5.3. In this case we have assumed that $\mathbf{g}\in C^{[\frac{J+3}{2}]+1}(0,2\pi;W^{3/2,2}(\sigma))$.

## 4.6 Method of asymptotic partial decomposition of the domain

The obtained asymptotic expansion of the solution to the time-periodic non-steady Navier–Stokes problem can be applied to justify the method of asymptotic partial decomposition of the domain (MAPDD) proposed for the steady case in [65, 66].

Let us describe the algorithm of the MAPDD for the non-steady Navier–Stokes problem set in a tube structure $B_\varepsilon$. Let $\delta$ be a small positive number much greater than $\varepsilon$. For any edge $e=\overline{O_iO_j}$ of the graph introduce two hyperplanes orthogonal to this edge and crossing it at the distance $\delta$ from its ends.

Denote the cross-section of the cylinders $\Pi_\varepsilon^{(e)}$ by these two hyperplanes respectively, by $S_{i,j}$ (the cross-section at the distance $\delta$ from $O_i$), and $S_{j,i}$ (the cross-section at the distance $\delta$ from $O_j$), and denote the part of the cylinder $\Pi_\varepsilon^{(e)}$ between these two cross-sections by $B_{i,j}^{\mathrm{dec},\varepsilon}$ (see Figure 4.1). Let $B_i^{\varepsilon,\delta}$ be the connected, truncated by the cross-sections $S_{i,j}$ part of $B_\varepsilon$, which contains the vertex or the node $O_i$ (see Figure 4.2).

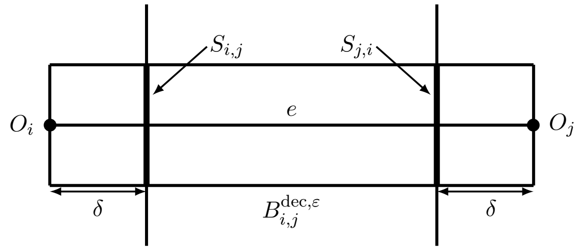


Figure 4.1 – Truncation of the cylinder $\Pi_\varepsilon^{(e)}$.

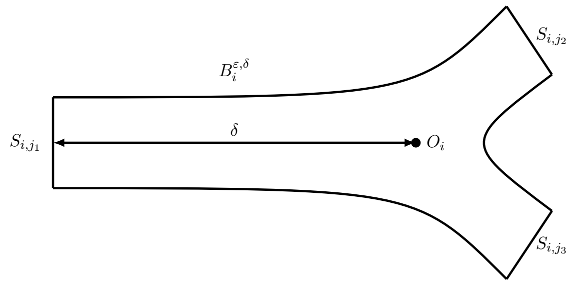


Figure 4.2 – Connected component $B_i^{\varepsilon,\delta}$.

Introduce the space $H^1_{\mathrm{div}=0}(B_\varepsilon)$ of all divergence free vector valued functions from the space $W^{1,2}(B_\varepsilon)$ vanishing for $x \in \partial B_\varepsilon \setminus (\bigcup_{j=N_1+1}^{N} \gamma_\varepsilon^j)$.

Define the subspace $H^1_{\mathrm{div}=0}(B_\varepsilon, \delta)$ of $H^1_{\mathrm{div}=0}(B_\varepsilon)$ such that every truncated cylinder $B_{i,j}^{\mathrm{dec},\varepsilon}$ its elements (vector-valued functions) coincide with the Womersley type flow. Here Womersley type flow is a vector-valued function $\mathbf{u}_W$ such that in local coordinates $x^{(e)}$ associated to the edge $e$, its "last" (longitudinal) component $u_{n,W}(x^{(e)}/\varepsilon)$ is independent of $x_n^{(e)}$, i.e., $\mathbf{u}_{n,W} = \mathbf{u}_{n,W}(x^{(e)\prime}/\varepsilon)$ while all transversal components of the velocity are equal to zero. We will consider as well the subspace $H^1_{0,\mathrm{div}=0}(B_\varepsilon, \delta)$ of the space $H^1_{\mathrm{div}=0}(B_\varepsilon, \delta)$ such that its elements vanish on the whole boundary $\partial B_\varepsilon$ and the subspace $L^2(B_\varepsilon, \delta)$ of the space $L^2(B_\varepsilon)$ such that its elements (vector-valued functions) coincide the Womersley type flows on every truncated cylinder $B_{i,j}^{\mathrm{dec},\varepsilon}$.

The method of asymptotic partial decomposition (MAPDD) replaces the problem (4.1) by its projection on $H^1_{\mathrm{div}=0}(B_\varepsilon, \delta)$: find $\hat{\mathbf{u}}_{\varepsilon,\delta}$ from $L^2_{\mathrm{per}}(0, 2\pi; H^1_{\mathrm{div}=0}(B_\varepsilon, \delta))$, such that $\hat{\mathbf{u}}_{\varepsilon,\delta} - \hat{\mathbf{g}} \in L^2_{\mathrm{per}}(0, 2\pi; H^1_{0,\mathrm{div}=0}(B_\varepsilon, \delta))$, $(\hat{\mathbf{u}}_{\varepsilon,\delta} - \hat{\mathbf{g}})_t \in L^2_{\mathrm{per}}(0, 2\pi; H^1_{0,\mathrm{div}=0}(B_\varepsilon, \delta))$, and for any test function $\boldsymbol{\eta} \in H^1_{0,\mathrm{div}=0}(B_\varepsilon, \delta)$ holds the integral identity

$$\int\limits_{B_\varepsilon} \Big( \frac{1}{\varepsilon^\beta} (\hat{\mathbf{u}}_{\varepsilon,\delta})_t \cdot \boldsymbol{\eta} + \nu \nabla \hat{\mathbf{u}}_{\varepsilon,\delta} : \nabla \boldsymbol{\eta} - ((\hat{\mathbf{u}}_{\varepsilon,\delta} + \hat{\mathbf{g}}) \cdot \nabla) \boldsymbol{\eta} \cdot \hat{\mathbf{u}}_{\varepsilon,\delta} \\ - (\hat{\mathbf{u}}_{\varepsilon,\delta} \cdot \nabla) \boldsymbol{\eta} \cdot \hat{\mathbf{g}} \Big) \, \mathrm{d}x = 0. \tag{4.62}$$

Here $\hat{\mathbf{g}}$ is an extension of the boundary function $\mathbf{g}$ constructed above as $\mathbf{u}^{(J+2)}$ with the modification described in Remark 4.5.4, i.e., $\delta = C_J \varepsilon |\ln \varepsilon| |e|_{\min}$.

**Theorem 4.6.1.** *Let* $\mathbf{g} \in C^{[\frac{J+3}{2}]+1}(0, 2\pi; W^{3/2,2}(\sigma))$. *Then there exists a unique solution* $\hat{\mathbf{u}}_{\varepsilon,\delta}$ *of the partially decomposed problem* (4.62) *and*

$$\sup_{t \in [0,2\pi]} \|\mathbf{v}(\cdot,t) - \hat{\mathbf{u}}_{\varepsilon,\delta}(\cdot,t)\|_{L^2(B_\varepsilon)} + \varepsilon^{\beta/2} \|\nabla(\mathbf{v} - \hat{\mathbf{u}}_{\varepsilon,\delta})\|_{L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))} \\ \leqslant c \varepsilon^{J+\beta/2} \sqrt{\mathrm{mes}(\mathrm{B}_\varepsilon)}.$$

The proof of this theorem may be done by using estimates (4.12) and (4.14), which remains valid for this problem. The Galerkin approximations are constructed in the space $H^1_{0,\mathrm{div}=0}(B_\varepsilon, \delta)$ instead of $H^1_{\mathrm{div}=0}(B_\varepsilon)$. For more details see [49].

# Chapter 5

# Steady state Navier–Stokes equations with the Bernoulli pressure

## 5.1 Formulation of the problem

Let us consider the following boundary value problem for the steady-state Navier–Stokes equations in a tube structure $B_\varepsilon$ (see Definition 3)

$$\begin{cases} -\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p=\mathbf{f}, & x\in B_\varepsilon,\\ \operatorname{div}\mathbf{v}=0, & x\in B_\varepsilon,\\ \mathbf{v}=0, & x\in \partial B_\varepsilon\setminus\cup_{j=N_1+1}^{N}\gamma_\varepsilon^j,\\ \mathbf{v}_{\boldsymbol{\tau}}=0, & x\in\gamma_\varepsilon^j,\\ -\nu\partial_n(\mathbf{v}\cdot\mathbf{n})+\Big(p+\dfrac{1}{2}|\mathbf{v}|^2\Big)=c_j/\varepsilon^2, & x\in\gamma_\varepsilon^j,\ j=N_1+1,\dots,N, \end{cases} \tag{5.1}$$

where $\nu$ is a positive constant, $\mathbf{n}$ is the unit normal vector to $\gamma_\varepsilon^j$, $\mathbf{v}_{\boldsymbol{\tau}}=\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of the vector $\mathbf{v}$, $\partial_n h=\nabla h\cdot\mathbf{n}$ is the normal derivative of $h$, $c_j$ are some constants. From the boundary condition $\mathbf{v}_{\boldsymbol{\tau}}|_{\gamma_\varepsilon^j}=0$ and the divergence equation $\operatorname{div}\mathbf{v}=0$, it follows that $-\nu\partial_n(\mathbf{v}\cdot\mathbf{n})|_{\gamma_\varepsilon^j}=0$. Thus, using the identity $\frac{1}{2}(\nabla\mathbf{v}^2)=\mathbf{v}\cdot(\nabla\mathbf{v})^t=\sum_{i=k}^{n}v_k\cdot\nabla v_k$, (here $(\nabla\mathbf{v})^t$ defines the transposed matrix) we can rewrite (5.1) with the right-hand side in the following form

$$
\begin{cases}
-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{v}\cdot(\nabla\mathbf{v})^t+\nabla\Phi=\mathbf{f}, & x\in B_\varepsilon,\\
\operatorname{div}\mathbf{v}=0, & x\in B_\varepsilon,\\
\mathbf{v}=0, & x\in\partial B_\varepsilon\backslash\cup_{j=N_1+1}^{N}\gamma_\varepsilon^j,\\
\mathbf{v}_{\boldsymbol{\tau}}=0, & x\in\gamma_\varepsilon^j,\\
\Phi=p_j, & x\in\gamma_\varepsilon^j,\ j=N_1+1,\dots,N,
\end{cases}
\tag{5.2}
$$

where $\Phi=(p+\frac{1}{2}|\mathbf{v}|^2)$ is the Bernoulli pressure, $p_j$ stand for the constants $c_j/\varepsilon^2$.

Let us define a weak solution of problem (5.2) as a vector field $\mathbf{v}\in\widehat{J}_\gamma^{1,2}(B_\varepsilon)=\{\boldsymbol{\eta}\in\widehat{W}_\gamma^{1,2}(B_\varepsilon):\operatorname{div}\boldsymbol{\eta}=0\}$, satisfying the integral identity

$$
\begin{aligned}
\nu\int\limits_{B_\varepsilon}\nabla\mathbf{v}:\nabla\boldsymbol{\eta}\,\mathrm{d}x+\int\limits_{B_\varepsilon}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x-\int\limits_{B_\varepsilon}(\boldsymbol{\eta}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x\\
=-\sum_{j=N_1+1}^{N}p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x
\end{aligned}
\tag{5.3}
$$

for every $\boldsymbol{\eta}\in\widehat{J}_\gamma^{1,2}(B_\varepsilon)$.

Introduce $p_j^*=p_j-p_N,\ j=N_1+1,\dots,N$. Consider an equivalent weak formulation: find a vector field $\mathbf{v}\in\widehat{J}_\gamma^{1,2}(B_\varepsilon)$ satisfying the integral identity

$$
\begin{aligned}
\nu\int\limits_{B_\varepsilon}\nabla\mathbf{v}:\nabla\boldsymbol{\eta}\,\mathrm{d}x+\int\limits_{B_\varepsilon}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x-\int\limits_{B_\varepsilon}(\boldsymbol{\eta}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x\\
=-\sum_{j=N_1+1}^{N-1}p_j^*\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x
\end{aligned}
\tag{5.4}
$$

for every $\boldsymbol{\eta}\in\widehat{J}_\gamma^{1,2}(B_\varepsilon)$. The equivalence of these formulations follows from the equality

$$
\sum_{j=N_1+1}^{N-1}p_j^*\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'=\sum_{j=N_1+1}^{N}p_j\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x',
$$

which follows form the relation

$$
\sum_{j=N_1+1}^{N}\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'=0
$$

for the solenoidal vector-valued function $\boldsymbol{\eta}$.

Let us explain this weak formulation heuristically; the rigorous analysis of the equivalence of the weak formulation and the classical one needs to study the regularity of the weak solution, see [13] for the methods.

Identity (5.3) follows from equations (5.2) after multiplying them by $\boldsymbol{\eta} \in \widehat{J}^{1,2}_{\gamma}(B_\varepsilon)$ and integrating by parts in $B_\varepsilon$. On the other hand, for a sufficiently regular solution $\mathbf{v}$ satisfying (5.3) there exists a pressure field $p$ such that the pair $(\mathbf{v}, p)$ satisfies equations $(5.2)_{1,2}$ a.e. in $B_\varepsilon$. Boundary conditions $(5.2)_{3,4,5}$ are satisfied in the sense of traces (see the definition of the space $\widehat{J}^{1,2}_{\gamma}(B_\varepsilon)$). More exactly, function $\Phi$ is defined up to an additive constant but this constant can be chosen so that $\Phi$ satisfies $(5.2)_5$. Indeed, take in (5.3) a smooth solenoidal function $\boldsymbol{\eta}$ satisfying the boundary conditions $\boldsymbol{\eta}|_\Gamma = 0,\ \boldsymbol{\eta}_{\boldsymbol{\tau}}|_{\gamma^j_\varepsilon} = 0,\ j = N_1+1, \ldots, N$. Integrating by parts (5.3) for smooth solutions yields

$$\begin{aligned}
&\int\limits_{B_\varepsilon} (-\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{v}\cdot(\nabla\mathbf{v})^t - \mathbf{f})\cdot\boldsymbol{\eta}\ \mathrm{d}x \\
&= -\nu \sum_{j=N_1+1}^{N} \int\limits_{\gamma^j_\varepsilon} \partial_n(\mathbf{v}\cdot\boldsymbol{\eta})\,\mathrm{d}S - \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma^j_\varepsilon} \boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x' \\
&= -\sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma^j_\varepsilon} \boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'.
\end{aligned} \tag{5.5}$$

If $\boldsymbol{\eta} \in J^\infty_0(B_\varepsilon) = \{\boldsymbol{\eta} \in C^\infty_0(B_\varepsilon) : \operatorname{div}\boldsymbol{\eta} = 0\}$, then it follows from (5.5) that

$$\int\limits_{B_\varepsilon} (L(\mathbf{v}) - \mathbf{f})\cdot\boldsymbol{\eta}\,\mathrm{d}x = 0 \quad \forall\boldsymbol{\eta} \in J^\infty_0(B_\varepsilon),$$

where

$$L(\mathbf{v}) = -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{v}\cdot(\nabla\mathbf{v})^t.$$

Hence, there exists a pressure function $\Phi$ such that (e.g. [49])

$$L(\mathbf{v}) + \nabla\Phi = \mathbf{f} \ \text{ a.e. in } B_\varepsilon.$$

Then

$$\int\limits_{B_\varepsilon} (L(\mathbf{v}) - \mathbf{f})\cdot\boldsymbol{\eta}\,\mathrm{d}x = -\int\limits_{B_\varepsilon} \nabla\Phi\cdot\boldsymbol{\eta}\,\mathrm{d}x = -\sum_{j=N_1+1}^{N} \int\limits_{\gamma^j_\varepsilon} \Phi\cdot\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'$$

for every solenoidal $\boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon)$. Therefore,

$$\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} \Phi \cdot \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = \sum_{j=N_1+1}^{N} p_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x'.$$

Thus,

$$\sum_{j=N_1+1}^{N} \int\limits_{\gamma_\varepsilon^j} (\Phi - p_j) \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = 0 \quad \forall \boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon). \tag{5.6}$$

Let us fix arbitrary $j \in \{N_1+1, \ldots, N\}$. Taking $\boldsymbol{\eta} \in J^{1,2}_\gamma(B_\varepsilon) = \{\boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon) : \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}S = 0, \ j = N_1+1, \ldots, N\}$ such that $\boldsymbol{\eta}|_{\gamma_\varepsilon^k} = 0$ for $k \neq j$, we get

$$(\Phi - p_j)|_{\gamma_\varepsilon^j} = \widetilde{c}_j,$$

where $\widetilde{c}_j$ is a constant (see [28], [45]). Using these relations and taking now in (5.6) a test function $\boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon)$ such that $\int\limits_{\gamma_\varepsilon^k} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = 0$ for $k \neq j$ and $k \neq N$, $\int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = 1$ and $\int\limits_{\gamma_\varepsilon^N} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = -1$, we get

$$\sum_{j=N_1+1}^{N} \widetilde{c}_j \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' = \widetilde{c}_j - c_N \Rightarrow \widetilde{c}_j = c_N.$$

Thus,

$$\widetilde{c}_j = c_N \quad \forall j = N_1+1, \ldots, N. \tag{5.7}$$

Since the Bernoulli pressure $\Phi$ in the weak formulation is defined up to an additive constant, we may set $\widetilde{c}_j = c_N = 0, \ j = N_1+1, \ldots, N$. Then from (5.7) we have

$$\Phi|_{\gamma_\varepsilon^j} = p_j, \quad j = N_1+1, \ldots, N.$$

## 5.2 Existence, uniqueness and stability of a solution

In this section we prove the existence and uniqueness of the solution to problem (5.1) with the right-hand side $\mathbf{f} \in L^2(B_\varepsilon)$.

**Theorem 5.2.1.** *For arbitrary* $\mathbf{f} \in L^2(B_\varepsilon)$ *and* $p_j^* \in \mathbb{R}$, $j = N_1+1, \dots, N-1$ *problem* (5.2) *admits at least one weak solution* $\mathbf{v} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$. *There holds the estimate*

$$\|\nabla \mathbf{v}\|_{L^2(B_\varepsilon)} \leqslant c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big) \tag{5.8}$$

*with the constant* $c$ *independent of* $\varepsilon$.

*Proof.* Define in $\widehat{J}_\gamma^{1,2}(B_\varepsilon)$ the inner product $[\mathbf{v}, \boldsymbol{\eta}] = \int\limits_{B_\varepsilon} \nabla \mathbf{v} : \nabla \boldsymbol{\eta}\, \mathrm{d}x$ corresponding to the Dirichlet norm. Using Hölder inequality and Lemmas 2.1.1, 2.1.6 we derive the estimates

$$\begin{aligned}
&\Big|\int\limits_{B_\varepsilon} (\boldsymbol{\eta}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x\Big| + \Big|\int\limits_{B_\varepsilon} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x\Big| \\
&\leqslant \Big(\int\limits_{B_\varepsilon} |\mathbf{v}|^4\,\mathrm{d}x\Big)^{1/4} \Big(\int\limits_{B_\varepsilon} |\nabla \mathbf{u}|^2\,\mathrm{d}x\Big)^{1/2} \Big(\int\limits_{B_\varepsilon} |\boldsymbol{\eta}|^4\,\mathrm{d}x\Big)^{1/4} \\
&\leqslant c\varepsilon^\alpha \|\nabla \mathbf{v}\|^2_{L^2(B_\varepsilon)} \|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)},
\end{aligned}$$

where $\alpha = 1$ for $n = 2$ and $\alpha = 1/2$ for $n = 3$. From Lemma 2.1.7 it follows that

$$\begin{aligned}
\Big|\sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'\Big| &\leqslant \sum_{j=N_1+1}^{N-1} |p_j^*| \Big(\int\limits_{\gamma_\varepsilon^j} |\boldsymbol{\eta}|^2\,\mathrm{d}x\Big)^{1/2} |\gamma_\varepsilon^j|^{1/2} \\
&\leqslant c\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| \|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)}.
\end{aligned} \tag{5.9}$$

Finally,

$$\Big|\int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x\Big| \leqslant \Big(\int\limits_{B_\varepsilon} |\mathbf{f}|^2\,\mathrm{d}x\Big)^{1/2} \Big(\int\limits_{B_\varepsilon} |\boldsymbol{\eta}|^2\,\mathrm{d}x\Big)^{1/2} \leqslant c\varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)} \|\nabla \boldsymbol{\eta}\|_{L^2(B_\varepsilon)}. \tag{5.10}$$

From above estimates and the Riesz theorem it follows that the integral identity (5.4) is equivalent to the operator equation in the space $\widehat{J}_\gamma^{1,2}(B_\varepsilon)$:

$$\mathbf{v} = \mathcal{A}\mathbf{v}, \tag{5.11}$$

where the operator $\mathcal{A}$ is defined by

$$\begin{aligned}
[\mathcal{A}\mathbf{v}, \boldsymbol{\eta}] = &\int\limits_{B_\varepsilon} \nu^{-1} \Big[-(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta} + (\boldsymbol{\eta}\cdot\nabla)\mathbf{v}\cdot\mathbf{v} + \mathbf{f}\cdot\boldsymbol{\eta}\Big]\,\mathrm{d}x \\
&-\nu^{-1} \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x' \quad \forall \boldsymbol{\eta} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon).
\end{aligned}$$

Using compactness of the embedding $W^{1,2}(B_\varepsilon) \hookrightarrow L^4(B_\varepsilon)$ it is standard to show that the operator $\mathcal{A}: \widehat{J}^{1,2}_\gamma(B_\varepsilon) \mapsto \widehat{J}^{1,2}_\gamma(B_\varepsilon)$ is compact (see [49]). Thus, the existence of at least one solution of (5.11) will follow from the Leray-Schauder fixed point theorem if we show that all possible solutions $\mathbf{v}^{(\lambda)}$ of the equation

$$\mathbf{v}^{(\lambda)} = \lambda \mathcal{A} \mathbf{v}^{(\lambda)}, \quad \lambda \in [0,1] \tag{5.12}$$

are uniformly (with respect to $\lambda$) bounded.

