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Approximation of analytic functions by shifts of certain zeta-functions

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Analizinių funkcijų aproksimavimas kai kurių dzeta funkcijų postūmiais

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Contents

Notation

j, k, l, m, n	natural numbers
p	prime number
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all natural numbers
$2\mathbb{N}$	set of all even natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it, \sigma, t \in \mathbb{R}$	complex variable
$\bigoplus\limits_m A_m$	direct sum of sets A_m
$A \times B$	Cartesian product of the sets A and B
$\prod\limits_m A_m$	Cartesian product of sets A_m
A^m	Cartesian product of m copies of the set A
$\mathrm{meas}A$	Lebesgue measure of the set $A \subset \mathbb{R}$
$\#A$	cardinality of the set A
$H(G)$	space of analytic functions on G
$\mathcal{B}(\mathbb{X})$	class of Borel sets of the space $\mathbb{X}$
$\mathbb{E}X$	expectation of the random variable
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\Gamma(s)$	Euler gamma-function
$\zeta(s)$	Riemann zeta-function
$\zeta(s, \alpha)$	Hurwitz zeta-function
$a \ll_\eta b, b > 0$	there exists a constant $C = C(\eta) > 0$ such that $\|a\| \leqslant Cb$

Chapter 1

Introduction

1.1 Research topic

In the dissertation, approximation properties of certain zeta-functions are investigated. Recall that zeta-functions are usually defined in a certain half-plane of the complex plane by Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

with coefficients $a_m \in \mathbb{C}$ having one or other arithmetical sense. The mother of zeta-functions is the Riemann zeta-function $\zeta(s),\ s = \sigma + it$, defined, for $\sigma > 1$, by a very simple Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

Moreover, the function $\zeta(s)$ has the analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue 1. The function $\zeta(s)$ is not only used in various problems of analytic number theory, but also in many other brunches of mathematics. Moreover, Riemann zeta-function has found interesting applications in physics (quantum electrodynamics), statistics, astronomy (astrophysics) and even in music, see, for example [47], [13], [14], [1]. Also, zeta-function is very useful for the investigation of the distribution of prime numbers. The asymptotic formula for the number of prime numbers p

$$\sum_{p \leqslant x} 1 \sim \int_2^x \frac{\mathrm{du}}{\log u}, \quad x \to \infty, \tag{1.1}$$

was proved using the fact that $\zeta(s) \neq 0$ for $\sigma \geqslant 1$. Moreover, the function has good approximation properties, its shifts $\zeta(s+i\tau), \tau \in \mathbb{R}$, approximate a wide class of analytic functions.

In the dissertation, we consider a generalization of the function $\zeta(s)$, the Hurwitz type zeta-functions. Let $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and, as $\zeta(s)$, can be analitically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s, \alpha)$ depends on the parameter α, and its analytic properties are governed by arithmetic of that parameter. It is easy to see that $\zeta(s, 1) = \zeta(s)$, and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. However, in general, the functions $\zeta(s),\ \zeta(s, \alpha)$ have different analytic properties and this difference is explained by the existence of the so-called Euler product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 1. \tag{1.2}$$

Also, we consider a generalization of the function $\zeta(s, \alpha)$ which is called the periodic Hurwitz zeta-function. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$, be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha, \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l+\alpha}{q}\right).$$

Therefore, the periodic Hurwitz zeta-function again has the analytic continuation to the whole complex plane, except for the point $s = 1$ that is a simple pole with residue

$$\hat{a} \stackrel{def}{=} \frac{1}{q} \sum_{l=0}^{q-1} a_l.$$

If $\hat{a} = 0$, then $\zeta(s, \alpha; \mathfrak{a})$ is an entire function. Denote $\{1\} = \{a_m : a_m \equiv 1\}$. Then we have $\zeta(s, \alpha; \{1\}) = \zeta(s, \alpha)$ and $\zeta(s, 1; \{1\}) = \zeta(\alpha)$. Thus, $\zeta(s, \alpha, \mathfrak{a})$ is a generalization of the classical zeta-functions $\zeta(s, \alpha)$ and $\zeta(s)$.

The function $\zeta(s)$ with complex s was introduced in [58], the function $\zeta(s, \alpha)$ in [21], and $\zeta(s, \alpha, \mathfrak{a})$ in [30].

One more object of the dissertation is zeta-functions of cusp forms. Let

$$SL(2, \mathbb{Z}) \stackrel{def}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

be the full modular group. Consider the function $F(z)$ which is analytic in the half-plane $\mathfrak{Im} z > 0$, and, for all elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

satisfies the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^{\kappa} F(z) \tag{1.3}$$

with certain $\kappa \in 2\mathbb{N}$. Then $F(z)$ is called a modular form of weight κ for the group $SL(2, \mathbb{Z})$. Clearly, $F(z)$ is periodic with period 1, therefore, it has the Fourier series expansion

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) \mathrm{e}^{2\pi i m z}.$$

If $c(m) = 0$ for all $m \leqslant 0$, then $F(z)$ is called a cusp form of weight κ for the full modular group. For $\sigma > \frac{\kappa+1}{2}$, define

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

Then the function $\zeta(s, F)$ has the analytic continuation to the entire complex plane and is called the zeta-function of a cusp form $F(z)$.

Let $\upsilon \in \mathbb{N}$ and

$$\Gamma_0(\upsilon) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 (\mathrm{mod}\ \upsilon) \right\}$$

be the Hecke subgroup of the full modular group of level υ. If $F(z)$ satisfies (1.3) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\upsilon)$, then the cusp form $F(z)$ is called a cusp form of weight κ and level υ. The zeta-functions of the latter cusp forms were very important in the process of the proof of the last Fermat theorem that the Diophantine equation

$$x^n + y^n = z^n$$

has no of nontrivial solutions for $n \geqslant 3$. Let E be an elliptic curve over the field $\mathbb{Q}$ given by the Weierstrass equation

$$y^2 = x^3 + ax + b, a, b \in \mathbb{Z},$$

with non-zero discriminant $D = -16(4a^3 + 27b^2)$. For a prime number p, denote by E_p the reduction of the curve E modulo p which is a curve over the finite field $\mathbb{F}_p$. Denote by $|E(\mathbb{F}_p)|$ the number of points of E_p and define the numbers $\lambda(p)$ by the equality

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p).$$

The zeta-function $\zeta_{\mathbb{E}}(s)$ of the curve $\mathbb{E}$ is defined, for $\sigma > \frac{3}{2}$, by the product

$$\zeta_{\mathbb{E}}(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1},$$

and can be continued analitically to the whole complex plane.

The Japanese mathematicians G. Shimura and Y. Taniyama stated a hypothesis that the zeta-function of a cusp form of certain weight and level coincides with that of the elliptic curve. Later, it was observed that the Shimura-Taniyana conjecture implies the last Fermat theorem. Finally, A.Wiles [67] proved a partial case of the Shimura-Taniyama hypothesis and obtained the proof of the last Fermat theorem.

Suppose that $F(z)$ is a cusp form of weight κ for the full modular group. We additionally require that $F(z)$ would be the Hecke - eigenform, i.e., that $F(z)$ would be eigenfunction of all Hecke operators

$$T_m f(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^\kappa} \sum_{b (\mathrm{mod} d)} f\left(\frac{az+b}{d}\right), m \in \mathbb{N}.$$

Then the form $F(z)$ can be normalized, thus, we may suppose that $c(1) = 1$. In this case, the zeta-function $\zeta(s, F)$ has, for $\sigma > \frac{\kappa+1}{2}$, the Euler product representation over primes

$$\zeta(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $\alpha(p) + \beta(p) = c(p)$.

The majority of zeta-functions satisfy the functional equations. For example, for the Riemann zeta-function the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

where $\Gamma(s)$ is the Euler gamma-function, is valid. From this equation, it follows that $\zeta(-2k) = 0$ for $k \in \mathbb{N}$. The numbers $s = -2k$ are called trivial zeros of $\zeta(s)$. Moreover, the function $\zeta(s)$ has infinitely many non-trivial complex zeros $\rho = \beta + i\gamma$ lying in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. Let $\{\gamma_l : \gamma_l > 0,\ l \in \mathbb{N}\}$ be the sequence of positive imaginary parts of non-trivial zeros of $\zeta(s)$.

The zeta-function $\zeta(s, F)$ also has the functional equation

$$(2\pi)^{-s} \Gamma(s) \zeta(s, F) = (-1)^{\frac{\kappa}{2}} (2\pi)^{s-\kappa} \zeta(\kappa - s, F).$$

In the dissertation, we approximate a wide class of analytic functions by using shifts $\zeta(s + ih\gamma_k, F), h > 0$. Also, we obtain joint approximation results by collections of shifts $\big(\zeta(s + i\tau, \alpha_1), \dots, \zeta(s + i\tau, \alpha_r)\big)$ and $\big(\zeta(s + ikh, \alpha_1), \dots, \zeta(s + ikh, \alpha_r)\big)$ for arbitrary parameters $\alpha_1, \dots, \alpha_r$. The last approximation results of analytic functions are devoted to shifts $\zeta(s + i\tau, \alpha; \mathfrak{a})$ with arbitrary parameter α.

1.2 Aims and problems

The aims of the dissertation are approximation of analytic functions by shifts of the zeta-functions of normalized Heckle-eigen cusp forms as well as by shifts of the Hurwitz type zeta-functions. The problems considered are the following:

1. Approximation of a wide class of analytic functions by zeta-functions of normalized Hecke-eigen cusp-forms twisted by non-trivial zeros of the Riemann zeta-function.