A solution $\mathbf{v}^{(\lambda)}$ of (5.12) satisfies the integral identity

$$\begin{aligned} &\nu \int\limits_{B_\varepsilon} \nabla \mathbf{v}^{(\lambda)} : \nabla \boldsymbol{\eta}\, \mathrm{d}x + \lambda \int\limits_{B_\varepsilon} (\mathbf{v}^{(\lambda)} \cdot \nabla) \mathbf{v}^{(\lambda)} \cdot \boldsymbol{\eta}\, \mathrm{d}x - \lambda \int\limits_{B_\varepsilon} (\boldsymbol{\eta} \cdot \nabla) \mathbf{v}^{(\lambda)} \cdot \mathbf{v}^{(\lambda)}\, \mathrm{d}x \\ &\qquad = -\lambda \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n}\, \mathrm{d}x' + \lambda \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta}\, \mathrm{d}x \quad \forall \boldsymbol{\eta} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon). \end{aligned} \tag{5.13}$$

Taking in (5.13) $\boldsymbol{\eta} = \mathbf{v}^{(\lambda)}$ we get

$$\nu \int\limits_{B_\varepsilon} |\nabla \mathbf{v}^{(\lambda)}|^2\, \mathrm{d}x = -\lambda \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \mathbf{v}^{(\lambda)} \cdot \mathbf{n}\, \mathrm{d}x' + \lambda \int\limits_{B_\varepsilon} \mathbf{f} \cdot \mathbf{v}^{(\lambda)}\, \mathrm{d}x.$$

Using (5.9), (5.10), we obtain

$$\|\nabla \mathbf{v}^{(\lambda)}\|^2_{L^2(B_\varepsilon)} \leqslant c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big) \|\nabla \mathbf{v}^{(\lambda)}\|_{L^2(B_\varepsilon)}.$$

Hence

$$\|\nabla \mathbf{v}^{(\lambda)}\|_{L^2(B_\varepsilon)} \leqslant c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big).$$

The constant $c$ in the last inequality is independent of $\lambda$ and $\varepsilon$. This finishes the proof of the theorem. □

**Theorem 5.2.2.** *1. There exists a positive constant $c_0$ independent of $\varepsilon$, such that if*

$$c_0 \varepsilon^\alpha \Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big) < \nu, \tag{5.14}$$

*where $\alpha = 1$ for $n = 2$ and $\alpha = 1/2$ for $n = 3$, then the weak solution $\mathbf{v} \in \widehat{J}^{1,2}_\gamma(B_\varepsilon)$ of (5.2) is unique.*

*2. Let $\{p^*_{1j}\}$ and $\{p^*_{2j}\}$, $j = N_1+1, \ldots, N$ be two sets of real constants, and $\mathbf{f}_1$, $\mathbf{f}_2$ be functions, $\mathbf{f}_i \in L^2(B_\varepsilon)$, $i = 1,2$, satisfying (5.14), and let $\mathbf{v}_i \in \widehat{J}^{1,2}_\gamma(B_\varepsilon)$*

*be weak solutions of problem* (5.2) *corresponding to* $\{p^*_{ij}\}$ *and* $\mathbf{f}_i$, $i = 1,2$. *Then there exists a constant* $C$ *independent of* $\varepsilon$ *such that*

$$\|\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2\|_{L^2(B_\varepsilon)} \leqslant \varepsilon C \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^2(B_\varepsilon)} + c\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p^*_{1j} - p^*_{2j}|. \quad (5.15)$$

*Proof.* 1. Suppose that there exist two weak solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfying (5.4). Subtracting identity (5.4) for $\mathbf{v}_2$ from the one for $\mathbf{v}_1$, we obtain

$$\begin{aligned} \nu \int\limits_{B_\varepsilon} \nabla \mathbf{w} : \nabla \boldsymbol{\eta}\, \mathrm{d}x + \int\limits_{B_\varepsilon} \left[ (\mathbf{w}\cdot\nabla)\mathbf{v}_1 \cdot \boldsymbol{\eta} + (\mathbf{v}_2 \cdot \nabla)\mathbf{w}\cdot\boldsymbol{\eta} \right] \mathrm{d}x \\ - \int\limits_{B_\varepsilon} \left[ (\boldsymbol{\eta}\cdot\nabla)\mathbf{v}_1 \cdot \mathbf{w} + (\boldsymbol{\eta}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}_2 \right] \mathrm{d}x = 0, \end{aligned} \quad (5.16)$$

where $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. Taking in (5.16) $\boldsymbol{\eta} = \mathbf{w}$, we get in virtue of Lemma 2.1.6 and estimate (5.8) for $\mathbf{v}_2$,

$$\begin{aligned} \nu \int\limits_{B_\varepsilon} |\nabla \mathbf{w}|^2\, \mathrm{d}x &= \int\limits_{B_\varepsilon} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}_2\, \mathrm{d}x - \int\limits_{B_\varepsilon} (\mathbf{v}_2\cdot\nabla)\mathbf{w}\cdot\mathbf{w}\, \mathrm{d}x \\ &\leqslant 2\|\nabla \mathbf{v}_2\|_{L^4(B_\varepsilon)} \|\nabla \mathbf{w}\|_{L^2(B_\varepsilon)} \|\mathbf{w}\|_{L^4(B_\varepsilon)} \\ &\leqslant c_0 \varepsilon^\alpha \Big( \varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p^*_j| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)} \Big) \|\nabla \mathbf{w}\|^2_{L^2(B_\varepsilon)}, \end{aligned}$$

where the constant $c_0$ is independent of $\varepsilon$. If condition (5.14) is valid, the last inequality yields

$$\int\limits_{B_\varepsilon} |\nabla \mathbf{w}|^2\, \mathrm{d}x = 0, \quad (5.17)$$

and, thus, $\mathbf{w} = 0$.

2. Subtracting identity (5.4) for $\mathbf{v}_2$ from the one for $\mathbf{v}_1$, we obtain

$$\begin{aligned} \nu \int\limits_{B_\varepsilon} \nabla \mathbf{w} : \nabla \boldsymbol{\eta}\, \mathrm{d}x + \int\limits_{B_\varepsilon} \left[ (\mathbf{w}\cdot\nabla)\mathbf{v}_1 \cdot \boldsymbol{\eta} + (\mathbf{v}_2 \cdot \nabla)\mathbf{w}\cdot\boldsymbol{\eta} \right] \mathrm{d}x \\ - \int\limits_{B_\varepsilon} \left[ (\boldsymbol{\eta}\cdot\nabla)\mathbf{v}_1 \cdot \mathbf{w} + (\boldsymbol{\eta}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}_2 \right] \mathrm{d}x \\ = \int\limits_{B_\varepsilon} (\mathbf{f}_2 - \mathbf{f}_1)\cdot\boldsymbol{\eta}\ \mathrm{d}x - \sum_{j=N_1+1}^{N-1} (p^*_{1j} - p^*_{2j}) \int\limits_{\gamma^j_\varepsilon} \boldsymbol{\eta}\cdot\mathbf{n}\, \mathrm{d}x', \end{aligned} \quad (5.18)$$

where $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. Taking in (5.18) $\boldsymbol{\eta} = \mathbf{w}$, we get in virtue of Lemma 2.1.6

and estimate (5.8) for $\mathbf{v}_2$,

$$\nu \int\limits_{B_\varepsilon} |\nabla \mathbf{w}|^2 \, \mathrm{d}x = \int\limits_{B_\varepsilon} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}_2 \, \mathrm{d}x - \int\limits_{B_\varepsilon} (\mathbf{v}_2\cdot\nabla)\mathbf{w}\cdot\mathbf{w} \, \mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{f}_2-\mathbf{f}_1)\cdot\mathbf{w} \, \mathrm{d}x$$

$$- \sum_{j=N_1+1}^{N-1} (p^*_{1j}-p^*_{2j}) \int\limits_{\gamma^j_\varepsilon} \mathbf{w}\cdot\mathbf{n} \, \mathrm{d}x'$$

$$\leqslant c\varepsilon^{\alpha/2}\|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}\|\mathbf{v}_2\|_{L^4(B_\varepsilon)} + \|\mathbf{f}_2-\mathbf{f}_1\|_{L^2(B_\varepsilon)}\|\mathbf{w}\|_{L^2(B_\varepsilon)}$$

$$+ c\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p^*_{1j}-p^*_{2j}| \|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}$$

$$\leqslant c_0\varepsilon^{\alpha}\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p^*_j| + \varepsilon\|\mathbf{f}_2\|_{L^2(B_\varepsilon)}\Big)\|\nabla\mathbf{w}\|^2_{L^2(B_\varepsilon)}$$

$$+ \varepsilon C\|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)}\|\mathbf{f}_2-\mathbf{f}_1\|_{L^2(B_\varepsilon)}$$

$$+ c\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p^*_{1j}-p^*_{2j}| \|\nabla\mathbf{w}\|_{L^2(B_\varepsilon)},$$

where $\varepsilon C$ is the Poincaré-Friedrich's constant for the domain $B_\varepsilon$ (constant $C$ is independent of $\varepsilon$). If condition (5.14) is valid, the last inequality yields (5.15). □

**Remark 5.2.3.** *Notice also that the weak solution* $\mathbf{v}$ *of problem* (5.2) *belongs to the space* $W^{2,2}(B_\varepsilon)$ *whenever* $\mathbf{f} \in L^2(B_\varepsilon)$*. The corresponding pressure belongs to* $W^{1,2}(B_\varepsilon)$*. This can be proved extending the solutions and the data by reflection over the sections* $\gamma^j_\varepsilon$ *to a larger domain (see [13]).*

## 5.3 Asymptotic expansion of the solution

In this section we describe the construction of the asymptotic expansion. Let $\zeta \in C^2(\mathbb{R})$ be even function independent of $\varepsilon$ such that, $\zeta(t) = 0$ if $|t| \leqslant 1/3$, and $\zeta(t) = 1$ if $|t| \geqslant 2/3$. Denote $e = e_{O_j}$ (the edge with the end $O_j$) and $x^{(e)}$ the Cartesian coordinates corresponding to the origin $O_j$ and the edge $e$, i.e., $x^{(e)} = \mathcal{P}^{(e)}(x - O_j)$, $\mathcal{P}^{(e)}$ is the orthogonal matrix relating the global coordinates $x$ with the local ones $x^{(e)}$.

The asymptotic expansion of the velocity field is sought in the form:

$$\begin{gathered}\mathbf{v}^{(J)}(x)=\sum_{O_l,\ l=N_1+1,\dots,N;\ e=\overline{O_lO_{i_l}}}\zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)\prime}}{\varepsilon}\Big)\\+\sum_{e=\overline{O_\alpha O_\beta};\ \alpha,\beta\leqslant N_1}\zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e|-x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)\prime}}{\varepsilon}\Big)\\+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\mathbf{V}^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big),\end{gathered}\tag{5.19}$$

where the first sum is taken over all edges having a vertex as an end point (and with the origin of the local coordinate system at the vertex), the second sum is taken over all remaining edges, and all terms in these sums are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$; the terms of the third sum are extended by zero out of the corresponding bundles;

$$\begin{cases}\mathbf{V}^{[e,J]}=\big(P^{(e)}\big)^t(0,\dots,0,\widetilde{V}^{[e,J]})^t,\\ \widetilde{V}^{[e,J]}(y^{(e)\prime})=\displaystyle\sum_{j=0}^{J}\varepsilon^j\widetilde{V}_j^{(e)}(y^{(e)\prime}),\\ \mathbf{V}^{[BLO_l,J]}(y)=\displaystyle\sum_{j=0}^{J}\varepsilon^j\mathbf{V}_j^{[BLO_l]}(y).\end{cases}\tag{5.20}$$

The asymptotic expansion of the pressure for every half-cylinder $\Pi_\varepsilon^{(e)}$, $x_n<|e|/2$, corresponding to the edge $e=\overline{O_lO_{i_l}}$, $l=N_1+1,\dots,N$, ($O_l$ is the origin of the local coordinate system) is sought in the form:

$$p^{(J)}(x)=-s^{(e)}x_n^{(e)}+a^{(e)}+\frac{1}{\varepsilon}\left(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\right)P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big),\tag{5.21}$$

and on every half-bundle $HB_{O_l}$, $l=1,\dots,N_1$, ( $O_l$ is the origin of the local coordinate system) we define:

$$\begin{gathered}p^{(J)}(x)=\sum_{e\subset\mathcal{B}_l}\zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\big(-s^{(e)}x_n^{(e)}+a^{(e)}-a^{(e_s)}\big)+a^{(e_s)}\\+\frac{1}{\varepsilon}\left(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\right)P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big),\end{gathered}\tag{5.22}$$

where the terms of the sum are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$,

$$s^{(e)}=\frac{1}{\varepsilon^2}\sum_{j=0}^{J}\varepsilon^j s_j^{(e)},\quad a^{(e)}=\frac{1}{\varepsilon^2}\sum_{j=0}^{J}\varepsilon^j a_j^{(e)}\tag{5.23}$$

and

$$P^{[BLO_l,J]}(y) = \sum_{j=0}^{J} \varepsilon^j P_j^{[BLO_l,J]}(y). \tag{5.24}$$

Here $e_s$ is the selected edge of the bundle (arbitrary chosen among edges of the bundle) and the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin $O_l$.

The algorithm of successive determination of the terms in asymptotic expansions (5.19), (5.21) is as follows.

**The base case**. Introduce the normalized Poiseuille type velocity $V_0^{(e)}(y^{(e')})$, the solution of the Dirichlet problem

$$\begin{cases} -\nu\Delta_{(y^{(e)\prime})} V_0^{(e)}(y^{(e)\prime}) = 1, & y^{(e)\prime} \in \sigma^{(e)}, \\ V_0^{(e)}(y^{(e)\prime}) = 0, & y^{(e)\prime} \in \partial\sigma^{(e)}, \end{cases}$$

and denote

$$\kappa_e = \int\limits_{\sigma^{(e)}} V_0^{(e)}(y^{(e)\prime}) \mathrm{d}y^{(e)\prime}.$$

Solve the conductivity problem on the graph for the function $p_0$:

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_0^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, & x_n^{(e)} \in (0,|e|), \\ -\sum\limits_{e: O_l \in e} \kappa_e \dfrac{\partial p_0^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l = 1,\dots,N_1, \\ p_0^{(e)}(0) = c_l, & l = N_1+1,\dots,N, \\ p_0^{(e)}(0) = p_0^{(e_s)}(0), \quad \forall e \subset \mathcal{B}_l. \end{cases} \tag{5.25}$$

Here $c_l$ are given constants, the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin $O_l$. So, $p_0$ is a continuous function on the graph. Indeed, the last condition of this problem means that the values of the function $p_0$ are the same for the all edges $e$ of the bundle $\mathcal{B}_l$ when the local variables $x_n^{(e)} = 0$. Note that applying the same Lax-Milgram lemma arguments as in the first part of [68] one can prove the existence and uniqueness of the solution to this problem.

Solving the above conductivity problem, we define for every edge $e$ the constants $s_0^{(e)}$ and $a_0^{(e)}$ such that

$$p_0^{(e)}(x^{(e)}) = -s_0^{(e)} x_n^{(e)} + a_0^{(e)} \tag{5.26}$$

and the velocity

$$\widetilde{V}_0^{(e)}(y^{(e)\prime}) = s_0^{(e)} V_0^{(e)}(y^{(e)\prime}),\ \mathbf{V}_0^{(e)}(y^{(e)\prime}) = (\mathcal{P}^{(e)})^t (0,\dots,0,\widetilde{V}_0^{(e)})^t (y^{(e)\prime}). \quad (5.27)$$

For $l = 1,\dots,N_1$ the boundary layer problem for $(\mathbf{V}_0^{[BLO_l]}(y), P_0^{[BLO_l]}(y))$ is:

$$\begin{cases} -\nu\Delta_y \mathbf{V}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} = \mathbf{f}_0^{[REGO_l]} + \mathbf{f}_0^{[BLO_l]}, & y \in \Omega_l, \\ \mathrm{div}_y \mathbf{V}_0^{[BLO_l]} = h_0^{[REGO_l]}, & y \in \Omega_l, \\ \mathbf{V}_0^{[BLO_l]} = 0, & y \in \partial\Omega_l, \end{cases} \quad (5.28)$$

where

$$\mathbf{f}_0^{[REGO_l]}(y) = -\sum_{e:O_l\in e} \Big\{ s_0^{(e)} \Big( -\nu\Delta_y \Big( \zeta\Big(\frac{y_n^{(e)}}{3r}\Big) (\mathcal{P}^{(e)})^t (0,\dots,0,V_0^{[e]}(y^{(e)\prime}))^* \Big) \\ -\nabla_y (\zeta\Big(\frac{y_n^{(e)}}{3r}\Big) y_n^{(e)}) \Big) + (a_1^{(e)} - a_1^{(e_s)})\ \nabla_y (\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)) \Big\}, \quad (5.29)$$

$$\mathbf{f}_0^{[BLO_l]}(y) = 0, \quad (5.30)$$

$$h_0^{[REGO_l]}(y) = \mathrm{div}_y \sum_{e:O_l\in e} \Big\{ s^{(e)} \zeta\Big(\frac{y_n^{(e)}}{3r}\Big) (\mathcal{P}^{(e)})^t (0,\dots,0,V_0^{[e]}(y^{(e)\prime}))^t \Big\}. \quad (5.31)$$

Here the sum $\sum\limits_{e:O_l\in e}$ is taken over all edges $e$ having ends in the node $O_l$ and the terms are extended by zero out of each cylinder $\Pi_\varepsilon^{(e_j)}$. Here we have an unknown quantity in the right-hand side, the constant $a_1^{(e)} - a_1^{(e_s)}$ is unknown. Let us denote by $(\mathbf{V}_0^{[BLO_l]}(y), \widehat{P}_0^{[BLO_l]}(y))$ the solution of problem (5.28) without the last term $(a_1^{(e)} - a_1^{(e_s)})\nabla_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big)$ in $\mathbf{f}_0^{[REGO_l]}(y)$ (since this term is of gradient form, the solutions differ only by the pressure components). The right-hand sides of system (5.28) have compact supports. Therefore, according to results of the Section 3 in [70], the pressure $\widehat{P}_0^{[BLO_l]}(y)$ exponentially stabilizes in each outlet (corresponding to the edge $e$) to a constant, say $\widehat{a}_0^{[BLO_l,e]}$, in the sense of integral estimates

$$\lim_{k\to+\infty} \int\limits_{\{y_n^{(e)}\in(k,k+1)\}\cap\Omega_l} (\widehat{P}_0^{[BLO_l]}(y) - \widehat{a}_0^{[BLO_l,e]})^2 \, \mathrm{d}y = 0. \quad (5.32)$$

For $l = N_1+1,\dots,N$, $(\mathbf{V}_0^{[BLO_l]}(y), P_0^{[BLO_l]}(y)) = (0,0)$.

Consider now the conductivity problem of rank 1 on the graph for the function $p_1$:

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_1^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, & x_n^{(e)} \in (0,|e|), \\ -\displaystyle\sum_{e:O_l\in e} \kappa_e \dfrac{\partial p_1^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l = 1,\dots,N_1, \\ p_1^{(e)}(x_n^{(e)} = 0) = 0, & l = N_1+1,\dots,N, \\ p_1^{(e)}(0) - p_1^{(e_s)}(0) = \widehat{a}_0^{[BLO_l,e]}, & \forall e \subset \mathcal{B}_l,\ e \neq e_s, \end{cases} \tag{5.33}$$

where $e_s$ is the selected edge of the bundle. So, in this problem on the graph the solution may be discontinuous at the nodes. Namely, at each node $O_l$ there are prescribed jumps of $p_1^{(e)}$ between the edges $e$ and $e_s$ of the bundle. This problem as well has a unique solution $p_1$.

Now, constants $s_1^{(e)}$ and $a_1^{(e)}$ are known:

$$p_1^{(e)}(x_n^{(e)}) = -s_1^{(e)} x_n^{(e)} + a_1^{(e)},$$

and we can completely determine the pressure in the boundary layer problem (5.28):

$$P_0^{[BLO_l]}(y) = \widehat{P}_0^{[BLO_l]}(y) - \sum_{e:O_l\in e,\, e\neq e_s} \zeta\Big(\frac{y_n^{(e_{\alpha_m})}}{3r}\Big)\widehat{a}_0^{[BLO_l,e]}.$$

Suppose that all terms of expansion (5.19)–(5.24) corresponding to the rank less or equal to $j-1$ are known, and that the macroscopic pressure on the graph $p_j$ is known as well. Let us describe the passage from the rank $j-1$ to the rank $j$.

**Step 1**. As the macroscopic pressure on the graph $p_j$ is known, define for every edge $e$ constants $s_j^{(e)}$ and $a_j^{(e)}$ such that

$$p_j^{(e)}(x^{(e)}) = -s_j^{(e)} x_n^{(e)} + a_j^{(e)}$$

and

$$\begin{aligned} \widetilde{V}_j^{(e)}(y^{(e)\prime}) &= s_j^{(e)} V_0^{(e)}(y^{(e)\prime}), \\ \mathbf{V}_j^{(e)} &= (\mathcal{P}^{(e)})^t (0,\dots,0,\widetilde{V}_j^{(e)})^t. \end{aligned} \tag{5.34}$$

**Step 2.** The boundary layer solution is a pair $(\mathbf{V}_j^{[BLO_l]}, P_j^{[BLO_l]})$ solving the following Stokes system in $\Omega_l$, $l=1,\dots,N_1$:

$$\begin{cases} -\nu\Delta_y \mathbf{V}_j^{[BLO_l]} + \nabla_y P_j^{[BLO_l]} = \mathbf{f}_j^{[REGO_l]} + \mathbf{f}_j^{[BLO_l]}, \\ \mathrm{div}_y \mathbf{V}_j^{[BLO_l]} = h_j^{[REGO_l]}, \\ \mathbf{V}_j^{[BLO_l]}|_{\partial\Omega_l} = 0, \quad j=0,\dots,J, \end{cases} \tag{5.35}$$

where

$$\begin{aligned} \mathbf{f}_j^{[REGO_l]}(y^{(e)}) = -\sum_{e:O_l\in e} \Big\{ -\nu\Delta_y\Big[\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_j^{[e]}(y^{(e)\prime})\Big] \\ +\nabla_y\Big[\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)(-s_j^{(e)}y_n^{(e)})\Big] \\ +\sum_{p+r=j-1}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)(\mathbf{V}_p^{[e]}(y^{(e)\prime})\cdot\nabla_y)\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_r^{[e]}(y^{(e)\prime})\Big) \\ +\widehat{a}_{j+1}^{(e)}\ \nabla_y\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big\} \end{aligned} \tag{5.36}$$

(for $j=J$ the coefficient $\widehat{a}_{j+1}^{(e)}(t)$ is omitted),

$$\begin{aligned} \mathbf{f}_j^{[BLO_l]}(y^{(e)}) = -\sum_{e:O_l\in e}\Big\{\sum_{p+r=j-1}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)(\mathbf{V}_p^{[e]}(y^{(e)\prime})\cdot\nabla_y)\mathbf{V}_r^{[BLO_l]}(y) \\ +\sum_{p+r=j-1}(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y)\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_r^{[e]}(y^{(e)\prime})\Big)\Big\} \\ -\sum_{p+r=j-1}(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y)\mathbf{V}_r^{[BLO_l]}(y), \end{aligned} \tag{5.37}$$

$$h_j^{[REGO_l]}(y,t) = -\sum_{e:O_l\in e}\mathrm{div}_y\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}_j^{[e]}(y^{(e)\prime},t)\Big). \tag{5.38}$$

Here the sum $\sum\limits_{e:O_l\in e}$ is taken over all edges $e$ having ends in the node $O_l$, the terms of the sum are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$ and by convention, the terms with the negative subscripts $j$ are equal to zero.

First, we find the couple $(\mathbf{V}_j^{[BLO_l]}, \widehat{P}_j^{[BLO_l]})$ which is the solution to the same problem (5.35) without the last term in the definition of $\mathbf{f}_j^{[REGO_l]}$ (see (5.36)). It can be proved by induction, using the results of Theorems 3.1 and Corollary 3.1 [70], that $\mathbf{V}_j^{[BLO_l]}$ exponentially tends to zero as $|y|\to+\infty$, while the corresponding pressure function $\widehat{P}_j^{[BLO_l]}$ stabilizes in outlets to infinity to some constants $\widehat{a}_j^{[BLO_l,e]}$ in the sense of (5.32); these constants may be different for different outlets. Since the pressure function is defined up to an additive constant, we can fix the limit constant equal to zero for the outlet corresponding to the selected edge $e_s$.

Then we solve the problems in half-cylinders $\Omega_l$, $l = N_1+1,\dots,N$:

$$\begin{cases} -\nu\Delta_y \mathbf{V}_j^{[BLO_l]} + \nabla_y \widehat{P}_j^{[BLO_l]} = \mathbf{f}_j^{[REGO_l]} + \mathbf{f}_j^{[BLO_l]}, \\ \operatorname{div}_y \mathbf{V}_j^{[BLO_l]} = 0, \\ \mathbf{V}_j^{[BLO_l]}|_{\partial\Omega_l\setminus\{y_n=0\}} = 0, \\ \mathbf{V}_{j\boldsymbol{\tau}}^{[BLO_l]}|_{y_n=0} = 0, \\ \Big(-\nu\dfrac{\partial \mathbf{V}_j^{[BLO_l]}\cdot\mathbf{n}}{\partial n} + \widehat{P}_j^{[BLO_l]}\Big)\Big|_{y_n=0} = -\dfrac{1}{2}\sum\limits_{p+r=j-1}\Big[(\mathbf{V}_p^{[e]}(y^{(e)\prime}) + \mathbf{V}_p^{[BLO_l]}|_{y_n=0}) \\ \qquad\qquad \cdot(\mathbf{V}_r^{[e]}(y^{(e)\prime}) + \mathbf{V}_r^{[BLO_l]}|_{y_n=0})\Big], \end{cases} \tag{5.39}$$

where $\dfrac{\partial \mathbf{V}_j^{[BLO_l]}\cdot\mathbf{n}}{\partial n}$ is fictive and $j = 0,\dots,J$.

$$\mathbf{f}_j^{[REGO_l]}(y^{(e)}) = 0, \tag{5.40}$$

$$\begin{aligned} \mathbf{f}_j^{[BLO_l]}(y^{(e)}) = &-\sum_{p+r=j-1}(\mathbf{V}_p^{[e]}(y^{(e)\prime})\cdot\nabla_y)\mathbf{V}_r^{[BLO_l]}(y) \\ &-\sum_{p+r=j-1}(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y)\mathbf{V}_r^{[e]}(y^{(e)\prime}) \\ &-\sum_{p+r=j-1}(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y)\mathbf{V}_r^{[BLO_l]}(y). \end{aligned} \tag{5.41}$$

The pressure here $\widehat{P}_j^{[BLO_l]}$ tends to a constant $\widehat{a}_j^{[BLO_l,e]}$.

If $j = J$ then the right-hand side of the boundary condition is replaced by

$$-\frac{1}{2}\sum_{J-1\leqslant p+r\leqslant 2J}(\mathbf{V}_p^{[e]}(y^{(e)\prime}) + \mathbf{V}_p^{[BLO_l]}|_{y_n=0})(\mathbf{V}_r^{[e]}(y^{(e)\prime}) + \mathbf{V}_r^{[BLO_l]}|_{y_n=0}). \tag{5.42}$$

**Step 3.** Solve the conductivity problem on the graph for the function $p_{j+1}^{(e)}$ $(j < J)$:

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_{j+1}^{(e)}}{\partial {x_n^{(e)}}^2}(x_n^{(e)}) = 0, & x_n^{(e)} \in (0,|e|), \\ -\sum\limits_{e:O_l\in e}\kappa_e \dfrac{\partial p_{j+1}^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l = 1,\dots,N_1, \\ p_{j+1}^{(e)}(0) = \widehat{a}_j^{[BLO_l,e]}, & l = N_1+1,\dots,N, \\ p_{j+1}^{(e)}(0) - p_{j+1}^{(e_s)}(0) = \widehat{a}_j^{[BLO_l,e]}, & \forall e\subset\mathcal{B}_l,\ e\neq e_s, \end{cases}$$

where the local coordinates $x^{(e)}$ are redefined so that all of them have the same origin $O_l$.

**Step 4.** Finally, we find the pressure $P_j^{[BLO_l]}(y)$ in boundary layer problem (5.35), (5.36) for $l=1,\dots,N_1$:

$$P_j^{[BLO_l]}(y)=\widehat{P}_j^{[BLO_l]}(y)-\sum_{e:O_l\in e,\, e\neq e_s}\zeta\Big(\frac{y_n^{(e_{\alpha_m})}}{3r}\Big)\widehat{a}_j^{[BLO_l,e]},$$

and $P_j^{[BLO_l]}(y)$ in boundary layer problem (5.39), (5.40) for $l=N_1+1,\dots,N$:

$$P_j^{[BLO_l]}(y)=\widehat{P}_j^{[BLO_l]}(y)-\widehat{a}_j^{[BLO_l,e]}.$$

This step finalizes the passage from $j$ to $j+1$.

## 5.4 Residual

Consider the asymptotic expansion $\big(\mathbf{v}^{(J)},p^{(J)}\big)$ of order $J$ (see (5.19), (5.21)). By construction,

$$\mathbf{v}^{(J)}\in W^{2,2}(B_\varepsilon),\quad \nabla p^{(J)}\in L^2(B_\varepsilon). \tag{5.43}$$

Moreover, $\|\mathbf{v}^{(J)}\|^4_{L^4(B_\varepsilon)}\leqslant c\varepsilon^{(n-1)/4}$. Indeed, the Poiseuille part of $\mathbf{v}^{(J)}$ satisfies this estimate. The $\|\cdot\|_{L^4(B_\varepsilon^{(i)})}-$norm of the boundary layer functions in each bundle $B_\varepsilon^{(i)}$ can be estimated by the $L^4-$norm in the unbounded dilated domain $\Omega_i$ multiplied by $\varepsilon^{n/4}$. Taking into consideration an exponential decay of the boundary layers, we get the desired estimate.

Put $\mathcal{L}(\mathbf{v},p)=-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p$. Let us calculate $\mathcal{L}(\mathbf{v}^{(J)},p^{(J)})$ in a half-bundle $HB_{O_l}$, $l=1,\dots,N_1$. We obtain

$$\mathbf{f}^{(J)}(x)=\mathcal{L}(\mathbf{v}^{(J)},p^{(J)})$$

$$=\sum_{J+1\leqslant j\leqslant 2J}\varepsilon^{j-2}\Bigg(\sum_{e:O_l\in e}\sum_{\alpha+\beta=j-1}\Big(\mathbf{V}^{(e)}_{(\alpha)}(y^{(e)\prime})\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\cdot\nabla_y\Big)\mathbf{V}^{(e)}_{(\beta)}(y^{(e)\prime})\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)$$

$$+\sum_{e:O_l\in e}\Big[\sum_{p+r=j-1}\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\Big(\mathbf{V}^{(e)}_{(p)}(y^{(e)\prime})\cdot\nabla_y\Big)\mathbf{V}_r^{[BLO_l]}(y)$$

$$+\sum_{p+r=j-1}\Big(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y\Big)\Big(\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)\mathbf{V}^{(e)}_{(r)}(y^{(e)\prime})\Big)$$

$$+\sum_{p+r=j-1}\Big(\mathbf{V}_p^{[BLO_l]}(y)\cdot\nabla_y\Big)\mathbf{V}_r^{[BLO_l]}(y)\Big]\Bigg)$$

$$+\varepsilon^{J-2}\sum_{e:O_l\in e}\widehat{a}_J^{[BLO_l,e]}\nabla_y\zeta\Big(\frac{y_n^{(e)}}{3r}\Big)$$

$$-\Big\{\mathcal{L}\Big(\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\mathbf{V}^{[BLO_l,J]}(y),\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)P^{[BLO_l,J]}(y)\Big)\chi(x)\Big\}.$$

Here $y=\dfrac{x-O_l}{\varepsilon}$; $y^{(e)}=\dfrac{x^{(e)}}{\varepsilon}$; $\chi=\chi_{\mathrm{supp}\big(1-\zeta\big(\frac{|x-O_l|}{|e|_{\min}}\big)\big)}$ is the characteristic function of the set $\mathrm{supp}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)$. As before the terms of the sums $\sum\limits_{e:O_l\in e}$ are extended by zero out of cylinders $\Pi_\varepsilon^{(e)}$.

Here the first four lines come from the inertial term and they contain all combinations of $\mathbf{V}_\beta^{(e)}$, and $\mathbf{V}_\beta^{[BLO_l]}$ having the order higher than $J-2$, the next line comes from the pressure gradient term; this term is the only one that was not compensated by the boundary layer-in-space problems. The last line is the residual generated by the multiplication of the boundary layer correctors by the cut-off function $\zeta\big(\frac{|x-O_l|}{|e|_{\min}}\big)$. Notice that terms appearing in this last line exponentially vanish because in the set $\mathrm{supp}\big(1-\zeta\big(\frac{|x-O_l|}{|e|_{\min}}\big)\big)$ (where $\chi\neq 0$) the order of this term in $L^2$-norm is $O(e^{-c_1/\varepsilon})$ with some positive constant $c_1$ (see the Appendix in [70]). From the obtained formulas it follows that

$$\|\mathbf{f}^{(J)}\|_{L^2(B_\varepsilon)}=\|\mathcal{L}(\mathbf{v}^{(J)},p^{(J)})\|_{L^2(B_\varepsilon)}=O(\varepsilon^{J-2}). \tag{5.44}$$

In the cylinders associated to the vertex $O_l$, $l=N_1+1,\dots,N$, the residual is simpler: it is without the factor $\zeta\Big(\dfrac{y_n^{(e)}}{3r}\Big)$.

Let us calculate the divergence of $\mathbf{v}^{(J)}$. In any half-bundle we have

$$\operatorname{div}\mathbf{v}^{(J)}=-\nabla\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\cdot\mathbf{V}^{[BLO_l,J]}(y)=h^{(J)}(y).$$

Obviously, $h^{(J)}\in W^{1,2}(B_\varepsilon)$. Since the support of the function $\nabla\zeta\big(\frac{|x-O_l|}{|e|_{\min}}\big)$ belongs to the middle third of every cylinder, there the relation

$$\|h^{(J)}\|_{W^{1,2}(B_\varepsilon)}=O(e^{-c_2/\varepsilon}) \tag{5.45}$$

holds for some $c_2>0$.

Finally, the boundary conditions are satisfied with residual $\varepsilon^{J-1}\widehat{a}_J^{[BLO_l,e]}$ on $\gamma_\varepsilon^l$. This residual appears as a result of subtraction of the constant $\widehat{a}_j^{[BLO_l,e]}$ from the boundary layer pressure $P_j^{[BLO_l]}(y)$ at the step 4 of the algorithm. For all $j < J$ it is compensated by the gaps of the pressure in the problem on the graph, but for $j = J$ it remains as a residual.

It is easy to see that

$$\int\limits_{B_\varepsilon} h^{(J)}(y)\,\mathrm{d}y = 0.$$

Therefore, by Lemma 3.7 in [69], there exists a vector field $\mathbf{w}^{(J)} \in \mathring{W}^{1,2}(B_\varepsilon)$ such that $\operatorname{div}\mathbf{w}^{(J)} = -h^{(J)}$. Moreover, the estimate

$$\|\mathbf{w}^{(J)}\|_{W^{1,2}(B_\varepsilon)} \leqslant \varepsilon^{-1} c \|h^{(J)}\|_{L^2(B_\varepsilon)} \tag{5.46}$$

holds.

Set $\widetilde{\mathbf{v}}^{(J)} = \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$. Then $\operatorname{div}\widetilde{\mathbf{v}}^{(J)} = 0$, $\widetilde{\mathbf{v}}^{(J)}$ satisfies the boundary conditions with the residual $-\varepsilon^{J-1}\widehat{a}_J^{[BLO_l,e]}$ on $\gamma_\varepsilon^l$, and because of (5.45) we have

$$\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)} = O(\varepsilon^{J-2}), \tag{5.47}$$

where $\mathbf{f}_1^{(J)} = \mathcal{L}(\widetilde{\mathbf{v}}^{(J)}, p^{(J)})$.

## 5.5 Error estimate

**Theorem 5.5.1.** *The following error estimate*

$$\|\mathbf{v} - \widetilde{\mathbf{v}}^{(J)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}) \tag{5.48}$$

*holds.*

*Proof.* Let $\mathbf{u} = \mathbf{v} - \widetilde{\mathbf{v}}^{(J)}$. Then the integral identity

$$\begin{aligned}
\nu \int\limits_{B_\varepsilon} \nabla\mathbf{u} : \nabla\boldsymbol{\eta}\,\mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{u}\cdot\nabla)\widetilde{\mathbf{v}}^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x \\
+ \int\limits_{B_\varepsilon} (\widetilde{\mathbf{v}}^{(J)}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\eta}\,\mathrm{d}x - \int\limits_{B_\varepsilon} (\boldsymbol{\eta}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x \\
- \int\limits_{B_\varepsilon} (\boldsymbol{\eta}\cdot\nabla)\mathbf{u}\cdot\widetilde{\mathbf{v}}^{(J)}\,\mathrm{d}x - \int\limits_{B_\varepsilon} (\boldsymbol{\eta}\cdot\nabla)\widetilde{\mathbf{v}}^{(J)}\cdot\mathbf{u}\,\mathrm{d}x \\
= \varepsilon^{J-1} \sum_{l=N_1+1}^{N} \widehat{a}_J^{[BLO_l,e]} \int\limits_{\gamma_\varepsilon^l} \boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x' - \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x
\end{aligned}$$

holds for every $\boldsymbol{\eta} \in \widehat{J}_\gamma^{1,2}(B_\varepsilon)$.