2. Joint continuous approximation of analytic functions by shifts of Hurwitz zeta-functions with arbitrary parameters.

3. Joint discrete approximation of analytic functions by shifts of Hurwitz zeta-functions with arbitrary parameters.

4. Approximation of analytic functions by shifts of periodic Hurwitz zeta-functions.

1.3 Actuality

Approximation of analytic functions is a very important branch of Mathematics with respect to both theoretical and practical applications. By the classical Mergelyan theorem, every analytic function can be approximated uniformly on compact sets with connected complements with a given accuracy by polynomials. Later it turned out that some zeta-functions are much more powerful than polynomials because a wide class of analytic functions can by approximated by shifts of one and the same zeta-function. This approximation phenomenon generalizes the Bohr-Courant denseness results for zeta-functions to the space of analytic functions and has a deep mathematical sense. Therefore, it is important to develop the theory approximation of analytic functions by zeta-functions, to search new zeta-functions and their shifts with approximating properties. Moreover, investigation of the approximation theory of zeta-functions is one of productive fields of the Lithuanian school of analytic number theory, and it is a duty of young mathematicians to continue the researches of their colleagues.

1.4 Methods

The proofs of approximation theorems for zeta-functions use the methods of Dirichlet series theory, zero-distribution theory of the Riemann zeta-function, Fourier analysis and weak convergence of probability measures theory.

1.5 Novelty

All the results presented in the dissertation are new. The non-trivial zeros of the Riemann zeta-function are applied for the first time in the approximation of analytic functions by shifts of zeta-functions of cusp forms. Approximation of analytic functions by shifts of Hurwitz type zeta-functions with arbitrary parameters is also a new direction of analytic number theory.

1.6 History of the problem and the main results

Zeta-functions and Dirichlet series in general are the principal analytic tools of analytic number theory. Let, for $\sigma > \sigma_0$,

$$A(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \tag{1.4}$$

Analytical properties of the function $A(s)$ allow to obtain a certain information on the coefficients a_m, in particular, for the mean value

$$M(x) = \sum_{m \leqslant x} a_m$$

as $x \to \infty$. For example, the classical inverse formula asserts that if the series (1.4) is absolutely convergent for $\sigma > 1$, $|a_m| \leqslant g(m)$ with positive monotonically increasing $g(m)$ and, for $\sigma \to 1+0$,

$$\sum_{m=1}^{\infty} \frac{|a_m|}{m^s} = O\big((\sigma-1)^a\big), a > 0,$$

then for every $b_0 \geqslant b > 1$, $T \geqslant 1$, $x = n + \frac{1}{2}$, the formula

$$M(x) = \frac{1}{2\pi} \int_{b-iT}^{b+iT} A(s) \frac{x^a}{s} ds + O\Big(\frac{x^b}{T(b-1)^a}\Big) + O\Big(\frac{xg(2x)\log x}{T}\Big) \tag{1.5}$$

is valid. The latter formula is applied to obtain the asymptotic distribution law of prime numbers. From the Euler product (1.2) for $\zeta(s)$, it follows that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}, \sigma > 1, \tag{1.6}$$

where $\Lambda(m)$ is the von Mangoldt function,

$$\Lambda(m) = \begin{cases} \log p \text{ if } m = p^k, k \geqslant 1, \\ 0 \text{ otherwise.} \end{cases}$$

Application of (1.5), (1.6) and the equality

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + const,$$

where ρ runs over all non-trivial zeros of $\zeta(s)$, leads to

$$\sum_{m \leqslant x} \Lambda(m) = x + o(x), x \to \infty,$$

and this proves the relation (1.1).

The function $\zeta(s, F)$ is also applied for investigations of the coefficients $c(m)$. For example, in [57] it was obtained that

$$\sum_{m \leqslant x} c^2(m) = A_F x^{\kappa} + O(x^{\kappa - \frac{2}{5}}), x \to \infty,$$

with a certain constant $A_F > 0$. The result of P. Deligne is very deep [10]

$$|c(m)| \leqslant d(m) m^{\frac{x-s}{2}},$$

where $d(m)$ is the divisor function. The function $\zeta(s, F)$ was also studied itself. A lot of attention was devoted to the moments

$$I_k(T, \sigma, F) = \int_0^T |\zeta(\sigma + it, F)|^{2k} dt.$$

The first result in this direction was obtained by H. S. A. Potter in [54]. He proved the asymptotic formula

$$I_1(T, \sigma, F) \sim T \sum_{m=1}^{\infty} \frac{c^2(m)}{2^{2\sigma}}, \sigma > \frac{\kappa}{2}, T \to \infty,$$

and in [55] he found the bound

$$I_1(T, \frac{\kappa}{2}, F) \leqslant T \log T.$$

A.Good improved the latter result till [19],

$$I_1(T, \frac{\kappa}{2}, F) \sim 2\kappa A_F T \log T,$$

and [20]

$$I_1(T, \frac{\kappa}{2}, F) = \begin{cases} 2\kappa A_F T \log T + O(T) \text{ if } \sigma = \frac{\kappa}{2}, \\ A_F(\sigma)T + O(T^{\kappa+1-2\sigma}) \text{ if } \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, \\ A_F(\sigma)T + O((\log T)^2) \text{ if } \sigma > \frac{\kappa+1}{2} \end{cases}$$

with a certain constant $A_F(\sigma)$. M. Jutila gave [24] the estimate

$$I_3(T, \frac{\kappa}{2}, F) \ll T^{2+\varepsilon}, \quad \varepsilon > 0$$

In [31], it was proved that, for $k = \frac{1}{n}, n \in \mathbb{N}$,

$$I_k(T, \frac{\kappa}{2}, F) \ll T(\log T)^{k^2}.$$

The idea of application of zeta-functions for approximation of analytic functions belongs to S.Voronin. In [66], he obtained the following important result.

Theorem A. *Suppose that $0 < r < \frac{1}{4}$, the function $f(s)$ is continuous non-vanishing on the disc $|s| \leqslant r$, and analytic in the interior of this disc. Then, for every $\varepsilon > 0$, there exists a number $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s| \leqslant r} |\zeta(s + \frac{3}{4} + i\tau) - f(s)| < \varepsilon.$$

Voronin called the latter property of $\zeta(s)$ the universality. Really, Theorem A has features of universality because a wide class of analytic functions is approximated by shifts of one and the same function $\zeta(s)$.

The Voronin theorem was improved and extended by various authors. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let $\text{meas}A$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then we have the following modification of Theorem A, see, for example, [26].

Theorem B. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Theorem B shows that the set of shifts $\zeta(s+i\tau)$ approximating a given function $f(s) \in H_0(K)$ has a positive lower density, hence it is infinite. On the other hand, any concrete value of τ is not known.

In [48] and [35] it was observed independently that a lower density in Theorem B can be replaced by density. Thus, the following complement of Theorem B is true.

Theorem C. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Since the space $H(D)$ is infinite-dimensional, the Voronin universality theorem can be considered as an infinite-dimensional generalization of the Bohr-Courant theorem [8] on the denseness, for every $\frac{1}{2} < \sigma < 1$, of the set

$$\{\zeta(\sigma+it) : t \in \mathbb{R}\}.$$

The universality of the zeta-function $\zeta(s,F)$ was begun to study in [25] under the hypothesis that there exists $\eta > 0$ such that, for $\delta > \frac{1}{2}$, the series

$$\sum_{\substack{p \\ |c|p<\eta}} \frac{1}{p^\delta}, c_p = c(p)p^{\frac{1-\kappa}{2}},$$

converges. In [27], the above requirement was removed. Thus, the following theorem is known. Denote $D_F = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$. Let $\mathcal{K}_F$ be the class of compact subsets of the strip D_F with connected complements and $H_{0F}(K)$ with $K \in \mathcal{K}_F$ be the class of continuous non-vanishing functions on K that are analytic in the interior of K.

Theorem D. *Suppose that $K \in \mathcal{K}_F$ and $f(s) \in H_{0F}(K)$. Then, for every*

$\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon \right\} > 0.$$

Theorem D has its discrete version. Let $\#A$ denote the cardinality of a set A and N runs over the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem E. *Suppose that $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$ and $h > 0$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + ikh, F) - f(s)| < \varepsilon \right\} > 0.$$

Theorem E for h such that the number $\exp\{\frac{2\pi m}{h}\}$ is irrational for all $m \in \mathbb{Z} \setminus \{0\}$ was obtained in [29] and the stated version in [38].

For approximation of analytic functions more general shifts $\zeta(s+i\varphi(\tau), F)$ in place of $\zeta(s + i\tau, F)$ can be used. This was done in [65]. Denote by $U(T_0), T_0 > 0$, the class of functions $\varphi(\tau)$ satisfying the conditions:

1° $\varphi(\tau)$ are differentiable real-valued positive increasing functions on $[T_o, \infty)$.

2° The derivatives $\varphi'(\tau)$ are monotonic continuous positive on $[T_o, \infty)$ and $(\varphi'(\tau))^{-1} = o(\tau)$ as $\tau \to \infty$.

3° The estimate

$$\varphi(2\tau) \max_{\tau \leqslant u \leqslant 2\tau} \frac{1}{\varphi'(u)} \ll \tau, \tau \to \infty,$$

is valid.

Theorem F. *Suppose that $\varphi(\tau) \in U(T_0)$. Let $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$, Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T - T_0} meas \left\{ \tau \in [T_0, T] : \sup_{s\in K} |\zeta(s + i\varphi(\tau), F) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T - T_0} meas \left\{ \tau \in [T_0, T] : \sup_{s\in K} |\zeta(s + i\varphi(\tau), F) - f(s)| < \varepsilon \right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem F has a discrete version. For its statement, the notion of uniform distribution modulo 1 of sequences is needed. Let $\{u\}$ denote the fractional part of $u \in \mathbb{R}$, and let χ_I be the indicator function of the set I. The sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every interval $I = [a, b) < [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = b - a.$$

As in Theorem F, a certain class of functions is involved. Let $k_0 \in \mathbb{N}$. A function $\varphi(t) \in U(k_0)$ if the following hypotheses are satisfied:

1° $\varphi(t)$ is a real-valued increasing positive function on $[k_0 - \frac{1}{2}, \infty)$;

2° $\varphi(t)$ has a continuous derivative on $[k_0 - \frac{1}{2}, \infty)$ satisfying the estimate

$$\varphi(2t)\Big(\max_{t\leqslant u\leqslant 2t} \frac{1}{\varphi'(u)} + \max_{t\leqslant u\leqslant 2t} \varphi'(u)\Big) \ll t, \quad t \to \infty;$$

3° A sequence $\{a\varphi(k) : k\geqslant k_0\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1.