Taking $\boldsymbol{\eta}=\mathbf{u}$ and integrating by parts, we obtain

$$\begin{aligned}\nu\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x &= \varepsilon^{J-1}\sum_{l=N_1+1}^{N}\widehat{a}_J^{[BLO_l,e]}\int\limits_{\gamma_\varepsilon^l}\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}x' \\ &+\int\limits_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\widetilde{\mathbf{v}}^{(J)}\,\mathrm{d}x-\int\limits_{B_\varepsilon}(\widetilde{\mathbf{v}}^{(J)}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x-\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\mathbf{u}\,\mathrm{d}x.\end{aligned}\tag{5.49}$$

From Lemma 2.1.7, it follows that

$$\left|\varepsilon^{J-1}\sum_{l=N_1+1}^{N}\widehat{a}_J^{[BLO_l,e]}\int\limits_{\gamma_\varepsilon^l}\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}x'\right|\leqslant c\varepsilon^{J-1+n/2}\sum_{l=N_1+1}^{N}\left|\widehat{a}_J^{[BLO_l,e]}\right|\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}.$$

Using Hölder inequality, (2.11), (2.12) and estimate $\|\widetilde{\mathbf{v}}^{(J)}\|_{L^4(B_\varepsilon)}\leqslant c\varepsilon^{(n-1)/4}$, we get

$$\begin{aligned}\Big|\int\limits_{B_\varepsilon}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\widetilde{\mathbf{v}}^{(J)}\,\mathrm{d}x\Big|,\Big|\int\limits_{B_\varepsilon}(\widetilde{\mathbf{v}}^{(J)}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x\Big| &\leqslant \|\mathbf{u}\|_{L^4(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)}\|\widetilde{\mathbf{v}}^{(J)}\|_{L^4(B_\varepsilon)} \\ &\leqslant c\varepsilon^{\alpha+\frac{n-1}{4}}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}\leqslant c\varepsilon^{\frac{3}{4}}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)},\end{aligned}$$

where $\alpha$ is the same as in Theorem 5.2.2. Besides

$$\Big|\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\mathbf{u}\,\mathrm{d}x\Big|\leqslant\varepsilon C\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)},$$

where $\varepsilon C$ is Poincaré-Friedrich's constant for the domain $B_\varepsilon$.

From these estimates and identity (5.49), we obtain

$$\begin{aligned}\nu\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x &\leqslant c\varepsilon^{\frac{2(J-1)+n}{2}}\sum_{l=N_1+1}^{N}\left|\widehat{a}_J^{[BLO_l,e]}\right|\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} \\ &+c\varepsilon^{\frac{3}{4}}\|\nabla\mathbf{u}\|^2_{L^2(B_\varepsilon)}+\varepsilon C\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)}\|\nabla\mathbf{v}\|_{L^2(B_\varepsilon)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^2(B_\varepsilon)} &\leqslant \frac{C}{\nu-c\varepsilon^{3/4}}\left(\varepsilon^{\frac{2(J-1)+n}{2}}\sum_{l=N_1+1}^{N}\left|\widehat{a}_J^{[BLO_l,e]}\right|+\varepsilon\|\mathbf{f}_1^{(J)}\|_{L^2(B_\varepsilon)}\right) \\ &\leqslant \frac{C}{\nu-c\varepsilon^{3/4}}\left(\varepsilon^{\frac{2(J-1)+n}{2}}\sum_{l=N_1+1}^{N}\left|\widehat{a}_J^{[BLO_l,e]}\right|+\varepsilon^{J-1}\right).\end{aligned}$$

If $\nu-c\varepsilon^{3/4}>\dfrac{\nu}{2}$, then

$$\|\mathbf{u}\|_{W^{1,2}(B_\varepsilon)}=\|\mathbf{v}-\tilde{\mathbf{v}}^{(J)}\|_{W^{1,2}(B_\varepsilon)}=O(\varepsilon^{J-1}).\tag{5.50}$$

Evaluating now the norm of the difference $\mathbf{v}^{(J)}$ and $\mathbf{v}^{(J+2)}$ we obtain:

$$\|\tilde{\mathbf{v}}^{(J)} - \tilde{\mathbf{v}}^{(J+2)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}).$$

Replacing $J$ by $J+2$ in (5.50) we obtain:

$$\|\mathbf{v} - \tilde{\mathbf{v}}^{(J+2)}\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+1}).$$

So, the triangle inequality gives estimate (5.48). □

# Chapter 6

# Conclusions

The aim of this dissertation is to analyse Navier–Stokes equations in different domains. Our purpose was to get the theoretical results, which may be used to create a simplified blood circulation model. This was done by the following steps:

— In Chapter 3, the existence and uniqueness of a weak solution for the time-periodic Stokes system in the domain with an outlet to infinity were proved by constructing an appropriate boundary value extension. This step gives the possibility to study the time-periodic Navier–Stokes equations.

— In Chapter 4, besides the proof of the existence and uniqueness of the solution, the main goal was to construct and justify the appropriate asymptotic expansion of the weak solution. This asymptotic expansion let us to develop the hybrid-dimension model, which is suitable for small vessels like arterioles and capillaries.

— The results in Chapter 4 obtained in the case of Dirichlet type boundary condition, however more natural boundary condition is Neumann type. Therefore in Chapter 5, we constructed the asymptotic expansion with the given Bernoulli pressure. We also prove the existence and uniqueness of the solution.

The obtained results may be used to create a simplified blood flow model for small vessels. The case with given Bernoulli pressure, may be developed with the time periodicity condition. That will allow to create a more realistic and complicated blood flow model.

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# Santrauka

## Tyrimo objektas

Disertacijoje nagrinėjami šie uždaviniai: laiko atžvilgiu periodinė Stokso sistema su nehomogenine kraštine sąlyga

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v}(x,t) = \boldsymbol{\varphi}(x), & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega, \end{cases} \tag{S.1}$$

laiko atžvilgiu periodinė Navjė ir Stokso sistema su nehomogenine kraštine sąlyga

$$\begin{cases} \frac{1}{\varepsilon^\beta}\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in B_\varepsilon \times (0,2\pi), \ \beta = 0,2, \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in B_\varepsilon \times (0,2\pi), \\ \mathbf{v}(x,t) = \mathbf{g}(x,t), & (x,t) \in \partial B_\varepsilon \times (0,2\pi), \\ \mathbf{v}(x,t) = \mathbf{v}(x,t+2\pi), \ x \in B_\varepsilon, \end{cases} \tag{S.2}$$

bei stacionarioji Navjė ir Stokso sistema su duotu Bernulio slėgiu

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & x \in B_\varepsilon, \\ \operatorname{div}\mathbf{v} = 0, & x \in B_\varepsilon, \\ \mathbf{v} = 0, & x \in \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}} = 0, & x \in \gamma_\varepsilon^j, \\ -\nu\partial_n(\mathbf{v}\cdot\mathbf{n}) + \left(p + \frac{1}{2}|\mathbf{v}|^2\right) = c_j/\varepsilon^2, \ x \in \gamma_\varepsilon^j, j = N_1+1, \dots, N, \end{cases} \tag{S.3}$$

čia $\mathbf{v}$ ir $p$ yra sistemos nežinomieji, $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \dots, v_n(x,t))$ greičio vektorius, $p = p(x,t)$ skysčio slėgis, $\mathbf{f} = \mathbf{f}(x,t) = (f_1(x,t), \dots, f_n(x,t))$ išorinės jėgos vektorius, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x) = (\varphi_1(x), \dots, \varphi_n(x))$ ir $\mathbf{g} = \mathbf{g}(x,t) = (g_1(x,t), \dots,$

$g_n(x,t)$) yra duoti skysčio greičiai ant srities krašto, $\mathbf{n}$ - išorinės vienetinės normalės vektorius, $\mathbf{v}_\tau = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ - greičio vektoriaus tangentinė komponentė, $\partial_n h = \nabla h \cdot \mathbf{n}$ - funkcijos $h$ normalinė išvestinė, o $\left(p + \frac{1}{2}|\mathbf{v}|^2\right)$ duotas Bernulio slėgis, $\nu > 0$ - skysčio klampumo koeficientas, $\varepsilon$ - koeficientas, kuris lygus cilindro diametro ir ilgio santykiui, $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ greičio periodiškumo laiko atžvilgiu sąlyga (su periodu $2\pi$).

## Mokslinės problemos istorija ir aktualumas

Stokso bei Navjė ir Stokso lygtys, aprašančios klampaus nespūdaus skysčio tekėjimą, yra nagrinėjamos sprendžiant įvairius hidrodinamikos uždavinius. Šios lygtys yra įdomios tiek teoriniu, tiek praktiniu požiūriu, dėl to, jos yra nagrinėjamos daugybėje mokslo bei technikos sričių. Šie tyrimai tampa ypač aktualūs šiuolaikiniame pasaulyje. Praktikoje įvairių sričių specialistai dažnai susiduria su skysčio tekėjimo uždaviniais, kuriuos aprašo Navjė ir Stokso lygčių sistema. Plačios šių lygčių taikymo galimybės leidžia plėtoti tarpdisciplininius tyrimus ir vystyti Navjė ir Stokso lygčių teoriją tiek teoriškai, tiek praktiškai.

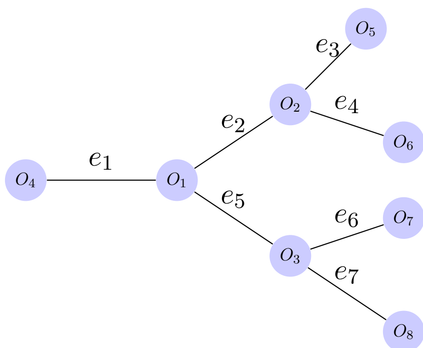
Viena iš sričių, kurioje reikalingas klampaus nespūdaus skysčio tekėjimo modeliavimas, yra medicina. Vienas iš šios disertacijos siekinių yra gauti medicinoje pritaikomus rezultatus. Tyrimas buvo atliekamas 2017 - 2021 dirbant jaunesniąja mokslo darbuotoja projekte “Klampaus tekėjimo sudėtingos geometrijos srityse daugiaskaliai modeliai ”.[1] Disertacijoje nagrinėjamos lygtys, kurios leidžia kurti supaprastintus kraujotakos sistemos modelius neprarandant tikslumo. Gauti rezultatai gali būti taikomi modeliuojant kraujo tekėjimą smulkiose kraujagyslėse, tokiose kaip arteriolės ir kapiliarai. Taip pat, šie rezultatai gali būti plėtojami kuriant sudėtingesnius modelius, kurie, pavyzdžiui, aprašytų kraujo tekėjimą širdyje ar kituose vidaus organuose.

Stokso bei Navjė ir Stokso lygtys griežtai matematiškai nagrinėjamos nuo XX a. pradžios. Per pastaruosius metus pasiektas reikšmingas postūmis sprendžiant Lerė problemą [45], tačiau nepaisant didelio susidomėjimo ir pastangų, daugybė klausimų susijusių su šiomis lygtimis, vis dar lieka atviri.

1. Šis projektas bendrai finansuotas iš Europos socialinio fondo (projekto Nr. 09.3.3-LMT-K-712-01-0012) pagal dotacijos sutartį su Lietuvos mokslo taryba (LMTLT).

Disertacijoje netiesinės Navjė ir Stokso sistemos nagrinėjimas yra pradedamas nuo tiesinės jos versijos, t.y. Stokso sistemos nagrinėjimo. Pirmiausia nagrinėjama laiko atžvilgiu periodinė Stokso sistema, srityje su išėjimu į begalybę [29]. Tačiau siekiant gauti rezultatus, kurie aprašytų kraujotakos sistemas Stokso sistema nėra pakankama, be to, sritis su išėjimu į begalybę neaprašo kraujagyslių tinklo struktūros. Dėl šių priežasčių, toliau disertacijoje nagrinėjama Navjė ir Stokso sistema cilindrinėje struktūroje, kurią sudaro plonų vamzdelių sąjunga.

Vienas iš disertacijos siekinių yra gauti rezultatus, kurie galėtų būti pritaikomi modeliuojant kraujotakos sistemas. Šioms sistemoms įtakos turi širdies plakimas, kuris matematiškai gali būti aprašomas, kaip laiko atžvilgiu periodinė funkcija, todėl yra nagrinėjami periodiniai uždaviniai. Norėdami aprašyti kraujo tekėjimą skirtingose kraujagyslėse, mes įvedame parametrą $\varepsilon$ ir nagrinėjame laiko atžvilgiu periodinę Navjė ir Stokso sistemą cilindrinėje struktūroje. Kiekvieną cilindrinę struktūrą atitinka jos vienmatis grafas, kuris gaunamas cilindrų diametrą artinant į nulį. Grafo, kuris turi tris mazgus $O_1, O_2, O_3$, penkias viršūnes $O_4, O_5, O_6, O_7, O_3$ ir septynias kraštines $e_1, \ldots, e_7$ pavyzdys pateiktas (1) pav. Vienmatė struktūra yra paprasta,




1 pav. – Cilindrinės struktūros grafas

tačiau ji turi trūkumų: šiuo metu egzistuojantys vienmačiai modeliai ir kodai negali užtikrinti reikiamo tikslumo srityse, kuriose formuojasi trombai, yra stentai arba kraujagyslės išsišakoja. Tikslumo problemą galima išspręsti taikant trimatį modelį visai kraujotakos sistemai, tačiau gausime uždavinį, kurio sprendimas reikalaus labai daug kompiuterių resursų. Todėl trimatis modelis gali būti taikomas tik tam tikrose kraujotakos sistemos dalyse. Dėl šių priežasčių, disertacijoje yra siūlomas hibridinis modelis, kuris sujungia vienmatį ir trimatį modelius, kai trimatis modelis taikomas tik tam tikrose struktūros vietose, kuriose vienmatis modelis yra netikslus (pavyzdžiui, kraujagyslių išsišakojime). Šis metodas pirmą kartą buvo pasiūlytas G. Panasenko 1998 metais sprendžiant stacionarią Navjė ir Stokso lygtį [65, 66]. Vėliau 2015 metais, šis metodas G. Panasenko ir K. Pilecko darbuose buvo išplėtotas sprendžiant nestacionariąją Navjė ir Stokso sistemą cilindrinėje struktūroje [70, 71]. Šioje disertacijoje, šis metodas yra pritaikomas sprendžiant laiko atžvilgiu periodinį uždavinį. Tam, jog galėtume sujungti vienmatį ir trimatį modelius, mes konstruojame uždavinio sprendinio asimptotinį skleidinį. Metodas, leidžiantis kurti hibridinės dimensijos modelį, yra vadinamas asimptotiniu dalinio srities išskaidymo metodu (method of asymptotic partial decomposition of the domain (MAPDD)). Apibendrintas MAPDD metodas pradiniam ir kraštiniam uždaviniui, pasitelkiant kompiuterinę simuliaciją, buvo nagrinėjamas [5] straipsnyje. Skaitiniai tyrimai parodė, jog lyginant su A. Quarteroni komandos sukurtu modeliu, kuris taip pat apjungia vienmačius ir trimačius modelius [15], MAPDD yra labiau tinkamas modeliuojant smulkias kraujagysles, tokias kaip arteriolės. Tuo tarpu, A. Quarteroni komandos modelis tiksliau aprašo stambias kraujagysles, tokias kaip arterijos.

Norėdami aprašyti skirtingų tipų smulkias kraujagysles, disertacijoje naudojame dvi skirtingas skales. Tam, prie greičio vektoriaus išvestinės laiko atžvilgiu, įvedame mažą parametrą $\varepsilon$, kuris yra lygus kraujagyslės diametro ir ilgio santykiui. Nagrinėjame du skirtingus atvejus: $\varepsilon^0$ ir $\varepsilon^{-2}$. Parametras $\varepsilon^{-2}$ generuoja didelį koeficientą prie greičio išvestinės laiko atžvilgiu. Šie atvejai aprašo smulkias ir labai smulkias kraujagysles, tokias kaip arteriolės ir kapiliarai.

Mūsų tikslas yra gauti rezultatus, kurie gali būti taikomi modeliuojant kraujotakos sistemas. Norint sumažinti skaičiavimo kaštus ir gauti reikiamą tikslumą, kompiuterinėje simuliacijoje reikia naudoti hibridinės dimensijos modelius. Kertinis žingsnis, leidžiantis kurti tokius modelius, yra sprendinio asimptotinio skleidinio konstravimas. Pagrindinis asimptotinio skleidinio narys priklauso nuo uždavinio ant grafo sprendinio. Toks uždavinys, kuriuo remiasi asimptotinio skleidinio formavimas cilindrinėse struktūrose, pirmą kartą buvo pasiūlytas G. Panasenko ir K. Pilecko [68, 72] straipsniuose. Šioje disertacijoje asimptotinis skleidinys konstruojamas remiantis G. Panasenko ir K. Pilecko [68, 72] bei E. Marušič-Paloka [55] straipsniuose gautais rezultatais, pritaikant juos laiko atžvilgiu periodiniam uždaviniui.

Laiko atžvilgiu periodinė Navjė ir Stokso sistema buvo išnagrinėta su Dirichlė tipo kraštine sąlyga, tačiau sprendžiant hemodinamikos uždavinius, geriau yra naudoti Noimano tipo sąlygas. Jos yra artimesnės realioms sąlygoms, tačiau tokie uždaviniai nėra plačiai išnagrinėti. Dėl šios priežasties, disertacijoje yra nagrinėjama stacionarioji Navjė ir Stokso sistema su Noimano tipo sąlygomis, t.y. su duotu Bernulio slėgiu kraštuose, per kuriuos skystis gali įtekėti arba ištekėti. Šiuo atveju, sukonstruotas apibendrintojo sprendinio asimptotinis skleidinys, kai turime netiesinę kraštinę sąlygą.

Visais atvejais, t.y., tiek nagrinėjant laiko atžvilgiu periodinę Stokso sistemą srityje su išėjimu į begalybę, tiek nagrinėjant laiko atžvilgiu periodinę bei stacionariąją Navjė ir Stokso sistemas cilindrinėje struktūroje, yra įrodomas apibendrintųjų sprendinių egzistavimas ir vienatis. Be to, Navjė ir Stokso lygčių atveju, sukonstruojamas apibendrintojo sprendinio asimptotinis skleidinys, kuris leidžia kurti hibridinės dimensijos modelius, mažinančiais modelio skaičiavimo kaštus.

## Disertacijos tikslai

Disertacijoje nagrinėjamos laiko atžvilgiu periodinės Stokso bei Navjė ir Stokso sistemos, taip pat, stacionarioji Navjė ir Stokso sistema skirtingose srityse. Pradedama nuo periodinės pagal laiką Stokso sistemos nagrinėjimo srityje su išėjimu į begalybę (žr. 2 pav.). Vėliau analizuojama laiko atžvilgiu periodinė bei stacionarioji Navjė ir Stokso sistemos cilindrinėje struktūroje (žr. 4. 3 pav.). Mūsų siekis - išnagrinėjus šiuos uždavinius, gauti rezultatus, kurie būtų taikomi kuriant supaprastintus kraujotakos sistemos modelius aprašančius kraujo tekėjimą smulkiose kraujagyslėse. Šios sistemos buvo nagrinėjamos tokiais etapais:

— įrodomas laiko atžvilgiu periodinio Stokso uždavinio apibendrintojo sprendinio egzistavimas ir vienatis sukonstruojant specialų kraštinės sąlygos pratęsimą, kai sritis turi išėjimą į begalybę,

— įrodomas laiko atžvilgiu periodinės ir stacionariosios Navjė ir Stokso sistemos sprendinio egzistavimas cilindrinėje struktūroje, kai cilindro skersmuo yra mažas,

— sukonstruojamas asimptotinis apibendrintojo sprendinio skleidinys, kuris leidžia kurti hibridinės dimensijos modelius. Jis pagrindžiamas tiek Navjė ir Stokso sistemai, kuri yra periodinė laiko atžvilgiu, tiek stacionariajai Navjė ir Stokso sistemai su duotu Bernulio slėgiu,

— išplėtojamas asimptotinis dalinis srities išskaidymo metodas (method of asymptotic partial decomposition of the domain (MAPDD)) laiko atžvilgiu periodiniam Navjė ir Stokso uždaviniui.

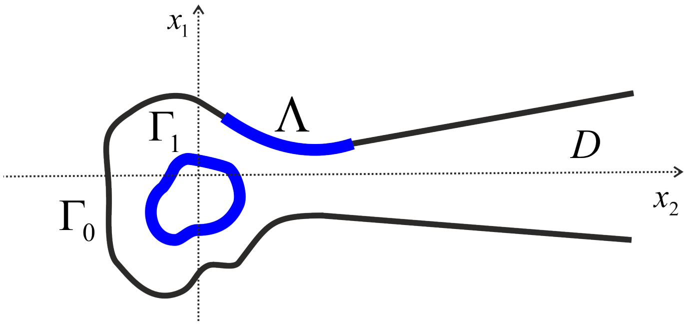
## Disertacijos struktūra ir apimtis

Disertacijos tekstą sudaro penki skyriai, išvados ir literatūros sąrašas. Pirmasis skyrius yra įvadinis. Jame apžvelgiama nagrinėjamos problemos istorija ir aktualumas bei sritys, kuriose nagrinėjamos sistemos. Antrame skyriuje pateikiami pagrindiniai žymėjimai ir pagalbiniai rezultatai, kurie naudojami disertacijoje. Trečiame skyriuje pateikiami rezultatai gauti analizuojant laiko atžvilgiu periodinę Stokso sistemą su nehomogenine kraštine sąlyga srityje su išėjimu į begalybę. Tuo tarpu, ketvirtame ir penktame skyriuose išnagrinėtos laiko atžvilgiu periodinė Navjė ir Stokso sistema su nehomogenine kraštine sąlyga bei stacionarioji Navjė ir Stokso sistema su duotu Bernulio slėgiu cilindrinėje struktūroje. Disertacijos pabaigoje pateikiamos išvados ir naudotos literatūros sąrašas.

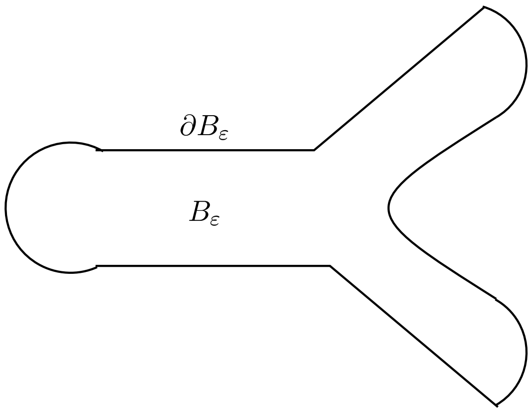
## Disertacijoje gautų rezultatų apžvalga

Nagrinėjamos Stokso bei Navjė ir Stokso sistemos skirtingose srityse. Tiesinė Stokso sistema (S.1) nagrinėjama dvimatėje srityje $\Omega$, kuri turi išėjimą į begalybę. Srities kraštas yra sudarytas iš skirtingų komponenčių, kurios sudaro vidinį ir išorinį kraštus (žr. 2 pav.).

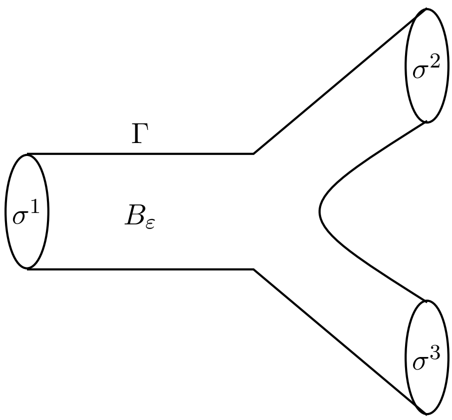
Netiesinės Navjė ir Stokso sistemos (S.2),(S.3) nagrinėjamos srityje $B_\varepsilon$ sudarytoje iš keleto atskirų vamzdelių (cilindrų). Tokia sritis vadinama cilindrine struktūra. Struktūros sudarytos iš trijų vamzdelių pavyzdžiai pateikiami 4, 3 paveikslėliuose.




2 pav. – Sritis $\Omega$




3 pav. – Sritis $B_\varepsilon$




4 pav. – Sritis $B_\varepsilon$

Disertacijoje įrodomas šių sistemų apibendrintųjų sprendinių egzistavimas ir vienatis. Navjė ir Stokso sistemos atveju, taip pat, sukonstruojami asimptotiniai skleidiniai, kurie leidžia kurti hibridinės dimensijos modelius.

Žemiau bus pateikiami pagrindiniai rezultatai gauti nagrinėjant šiuos uždavinius. Visų ten naudojamų funkcijų erdvių apibrėžimus (jei nenurodyta kitaip) galima rasti disertacijos 2 skyriuje.

## Laiko atžvilgiu periodinė Stokso sistema srityje su išėjimu į begalybę

Laiko atžvilgiu periodinė Stokso sistema (S.1) nagrinėjama dvimatėje srityje $\Omega = \Omega_0 \cup D$, kuri sudaryta iš aprėžtosios dalies $\Omega_0 = \Omega \cap B_{R_0}(0) = \Omega \cap \{x \in \mathrm{R}^2 : |x| \leqslant R_0\}$ ir turinti išėjimą į begalybę $D = \{x \in \mathbb{R}^2 : |x_1| < g(x_2),\ x_2 > R_0\}$ (žr. 2 pav.). Funkcija $g$ tenkina Lipšico sąlygą

$$|g(t_1) - g(t_2)| \leqslant L|t_1 - t_2|,\ t_1, t_2 > R_0,\ g(t) \geqslant \mathrm{const} > 0.$$

Srities kraštas $\partial\Omega \in C^2$ yra sudarytas iš vidinio krašto $\Gamma_1$, ir išorinio krašto $\Gamma_0$. Kraštinė sąlyga $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ turi kompaktinę atramą ir $\Lambda = \operatorname{supp} \boldsymbol{\varphi} \cap \Gamma_0 \subset \Gamma_0 \cap B_{R_0}(0)$.

**Apibrėžimas.** Apibendrintuoju (S.1) uždavinio sprendiniu vadinsime solenoidinį, laiko atžvilgiu periodinį vektorinį lauką $\mathbf{v}$, kai $\nabla \mathbf{v}, \mathbf{v}_t \in L^2(0, 2\pi; L^2(\Omega))$, tenkinantį kraštinę sąlygą $\mathbf{v}|_{\partial\Omega} = \boldsymbol{\varphi}$, periodiškumo sąlygą $\mathbf{v}(x,0) = \mathbf{v}(x, 2\pi)$ ir integralinę tapatybę

$$\int_0^{2\pi} \int_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\ \mathrm{d}x\mathrm{d}t + \nu \int_0^{2\pi} \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\eta}\ \mathrm{d}x\mathrm{d}t = \int_0^{2\pi} \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\ \mathrm{d}x\mathrm{d}t$$

kiekvienai laiko atžvilgiu periodinei funkcijai $\boldsymbol{\eta} \in L^2(0, 2\pi; H(\Omega))$.

Kadangi vektorinis laukas $\mathbf{v}$ yra solenoidinis ($\operatorname{div} \mathbf{v} = 0$), tai reiškia, jog skystis yra nespūdus (įtekančio skysčio kiekis lygus ištekančio skysčio kiekiui), todėl

$$\int_{\sigma(R)} \mathbf{v} \cdot \mathbf{n}\ \mathrm{d}S = -\big(\mathcal{F}^{(\mathrm{inn})} + \mathcal{F}^{(\mathrm{out})}\big),$$

čia $\sigma(R) = (-g(R), g(R))$ yra išėjimo $D$ skerspjūvis (kertama tiese $x_2 = R$), $\mathcal{F}^{(\mathrm{inn})} = \int_{\Gamma_1} \boldsymbol{\varphi} \cdot \mathbf{n}\ \mathrm{d}S$ ir $\mathcal{F}^{(\mathrm{out})} = \int_{\Lambda} \boldsymbol{\varphi} \cdot \mathbf{n}\ \mathrm{d}S$ srautai per srities vidinį ir išorinį kraštus.

Laiko atžvilgiu periodinės Stokso sistemos (S.1) apibendrintojo sprendinio ieškome tokiu pavidalu:

$$\mathbf{v}(x,t) = \mathbf{A}(x) + \mathbf{u}(x,t),$$

čia $\mathbf{A}$ disertacijoje (žr. 3.2 poskyrį) sukonstruotas specialus kraštinės sąlygos pratęsimas. Šis pratęsimas leidžia suvesti nagrinėjamą nehomogeninį uždavinį į uždavinį su homogenine kraštine sąlyga, kurioje naujas nežinomasis yra laiko atžvilgiu periodinis vektorinis laukas $\mathbf{u}$:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \nu\Delta\mathbf{A} + \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{u} = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{u} = \mathbf{0}, & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), & x \in \Omega. \end{cases} \tag{S.4}$$

Rezultatas apie laiko atžvilgiu periodinės Stokso sistemos apibendrintojo sprendinio egzistavimą ir vienatį, srityje su begaliniu išėjimu, pateikiamas žemiau esančioje teoremoje.

**1 teorema.** *(Theorem 3.3.1) Tarkime, kad* $\Omega \subset \mathbb{R}^2$ *turi išėjimą į begalybę, kraštinė sąlyga* $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ *turi kompaktinę atramą ir* $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi; L^2_1(\Omega))$ *(svorinės erdvės* $L^2_1(\Omega)$ *apibrėžimą galima rasti disertacijos 3.3 poskyryje). Jeigu* $\int\limits_{R_0}^{+\infty} \frac{\mathrm{d}x_2}{g^3(x_2)} < +\infty$*, tuomet* (S.1) *sistema turi vienintelį apibendrintąjį sprendinį* $\mathbf{v} = \mathbf{A} + \mathbf{u}$*, kuris tenkina įvertį:*

$$\begin{aligned} &\|\mathbf{v}_t\|_{L^2(0,2\pi;L^2(\Omega))} + \|\nabla\mathbf{v}\|_{L^2(0,2\pi;L^2(\Omega))} \\ &\leqslant c\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1 + \int\limits_{R_0}^{+\infty} \frac{1}{g^3(x_2)}\, \mathrm{d}x_2\right)\right)^{1/2} + \|\mathbf{f}\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))}\right). \end{aligned}$$

## Navjė ir Stokso lygtys cilindrinėje struktūroje

Laiko atžvilgiu periodinė Navjė ir Stokso sistema (S.2) ir stacionarioji Navjė ir Stokso sistema (S.3) nagrinėjamos cilindrinėje struktūroje $B_\varepsilon$. Joje yra įrodomas apibendrintojo sprendinio egzistavimas ir vienatis bei sukonstruojamas sprendinio asimptotinis skleidinys, kuris leidžia kurti hibridinės dimensijos modelius.

Nagrinėdami laiko atžvilgiu periodinę Navjė ir Stokso sistemą (S.2) srityje $B_\varepsilon$ (žr. 3 pav.), tariame, jog kraštinė funkcija $\mathbf{g}$ yra periodinė laiko atžvilgiu, lygi nuliui ant krašto visur, išskyrus cilindrų galus $\gamma_\varepsilon^j = \partial B_\varepsilon \cap \partial \omega_\varepsilon^j$, $j = N_1+1,\dots,N$ (čia $j$ yra grafo viršūnių taškai (žr. 1 pav.)) bei tenkina šias sąlygas

$$\widetilde{F}^j(t) = \int\limits_{\gamma_\varepsilon^j} \mathbf{g}\cdot\mathbf{n}\ \mathrm{d}S \equiv \varepsilon^{n-1}F^j(t), \quad j = N_1+1,\dots,N \tag{S.5}$$

ir

$$\sum_{j=N_1+1}^{N} F^j(t) = 0 \quad \forall t \in [0,2\pi]. \tag{S.6}$$

Kraštinės funkcijos pratęsimą pažymėkime $\mathbf{g}$ (taip pat kaip ir kraštinę funkciją) ir tarkime, jog jis yra solenoidinis, laiko atžvilgiu periodinis bei tenkina šiuos asimptotinius įverčius

$$\begin{aligned} &\sup_{x\in B_\varepsilon} |\mathbf{g}(x,t)| \leqslant c, \qquad && \|\nabla \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-3}{2}} \quad \forall t\in[0,2\pi],\\ &\|\mathbf{g}_t\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-1}{2}}, && \|\nabla^2 \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-5}{2}} \quad \forall t\in[0,2\pi]. \end{aligned} \tag{S.7}$$

Disertacijoje įrodomas laiko atžvilgiu periodinės Navjė ir Stokso sistemos (S.2) apibendrintojo sprendinio egzistavimas dvimačiu bei trimačiu atvejais, pasinaudojant Stokso operatoriaus savybėmis (žr. 2.2 poskyrį). (S.2) sistemos apibendrintojo sprendinio egzistavimas ir vienatis įrodomi sprendžiant šį variacinį uždavinį: ieškome vektorinio lauko $\mathbf{v} = \mathbf{u}+\mathbf{g}$, kai $\operatorname{div}\mathbf{u} = 0$, $\mathbf{u} \in L^\infty_{\mathrm{per}}(0,2\pi; \mathring{W}^{1,2}(B_\varepsilon)\cap W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t \in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))$ bei tenkinama integralinė tapatybė

$$\begin{aligned} \int\limits_{B_\varepsilon} \left( \frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\eta} - ((\mathbf{u}+\mathbf{g})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{g}\right)\mathrm{d}x \\ = \int\limits_{B_\varepsilon} \mathbf{f}\cdot\boldsymbol{\eta}\ \mathrm{d}x, \end{aligned} \tag{S.8}$$

kiekvienam solenoidiniam vektoriniam laukui $\boldsymbol{\eta} \in H(B_\varepsilon)$. Čia $\mathbf{g}$ yra pratęsimas tenkinantis (S.7) sąlygas, o $\mathbf{f}$ laiko atžvilgiu periodinė funkcija, tokia, kad $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi; L^2(B_\varepsilon))$. Pažymėkime, kad

$$A_1(t) = \|\mathbf{f}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)}. \tag{S.9}$$

Laiko atžvilgiu periodinės Navjė ir Stokso sistemos (S.2) apibendrintojo sprendinio egzistavimas ir vienatis dvimačiu ir triamčiu atvejais suformuluoti žemiau esančiose teoremose.

**2 teorema.** *(Theorem 4.2.1) Tegul* $B_\varepsilon \subset \mathbb{R}^2$, $\partial B_\varepsilon \in C^2$. *Tarkime, jog pratęsimo funkcija* $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{2,2}(B_\varepsilon))$ *bei tenkina* (S.5), (S.6) *ir* (S.7) *sąlygas. Be to,* $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))$ *ir tenkina* (S.9) *sąlygą. Tuomet pakankamai mažam* $\varepsilon$ (S.8) *variacinis uždavinys turi vienintelį sprendinį* $\mathbf{u}$, *kuris tenkina įverčius*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla \mathbf{u}(x,t)|^2 \, \mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2+\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t,$$

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t(x,t)|^2 \, \mathrm{d}x\,\mathrm{d}t + \varepsilon^\beta \int_0^{2\pi}\int_{B_\varepsilon} |\nabla^2 \mathbf{u}(x,t)|^2 \, \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c\varepsilon^{\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t,$$

*čia konstanta* $c$ *nepriklauso nuo* $\varepsilon$, $\beta = 0,2$.

**3 teorema.** *(Theorem 4.2.3) Tegul* $B_\varepsilon \subset \mathbb{R}^3$, $\partial B_\varepsilon \in C^2$. *Tarkime, jog pratęsimo funkcija* $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0,2\pi;W^{2,2}(B_\varepsilon))$ *bei tenkina* (S.5), (S.6) *ir* (S.7) *sąlygas. Be to,* $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))$ *ir* $\|\mathbf{f}\|_{L^2(B_\varepsilon)} \leqslant c_0$, *čia konstanta* $c_0$ *yra pakankamai maža ir nepriklauso nuo* $\varepsilon$. *Tuomet pakankamai mažam* $\varepsilon$ (S.8) *variacinis uždavinys turi vienintelį sprendinį* $\mathbf{u}$, *kuris tenkina įvertį*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int_0^{2\pi} \|\nabla \mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} \mathrm{d}t \leqslant c\varepsilon^{2+\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t,$$

*Jeigu* $c_0$ *yra pakankamai maža konstanta (neprilkauso nuo* $\varepsilon$*) tuo atveju, kai* $\beta = 0$ *arba* $\beta = 2$, *taip pat galioja įvertis*

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int_0^{2\pi}\int_{B_\varepsilon} |\mathbf{u}_t(x,t)|^2 \, \mathrm{d}x\,\mathrm{d}t$$
$$+\varepsilon^\beta \int_0^{2\pi} \|\nabla^2 \mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} \mathrm{d}t \leqslant c\varepsilon^{\beta}\int_0^{2\pi} A_1(t)\,\mathrm{d}t.$$

Gautam apibendrintajam sprendiniui konstruojamas asimptotinis skleidinys, kuris leidžia kurti hibridinės dimensijos modelius. Šis skleidinys konstruojamas remiantis matematine indukcija. Pirmiausia sprendžiame laiko atžvilgiu periodinį uždavinį ant grafo ir randame makroskopinį slėgį, kuris yra tiesinė, laiko atžvilgiu periodinė funkcija ant kiekvienos grafo kraštinės bei priklauso tik nuo paskutinės lokalios koordinačių sistemos komponentės $x_n^{(e)}$ (plačiau apie skirtingas koordinačių sistemas galima paskaityti disertacijos įvade). Mazguose (taškuose, kuriuose susikerta kelios grafo kraštinės (žr. 1 pav.)), šis uždavinys tenkina Kirhofo tipo susikirtimo sąlygas. Tuo tarpu kiekviename cilindre $\Pi_\varepsilon^{(e)}$ (cilindro $\Pi_\varepsilon$ apibrėžimą taip pat galima rasti disertacijos įvade) turime Puazeilio tipo greitį, kuris priklauso tik nuo $x^{(e)\prime}$, t.y. nepriklauso nuo paskutinės erdvės komponentės. Tuomet dauginame Puazeilio tipo greitį ir slėgį iš nupjautinės funkcijos $\zeta$, kuri lygi vienam viduriniame cilindrų trečdalyje ir nykstamai mažėja mazgų $O(\varepsilon)$ aplinkoje. Tačiau ši sandauga Navjė ir Stokso lygties dešinėje pusėje generuoja paklaidas, kurių atrama priklauso tai pačiai $O(\varepsilon)$ aplinkai. Šių paklaidų kompensavimui įvedamas pasienio sluoksnių korektorius. Korektoriai yra Stokso sistemos sprendiniai, kuri sprendžiama struktūroje sudarytoje iš cilindrų, turinčių begalinius išėjimus.

Taip konstruojami asimptotiniai skleidiniai greičio vektoriui ir slėgiui turi tokias išraiškas:

$$\begin{aligned}\mathbf{v}^{(J)}(x,t) = &\sum_{i=1}^{M}\zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i|-x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J}\varepsilon^j\mathbf{V}_j^{(e_i)}(y^{(e_i)\prime},t)\\ &+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\sum_{j=-1}^{J}\varepsilon^j\mathbf{V}_j^{[BLO_l]}(y,t),\end{aligned}$$

$$\begin{aligned}p^{(J)}(x,t) = &\sum_{i=1}^{M}\zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i|-x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J}\varepsilon^{j-2}\big(-s_j^{(e_i)}(t)x_n^{(e_i)}+a_j^{(e_i)}(t)\big)\\ &+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\sum_{j=-1}^{J}\varepsilon^{j-1}P_j^{[BLO_l]}(y,t),\end{aligned}$$

čia $y=\frac{x^{(e)}}{\varepsilon}$, $0\leqslant\zeta(\tau)\leqslant 1$ yra glodi nupjautinė funkcija, be to, $\zeta(\tau)=0$, kai $\tau\leqslant 1/3$ ir $\zeta(\tau)=1$, kai $\tau\geqslant 1/3$. Pora $(\mathbf{V}_j^{[BLO_l]}(y,t),P_j^{[BLO_l]}(y,t))$ aprašo greičio vektorių ir slėgį pasienio sluoksniuose. Jie randami sprendžiant laiko atžvilgiu periodinę Stokso sistemą pusiau begaliniuose cilindruose. Narys $\mathbf{V}_j^{(e_i)}(y^{(e)\prime},t)$ aprašo Puazeilio tipo greitį (vamzdelyje įgyja parabolinę formą), o narys $p_j^{(e_i)}=-s_j^{(e_i)}(t)x_n^{(e_i)}+a_j^{(e_i)}(t)$ - makroskopinį slėgį. Šie nariai randami sprendžiant laiko atžvilgiu periodinę sistemą ant grafo. Kadangi asimptotinė išraiška konstruojama remiantis matematine indukcija pagal $j$, tai pradiniame žingsnyje, kai $j=0$, ieškome funkcijos $p_0 \in L^2_{\text{per}}(0,2\pi;W^{1,2}(\mathcal{B}))$, kuri tenkina sistemą

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big)=0, \quad x_n^{(e)} \in (0,|e|),\ \forall e=e_j,\ j=1,\dots,M, \\ -\sum\limits_{e:O_l\in e}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=0, \qquad l=1,\dots,N_1, \\ \qquad -\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t)=\Psi_l(t), \quad l=N_1+1,\dots,N, \end{cases}$$

čia $\Psi_l(t)=\int\limits_{\gamma^l}\mathbf{g}_l\cdot\mathbf{n}\,\mathrm{d}S$. Operatorius $L^{(e)}$ susieja slėgio krypties koeficientą $\mathcal{S}$ ir srautą $\mathcal{H}$ begaliniame cilindre, kurio skerspjūvis yra $\sigma^{(e)}$. Operatoriaus $L^{(e)}$ ieškome spendžiant laiko atžvilgiu periodinę šilumos laidumo lygtį: duotam $\mathcal{S}\in L^2_{\text{per}}(0,2\pi)$ ieškome $\mathcal{V}\in L^2_{\text{per}}(0,2\pi;\mathring{W}^{1,2}(\sigma^{(e)}))$, kai $\dfrac{\partial\mathcal{V}}{\partial t}\in L^2_{\text{per}}(0,2\pi;L^2(\sigma^{(e)}))$ ir

$$\begin{cases} \dfrac{\partial\mathcal{V}}{\partial t}(y^{(e)\prime},t)-\nu\Delta'_{y^{(e)\prime}}\mathcal{V}(y^{(e)\prime},t)=S(t), \quad y^{(e)\prime}\in\sigma^{(e)},\ t>0, \\ \qquad \mathcal{V}(y^{(e)\prime},t)|_{\partial\sigma^{(e)}}=0, \quad \mathcal{V}(y^{(e)\prime},t)=\mathcal{V}(y^{(e)\prime},t+2\pi). \end{cases}$$

Tuomet, tiesinio aprėžtojo operatoriaus išraiška yra tokia

$$L^{(e)}\mathcal{S}(t)=\int\limits_{\sigma^{(e)}}\mathcal{V}(y^{(e)\prime},t)\,\mathrm{d}y^{(e)\prime}=\mathcal{H}(t).$$

Plačiau apie asimptotinio skleidinio konstravimo žingsnius galima paskaityti disertacijos 4 skyriuje.