For example, the function $\varphi(t) = t \log^\alpha(t)$ with $0 < \alpha < 1$ is an element of the class $U(2)$.

The paper [40] contains the following statement.

Theorem G. *Suppose that $\varphi(t) \in U(k_0)$. Let $K \in \mathcal{K}_F$, and $f(s) \in H_{0F}(K)$, Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s + i\varphi(k), F) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{N - k_0 + 1} \# \left\{ k_0\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s + i\varphi(k), F) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Let $0 < \gamma_1 < \gamma_2 < ...\leqslant\gamma_k\leqslant\gamma_{k+1}...$ be the sequence of imaginary parts of non-trivial zeros of the Riemann zeta-function. There are known discrete universal theorems with shifts of zeta-functions involving the sequence $\{\gamma_k\}$. The first result in this direction was obtained in [17]. The distribution of non-trivial zero is mysterious, therefore, we have no sufficient information on the sequence $\{\gamma_k\}$. In [17], it was used the following conjecture that, for $c > 0$,

$$\sum_{\substack{\gamma_k \leqslant T \\ |\gamma_k - \gamma_l| < \frac{c}{\log T}}} \sum_{\gamma_l \leqslant T} 1 \ll T \log T, T \to \infty. \tag{1.7}$$

The latter estimate follows from the well-known Montgomery pair correlation conjecture [52] that, for $\alpha < \beta$,

$$\sum_{\substack{\gamma_k \leqslant T \\ \frac{2\pi\alpha}{\log T} < \gamma_k - \gamma_l < \frac{2\pi\beta}{\log T}}} \sum_{\gamma_l \leqslant T} 1 \sim \left(\int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) du + \delta(\alpha, \beta) \right) \frac{T \log T}{2\pi}$$

as $T \to \infty$, where

$$\delta(\alpha, \beta) = \begin{cases} 1 \text{ if } 0 \in [\alpha, \beta], \\ 0 \text{ otherwise.} \end{cases}$$

Thus, under the hypothesis (1.7) in [17] it was proved that the shifts $\zeta(s + ih\gamma_k)$ with every $h > 0$ approximate the functions of the class $H_0(K), K \in \mathcal{K}$. In the paper [15], the same result was obtained by using the Riemann hypothesis and the moment estimates from [16]. The paper [44] is devoted to the approximation of analytic function by shifts $\zeta(s + i\gamma_k h, \alpha)$ of the Hurwitz zeta-function with linearly independent set $L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$ over $\mathbb{Q}$.

Chapter 2 of the dissertation is devoted to the approximation of analytic functions by shifts $\zeta(s + i\gamma_k h, F)$. The main result of the chapter is the following theorem.

Theorem 2.1. *Suppose that the estimate* (1.7) *is true. Let* $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$ *and* $h > 0$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The proof of Theorem 2.1 is based on the weak convergence of

$$\frac{1}{N}\#\left\{1\leqslant k\leqslant N : \zeta(s+i\gamma_k h, F)\in A\right\}, A\in\mathcal{B}(H(D_F)),$$

as $N\to\infty$. Here $H(D_F)$ is the space of analytic on D_F functions endowed with the topology of uniform convergence on compacta, $\mathcal{B}(H(D_F))$ denotes the Borel σ-field of the space $H(D_F)$.

Recently, Professor Joern Steuding has informed that the condition (1.7) in the above theorem can be removed [61].

The results of Chapter 2 are published in [5].

The remaining chapters of the dissertation are devoted to approximation of analytic functions by shifts of Hurwitz type zeta-functions. Denote by $H(K)$ with $K\in\mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Thus, $H_0(K)\subset H(K)$. The approximation properties of the function $\zeta(s,\alpha)$ (or universality) are described by the following theorem. We recall that the number α is transcendental if there is no any polynomial $p(s)\not\equiv 0$ with rational coefficient such that $p(\alpha)=0$. Otherwise, α is algebraic.

Theorem H. *Suppose that the parameter α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T] : \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0. \quad (1.8)$$

Theorem H by different methods was obtained by B.Bagchi [2], S.M.Gonek [18], see also [28]. In case of rational α the proof is usually reduced to the joint universality of Dirichlet L-functions, while in case of transcendental α a probabilistic approach is convenient.

A weighted generalization of Theorem H is given in [64].

In case of algebraic irrational α, the approximation of all functions from the class $H(K)$ (universality) is not known. In this case, the following result is known [3]. Let $H(D)$ denote the space of analytic on D functions.

Theorem I. *Suppose that the parameter α is algebraic irrational. Then there exists a closed non-empty subset $F_\alpha\subset H(D)$ such that, for every compact set $K\subset D$, $f(s)\in F_\alpha$ and $\varepsilon>0$, the inequality* (1.8) *is valid. Moreover, the limit*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T] : \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon > 0$.

In the latter theorem, the set F_α is not explicitly given, only its existence is proved.

The most important and very complicated result on universality of $\zeta(s, \alpha)$ with algebraic irrational α was recently obtained in [62] for all but finitely many α of bounded degree. Moreover, the interval $[T, 2T]$ with τ satisfying (1.8) was given when K is a disc.

Theorem H has a certain joint generalization. The numbers $\alpha_1, ..., \alpha_r$ are called algebraically independent over $\mathbb{Q}$ if there is no any polynomial $p(s_1, ..., s_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1, ..., \alpha_r) = 0$.

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}.$$

Then the following extension of Theorem H is known [32].

Theorem J. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

It is not difficult to see that the algebraic independence of the parameters $\alpha_1, ..., \alpha_r$ implies the linear independence of the set $L(\alpha_1, \dots, \alpha_r)$.

In Chapter 3 of the dissertation a joint generalization of Theorem I is given.

Theorem 3.1. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Theorem 3.1 has a modification in terms of density.

Theorem 3.2. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$ and $(f_1, \dots, f_r) \in$*

$F_{\alpha_1,\dots,\alpha_r}$, the limit

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

For the proof of Theorems 3.1 and 3.2 the probabilistic approach is applied. These theorems are published in [11].

The approximation of analytic functions by discrete shifts of the Hurwitz zeta-function is also considered. In this case, an additional parameter h appeares in shifts $\zeta(s+ikh,\alpha)$. In [2], the following discrete theorem has been obtained.

Theorem K. *Suppose that α is a rational number, $\alpha<1$, $\alpha\neq\frac{1}{2}$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}|\zeta(s+ikh,\alpha)-f(s)|<\varepsilon\right\}>0. \tag{1.9}$$

For the proof of Theorem K a probabilistic method and the representation of $\zeta(s,\alpha)$ by Dirichlet L-functions is used. By a different method, Theorem K was proved in [59].

The discrete case with transcendental α is more complicated than that of rational α and requires some relation between h and π.

Theorem L. *[33]. Suppose that the number $\exp\{\frac{2\pi}{h}\}$, $h>0$, is rational and α is transcendental. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then the inequality* (1.9) *is true.*

In [36], the rationality of $\exp\{\frac{2\pi}{h}\}$ was replaced by a weaker condition. Define the set

$$L(\alpha,h,\pi)=\left\{(\log(m+\alpha):m\in\mathbb{N}_0),\frac{2\pi}{h}\right\}.$$

Then the following extention of Theorem L is valid [32].

Theorem M. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then the inequality* (1.9) *is valid.*

If the numbers α and $\exp\{\frac{2\pi}{h}\}$ are algebraically independent over $\mathbb{Q}$, then it is not difficult to see that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Therefore, by the Nesterenko theorem (π and e^{π} are algebraically independent), Theorem M is valid with $\alpha = \frac{1}{\pi}$ and rational h.

In place of the shifts $\zeta(s + ikh, \alpha)$, generalized discrete shifts $\zeta(s + i\varphi(k), \alpha)$ can also be used. Let $U(k_0)$ be the same class of functions as in Theorem G. Then in [39], the following theorem is given.

Theorem N. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $\varphi(k) \in U(k_0)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \# \left\{ k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + i\varphi(k), \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \# \left\{ k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s + i\varphi(k), \alpha) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For positive $h_1, ..., h_r$, define the set

$$L(\alpha_1, ..., \alpha_r; h_1, ..., h_r; 2\pi) = \big\{ \big(h_1 \log(m + \alpha_1) : m \in \mathbb{N}_0\big),$$

$$..., \big(h_r \log(m + \alpha_r) : m \in \mathbb{N}_0\big), 2\pi \big\}$$

Then the joint discrete theorem for Hurwitz zeta-functions follows [37].

Theorem O. *Suppose that the set $L(\alpha_1, ..., \alpha_r; h_1, ..., h_r; 2\pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ikh_j, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In Chapter 4, the discrete versions of Theorem 3.1 and 3.2 are obtained. These results in a certain sense extend Theorem O for arbitrary $\alpha_1, ...\alpha_r$.

Let, for brevity, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\underline{h} = (h_1, \ldots, h_r)$.

Theorem 4.1. *Suppose that the numbers $0 < \alpha_j < 1$, ($\alpha_j \neq \frac{1}{2}$) and positive numbers h_j, $j = 1, \ldots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ such that, for every compact sets $K_1, \ldots, K_r \subset D$,*

$(f_1, \ldots, f_r) \in F_{\underline{\alpha},\underline{h}}$ and $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

A discrete version of Theorem 3.2 is of the following form.

Theorem 4.2. *Suppose that the numbers $0 < \alpha_j < 1$, ($\alpha_j \neq \frac{1}{2}$) and positive numbers h_j, $j = 1, \ldots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ such that, for every compact sets $K_1, \ldots, K_r \subset D$, $(f_1, \ldots, f_r) \in F_{\underline{\alpha},\underline{h}}$, the limit*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 4.1 and 4.2 are published in [6].

Chapter 5 of the dissertation is devoted to a generalization of Theorem I for the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$.

The first result on the approximation of analytic functions by shifts $\zeta(s + i\tau, \alpha; \mathfrak{a})$ was obtained in [22] with transcendental parameter α.

Theorem P. *Suppose that the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

A discrete version of Theorem P is given in [33].

Theorem Q. *Suppose that the parameter α is transcendental number, and $h > 0$ such that $\exp\{\frac{2\pi}{h}\}$ is a rational number. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Joint universality theorems for periodic Hurwitz zeta-functions have been studied in many works. We mention the paper [30]. In the joint case, the properties of the periodic sequences $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}$, $j = 1, ..., r$, play an important role. Let q be the period of $\mathfrak{a}_j, r \leqslant q$. Define the matrix

$$A = \begin{pmatrix} a_{01} & a_{02} & \dots & a_{0r} \\ a_{11} & a_{12} & \dots & a_{1r} \\ \dots & \dots & \dots & \dots \\ a_{q1} & a_{q2} & \dots & a_{qr} \end{pmatrix}.$$

Theorem R. *Suppose that the α is a transcendental number and $rank(A) = r$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau,\alpha;\mathfrak{a}_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Generalizations of Theorem R with different parameters $\alpha_1, ...\alpha_r$ and sequences $\mathfrak{a}_j$ with different periods were given in [60] and [43]. In this case, it is required that the numbers $\alpha_1, ...\alpha_r$ would be algebraically independent over $\mathbb{Q}$, and q in matrix A is replaced by the least common multiple of the periods $q_1, ...q_r$.

In the dissertation the following general results are proved.

Theorem 5.1. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that, for every compact subset $K \subset D$, $f(s) \in F_{\alpha,\mathfrak{a}}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

The next theorem is a version of Theorem 5.1 in terms of density.

Theorem 5.2. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that, for every compact subset $K \subset D$ and $f(s) \in F_{\alpha,\mathfrak{a}}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorems 5.1 and 5.2 are generalized for certain compositions. The first approximations by shifts of compositions $F(\zeta(s+i\tau)), F : H(D) \to H(D)$, were discussed in [34]. For example, the shifts $\sin(\zeta(s+i\tau))$ approximate the analytic functions on D which do not take values 1 and -1.

In the dissertation, the following examples of approximations are presented.

Theorem 5.3. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that if $\Phi : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty, then, for every compact subset $K \subset D$, $f(s) \in \Phi\left(F_{\alpha,\mathfrak{a}}\right)$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|\Phi\left(\zeta(s+i\tau,\alpha;\mathfrak{a})\right) - f(s)\right| < \varepsilon\right\} > 0.$$

The next theorem is a generalization of Theorem 5.2 for compositions.

Theorem 5.4. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that if $\Phi : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty, then, for every compact subset $K \subset D$ and $f(s) \in \Phi\left(F_{\alpha,\mathfrak{a}}\right)$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|\Phi\left(\zeta(s+i\tau,\alpha;\mathfrak{a})\right) - f(s)\right| < \varepsilon\right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of above theorems, a probabilistic method is applied. The results of Chapter 5 are published in [12].

1.7 Approbation

The results of the dissertation were presented at these conferences: the International MMA (Mathematical Modelling and Analysis) conference (MMA-2019, May 28 – 31, 2019, Tallinn, Estonia), at the XVII and XX International Conferences «Algebra, Number Theory and Discrete Geometry and Multiscale Modeling: modern problems, applications and problems of history» (September 23 – 27, 2019, Tula, Russia), (September 21 – 24, 2021, online), at the conferences of Lithuanian Mathematical Society (LMS 2019, June 19 – 20, 2019, Vilnius, Lithuania), (LMS 2021, June 16 – 17, online), (LMS 2022, June 16 – 17, Lithuania) as well as at the Number Theory Seminar of Vilnius University.

1.8 Main publications

The results of the dissertation are published in the following papers:

1. V. Franckevič, A. Laurinčikas, D. Šiaučiūnas, On joint value distribution of Hurwitz zeta-functions, *Chebysh. Sbr.* **19(3)** (2018), 219–230.

2. V. Franckevič, A. Laurinčikas, D. Šiaučiūnas, On approximation of analytic functions by periodic Hurwitz zeta-functions, *Math. Model. Anal.* **24(1)** (2019), 20–33.

3. A. Balčiūnas, V. Franckevič, V. Garbaliauskienė, R. Macaitienė, A. Rimkevičienė, Universality of zeta-functions of cusp forms and non-trivial zeros of the Riemann zeta-function, *Math. Model. Anal.* **26(1)** (2021), 82–93.

4. A. Balčiūnas, V. Garbaliauskienė, V. Lukšienė, R. Macaitienė, A. Rimkevičienė, Joint discrete approximation of analytic functions by Hurwitz zeta-functions, *Math. Model. Anal.* **27(1)** (2022), 88–100.

1.9 Abstracts for conferences

1. V. Franckevič, A. Laurinčikas. On joint universality of Hurwitz zeta - functions, Abstracts of MMA2019, May 28–31, 2019, Talinn, Estonia, pp.

2. V. Franckevič. Universality of the periodic Hurwitz zeta - function. Abstracts of XVII International Conference «Algebra, Number Theory and Discrete Geometry: modern problems, applications and problems of history», September 23-28, 2019, Tula, Russia, pp.

3. V. Franckevič. Approximation of analytic functions by discrete shifts of periodic Hurwitz zeta - functions, Abstracts of MMA2021.

Chapter 2

Approximation of analytic functions by zeta-functions of cusp forms

It is known that zeta-functions $\zeta(s, F)$ of normalized Hecke-eigen cusp forms F are universal in the Voronin sense, i.e., their shifts $\zeta(s+i\tau, F)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. In this chapter, under a weak form of the Montgomery pair correlation conjecture, i.e., that with $c > 0$ the estimate

$$\sum_{\substack{\gamma_k \leqslant T \\ |\gamma_k - \gamma_2| < \frac{c}{\log T}}} \sum_{\gamma_2 \leqslant T} 1 \ll T \log T, T \to \infty, \tag{1.5}$$

is valid, it is proved that the shifts $\zeta(s + i\gamma_k h, F)$, where $\gamma_1 < \gamma_2 < \dots$ is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and $h > 0$, also approximate a wide class of analytic functions.

We recall that $\mathcal{K}_F$ is the class of compact subsets of the strip D_F with connected complements, and $H_{0F}(K)$ with $K \in \mathcal{K}_F$ is the class of continuous non-vanishing functions on K that are analytic in the interior of K.

2.1 Statements of the Theorems

The main result of this chapter is the following theorem.

Theorem 2.1. *Suppose that the estimate* (1.5) *is true. Let* $K \in \mathcal{K}_F$, $f(s) \in$

$H_{0F}(K)$ and $h > 0$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \#\left\{ 1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\left\{ 1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+i\gamma_k h, F) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

We recall that the condition (1.5) for the first-time was applied in [17] for the approximation by shifts $\zeta(s+i\gamma_k h)$, and in [42] for joint approximation by shifts $(\zeta(s+i\gamma_k h), \zeta(s+i\gamma_k h, \alpha))$, where $\zeta(s, \alpha)$ is the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

with transcendental parameter α. In [41], the joint approximation by shifts of Dirichlet L-functions involving the sequence $\{\gamma_k\}$ was discussed. Finally, the paper [4] is devoted to a generalization of [42] for shifts of the periodic and periodic Hurwitz zeta-functions.

For the proof of Theorem 2.1, we will apply some results from [17] and [27]. On the mentioned results, we will construct a probabilistic model.

2.2 Probabilistic model

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and let $H(D_F)$ be the space of analytic functions on D_F endowed with the topology of uniform convergence on compacta. In this section, we will consider the weak convergence as $N \to \infty$ for

$$P_{N,F}(A) \stackrel{\text{def}}{=} \frac{1}{N} \#\left\{ 1\leqslant k\leqslant N : \zeta(s+i\gamma_k h, F) \in A \right\}, \quad A \in \mathcal{B}(H(D_F)).$$

To state a limit theorem for $P_{N,F}$, we need some notation. Denote by γ the unit circle on the complex plane, by $\mathbb{P}$ the set of all prime numbers, and define the set

$$\Omega = \prod_{p\in\mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and operation of pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega = (\omega(p) : p \in \mathbb{P})$ the elements of the torus Ω, and on the above probability space define the $H(D_F)$-valued random element

$$\zeta(s, \omega, F) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

We note that the latter infinite product is uniformly convergent on compact subsets of the strip D_F for almost all $\omega \in \Omega$, thus, it defines an $H(D_F)$-valued random element. Denote by $P_{\zeta,F}$ the distribution of the random element $\zeta(s, \omega, F)$, i.e., for $A \in \mathcal{B}(H(D_F))$,

$$P_{\zeta,F}(A) = m_H \left\{\omega \in \Omega : \zeta(s, \omega, F) \in A\right\}.$$

We will prove the following statement

Theorem 2.2. *Suppose that the estimate* (1.5) *is true. Then $P_{N,F}$ converges weakly to the measure $P_{\zeta,F}$ as $N \to \infty$.*

The proof of Theorem 2.2 consists from three limit theorems that will be stated as separate lemmas.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant \mathrm{N} : \left(p^{-i\gamma_k h} : p \in \mathbb{P}\right) \in A \right\}.$$

Lemma 2.1. *Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

The lemma is proved in [17] by using the Fourier transform method. For this, the uniform distribution modulo 1 of the sequence $\{a\gamma_k : k \in \mathbb{N}\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is applied.