Sukonstravę asimptotinius skleidinius, konstruojame laiko atžvilgiu periodinės Navjė ir Stokso sistemos (S.2) apibendrintąjį sprendinį $(\mathbf{v},p)$ tokiu pavidalu: $\mathbf{v}=\mathbf{u}+\mathbf{u}^{(J)}=\mathbf{u}+\mathbf{v}^{(J)}+\mathbf{w}^{(J)}$, $p=q+p^{(J)}$, čia $\mathbf{v}^{(J)}$ sprendinio asimptotinis skleidinys, $\mathbf{w}^{(J)}\in L^2_{\text{per}}(0,2\pi;W^{2,2}(B_\varepsilon)\cap\mathring{W}^{1,2}(B_\varepsilon))$ vektorinis laukas, toks kad $\operatorname{div}\mathbf{w}^{(J)}=-h^{(j)}=-\operatorname{div}\mathbf{v}^{(J)}$, o $p^{(J)}$ slėgio funkcijos $p$ skleidinys. Tuomet $\mathbf{u}^{(J)}\in L^2_{\text{per}}(0,2\pi;W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t^{(J)}\in L^2_{\text{per}}(0,2\pi;L^2(B_\varepsilon))$.

**Apibrėžimas.** Solenoidinis laiko atžvilgiu periodinis vektorinis laukas $\mathbf{u}=\mathbf{v}-\mathbf{u}^{(J)}$ vadinamas apibendrintuoju sprendiniu, kai $\mathbf{u}\in L^2_{\text{per}}(0,2\pi;W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t\in L^2_{\text{per}}(0,2\pi;L^2(B_\varepsilon))$, tenkina kraštinę sąlygą $\mathbf{u}(x,t)|_{\partial B_\varepsilon}=0$ ir integralinę tapatybę

$$\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\eta}-((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x$$

$$=\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x,\quad \forall\boldsymbol{\eta}\in H(B_\varepsilon).$$

Apibendrintojo sprendinio $\mathbf{u}$ egzistavimas ir vienatis išplaukia iš 2 teoremos. Kai $n=2$ apibendrintasis sprendinys $\mathbf{u}$ tenkina įverčius, kuriuos suformuluosime teoremos pavidalu.

**4 teorema.** *Tegul* $n=2$, $\beta=0,2$. *Tuomet galioja šie įverčiai*

$$\sup_{t\in[0,2\pi]}\|\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)}+\varepsilon^\beta\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\varepsilon^{2J-2+\beta},$$

$$\sup_{t\in[0,2\pi]}\|\nabla\mathbf{u}(\,\cdot\,,t)\|^2_{L^2(B_\varepsilon)}+\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^\beta\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|\nabla^2\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant c\varepsilon^{2J-4+\beta}.$$

*Be to, egzistuoja tokia slėgio funkcija* $q\in L^2_{\mathrm{per}}(0,2\pi;L^2(B_\varepsilon))$, *kad* $\int\limits_{B_\varepsilon}q(x,t)\mathrm{d}x=0$ *ir*

$$\int\limits_{B_\varepsilon}\Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta}+\nu\nabla\mathbf{u}:\nabla\boldsymbol{\eta}-((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x$$

$$=\int\limits_{B_\varepsilon}q\,\mathrm{div}\,\boldsymbol{\eta}\,\mathrm{d}x+\int\limits_{B_\varepsilon}\mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x,\quad \forall\boldsymbol{\eta}\in\mathring{W}^{1,2}(B_\varepsilon).$$

*Tuomet, jei* $J\geqslant 2$, *tai galioja įvertis*

$$\int\limits_0^{2\pi}\int\limits_{B_\varepsilon}|q|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c\varepsilon^{2J-4-\beta}.$$

Stacionarioji Navjė ir Stokso sistema (S.3) cilindrinėje struktūroje $B_\varepsilon$ (žr. 4 pav.) nagrinėjama ją perrašant tokiu pavidalu:

$$\begin{cases}-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{v}\cdot(\nabla\mathbf{v})^t+\nabla\Phi=\mathbf{f}, & x\in B_\varepsilon,\\ \mathrm{div}\,\mathbf{v}=0, & x\in B_\varepsilon,\\ \mathbf{v}=0, & x\in\partial B_\varepsilon\backslash\cup_{j=N_1+1}^N\gamma_\varepsilon^j,\\ \mathbf{v}_{\boldsymbol{\tau}}=0, & x\in\gamma_\varepsilon^j,\\ \Phi=p_j, & x\in\gamma_\varepsilon^j,\ j=N_1+1,\dots,N,\end{cases}\tag{S.10}$$

čia $\Phi = (p + \frac{1}{2}|\mathbf{v}|^2)$ yra Bernulio slėgis, o $p_j$ atitinka konstantą $c_j/\varepsilon^2$.

**Apibrėžimas.** Stacionariosios Navjė ir Stokso sistemos (S.10) apibendrintuoju sprendiniu vadiname vektorinį lauką $\mathbf{v} \in \widehat{J}^{1,2}_{\gamma}(B_\varepsilon)$ kuris tenkina integralinę tapatybę

$$\nu \int\limits_{B_\varepsilon} \nabla \mathbf{v} : \nabla \boldsymbol{\eta} \, \mathrm{d}x + \int\limits_{B_\varepsilon} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\eta} \, \mathrm{d}x - \int\limits_{B_\varepsilon} (\boldsymbol{\eta} \cdot \nabla)\mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x$$
$$= - \sum_{j=N_1+1}^{N-1} p_j^* \int\limits_{\gamma_\varepsilon^j} \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}x' + \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x, \quad \forall \boldsymbol{\eta} \in \widehat{J}^{1,2}_{\gamma}(B_\varepsilon).$$

Teorema apie apibendrintojo sprendinio egzistavimą pateikiama žemiau.

**5 teorema.** *Tarkime* $\mathbf{f} \in L^2(B_\varepsilon)$ *ir* $p_j^* \in \mathbb{R}$, $j = N_1 + 1, \ldots, N-1$. *Tuomet* (S.10) *sistema turi bent vieną apibendrintąjį sprendinį* $\mathbf{v} \in \widehat{J}^{1,2}_{\gamma}(B_\varepsilon)$. *Be to, galioja įvertis*

$$\|\nabla \mathbf{v}\|_{L^2(B_\varepsilon)} \leqslant c\Big(\varepsilon^{n/2} \sum_{j=N_1+1}^{N-1} |p_j^*| + \varepsilon \|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big), \tag{S.11}$$

*čia konstanta* $c$ *nepriklauso nuo* $\varepsilon$.

Asimptotinis skleidinys stacionariu atveju konstruojamas panašiai, kaip ir prieš tai. Greičio vektoriaus asimptotinis skleidinys konstruojamas tokiu pavidalu:

$$\mathbf{v}^{(J)}(x) = \sum_{O_l,\ l=N_1+1,\ldots,N;\ e=\overline{O_l O_{i_l}}} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big) \mathbf{V}^{[e,J]}\Big(\frac{x^{(e)\prime}}{\varepsilon}\Big)$$
$$+ \sum_{e=\overline{O_\alpha O_\beta};\ \alpha,\beta \leqslant N_1} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big) \zeta\Big(\frac{|e| - x_n^{(e)}}{3r\varepsilon}\Big) \mathbf{V}^{[e,J]}\Big(\frac{x^{(e)\prime}}{\varepsilon}\Big)$$
$$+ \sum_{l=1}^{N} \Big(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\Big) \mathbf{V}^{[BLO_l,J]}\Big(\frac{x - O_l}{\varepsilon}\Big).$$

Čia pirma suma apima viršūnes, antroji suma mazgus, o trečioji suma kompensuoja paklaidas pasienio sluoksniuose.

Slėgio funkcijos asimptotinis skleidinys viršūnėse ir mazguose įgyja atitinkamas išraiškas:

$$p^{(J)}(x) = -s^{(e)} x_n^{(e)} + a^{(e)} + \frac{1}{\varepsilon} \bigg(1 - \zeta\Big(\frac{|x - O_l|}{|e|_{\min}}\Big)\bigg) P^{[BLO_l,J]}\Big(\frac{x - O_l}{\varepsilon}\Big),$$

ir

$$p^{(J)}(x) = \sum_{e\subset\mathcal{B}_l} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\big(-s^{(e)}x_n^{(e)} + a^{(e)} - a^{(e_s)}\big) + a^{(e_s)}$$
$$+\frac{1}{\varepsilon}\left(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\right)P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big).$$

Panašiai kaip laiko atžvilgiu periodinės Navjė ir Stokso sistemos atveju pasienio sluoksnių asimptotikos ($\mathbf{V}^{[BLO_l,J]}, P^{[BLO_l,J]}$) randamos sprendžiant stacionariąją Stokso sistemą. Pradiniu atveju (kai $j=0$) sistema atrodo taip:

$$\begin{cases} -\nu\Delta_y\mathbf{V}_0^{[BLO_l]} + \nabla_y P_0^{[BLO_l]} = \mathbf{f}_0^{[REGO_l]} + \mathbf{f}_0^{[BLO_l]}, & y\in\Omega_l,\\ \operatorname{div}_y\mathbf{V}_0^{[BLO_l]} = h_0^{[REGO_l]}, & y\in\Omega_l,\\ \mathbf{V}_0^{[BLO_l]} = 0, & y\in\partial\Omega_l. \end{cases}$$

Nariai $\mathbf{V}^{[e,J]}$ ir $p_j = (-s^{(e)}x_n^{(e)} + a^{(e)} - a^{(e_s)}) + a^{(e_s)}$ randami sprendžiant uždavinį ant grafo. Jų radimo algoritmą galima rasti disertacijos 5.3 poskyryje. Pradiniame žingsnyje funkcija $p_0$ randama iš uždavinio:

$$\begin{cases} -\kappa_e\dfrac{\partial^2 p_0^{(e)}}{\partial x_n^{(e)2}}(x_n^{(e)}) = 0, & x_n^{(e)}\in(0,|e|),\\ -\displaystyle\sum_{e:O_l\in e}\kappa_e\frac{\partial p_0^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l=1,\dots,N_1,\\ p_0^{(e)}(0) = c_l, & l=N_1+1,\dots,N,\\ p_0^{(e)}(0) = p_0^{(e_s)}(0), \quad \forall e\subset\mathcal{B}_l, \end{cases}$$

čia $\kappa_e$ randama sprendžiant Dirichlė uždavinį (žiūrėti disertacijos 5.3 poskyrį), o $c_l$ žinomos konstantos.

Disertacijoje (žr. 5 skyrių) pateikiamas sprendimo algoritmas bei įrodomas paklaidos įvertis. Tegul $\mathbf{v} = \mathbf{u} + \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$, čia $\mathbf{v}^{(J)}$ yra greičio vektoriaus asimptotinis skleidinys, o $\mathbf{w}^{(J)}\in\mathring{W}^{1,2}(B_\varepsilon)$ vektorinis laukas, toks kad $\operatorname{div}\mathbf{w}^{(J)} = -h^{(j)} = -\operatorname{div}\mathbf{v}^{(J)}$.

**6 teorema.** *Sukonstruotas pratęsimas tenkina paklaidos įvertį*

$$\|\mathbf{v}-(\mathbf{v}^{(J)}+\mathbf{w}^{(J)})\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}).$$

# Išvados

Disertacijoje buvo išnagrinėti šie uždaviniai: laiko atžvilgiu periodinė Stokso sistema su nehomogenine kraštine sąlyga (priklausančia tik nuo erdvės kintamojo) [2], laiko atžvilgiu periodinė Navjė ir Stokso sistema bei stacionarioji Navjė ir Stokso sistema su duotu Bernulio slėgiu. Rezultatai gauti nagrinėjant Navjė ir Stokso sistemą cilindrinėje struktūroje, gali būti panaudojami kuriant supaprastintą hibridinės dimensijos modelį, aprašantį kraujo tekėjimą smulkiose ir labai smulkiose kraujagyslėse. Stacionarioji Navjė ir Stokso sistema su duotu Bernulio slėgiu, gali būti toliau nagrinėjama sprendžiant laiko atžvilgiu periodinį uždavinį ir kuriant realistiškesnį kraujotakos modelį.

2. Laiko atžvilgiu periodinė Stokso sistema srityje su išėjimu į begalybę, kai kraštinė sąlyga priklauso ne tik nuo erdvės kintamųjų, bet ir nuo laiko, buvo apibendrinta K. Kaulakytės ir K. Pilecko straipsnyje [33].

# Résumé

## Objet de la recherche

Dans la thèse nous examinons les problèmes suivants : l'équation de Stokes $2\pi$-périodique en temps

$$\begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v} = \boldsymbol{\varphi}, & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega, \end{cases} \tag{R.12}$$

l'équation de Navier–Stokes $2\pi$-périodique en temps

$$\begin{cases} \frac{1}{\varepsilon^\beta}\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in B_\varepsilon \times (0,2\pi), \\ \operatorname{div}\mathbf{v} = 0, & (x,t) \in B_\varepsilon \times (0,2\pi), \\ \mathbf{v} = \boldsymbol{\varphi}, & (x,t) \in \partial B_\varepsilon \times (0,2\pi), \\ \mathbf{v}(x,t) = \mathbf{v}(x,t+2\pi), \; x \in B_\varepsilon. \end{cases} \tag{R.13}$$

et l'équation de Navier–Stokes stationnaire avec une pression donnée de Bernoulli

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, & x \in B_\varepsilon, \\ \operatorname{div}\mathbf{v} = 0, & x \in B_\varepsilon, \\ \mathbf{v} = 0, & x \in \partial B_\varepsilon \setminus \cup_{j=N_1+1}^{N} \gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}} = 0, & x \in \gamma_\varepsilon^j, \\ -\nu\partial_n(\mathbf{v}\cdot\mathbf{n}) + \left(p + \frac{1}{2}|\mathbf{v}|^2\right) = c_j/\varepsilon^2, & x \in \gamma_\varepsilon^j, j = N_1+1,\dots,N, \end{cases} \tag{R.14}$$

où $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t),\dots,v_n(x,t))$ est la vitesse de la fluide, $p = p(x,t)$ est la pression, $\mathbf{f} = \mathbf{f}(x,t) = (f_1(x,t),\dots,f_n(x,t))$ est la force extérieure et $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x,t) = (\varphi_1(x,t),\dots,\varphi_n(x,t))$ est la vitesse donnée sur la frontière du domaine, $\nu > 0$ représente la viscosité de la fluide. La condition $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ donne la périodicité temporelle (avec une période $2\pi$) et les ouverts sur lesquels sont posés ces trois problèmes sont représentés aux figures 6, 7 et 8.

## Histoire et pertinence du problème scientifique

Les équations de Stokes et de Navier–Stokes décrivant l'écoulement d'un fluide visqueux incompressible sont prises en compte dans la résolution de divers problèmes hydrodynamiques. Ces équations sont intéressantes à la fois du point de vu théorique et pratique, c'est pourquoi elles sont considérées dans de nombreux domaines de la science et de la technologie. Ces études deviennent particulièrement pertinentes dans le monde moderne. En pratique, les praticiens de divers domaines sont souvent confrontés aux problèmes d'écoulement de fluide décrits par le système des équations de Navier–Stokes. Le large éventail des applications de ces équations permet de développer des recherches interdisciplinaires et de développer la théorie des équations de Navier–Stokes tant sur le plan théorique que pratique.

Une application importante de la modélisation de l'écoulement du fluide visqueux incompressible est la médecine. L'un des objectifs de cette thèse est d'obtenir des résultats médicalement applicables. L'étude a été réalisée en 2017-2021 en tant que chercheur junior dans le projet “Multiscale Modeling for Viscous Flows in Domains with Complex Geometry”.[3] La thèse traite les équations qui permettent le développement des modèles simplifiés du système circulatoire sans perte de précision. Les résultats obtenus peuvent être appliqués à la modélisation du flux sanguin dans les petits vaisseaux sanguins tels que les artérioles et les capillaires. De plus, ces résultats peuvent être développés davantage en développant des modèles plus sophistiqués qui décrivent, par exemple, le flux sanguin dans le cœur ou d'autres organes internes.

Les équations de Stokes et de Navier–Stokes est un sujet difficile et important de mathématique. Des progrès significatifs ont été réalisés ces

3. This project has received funding from European Social Fund (project No. 09.3.3-LMT-K-712-01-0012) under grant agreement with the Research Council of Lithuania (LMTLT).

dernières années dans la résolution du problème de Leray [45], mais malgré un intérêt et des efforts considérables, de nombreuses questions liées à ces équations restent ouvertes.

Dans la thèse, l'analyse du système non-linéaire de Navier–Stokes commence par sa version linéaire, i.e. par l'examen du système de Stokes. Considérons d'abord le système de Stokes périodique en temps, dans le domaine avec la sortie vers l'infini [29]. Cependant, le système de Stokes n'est pas suffisant pour obtenir des résultats décrivant les systèmes circulatoires, et le domaine à la sortie infinie ne décrit pas la structure du réseau vasculaire. Pour ces raisons, le système de Navier–Stokes dans une structure tubulaire constituée d'une union de tubes minces est examiné plus en détail dans la thèse.