The next lemma deals with absolutely convergent Dirichlet series. For its proof, one property of weak convergence of probability measures is applied. Suppose P is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and a mapping $u : \mathbb{X}_1 \to \mathbb{X}_2$. This mapping is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$ - measurable when $u^{-1}\mathcal{B}(\mathbb{X}_2) \subset \mathcal{B}(\mathbb{X}_1)$, i. e., for every $A \in \mathcal{B}(\mathbb{X}_2)$,

$$u^{-1}A \in \mathcal{B}(\mathbb{X}_1).$$

Then P induces on $(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2))$ the unique probability measure Pu^{-1} which is defined by

$$Pu^{-1}(A) = P(u^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_1)$$

where $u^{-1}A$ is the preimage of A. If mapping $u : \mathbb{X}_1 \to \mathbb{X}_2$ is continuous then it is $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable. What is more, the following lemma is valid [7].

Lemma 2.2. *Suppose that $u : \mathbb{X}_1 \to \mathbb{X}_2$ is a continuous mapping, and P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$. If P_n converges weakly to P as $n \to \infty$, then also $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$.*

Let $\theta > \frac{1}{2}$ be a fixed number, for $m, n \in \mathbb{N}$, let

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\},$$

and

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m) v_n(m)}{m^s}.$$

Then it is known [27] that the latter series is absolutely convergent for $\sigma > \frac{\kappa}{2}$. Consider the mapping $u_{n,F} : \Omega \to H(D_F)$ given by $u_{n,F}(\omega) = \zeta_n(s, \omega, F)$, where

$$\zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m) v_n(m)}{m^s},$$

and

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Obviously, the series for $\zeta_n(s, \omega, F)$ is also absolutely convergent for $\sigma > \frac{\kappa}{2}$. Therefore, the mapping $u_{n,F}$ is continuous, hence it is $(\mathcal{B}(\Omega), \mathcal{B}(H(D_F)))$-measurable. Therefore, the Haar measure m_H defines the unique probability measure $V_{n,F} = m_H u_{n,F}^{-1}$ on $(H(D_F), \mathcal{B}(H(D_F)))$, where, for $A \in \mathcal{B}(H(D_F))$,

$$V_{n,F}(A) = m_H u_{n,F}^{-1}(A) = m_H(u_{n,F}^{-1} A).$$

For $A \in \mathcal{B}(H(D_F))$, set

$$P_{N,n,F}(A) = \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \zeta_n(s + i\gamma_k h, F) \in A \right\}.$$

Lemma 2.3. *$P_{N,n,F}$ converges weakly to the measure $V_{n,F}$ as $N \to \infty$.*

Proof. By the definitions of Q_N and $P_{N,n,F}$, we have

$$P_{N,n,F}(A) = \frac{1}{N}\#\left\{1 \leqslant k \leqslant N : \left(p^{-i\gamma_k h} : p \in \mathbb{P}\right) \in u_{n,F}^{-1}A\right\} = Q_N(u_{n,F}^{-1}A).$$

Thus, $P_{N,n,F} = Q_N u_{n,F}^{-1}$.

Therefore, the lemma is a corollary of Lemma 2.1, continuity of $u_{n,F}$ and Lemma 2.2. □

The weak convergence of the measure $V_{n,F}$ as $n \to \infty$ is very important for the proof of Theorem 2.2. The following assertion is true. We recall that the support of $P_{\zeta,F}$ is a minimal closed set $S \subset H(D_F)$ such that $P_{\zeta,F}(S) = 1$. The set S consists of all $g \in H(D_F)$ such that, for every open neighbourhood G of g, the inequality $P_{\zeta,F}(G) > 0$ is satisfied.

Lemma 2.4. *$V_{n,F}$ converges weakly to the measure $P_{\zeta,F}$ as $n \to \infty$. Moreover, the support of $P_{\zeta,F}$ is the set*

$$S_F = \left\{g \in H(D_F) : g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\right\}.$$

Proof. The lemma is a result of [25] and [27] because $V_{n,F}$, as $n \to \infty$, and

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau, F) \in A\right\}, \ \ A \in \mathcal{B}(H(D_F)),$$

as $T \to \infty$, have the same limit measure $P_{\zeta,F}$. □

To prove Theorem 2.2, it remains to show that the limit measure of $P_{N,F}$ as $N \to \infty$ coincides with that of $V_{n,F}$ as $n \to \infty$. For this, some mean square estimates will be applied. For convenience, we recall the Gallagher lemma which connects discrete and continuous mean squares of certain functions.

Lemma 2.5. *Let T_0 and $T \geqslant \delta > 0$ be real numbers, and $\mathcal{T} \neq \varnothing$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1,$$

and let $S(t)$ be a complex-valued continuous function on $[T_0, T_0+T]$ having *a continuous derivative on* (T_0, T_0+T). *Then*

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int\limits_{T_0}^{T_0+T}|S(t)|^2\mathrm{d}t + \left(\int\limits_{T_0}^{T_0+T}|S(t)|^2\mathrm{d}t \int_{T_0}^{T_0+T}|S'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

The proof of the lemma can be found in [51, Lemma 1.4].

Now, we recall a metric in the space $H(D_F)$. For $g_1, g_2 \in H(D_F)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup\limits_{s\in K_l}|g_1(s)-g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D_F such that

$$D_F = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if K is a compact subset of D_F, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric on $H(D_F)$ that induces the topology of uniform convergence on compacta.

Lemma 2.6. *Suppose that the estimate* (1.5) *is true. Then the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\rho\left(\zeta(s+i\gamma_k h, F), \zeta_n(s+i\gamma_k h, F)\right) = 0$$

holds.

Proof. We start with some remarks on the mean squares of the function $\zeta(s,F)$. It is well known that, for fixed σ, $\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}$, the bound

$$\int\limits_0^T |\zeta(\sigma+it, F)|^2\mathrm{d}t \ll_\sigma T$$

is true. Hence, it follows for the same σ that, for $\tau \in \mathbb{R}$,

$$\int\limits_0^T |\zeta(\sigma+i\tau+it, F)|^2\mathrm{d}t \ll_\sigma T(1+|\tau|). \qquad (2.1)$$

Moreover, the Cauchy integral formula together with (2.1) leads to

$$\int_0^T |\zeta'(\sigma + i\tau + it, F)|^2 \mathrm{d}t \ll_\sigma T(1 + |\tau|). \tag{2.2}$$

Now, we apply Lemma 2.5. It is known that $\gamma_k \sim \frac{2\pi k}{\log k}$ as $k \to \infty$. Therefore, $\gamma_k \leqslant \frac{ck}{\log k}$ with some $c > 0$ for all $k \geqslant 2$. In Lemma 2.5, we take $\mathcal{T} = \{\gamma_1 h, \ldots, \gamma_N h\}$, $\delta = h\left(\log \frac{N}{c \log N}\right)^{-1}$, $T_0 = \gamma_1 h - \frac{\delta}{2}$ and $T = \gamma_N h - T_0 + \frac{\delta}{2}$. Then, in view of (1.5), we find that

$$\sum_{k=1}^N N_\delta(\gamma_k h) = \sum_{k=1}^N \sum_{\substack{\gamma_l \leqslant \frac{cN}{h \log N} \\ |\gamma_k - \gamma_l| < \frac{\delta}{h}}} 1 = \sum_{\substack{0 < \gamma_l,\ \gamma_k \leqslant \frac{cN}{h \log N} \\ |\gamma_l - \gamma_k| < \frac{\delta}{h}}} \sum 1 \ll_h N.$$

Thus, applying Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\sum_{k=1}^N |\zeta(\sigma + i\tau + i\gamma_k h, F)| \\
&= \sum_{k=1}^N \left(N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)\right)^{\frac{1}{2}} |\zeta(\sigma + i\tau + i\gamma_k h, F)| \\
&\leqslant \left(\sum_{k=1}^N N_\delta(\gamma_k h) \sum_{k=1}^N N_\delta^{-1}(\gamma_k h) |\zeta(\sigma + i\tau + i\gamma_k h, F)|^2\right)^{\frac{1}{2}} \\
&\ll_h N^{\frac{1}{2}} \left(\sum_{k=1}^N N_\delta^{-1}(\gamma_k h) |\zeta(\sigma + i\tau + i\gamma_k h, F)|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Now, by Lemma 2.5 for the function $\zeta(\sigma + i\tau + i\gamma_k h, F)$, and, taking into account the estimates (2.1) and (2.2), we find

$$
\begin{aligned}
\sum_{k=1}^{N} |\zeta(\sigma+i\tau+i\gamma_k h,F)| \ll_h\ & N^{\frac12}\Bigg(\log N \int\limits_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta(\sigma+i\tau+it,F)|^2\mathrm{d}t \\
&+ \Bigg(\int\limits_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta(\sigma+i\tau+it,F)|^2\mathrm{d}t \int\limits_{\gamma_1 h-\frac{\delta}{2}}^{\gamma_N h-\gamma_1 h+\delta} |\zeta'(\sigma+i\tau+it,F)|^2\mathrm{d}t\Bigg)^{\frac12}\Bigg)^{\frac12} \\
\ll_h & N^{\frac12}\Bigg(\log N \int\limits_{0}^{\frac{c(h)N}{\log N}} |\zeta(\sigma+i\tau+it,F)|^2\mathrm{d}t \\
&+ \Bigg(\int\limits_{0}^{\frac{c(h)N}{\log N}} |\zeta(\sigma+i\tau+it,F)|^2\mathrm{d}t \int\limits_{0}^{\frac{c(h)N}{\log N}} |\zeta'(\sigma+i\tau+it,F)|^2\mathrm{d}t\Bigg)^{\frac12}\Bigg)^{\frac12} \\
\ll_h & N^{\frac12}\left(\log N\frac{c(h)N}{\log N}(1+|\tau|)\right)^{\frac12} + N^{\frac12}\left(\frac{c(h)N}{\log N}(1+|\tau|)\right)^{\frac12} \\
\ll_h & N(1+|\tau|)^{\frac12} \ll_h N(1+|\tau|).
\end{aligned}
\tag{2.3}
$$

Here $c(h)$ is a certain positive constant depending of h.

Let the number θ is the same as in the definition of $v_n(m)$, and

$$
l_n(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)n^s,
$$

where $\Gamma(s)$ denotes the Euler gamma-function. Then we have [25]

$$
\zeta_n(s,F) = \frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,F)l_n(z)\frac{\mathrm{d}z}{z}.
$$

Hence, taking $\theta_1 > 0$, we obtain

$$
\zeta_n(s,F) - \zeta(s,F) = \frac{1}{2\pi i}\int\limits_{-\theta_1-i\infty}^{-\theta_1+i\infty} \zeta(s+z,F)l_n(z)\frac{\mathrm{d}z}{z}. \tag{2.4}
$$

We take an arbitrary fixed compact subset K of the strip D_F, denote the points of K by $s = \sigma + iv$, fix $\varepsilon > 0$ such $\frac{\kappa}{2} + 2\varepsilon \leqslant \sigma \leqslant \frac{\kappa+1}{2} - \varepsilon$ for $s \in K$, and choose

$$\theta_1 = \sigma - \varepsilon - \frac{\kappa}{2} \quad \text{and} \quad \theta = \frac{\kappa}{2} + \varepsilon.$$

Then the representation (2.4) shows that, for $s \in K$,

$$\zeta(s+i\gamma_k h, F) - \zeta_n(s+i\gamma_k h, F) \ll \int_{-\infty}^{\infty} \left|\zeta(s+i\gamma_k h - \theta_1 + i\tau, F)\right| \frac{|l_n(-\theta_1 + i\tau)|}{|-\theta_1 + i\tau|} \mathrm{d}\tau.$$

Hence, after a shift $\tau + v \to \tau$, we have

$$\begin{aligned} \zeta(s+i\gamma_k h, F) - \zeta_n(s+i\gamma_k h, F) \ll & \int_{-\infty}^{\infty} \left|\zeta\left(\frac{\kappa}{2} + \varepsilon + i(\tau + \gamma_k h), F\right)\right| \\ & \times \frac{|l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)|}{|\frac{\kappa}{2} + \varepsilon - s + i\tau|} \mathrm{d}\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s+i\gamma_k h, F) - \zeta_n(s+i\gamma_k h, F)| \\ & \ll \int_{-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=1}^{N} \left|\zeta\left(\frac{\kappa}{2} + \varepsilon + i(\tau + \gamma_k h), F\right)\right| \sup_{s \in K} \frac{|l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)|}{|\frac{\kappa}{2} + \varepsilon - s + i\tau|} \right) \mathrm{d}\tau. \end{aligned} \tag{2.5}$$

It is well known that, uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$,

$$\Gamma(\sigma + i\tau) \ll \exp\{-c|\tau|\}, \quad c > 0.$$

Thus, taking into account the definition of the function $l_n(s)$, we find that, for $s \in K$,

$$\frac{l_n(\frac{\kappa}{2} + \varepsilon - s + i\tau)}{\frac{\kappa}{2} + \varepsilon - s + i\tau} \ll n^{-\varepsilon} \exp\left\{-\frac{c|\tau - v|}{\theta}\right\} \ll_K n^{-\varepsilon} \exp\{-c|\tau|\}.$$

Therefore, by (2.5) and (2.3),

$$\frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} |\zeta(s+i\gamma_k h, F) - \zeta_n(s+i\gamma_k h, F)|$$

$$\ll_{K,h} \quad n^{-\varepsilon} \int\limits_{-\infty}^{\infty} (1+|\tau|) \exp\{-c|\tau|\} \mathrm{d}\tau \ll_{K,h} n^{-\varepsilon}.$$

This shows that, for every compact set $K \subset D_F$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} \left|\zeta(s+i\gamma_k h, F) - \zeta_n(s+i\gamma_k h, F)\right| = 0,$$

and the assertion of the lemma follows from the definition of the metric ρ. □

Now, we are in position to prove Theorem 2.2.

For convenience, we recall one properthy of convergence in distribution. Denote by $\xrightarrow{\mathcal{D}}$ the convergence of random elements in distribution, i. e., the weak convergence of their distributions.

Lemma 2.7. *Let the space $(\mathbb{X}, d)$ be separable, and the $\mathbb{X}$ - valued random elements Y_n, X_{kn}, $n \in \mathbb{N}$, $k \in \mathbb{N}$, be defined on the same probability space with the measure μ. Suppose that for every $k \in \mathbb{N}$,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0.$$

Then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

This lemma is Theorem 4.2 from [7].

Proof of Theorem 2.2. Let ξ_N be a random variable on a certain probability space $(\hat{\Omega}, \mathcal{A}, \mu)$ with the distribution

$$\mu\{\xi_N = \gamma_k h\} = \frac{1}{N}, \quad k = 1, ..., N.$$

Denote by $X_{n,F}$ the $H(D_F)$-valued random element with the distribution $V_{n,F}$, where $V_{n,F}$ is the limit measure in Lemma 2.3, and, on the probability space

$(\hat{\Omega}, \mathcal{A}, \mu)$, define the $H(D_F)$-valued random element

$$X_{N,n,F} = X_{N,n,F}(s) = \zeta_n(s + i\xi_N, F).$$

Then, in view of Lemma 2.3,

$$X_{N,n,F} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,F}. \tag{2.6}$$

By Lemma 2.4, the measure $V_{n,F}$ is weakly convergent to $P_{\zeta,F}$ as $n \to \infty$. Thus,

$$X_{n,F} \xrightarrow[n\to\infty]{\mathcal{D}} P_{\zeta,F}. \tag{2.7}$$

On the above probability space, define one more $H(D_F)$-valued random element

$$Y_{N,F} = Y_{N,F}(s) = \zeta(s + i\xi_N, F).$$

Then, applying Lemma 2.6, we find that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{N\to\infty} \mu\big\{\rho(Y_{N,F}, X_{N,n,F}) \geqslant \varepsilon\big\} \\ \leqslant\ &\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \rho\big(\zeta(s + i\gamma_k h, F), \zeta_n(s + i\gamma_k h, F)\big) = 0. \end{aligned}$$

This equality together with (2.6) and (2.7) shows that all hypotheses of lemma 2.7 are satisfied. Therefore, we have

$$Y_{N,F} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\zeta,F},$$

in other words, $P_{N,F}$ converges weakly to $P_{\zeta,F}$ as $N \to \infty$. The theorem is proved.

2.3 Proof of Theorem 2.1

The proof of Theorem 2.1 is quite standard, and is based on Theorem 2.2 and the Mergelyan theorem on the approximation of analytic functions by polynomials [50]. For convenience we state it as lemma.

Lemma 2.8. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement and $g(s)$ is a continuous function on K which is analytic in the interior* *of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s\in K} |g(s) - p_\varepsilon(s)| < \varepsilon$$

Proof of Theorem 2.1. By the mentioned Mergelyan theorem, there exists a polynomial $p_\varepsilon(s)$ such that

$$\sup_{s\in K} \left| f(s) - e^{p_\varepsilon(s)} \right| < \frac{\varepsilon}{2}. \tag{2.8}$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D_F) : \sup_{s\in K} |g(s) - e^{p_\varepsilon(s)}| < \frac{\varepsilon}{2} \right\}.$$

Clearly, $e^{p_\varepsilon(s)} \in S$. Therefore, in virtue of Lemma 2.4, the set G_ε is an open neighbourhood of an element of the support of the measure $P_{\zeta,F}$. Hence, by a property of the support,

$$P_{\zeta,F}(G_\varepsilon) > 0, \tag{2.9}$$

and Theorem 2.2 together with the equivalent of weak convergence of probability measures in terms of open sets [7, Theorem 2.1] implies

$$\liminf_{N\to\infty} P_{N,F}(G_\varepsilon) \geqslant P_{\zeta,F}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,F}$ and G_ε, and (2.8) prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D_F) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then the boundary $\partial\hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D_F) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\},$$

therefore, $\partial G_{\varepsilon_1} \bigcap \partial G_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. This remark implies that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,F}$, i.e., $P_{\zeta,F}(\partial\hat{G}_\varepsilon) = 0$, for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 2.2 together with the equivalent of weak convergence of probability measures in terms of continuity sets [7, Theorem 2.1] gives the equality

$$\lim_{N\to\infty} P_{N,F}(\hat{G}_\varepsilon) = P_{\zeta,F}(\hat{G}_\varepsilon) \tag{2.10}$$

for all but at most countably many $\varepsilon > 0$. The definitions of the sets G_ε and $\hat{G}_\varepsilon$, and inequality (2.8) imply the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Hence, in view of (2.9), we have $P_{\zeta,F}(\hat{G}_\varepsilon) > 0$. The latter inequality, the definitions of $P_{N,F}$ and $\hat{G}_\varepsilon$, and (2.10) prove the second assertion of the theorem. The theorem is proved.