L'un des objectifs de la thèse est d'obtenir des résultats pouvant être appliqués à la modélisation des systèmes circulatoires. Ces systèmes sont affectés par le rythme cardiaque, qui peut-être décrit mathématiquement comme une fonction périodique en temps, et donc des problèmes périodiques sont pris en compte. Pour décrire le flux sanguin dans différents vaisseaux sanguins, nous introduisons le paramètre $\varepsilon$ et considérons le système de Navier–Stokes périodique en temps dans une structure tubulaire. Cette structure est une réunion des tubes cylindriques fins avec le rapport entre le diamètre de la section et la hauteur du cylindre égal à $\varepsilon$. Ce paramètre représente le rapport entre le diamètre d'un vaisseau et sa longueur. À chaque structure tubulaire correspond un graphe unidimensionnel, qui est obtenu en approchant le diamètre des tubes à zéro. Un exemple de graphe avec trois nœuds $O_1, O_2, O_3$, cinq sommets $O_4, O_5, O_6, O_7, O_3$ et sept arêtes $e_1, \ldots, e_7$ est représenté à la figure 5. La structure unidimensionnelle est simple, mais elle présente des inconvénients : les modèles et codes unidimensionnels existants actuellement ne peuvent pas fournir la précision requise dans les zones où des thrombus se forment, des stents sont présents ou des branches de vaisseaux sanguins. Le problème de précision peut-être résolu en appliquant un modèle tridimensionnel à l'ensemble du système circulatoire, mais nous obtiendrions une tâche qui nécessiterait beaucoup de ressources informatiques. Par conséquent, le modèle tridimensionnel ne peut-être appliqué qu'à certaines parties du système circulatoire. Pour ces raisons, la thèse

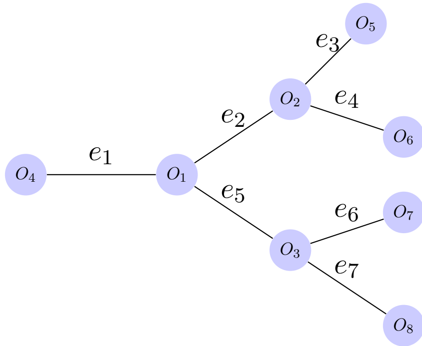



FIGURE 5 – Le graphe d'une structure tubulaire

propose un modèle hybride qui combine des modèles unidimensionnels et tridimensionnels, où le modèle tridimensionnel est appliqué uniquement à certains zones de la structure où le modèle unidimensionnel est inexact (par exemple, dans la ramification vasculaire). Cette méthode a été proposée pour la première fois par G. Panasenko en 1998 en résolvant l'équation stationnaire de Navier–Stokes [65, 66]. Plus tard en 2015, cette méthode a été développée par G. Panasenko et K. Pileckas pour résoudre le système non-stationnaire de Navier–Stokes dans une structure tubulaire [70, 71]. Dans cette thèse, cette méthode est appliquée pour résoudre un problème périodique en temps. Afin de combiner des modèles unidimensionnels et tridimensionnels, nous construisons un développement asymptotique de la solution du problème. La méthode qui permet le développement d'un modèle de dimension hybride est appelée la méthode de décomposition partielle asymptotique du domaine (method of asymptotic partial decomposition of the domain (MAPDD)). Une version généralisée de cette méthode avec une simulation numérique a été discutée dans [5]. Des études numériques ont montré que MAPDD est plus adapté à la modélisation des petits vaisseaux sanguins tels que les artérioles par rapport au modèle développé par l'équipe A. Quarteroni, qui combine également des modèles unidimensionnels et tridimensionnels [15]. Le modèle de l'équipe A. Quarteroni, quant à lui, décrit plus précisément les gros vaisseaux sanguins tels que les artères.

Pour décrire différents types de petits vaisseaux sanguins, nous utilisons deux échelles différentes dans la thèse. Pour ce faire, nous ajoutons un coefficient $\varepsilon^{-\beta}$ devant le terme $\mathbf{v}_t$. Nous considérons deux cas différents : $\varepsilon^0$

et $\varepsilon^{-2}$. Le paramètre $\varepsilon^{-2}$ génère un grand coefficient à la dérivée par rapport au temps. Ces cas décrivent des vaisseaux sanguins petits et très petits tels que les artérioles et les capillaires.

Notre objectif est d'obtenir des résultats applicables à la modélisation des systèmes tubulaires. Les modèles de dimension hybride doivent être utilisés dans la simulation informatique pour réduire les coûts de calcul et obtenir la précision requise. Une étape clé dans le développement de tels modèles est la construction d'un développement asymptotique de la solution. Le terme principal du développement asymptotique dépend de la solution du problème sur le graphe. Un tel problème, sur lequel repose le développement asymptotique dans les structures tubulaires, a été proposé pour la première fois dans l'article de G. Panasenko et K. Pileckas [68, 72]. Dans cette thèse, le développement asymptotique est construit sur la base des résultats obtenus dans l'article de G. Panasenko et K. Pileckas [68, 72] et l'article de E. Marušić-Paloka [55], en les adaptant au problème périodique en temps.

Le système de Navier-Stokes périodique en temps a été étudié avec une condition aux limites de type Dirichlet, mais il est préférable d'utiliser à la sortie de la structure des conditions de type Neumann pour résoudre les problèmes hémodynamiques. Elles sont plus proches tolérées par les codes numériques, mais ces défis ne sont pas largement explorés. Pour cette raison, la thèse traite le système stationnaire de Navier–Stokes avec des conditions de type Neumann, i.e. avec une pression de Bernoulli donnée aux bords par lesquels le liquide peut entrer ou sortir. Dans ce cas, un développement asymptotique de la solution faible est construit lorsque nous avons une condition aux limites non-linéaire.

Dans tous les cas, c'est-à-dire à la fois dans l'étude du système de Stokes périodique en temps dans le domaine de l'infini et dans l'analyse des systèmes de Navier–Stokes périodiques en temps et stationnaires dans la structure tubulaire, l'existence et l'unicité des solutions faibles sont prouvées. De plus, dans le cas des équations de Navier–Stokes, un développement asymptotique de la solution faible est construit, ce qui permet le développement de modèles de dimension hybride qui réduisent le coût de calcul du modèle.

## Objectifs de la thèse

La thèse porte sur les systèmes de Stokes et de Navier–Stokes périodiques en temps, ainsi que sur le système stationnaire de Navier–Stokes dans différents domaines. Il commence par une étude du système de Stokes $2\pi$-périodique en temps avec une sortie à l'infini (voir Figure 6). Ensuite, les systèmes de Navier–Stokes périodiques en temps et stationnaires dans une structure tubulaire sont analysés (voir Figure 8). Notre objectif est d'examiner les résultats de ces tâches pour développer des modèles simplifiés du système circulatoire qui décrivent le flux sanguin dans les petits vaisseaux sanguins. Les résultats principaux de l'étude sont :

- l'existence et l'unicité de la solution faible du problème de Stokes périodique en temps sont prouvées en construisant une extension spéciale de la condition aux limites lorsque le domaine a une sortie à l'infini,
- l'existence d'une solution d'un système de Navier–Stokes périodique en temps ou stationnaire dans une structure tubulaire de petit diamètre de cylindre est démontrée,
- un développement asymptotique de la solution faible est construit, ce qui permet le développement de modèles de dimension hybride. Il est basé à la fois sur le système de Navier–Stokes, qui est périodique en temps, et sur le système de Navier–Stokes stationnaire avec une pression de Bernoulli donnée à l'entrée et à la sortie,
- la méthode de décomposition partielle asymptotique du domaine (method of asymptotic partial decomposition of the domain (MAPDD)) pour le problème de Navier–Stokes périodique en temps est développée.

## Structure et portée de la thèse

Le texte de la thèse se compose de cinq chapitres, conclusions et références. Le premier chapitre est introductif. Il passe en revue l'historique et la pertinence du problème en question et les domaines dans lesquels les systèmes sont abordés. Le deuxième chapitre présente les principales notations et les résultats à l'appui utilisés dans la thèse. La troisième section présente les résultats obtenus en analysant le système de Stokes périodique en temps avec une condition aux limites in-homogène dans un domaine avec une sortie à l'infini. Dans le quatrième et le cinquième chapitres, le système de Navier–Stokes périodique en temps avec conditions aux limites in-homogènes et le système de Navier–Stokes stationnaire avec une pression de Bernoulli donnée dans une structure tubulaire sont examinés. À la fin de la thèse, les conclusions et la liste la bibliographie utilisée sont présentées.

## Examen des résultats obtenus dans la thèse

Les systèmes de Stokes et de Navier–Stokes dans différents domaines sont examinés. Le système linéaire de Stokes (R.12) est considéré dans le domaine bidimensionnel $\Omega$, qui a une sortie à l'infini. Le bord d'une zone est composé de différents composants qui constituent les bords intérieur et extérieur (voir Figure 6).

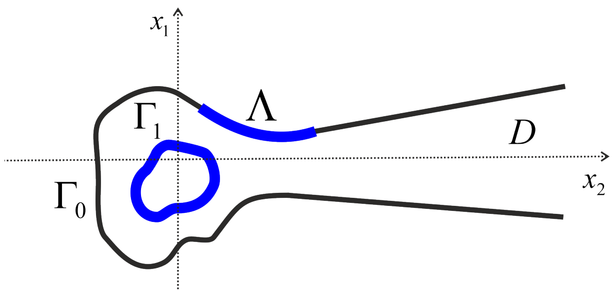



FIGURE 6 – Un domaine $\Omega$

Les systèmes non-linéaires de Navier–Stokes (R.13), (R.14) sont considérés dans le domaine $B_\varepsilon$ constituée de plusieurs tubes (cylindres) séparés. Un tel domaine est appelé une structure tubulaire. Un exemple de structure à trois tuyaux est illustré dans les figures 8,7.

La thèse prouve l'existence et l'unité des solutions faibles de ces systèmes. Dans le cas des systèmes de Navier–Stokes, des développement asymptotiques sont également construits. Ils permettent le développements des modèles de dimension hybride.

Les principaux résultats obtenus pour relever ces défis seront présentés ci-dessous. Les définitions de tous les espaces fonctionnels qui y sont utilisés (sauf indication contraire) se trouvent au chapitre 2 de la thèse.

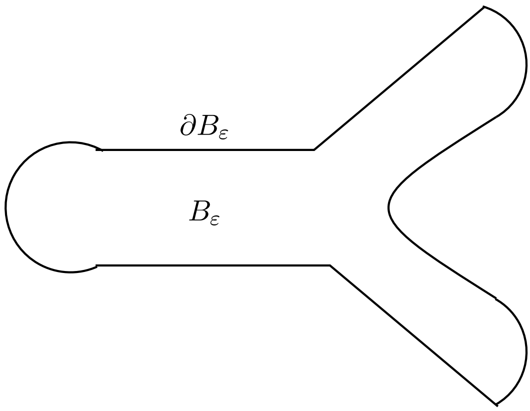



FIGURE 7 – Un domaine $B_\varepsilon$

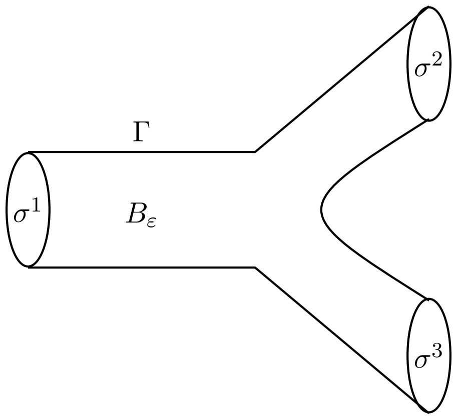



FIGURE 8 – Un domaine $B_\varepsilon$

## Système de Stokes périodique en temps dans le domaine avec une sortie à l'infini

Le système de Stokes périodique en temps (R.12) est considéré dans le domaine à deux dimensions $\Omega = \Omega_0 \cup D$ oú $\Omega_0 = \Omega \cap B_{R_0}(0) = \Omega \cap \{x \in \mathrm{R}^2 : |x| \leqslant R_0\}$ et avec une sortie à l'infini $D = \{x \in \mathbb{R}^2 : |x_1| < g(x_2),\ x_2 > R_0\}$ (voir Figure 6). La fonction $g$ satisfait la condition de Lipschitz

$$|g(t_1) - g(t_2)| \leqslant L|t_1 - t_2|,\ t_1, t_2 > R_0,\ g(t) \geqslant \text{const} > 0.$$

Le bord de la région $\partial\Omega \in C^2$ se compose du bord intérieur $\Gamma_1$ et du bord extérieur $\Gamma_0$. La vitesse donnée dans la condition aux limites $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ a un support compact et $\Lambda = \operatorname{supp} \boldsymbol{\varphi} \cap \Gamma_0 \subset \Gamma_0 \cap B_{R_0}(0)$.

**Définition.** La solution faible du problème (R.12) est le champ vectoriel solénoïdal $\mathbf{v}$ tel que $\nabla\mathbf{v}, \mathbf{v}_t \in L^2(0,2\pi;L^2(\Omega))$ satisfaisant la condition aux limites $\mathbf{v}|_{\partial\Omega} = \boldsymbol{\varphi}$, la condition de périodicité $\mathbf{v}(x,0) = \mathbf{v}(x,2\pi)$ et l'identité intégrale

$$\int_0^{2\pi}\int_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t + \nu \int_0^{2\pi}\int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t = \int_0^{2\pi}\int_\Omega \mathbf{f}\cdot\boldsymbol{\eta}\, \mathrm{d}x\mathrm{d}t$$

pour chaque fonction $2\pi$-périodique en temps $\boldsymbol{\eta} \in L^2(0,2\pi;H(\Omega))$.

Puisque le champ vectoriel $\mathbf{v}$ est solénoïdal ($\operatorname{div}\mathbf{v} = 0$), cela signifie que le liquide est incompressible (la quantité d'afflux est égale à la quantité d'effluent), donc

$$\int_{\sigma(R)} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S = -\big(\mathcal{F}^{(\mathrm{inn})} + \mathcal{F}^{(\mathrm{out})}\big),$$

où $\sigma(R) = (-g(R), g(R))$ est la section transversale de la sortie $D$ (par ligne $x_2 = R$), $\mathcal{F}^{(\mathrm{inn})} = \int_{\Gamma_1} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S$ et $\mathcal{F}^{(\mathrm{out})} = \int_{\Lambda} \boldsymbol{\varphi}\cdot\mathbf{n}\, \mathrm{d}S$ traverse les bords intérieur et extérieur du domaine.

On cherche une solution faible du système de Stokes $2\pi$-périodique en temps (R.12) sous la forme suivante :

$$\mathbf{v}(x,t) = \mathbf{A}(x) + \mathbf{u}(x,t).$$

Une extension spéciale $\mathbf{A}$ de la condition aux limites est construite dans la thèse (voir la section 3.2). Cette extension permet de combiner le problème inhomogène en question en un problème avec une condition aux limites homogène dans lequel la nouvelle inconnue est un champ vectoriel périodique $\mathbf{u}$, vérifiant

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \nu\Delta\mathbf{A} + \mathbf{f}, & (x,t) \in \Omega\times(0,2\pi), \\ \operatorname{div}\mathbf{u} = 0, & (x,t) \in \Omega\times(0,2\pi), \\ \mathbf{u} = \mathbf{0}, & (x,t) \in \partial\Omega\times(0,2\pi), \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), & x \in \Omega. \end{cases} \tag{R.15}$$

Le résultat d'existence et d'unicité de la solution faible du système de Stokes dans un domaine avec une sortie à l'infinie est donné dans le théorème ci-dessous.

**Théorème 1.** *(Theorem 3.3.1) Supposons que $\Omega \subset \mathbb{R}^2$ ait une sortie à l'infini, que la condition aux limites $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ ait un support compact et que $\mathbf{f} \in L^2_{\mathrm{per}}(0,2\pi; L^2_1(\Omega))$ (l'espace pondéré $L^2_1(\Omega)$ est dénifi à la section 3.3 de la thèse). Si $\int\limits_{R_0}^{+\infty} \frac{\mathrm{d}x_2}{g^3(x_2)} < +\infty$, alors le système* (R.12) *admet une unique solution faible $\mathbf{v} = \mathbf{A} + \mathbf{u}$, qui satisfait l'estimation :*

$$\|\mathbf{v}_t\|_{L^2(0,2\pi;L^2(\Omega))} + \|\nabla \mathbf{v}\|_{L^2(0,2\pi;L^2(\Omega))}$$
$$\leqslant c\Bigg(\Bigg(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\Bigg(1 + \int\limits_{R_0}^{+\infty} \frac{1}{g^3(x_2)}\,\mathrm{d}x_2\Bigg)\Bigg)^{1/2} + \|\mathbf{f}\|_{L^2_{\mathrm{per}}(0,2\pi;L^2_1(\Omega))}\Bigg).$$

## Équations de Navier–Stokes dans une structure tubulaire

Le système de Navier–Stokes périodique en temps (R.13) et le système de Navier–Stokes stationnaire (R.14) sont considérés dans la structure tubulaire $B_\varepsilon$. On prouve l'existence et l'unicité de la solution faible et on construit un développement asymptotique de la solution, ce qui permet le développement des modèles de dimension hybride.

En examinant le système non-stationnaire de Navier–Stokes (R.13) dans la structure tubulaire (voir figure 7), nous supposons que la fonction à la frontière $\mathbf{g}$ est périodique en temps et que $\mathbf{g} = 0$ sur le bord partout sauf aux extrémités du cylindre $\gamma_\varepsilon^j = \partial B_\varepsilon \cap \partial \omega_\varepsilon^j$, $j = N_1+1,\dots,N$ (où $j$ sont les sommets du graphe (voir Figure 5)) et que pour les débits

$$\widetilde{F}^j(t) = \int\limits_{\gamma_\varepsilon^j} \mathbf{g}\cdot\mathbf{n}\,\mathrm{d}S \equiv \varepsilon^{n-1} F^j(t), \quad j = N_1+1,\dots,N \tag{R.16}$$

on a

$$\sum_{j=N_1+1}^{N} F^j(t) = 0 \quad \forall t \in [0,2\pi]. \tag{R.17}$$

Notons l'extension de la fonction à la frontière $\mathbf{g}$ par la même notation $\mathbf{g}$ et supposons qu'elle est solénoïdale, périodique en temps, et satisfait les estimations asymptotiques :

$$\begin{aligned} \sup_{x\in B_\varepsilon} |\mathbf{g}(x,t)| \leqslant c, \qquad & \|\nabla \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-3}{2}} \quad \forall t \in [0,2\pi], \\ \|\mathbf{g}_t\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-1}{2}}, \quad & \|\nabla^2 \mathbf{g}\|_{L^2(B_\varepsilon)} \leqslant c\varepsilon^{\frac{n-5}{2}} \quad \forall t \in [0,2\pi]. \end{aligned} \tag{R.18}$$

La thèse prouve l'existence d'une solution faible du système de Navier–Stokes périodique en temps (R.13) dans des cas bidimensionnels et tridimensionnels, en utilisant les propriétés de l'opérateur de Stokes (voir la section 2.2). L'existence et l'unicité de la solution faible de (R.13) sont prouvées en résolvant le problème variationnel suivant : on cherche un champ vectoriel $\mathbf{v} = \mathbf{u} + \mathbf{g}$, lorsque $\operatorname{div} \mathbf{u} = 0$, $\mathbf{u} \in L^\infty_{\text{per}}(0, 2\pi; \mathring{W}^{1,2}(B_\varepsilon) \cap W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t \in L^2_{\text{per}}(0, 2\pi; L^2(B_\varepsilon))$ et une identité intégrale est satisfaite

$$\begin{aligned}\int\limits_{B_\varepsilon} \left( \frac{1}{\varepsilon^\beta} \mathbf{u}_t \cdot \boldsymbol{\eta} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\eta} - ((\mathbf{u} + \mathbf{g}) \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla) \boldsymbol{\eta} \cdot \mathbf{g} \right) \mathrm{d}x \\ = \int\limits_{B_\varepsilon} \mathbf{f} \cdot \boldsymbol{\eta} \, \mathrm{d}x,\end{aligned} \tag{R.19}$$

pour chaque champ vectoriel solénoïdal $\boldsymbol{\eta} \in \mathring{W}^{1,2}(B_\varepsilon)$. Ici $\mathbf{g}$ est une extension satisfaisant les conditions de (R.18), et $\mathbf{f}$ est une fonction $2\pi$-périodique en temps telle que $\mathbf{f} \in L^2_{\text{per}}(0, 2\pi; L^2(B_\varepsilon))$. Définissons

$$A_1(t) = \|\mathbf{f}(\,\cdot\,, t)\|^2_{L^2(B_\varepsilon)}. \tag{R.20}$$

L'existence et l'unicité de la solution faible du système de Navier–Stokes périodique en temps (R.13) dans le cas bidimensionnel sont formulées dans le théorème ci-dessous (un théorème tridimensionnel similaire peut-être trouvé au chapitre 4, théorème 4.2.3 ).