Chapter 3

Joint approximation of analytic functions by Hurwitz zeta-functions

It is well known that some zeta and L-functions are universal in the Voronin sense, i.e., they approximate a wide class of analytic functions. Also, some of them are jointly universal. In this case, a collection of analytic functions is simultaneously approximated by a collection of zeta-functions. In this chapter, a problem related to joint universality of Hurwitz zeta-functions is discussed. It is known that the Hurwitz zeta-functions $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ are jointly universal if the parameters $\alpha_1,\dots,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, or, more generally, if the set $\{\log(m+\alpha_j) : m \in \mathbb{N}_0,\ j=1,\dots,r\}$ is linearly independent over $\mathbb{Q}$. We consider the case of arbitrary parameters $\alpha_1,\dots,\alpha_r$ and obtain that there exists a non-empty closed set of analytic functions $F_{\alpha_1,\dots,\alpha_r}$ on the strip $D=\left\{s\in\mathbb{C} : \frac{1}{2}<\sigma<1\right\}$ such that, for every compact sets $K_1,\dots,K_r\subset D$, $f_1,\dots,f_r\in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon>0$, the set $\left\{\tau\in\mathbb{R} : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}$ has a positive lower density. Also, the case of positive density of the latter set is discussed.

3.1 Statements of the theorems

In this chapter we will prove a joint generalization of Theorem J, i.e., we will prove a certain theorem on joint approximation by the functions $\zeta(s,\alpha_1)$, ..., $\zeta(s,\alpha_r)$ without using any independence condition.

Theorem 3.1. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$, $f_1, \dots, f_r \in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Theorem 3.1 has the following modification.

Theorem 3.2. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$ and $f_1, \dots, f_r \in F_{\alpha_1,\dots,\alpha_r}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of above theorems we will apply the probabilistic approach. This is influenced in a certain sense by Yu. V. Linnik who was an expert not only in number theory but also in probability theory and mathematical statistics.

3.2 Auxiliary results

In this section, a joint limit theorem for the functions $\zeta(s,\alpha_1), \dots, \zeta(s,\alpha_r)$ in the space of analytic functions will be proved. For $A \subset \mathcal{B}(H^r(D))$, define

$$P_{T,\underline{\alpha}} = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \underline{\alpha}) \in A\right\},$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and

$$\underline{\zeta}(s, \underline{\alpha}) = (\zeta(s,\alpha_1), \dots, \zeta(s,\alpha_r)).$$

Theorem 3.3. *Suppose that the numbers α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, are arbitrary. Then, on $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure $P_{\underline{\alpha}}$ such that $P_{T,\underline{\alpha}}$ converges weakly to $P_{\underline{\alpha}}$ as $T \to \infty$.*

We divide the proof of Theorem 3.3 into lemmas.

Denote by γ the unit circle on the complex plane, and define the set

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the classical Tikhonov theorem, the infinite-dimensional torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Define one more set

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then again by the Tikhonov theorem, Ω^r is a compact topological Abelian group. Denote by $\underline{\omega} = (\omega_1, \dots, \omega_r)$, $\omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r$, the elements of Ω^r Moreover, let $\omega_j(m)$ be the m-th component of the element $\omega_j \in \Omega$, $j = 1, \dots, r$, $m \in \mathbb{N}_0$.

For $A \in \mathcal{B}(\Omega^r)$, define

$$\begin{aligned} Q_{T,\underline{\alpha}} = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \\ \big(\big((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0 \big), \dots, \big((m + \alpha_r)^{-i\tau} : m \in \mathbb{N}_0 \big) \big) \in A. \end{aligned}$$

Lemma 3.1. *On $(\Omega^r, \mathcal{B}(\Omega^r))$, there exists a probability measure $Q_{\underline{\alpha}}$ such that $Q_{T,\underline{\alpha}}$ converges weakly to $Q_{\underline{\alpha}}$ as $T \to \infty$.*

Proof. We apply the Fourier transform method. The dual group of Ω^r is isomorphic to

$$\mathcal{G} = \bigoplus_{j=1}^{r} \bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{mj},$$

where $\mathbb{Z}_{mj} = \mathbb{Z}$ for all $j = 1, \dots, r$, $m \in \mathbb{N}_0$. The element $\underline{k} = (k_{mj} : k_{mj} \in \mathbb{Z}, m \in \mathbb{N}_0, j = 1, \dots, r) \in \mathcal{G}$, where only a finite number of integers k_{mj} are distinct from zero, acts on Ω^r by

$$\omega \to \omega^{\underline{k}} = \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} \omega_j^{k_{mj}}(m).$$

Therefore, the Fourier transform $g_T(\underline{k})$ of $Q_{T,\underline{\alpha}}$ is of the form

$$g_T(\underline{k}) = \int_{\Omega^r} \left(\prod_{j=1}^{r} {\prod_{m\in\mathbb{N}_0}}' \omega_j^{k_{jm}}(m) \right) \mathrm{d}Q_{T,\underline{\alpha}},$$

where the sign "$'$" shows that only a finite number of integers k_{mj} are distinct from zero. Thus, by the definition of $Q_{T,\underline{\alpha}}$,

$$\begin{aligned} g_T(\underline{k}) &= \frac{1}{T}\int_0^T \prod_{j=1}^{r} {\prod_{m\in\mathbb{N}_0}}' (m+\alpha_j)^{-i\tau k_{mj}} \mathrm{d}\tau \\ &= \frac{1}{T}\int_0^T \exp\left\{ -i\tau \sum_{j=1}^{r} {\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j) \right\} \mathrm{d}\tau. \quad (3.1) \end{aligned}$$

Define two collections of integers

$$\{\underline{k}'\} = \left\{ k_{mj} : \sum_{j=1}^{r} {\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j) = 0 \right\}$$

and

$$\{\underline{k}''\} = \left\{ k_{mj} : \sum_{j=1}^{r} {\sum_{m\in\mathbb{N}_0}}' k_{mj}\log(m+\alpha_j) \neq 0 \right\}.$$

Obviously, in view of (3.1),

$$g_T(\underline{k}) = 1 \quad (3.2)$$

for $\underline{k} \in \{\underline{k}'\}$. If $\underline{k} \in \{\underline{k}''\}$, then integrating in (3.1), we find that

$$g_T(\underline{k}) = \frac{1 - \exp\left\{ -iT\sum_{j=1}^{r} {\sum}'_{m\in\mathbb{N}_0} k_{mj}\log(m+\alpha_j) \right\}}{iT\sum_{j=1}^{r} {\sum}'_{m\in\mathbb{N}_0} k_{mj}\log(m+\alpha_j)}.$$

This and (3.2) show that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in \{\underline{k}'\}, \\ 0 & \text{if } \underline{k} \in \{\underline{k}''\}. \end{cases}$$

The right-hand side of the later equality is continuous in the discrete topology. Therefore, by a continuity theorem for probability measures on compact groups, we obtain that $Q_{T,\underline{\alpha}}$, as $T\to\infty$, converges weakly to a probability measure $Q_{\underline{\alpha}}$ on $(\Omega^r, \mathcal{B}(\Omega^r))$ defined by the Fourier transform

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in \{\underline{k}'\}, \\ 0 & \text{if } \underline{k} \in \{\underline{k}''\}. \end{cases}$$

The lemma is proved. □

Unfortunately, the limit measure $Q_{\underline{\alpha}}$ in Lemma 3.1 is given by its Fourier transform, we do not know the explicit form of $Q_{\underline{\alpha}}$, and this reflects in Theorems 3.1 and 3.2 with non-effective set $F_{\alpha_1,\dots,\alpha_r}$. For example, if the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, then

$$g(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and we have that the limit measure $Q_{\underline{\alpha}}$ coincides with the Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$.

The next lemma is a joint limit theorem in the space $H^r(D)$ for absolutely convergent Dirichlet series.

Let σ_0 be a fixed number. For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, set

$$v_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m + \alpha_j}{n + \alpha_j} \right)^{\sigma_0} \right\}, \quad j = 1, \dots, r,$$

and define the functions

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

It is known [28] that the series for $\zeta_n(s, \alpha_j)$ are absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, let

$$\underline{\zeta}_n(s, \underline{\alpha}) = \left(\zeta_n(s, \alpha_1), \dots, \zeta_n(s, \alpha_r) \right),$$

and

$$P_{T,n,\underline{\alpha}} = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(s + i\tau, \underline{\alpha}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Lemma 3.2. *On $(\Omega^r, \mathcal{B}(\Omega^r))$, there exists a probability measure $P_{n,\underline{\alpha}}$ such that $P_{T,n,\underline{\alpha}}$ converges weakly to $P_{n,\underline{\alpha}}$ as $T \to \infty$.*

Proof. For $\omega_j \in \Omega_j$, define the functions

$$\zeta_n(s, \omega_j, \alpha_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

Since $|\omega_j(m)| = 1$, the series for $\zeta_n(s, \omega_j, \alpha_j)$ is also absolutely convergent for $\sigma > \frac{1}{2}$. Let

$$\underline{\zeta}_n(s, \omega, \underline{\alpha}) = (\zeta_n(s, \omega_1, \alpha_1), \dots, \zeta_n(s, \omega_r, \alpha_r)).$$

Consider the function $u_{n,\underline{\alpha}} : \Omega^r \to H^r(D)$ given by the formula

$$u_{n,\underline{\alpha}}(\omega) = \underline{\zeta}_n(s, \omega, \underline{\alpha}).$$

In virtue of the absolute convergence of the series for $\zeta_n(s, \omega_j, \alpha_j)$, $j = 1, \dots, r$, the function $u_{n,\underline{\alpha}}$ is continuous. Moreover,

$$u_{n,\underline{\alpha}} \left(\left((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0 \right), \dots, \left((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0 \right) \right) \\ = \underline{\zeta}_n(s + i\tau, \underline{\alpha}).$$