**Théorème 2.** *(Theorem 4.2.1) Soit $B_\varepsilon \subset \mathbb{R}^2$, $\partial B_\varepsilon \in C^2$. Supposons que l'extension $\mathbf{g} \in C^{[\frac{J+1}{2}]+1}(0, 2\pi; W^{2,2}(B_\varepsilon))$ satisfait* (R.16)*,* (R.17) *et* (R.18)*. De plus, supposons que $\mathbf{f} \in L^2_{\text{per}}(0, 2\pi; L^2(B_\varepsilon))$ et satisfait la condition* (R.20)*. Alors pour tout $\varepsilon$ suffisamment petit le problème* (R.19) *a la seule solution $\mathbf{u}$ qui satisfait les estimations*

$$\sup_{t \in [0, 2\pi]} \|\mathbf{u}(\,\cdot\,, t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int\limits_0^{2\pi} \int\limits_{B_\varepsilon} |\nabla \mathbf{u}(x, t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c \varepsilon^{2+\beta} \int\limits_0^{2\pi} A_1(t) \, \mathrm{d}t,$$

$$\begin{aligned}\sup_{t \in [0, 2\pi]} \|\nabla \mathbf{u}(\,\cdot\,, t)\|^2_{L^2(B_\varepsilon)} + \int\limits_0^{2\pi} \int\limits_{B_\varepsilon} |\mathbf{u}_t(x, t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \varepsilon^\beta \int\limits_0^{2\pi} \int\limits_{B_\varepsilon} |\nabla^2 \mathbf{u}(x, t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ \leqslant c \varepsilon^\beta \int\limits_0^{2\pi} A_1(t) \, \mathrm{d}t,\end{aligned}$$

*ici la constante $c$ ne dépend pas de $\varepsilon$, $\beta = 0$ ou 2.*

Un développement asymptotique est construit pour la solution faible résultante, ce qui permet le développement de modèles de dimension hybride. Ce développement est construit par récurrence. Nous résolvons d'abord le problème périodique en temps sur le graphe et trouvons la pression macroscopique, qui est une fonction périodique en temps, et une fonction affine de la variable longitudinale $x_n^{(e)}$ dans chaque cylindre $\Pi_\varepsilon^{(e)}$ d'axe $e$ (défini dans l'introduction, et muni d'un système de coordonnées locales $x^{(e)'}$ pour les directions transverses et $x_n^{(e)}$ pour la direction longitudinale), indépendante de la variable transversale $x^{(e)'}$. Aux nœuds (aux points d'intersection de plusieurs arêtes d'un graphe (voir Figure 5), ce problème satisfait les conditions de jonction de type Kirchhoff. Pendant ce temps, dans chaque cylindre $\Pi_\varepsilon^{(e)}$ (la définition du cylindre $\Pi_\varepsilon^{(e)}$ se trouve également dans l'introduction de la thèse) nous avons une vitesse de type Poiseuille qui ne dépend que la variable transversale $x^{(e)'}$ et $t$, c'est-à-dire ne dépend pas de la dernière composante de l'espace. On multiplie alors la vitesse et la pression de type Poiseuille par une fonction troncature, nulle sur un voisinage diamètre $O(\varepsilon)$ des jonctions et égale à 1 en dehors d'un ce voisinage $O(\varepsilon)$. Cependant, cette multiplication provoque un grand résidu dans le second membre du système de Navier–Stokes. Le support du résidu appartient au même voisinage de diamètre $O(\varepsilon)$. Pour compenser ces erreurs, un correcteur de couche limite est introduit. Les correcteurs sont des solutions du système de Stokes qui sont résolues dans un domaine composée des cylindres à sorties infinies.

Les développements asymptotiques du vecteur vitesse et de la pression ainsi construites ont les expressions suivantes :

$$\begin{aligned}\mathbf{v}^{(J)}(x,t) = &\sum_{i=1}^{M} \zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i| - x_n^{(e_i)}}{3r\varepsilon}\Big) \sum_{j=0}^{J} \varepsilon^j \mathbf{V}_j^{(e_i)}(y^{(e_i)'},t)\\ &+\sum_{l=1}^{N} \Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big) \sum_{j=-1}^{J} \varepsilon^j \mathbf{V}_j^{[BLO_l]}(y,t),\end{aligned}$$

$$\begin{aligned}p^{(J)}(x,t) =&\sum_{i=1}^{M} \zeta\Big(\frac{x_n^{(e_i)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e_i| - x_n^{(e_i)}}{3r\varepsilon}\Big)\sum_{j=0}^{J} \varepsilon^{j-2}\big(-s_j^{(e_i)}(t)x_n^{(e_i)} + a_j^{(e_i)}(t)\big)\\ &+\sum_{l=1}^{N} \Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big) \sum_{j=-1}^{J} \varepsilon^{j-1} P_j^{[BLO_l]}(y,t),\end{aligned}$$

où $y = \frac{x^{(e)}}{\varepsilon}$, $r$ est le maximum des diamètres des sections transverses des cylindres, $0 \leqslant \zeta(\tau) \leqslant 1$ est une fonction-troncature régulière, et $\zeta(\tau) = 0$ pour

$\tau \leqslant 1/3$, et $\zeta(\tau)=1$ pour $\tau \geqslant 1/3$. Le couple $(\mathbf{V}_j^{[BLO_l]}(y,t), P_j^{[BLO_l]}(y,t))$ décrit le vecteur de vitesse et la pression dans les couches limites. Ils sont trouvés en résolvant le système de Stokes périodique en temps dans des cylindres semi-infinis. Le terme $\mathbf{V}_j^{(e_i)}(y^{(e)'},t)$ décrit la vitesse du type de Poiseuille, et le terme $p_j^{(e_i)} = -s_j^{(e_i)}(t)x_n^{(e_i)} + a_j^{(e_i)}(t)$ - pression macroscopique. Ces termes sont trouvés en résolvant un système sur le graphe périodique en temps. Étant donné que le développement asymptotique est construit par récurrence sur $j$, dans l'étape initiale, lorsque $j=0$, nous recherchons la fonction $p_0 \in L^2_{\text{per}}(0,2\pi; W^{1,2}(\mathcal{B}))$, qui satisfait le système

$$\begin{cases} -\dfrac{\partial}{\partial x_n^{(e)}}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)},t)\Big) = 0, \quad x_n^{(e)} \in (0,|e|),\ \forall e = e_j,\ j=1,\dots,M, \\ -\sum\limits_{e:O_l\in e}\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t) = 0, \qquad l=1,\dots,N_1, \\ \qquad -\Big(L^{(e)}\dfrac{\partial p_0}{\partial x_n^{(e)}}\Big)(0,t) = \Psi_l(t), \quad l=N_1+1,\dots,N, \end{cases}$$

où $\Psi_l(t) = \int\limits_{\gamma^l} \mathbf{g}_l\cdot\mathbf{n}\,\mathrm{d}S$. L'opérateur $L^{(e)}$ relie le coefficient de direction de la pression $\mathcal{S}$ et le débit $\mathcal{H}$ dans un cylindre infini de section transversale $\sigma^{(e)}$. On cherche l'opérateur $L^{(e)}$ en résolvant l'équation de la chaleur $2\pi$-periodique en temps : pour un donné $\mathcal{S} \in L^2_{\text{per}}(0,2\pi)$ nous recherchons $\mathcal{V} \in L^2_{\text{per}}(0,2\pi; \mathring{W}^{1,2}(\sigma^{(e)}))$ tel que $\dfrac{\partial \mathcal{V}}{\partial t} \in L^2_{\text{per}}(0,2\pi; L^2(\sigma^{(e)}))$ et

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(y^{(e)'},t) - \nu\Delta'_{y^{(e)'}}\mathcal{V}(y^{(e)'},t) = S(t), \quad y^{(e)'} \in \sigma^{(e)},\ t>0, \\ \qquad \mathcal{V}(y^{(e)'},t)|_{\partial\sigma^{(e)}} = 0, \quad \mathcal{V}(y^{(e)'},t) = \mathcal{V}(y^{(e)'},t+2\pi). \end{cases}$$

Ensuite, l'expression de l'opérateur linéaire $L^{(e)}$ est la suivante

$$L^{(e)}\mathcal{S}(t) = \int\limits_{\sigma^{(e)}} \mathcal{V}(y^{(e)'},t)\,\mathrm{d}y^{(e)'} = \mathcal{H}(t).$$

Plus d'informations sur les étapes de construction du développement asymptotique peuvent être trouvées dans le chapitre 4 de la thèse.

Pour justifier les développements asymptotiques, nous considérons la solution faible $(\mathbf{v},p)$ du système de Navier–Stokes périodique en temps (R.13) sous la forme suivante : $\mathbf{v} = \mathbf{u} + \mathbf{u}^{(J)} = \mathbf{u} + \mathbf{v}^{(J)} + \mathbf{w}^{(J)},\ p = q + p^{(J)}$, où $\mathbf{v}^{(J)}$ est l'approximation asymptotique de la solution d'ordre $J$, $\mathbf{w}^{(J)} \in$

$L^2_{\rm per}(0,2\pi;W^{2,2}(B_\varepsilon)\cap \mathring{W}^{1,2}(B_\varepsilon))$ champ vectoriel tel que $\operatorname{div}\mathbf{w}^{(J)} = -h^{(j)} = -\operatorname{div}\mathbf{v}^{(J)}$, et $p^{(J)}$ est l'approximation de la fonction de pression $p$. Alors $\mathbf{u}^{(J)} \in L^2_{\rm per}(0,2\pi;W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t^{(J)} \in L^2_{\rm per}(0,2\pi;L^2(B_\varepsilon))$.

**Définition.** Le champ vectoriel solénoïdal périodique en temps $\mathbf{u} = \mathbf{v} - \mathbf{u}^{(J)}$ est appelé la solution faible lorsque $\mathbf{u} \in L^2_{\rm per}(0,2\pi;W^{2,2}(B_\varepsilon))$, $\mathbf{u}_t \in L^2_{\rm per}(0,2\pi;L^2(B_\varepsilon))$, satisfait la condition aux limites $\mathbf{u}(x,t)|_{\partial B_\varepsilon} = 0$ et l'identité intégrale

$$\int\limits_{B_\varepsilon} \Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\eta} - ((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x$$
$$= \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x, \quad \forall\boldsymbol{\eta}\in H(B_\varepsilon),$$

avec $\mathbf{f}_1^{(J)}$ est le reste qu'on obtient en substituant $\mathbf{u}^{(J)}$ dans Navier–Stokes.

L'existence et l'unicité de la solution faible $\mathbf{u}$ découle du théorème 2. Lorsque $n = 2$, la solution $\mathbf{u}$ satisfait les estimations, que nous formulerons sous la forme d'un théorème.

**Théorème 3.** *Soit $n = 2$, $\beta = 0,2$. Alors les estimations suivantes ont lieu :*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \varepsilon^\beta \int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-2+\beta},$$

$$\sup_{t\in[0,2\pi]} \|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(B_\varepsilon)} + \int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |\mathbf{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^\beta\int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |\nabla^2\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c\varepsilon^{2J-4+\beta}.$$

*De plus, il existe une fonction de pression $q \in L^2_{\rm per}(0,2\pi;\,L^2(B_\varepsilon))$ telle que $\int\limits_{B_\varepsilon} q(x,t)\mathrm{d}x = 0$ et*

$$\int\limits_{B_\varepsilon} \Big(\frac{1}{\varepsilon^\beta}\mathbf{u}_t\cdot\boldsymbol{\eta} + \nu\nabla\mathbf{u} : \nabla\boldsymbol{\eta} - ((\mathbf{u}+\mathbf{u}^{(J)})\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\eta}\cdot\mathbf{u}^{(J)}\Big)\,\mathrm{d}x$$
$$= \int\limits_{B_\varepsilon} q\operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x + \int\limits_{B_\varepsilon} \mathbf{f}_1^{(J)}\cdot\boldsymbol{\eta}\,\mathrm{d}x, \quad \forall\boldsymbol{\eta}\in \mathring{W}^{1,2}(B_\varepsilon).$$

*Alors, si $J \geqslant 2$, alors l'estimation est valide*

$$\int\limits_0^{2\pi}\int\limits_{B_\varepsilon} |q|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon^{2J-4-\beta}.$$

Considérons maintenant le système de Navier–Stokes stationnaire (R.14) dans la structure tubulaire $B_\varepsilon$ (voir Figure 8) est réécrivons-le comme suit :

$$\begin{cases} -\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{v}\cdot(\nabla\mathbf{v})^t+\nabla\Phi=\mathbf{f}, & x\in B_\varepsilon, \\ \operatorname{div}\mathbf{v}=0, & x\in B_\varepsilon, \\ \mathbf{v}=0, & x\in\partial B_\varepsilon\backslash\cup_{j=N_1+1}^{N}\gamma_\varepsilon^j, \\ \mathbf{v}_{\boldsymbol{\tau}}=0, & x\in\gamma_\varepsilon^j, \\ \Phi=p_j, & x\in\gamma_\varepsilon^j,\ j=N_1+1,\dots,N, \end{cases} \tag{R.21}$$

où $\Phi=(p+\dfrac{1}{2}|\mathbf{v}|^2)$ est la pression de Bernoulli, et $p_j$ correspond à la constante $c_j/\varepsilon^2$.

**Définition.** En tant que solution faible du système stationnaire de Navier–Stokes (R.21) nous appelons le champ vectoriel $\mathbf{v}\in\widehat{J}^{1,2}_\gamma(B_\varepsilon)$ qui satisfait l'identité intégrale

$$\nu\int\limits_{B_\varepsilon}\nabla\mathbf{v}:\nabla\boldsymbol{\eta}\,\mathrm{d}x+\int\limits_{B_\varepsilon}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,\mathrm{d}x-\int\limits_{B_\varepsilon}(\boldsymbol{\eta}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}x$$
$$=-\sum_{j=N_1+1}^{N-1}p_j^*\int\limits_{\gamma_\varepsilon^j}\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}x'+\int\limits_{B_\varepsilon}\mathbf{f}\cdot\boldsymbol{\eta}\,\mathrm{d}x,\quad\forall\boldsymbol{\eta}\in\widehat{J}^{1,2}_\gamma(B_\varepsilon).$$

Le théorème sur l'existence d'une solution faible est donné ci-dessous.

**Théorème 4.** *Supposons que* $\mathbf{f}\in L^2(B_\varepsilon)$ *et* $p_j^*\in\mathbb{R}$, $j=N_1+1,\dots,N-1$. *Le système* (R.21) *admet alors au moins une solution faible* $\mathbf{v}\in\widehat{J}^{1,2}_\gamma(B_\varepsilon)$. *De plus, l'estimation suivant a lieu :*

$$\|\nabla\mathbf{v}\|_{L^2(B_\varepsilon)}\leqslant c\Big(\varepsilon^{n/2}\sum_{j=N_1+1}^{N-1}|p_j^*|+\varepsilon\|\mathbf{f}\|_{L^2(B_\varepsilon)}\Big), \tag{R.22}$$

*ici la constante $c$ ne dépend pas de $\varepsilon$.*

Le développement asymptotique dans le cas stationnaire est construit de la même manière que précédemment. Le développement de la vitesse a la forme :

$$\mathbf{v}^{(J)}(x) = \sum_{O_l,\ l=N_1+1,\dots,N;\ e=\overline{O_lO_{i_l}}} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)'}}{\varepsilon}\Big)$$

$$+ \sum_{e=\overline{O_\alpha O_\beta};\ \alpha,\beta\leqslant N_1} \zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\zeta\Big(\frac{|e|-x_n^{(e)}}{3r\varepsilon}\Big)\mathbf{V}^{[e,J]}\Big(\frac{x^{(e)'}}{\varepsilon}\Big)$$

$$+\sum_{l=1}^{N}\Big(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\Big)\mathbf{V}^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big).$$

Ici, le premier terme inclut les sommets, le deuxième terme inclut les nœuds et le troisième terme compense les erreurs dans les couches limites.

L'approximation asymptotique de la fonction de pression aux sommets et aux nœuds acquiert les expressions appropriées :

$$p^{(J)}(x) = -s^{(e)}x_n^{(e)} + a^{(e)} + \frac{1}{\varepsilon}\left(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\right)P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big),$$

et

$$p^{(J)}(x) = \sum_{e\subset\mathcal{B}_l}\zeta\Big(\frac{x_n^{(e)}}{3r\varepsilon}\Big)\big(-s^{(e)}x_n^{(e)}+a^{(e)}-a^{(e_s)}\big)+a^{(e_s)}$$

$$+\frac{1}{\varepsilon}\left(1-\zeta\Big(\frac{|x-O_l|}{|e|_{\min}}\Big)\right)P^{[BLO_l,J]}\Big(\frac{x-O_l}{\varepsilon}\Big).$$

Semblable au système non-stationnaire de Navier–Stokes, les asymptotiques de la couche limite $(\mathbf{V}^{[BLO_l,J]}, P^{[BLO_l,J]})$ se trouvent par la résolution du système stationnaire de Stokes. Pour le terme initial (quand $j=0$) le système ressemble à ceci :

$$\begin{cases} -\nu\Delta_y\mathbf{V}_0^{[BLO_l]}+\nabla_y P_0^{[BLO_l]} = \mathbf{f}_0^{[REGO_l]}+\mathbf{f}_0^{[BLO_l]}, & y\in\Omega_l,\\ \operatorname{div}_y\mathbf{V}_0^{[BLO_l]} = h_0^{[REGO_l]}, & y\in\Omega_l,\\ \mathbf{V}_0^{[BLO_l]} = 0, & y\in\partial\Omega_l. \end{cases}$$

Les termes $\mathbf{V}^{[e,J]}$ et $p_j = -s^{(e)}x_n^{(e)}+a^{(e)}$ sont trouvés en résolvant le problème sur le graphe. L'algorithme pour leur détection peut-être trouvé dans la section 5.3 de la thèse. Dans l'étape initiale, la fonction $p_0$ est trouvée à partir du problème :

$$\begin{cases} -\kappa_e \dfrac{\partial^2 p_0^{(e)}}{\partial x_n^{(e)2}}(x_n^{(e)}) = 0, & x_n^{(e)} \in (0, |e|), \\ -\displaystyle\sum_{e:O_l \in e} \kappa_e \dfrac{\partial p_0^{(e)}}{\partial x_n^{(e)}}(0) = 0, & l = 1, \dots, N_1, \\ p_0^{(e)}(0) = c_l, & l = N_1 + 1, \dots, N, \\ p_0^{(e)}(0) = p_0^{(e_s)}(0), \quad \forall e \subset \mathcal{B}_l, \end{cases}$$

ici $\kappa_e$ se trouve dans la résolution du problème de Dirichlet (voir la section 5.3 de la thèse), et $c_l$ sont des constantes connues.

La thèse (voir le chapitre 5) présente l'algorithme et prouve l'estimation de l'erreur. Soit $\mathbf{v} = \mathbf{u} + \mathbf{v}^{(J)} + \mathbf{w}^{(J)}$, avec $\mathbf{v}^{(J)}$ l'approximation asymptotique d'ordre $J$ de la vitesse, et $\mathbf{w}^{(J)} \in \mathring{W}^{1,2}(B_\varepsilon)$ un champ vectoriel tel que $\operatorname{div} \mathbf{w}^{(J)} = -h^{(j)} = -\operatorname{div} \mathbf{v}^{(J)}$.

**Théorème 5.** *La relation suivante a lieu :*

$$\|\mathbf{v} - (\mathbf{v}^{(J)} + \mathbf{w}^{(J)})\|_{W^{1,2}(B_\varepsilon)} = O(\varepsilon^{J+(n-1)/2}).$$

# Conclusion

Les problèmes suivants ont été examinés dans la thèse : un système de Stokes périodique en temps avec une condition aux limites inhomogène (dépendant uniquement de la variable d'espace). [4] Les résultats obtenus en examinant le système Navier-Stokes dans une structure tubulaire peuvent être utilisés pour développer un modèle de dimension hybride simplifié décrivant le flux sanguin dans les petits et très petits vaisseaux. L'étude du système stationnaire de Navier–Stokes avec une pression de Bernoulli donnée à l'entrée et à la sortie de la structure tubulaire permet simplifier l'implémentation numérique de la méthode de décomposition asymptotique partielle de domaine.

4. Les résultats ont été généralisés dans l'article de K. Kaulakytė et K. Pileckas [33].

# Acknowledgements

I would like to especially thank Professors Konstantinas Pileckas and Grigory Panasenko for all their patience and support during my Doctoral studies. I am grateful to Doctor Kristina Kaulakytė for being an amazing supervisor during my master's studies and also for her valuable advice during my Doctoral studies. Finally, I am thankful to my family for believing in me all this time.

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Vilnius University Press
9 Saulėtekio Ave., Building III, LT - 10222 Vilnius
Email: info@leidykla.vu.lt, www.leidykla.vu.lt
bookshop.vu.lt, journals.vu.lt
Print run of 10 copies