Therefore, for every $A \in \mathcal{B}(H^r(D))$,

$$P_{T,n,\underline{\alpha}} = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \left\{ \left((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0 \right), \dots, \right. \right. \\ \left. \left((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0 \right) \in u_{n,\underline{\alpha}}^{-1} A = Q_{T,\underline{\alpha}}(u_{n,\underline{\alpha}}^{-1} A).$$

Hence, $P_{T,n,\underline{\alpha}} = Q_{T,\underline{\alpha}} u_{n,\underline{\alpha}}^{-1}$. Therefore, Lemma 2.2, Lemma 3.1 and the continuity of the function $u_{n,\underline{\alpha}}$ imply that $P_{T,n,\underline{\alpha}}$ converges weakly to the measure $P_{n,\underline{\alpha}} = Q_{\underline{\alpha}} u_{n,\underline{\alpha}}^{-1}$ as $T \to \infty$, where $Q_{\underline{\alpha}}$ is the limit measure in Lemma 3.1. □

The next step of the proof of Theorem 3.3 consists of the approximation of $\underline{\zeta}(s, \underline{\alpha})$ by $\underline{\zeta}_n(s, \underline{\alpha})$. For this, we recall the metric in the space $H^r(D)$. It is known, see, for example, [9], that there exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset D$ such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and, for every compact set $K \subset D$, there exists K_l

such that $K \subset K_l$. Let, for $g_1, g_2 \in H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric in the space $H(D)$ inducing the topology of uniform convergence on compacta. Now, setting, for $\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j})$$

gives a metric in the space $H^r(D)$ inducing its product topology.

Lemma 3.3. *The equality*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \underline{\rho}\left(\underline{\zeta}(s + i\tau, \underline{\alpha}), \underline{\zeta}_n(s + i\tau, \underline{\alpha})\right) \mathrm{d}\tau = 0$$

holds.

Proof. The proof of the lemma does not depend on the arithmetic of the numbers $\alpha_1, \dots, \alpha_r$, and can be found in [32], Lemma 7. □

Now, we consider the sequence $\{P_{n,\underline{\alpha}} : n \in \mathbb{N}\}$, where $P_{n,\underline{\alpha}}$ is the limit measure in Lemma 3.2.

Lemma 3.4. *The sequence $P_{n,\underline{\alpha}}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K_\varepsilon \subset H^r(D)$ such that*

$$P_{n,\underline{\alpha}}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Proof. For an arbitrary α, $0 < \alpha < 1$, define

$$P_{T,n,\alpha}(A) = \frac{1}{T} \mathrm{meas}\left\{\tau \in [0, T] : \zeta_n(s + i\tau, \alpha) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

and denote by $P_{n,\alpha}$ the limit measure of $P_{T,n,\alpha}$ as $T \to \infty$. Then, in [3], it was obtained that the sequences $\{P_{n,\alpha} : n \in \mathbb{N}\}$ is tight. Hence, the sequences

$$\{P_{n,\alpha_j} : n \in \mathbb{N}\}, \quad j = 1, \dots, r,$$

are tight. Clearly, P_{n,α_j} are the marginal measures of the measure $P_{n,\underline{\alpha}}$, i.e.,

$$P_{n,\alpha_j}(A) = P_{n,\underline{\alpha}}\left(\underbrace{H(D)\times\cdots\times H(D)}_{j-1}\times A\times H(D)\times\cdots\times H(D)\right),$$
$$A \in \mathcal{B}(H(D)), \quad (3.3)$$

$j = 1, \ldots, r$. Since the sequence $\{P_{n,\alpha_j}\}$ is tight, for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that

$$P_{n,\alpha_j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \ldots, r, \quad (3.4)$$

for all $n \in \mathbb{N}$. We put $K = K_1 \times \cdots \times K_r$. Then the set K is compact in the space $H^r(D)$. Moreover, in view of (3.3) and (3.4),

$$\begin{aligned}
&P_{n,\underline{\alpha}}(H^r(D)\setminus K)\\
&= P_{n,\underline{\alpha}}\left(\bigcup_{j=1}^{r}\left(\underbrace{H(D)\times\cdots\times H(D)}_{j-1}\right)\times(H(D)\setminus K_j)\times H(D)\times\cdots\times H(D)\right)\\
&\leqslant \sum_{j=1}^{r} P_{n,\underline{\alpha}}\left(\underbrace{H(D)\times\cdots\times H(D)}_{j-1}\times(H(D)\setminus K_j)\times H(D)\times\cdots\times H(D)\right)\\
&= \sum_{j=1}^{r} P_{n,\alpha_j}(H(D)\setminus K_j) \leqslant \sum_{j=1}^{r}\frac{\varepsilon}{r} = \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$P_{n,\underline{\alpha}}(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The lemma is proved. □

Proof of Theorem 3.3. We will use the language of convergence in distribution ($\xrightarrow{\mathcal{D}}$). Let the random variable θ be defined on a certain probability space with measure μ, and be uniformly distributed on $[0,1]$. Define the $H^r(D)$-valued random element by the formula

$$X_{T,n,\underline{\alpha}} = X_{T,n,\underline{\alpha}}(s) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}).$$

Moreover, let $X_{n,\underline{\alpha}} = X_{n,\underline{\alpha}}(s)$ be the $H^r(D)$-valued random element having the distribution $P_{n,\underline{\alpha}}$. Then the assertion of Lemma 3.2 can be written in the form

$$X_{T,n,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,\underline{\alpha}}. \tag{3.5}$$

Since the sequence $\{P_{n,\underline{\alpha}} : n \in \mathbb{N}\}$ is tight, by the Prokhorov theorem ([7, Theorem 6.1]), it is relatively compact. Therefore, there is a subsequence $\{P_{n_k,\underline{\alpha}}\} \subset \{P_{n,\underline{\alpha}}\}$ such that $P_{n_k,\underline{\alpha}}$ converges weakly to a certain probability measure $P_{\underline{\alpha}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \to \infty$. In other words, we have the relation

$$X_{n_k,\underline{\alpha}} \xrightarrow[k\to\infty]{\mathcal{D}} P_{\underline{\alpha}}. \tag{3.6}$$

Define one more $H^r(D)$-valued random element $X_{T,\underline{\alpha}}$ by the formula

$$X_{T,\underline{\alpha}} = X_{T,\underline{\alpha}}(s) = \underline{\zeta}(s + i\theta T, \underline{\alpha}).$$

Then, the application of Lemma 3.3 shows that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}\left(X_{T,\underline{\alpha}}, X_{T,n,\underline{\alpha}}\right) \geqslant \varepsilon\right\} \\
&\quad= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\rho}\left(\underline{\zeta}(s+i\tau,\underline{\alpha}), \underline{\zeta}_n(s+i\tau,\underline{\alpha})\right) \geqslant \varepsilon\right\} \\
&\quad\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T}\int_0^T \underline{\rho}\left(\underline{\zeta}(s+i\tau,\underline{\alpha}), \underline{\zeta}_n(s+i\tau,\underline{\alpha})\right) \mathrm{d}\tau = 0.
\end{aligned}$$

The latter equality together with relations (3.5) and (3.6) shows that all hypotheses of lemma 2.7 are satisfied. Therefore, we obtain the relation

$$X_{T,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\alpha}},$$

which is equivalent to the weak convergence of $P_{T,\underline{\alpha}}$ to $P_{\underline{\alpha}}$ as $T \to \infty$. The theorem is proved. □

3.3 Proof of Theorems 3.1 and 3.2

Theorems 3.1 and 3.2 follow easily from Theorem 3.3. For this, the notion of the support of a probability measure is applied. Denote by $F_{\alpha_1,\dots,\alpha_r}$ the support of the limit measure $P_{\underline{\alpha}}$ in Theorem 3.3. We remind that $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ is a minimal closed set such that $P_{\underline{\alpha}}(F_{\alpha_1,\dots,\alpha_r}) = 1$. The set $F_{\alpha_1,\dots,\alpha_r}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighborhood G of $\underline{g}$, the inequality $P_{\underline{\alpha}}(G) > 0$ is satisfied.

Also, we will use two equivalents of the weak convergence of probability measures. We recall that a set A is a continuity set of the probability measure P if $P(\partial A) = 0$, where ∂A is the boundary of the set A.

Lemma 3.5. *Let P_n, $n \in \mathbb{N}$, and P be the probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then the following statements are equivalent:*

1° *P_n converges weakly to P as $n \to \infty$;*

2° *For every open set $G \subset \mathbb{X}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° *For every continuity set A of the measure P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

The lemma is Theorem 2.1 of [7].

Proof of Theorem 3.1. Suppose that $F_{\alpha_1,\dots,\alpha_r}$ is the support of the measure $P_{\underline{\alpha}}$. Then $F_{\alpha_1,\dots,\alpha_r}$ is non-empty closed set of the space $H^r(D)$.

Let $f_1, \dots, f_r \in F_{\alpha_1,\dots,\alpha_r}$, $K_1, \dots, K_r$ are compact sets of the strip D and $\varepsilon > 0$. Define

$$G_\varepsilon = \left\{ (g1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the set G_ε is an open neighborhood of the element $(f_1, \dots, f_r)$ which belongs to the support of the measure $P_{\underline{\alpha}}$. Therefore,

$$P_{\underline{\alpha}}(G_\varepsilon) > 0. \tag{3.7}$$

Moreover, in view of Theorem 3.3, and 1° and 2° of Lemma 3.5, we have that

$$\liminf_{T\to\infty} P_{T,\underline{\alpha}}(G_\varepsilon) \geqslant P_{\underline{\alpha}}(G_\varepsilon).$$

This, the definitions of $P_{T,\underline{\alpha}}$ and G_ε, and (3.6) show that

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

□

Proof of Theorem 3.2. We use the same notation as in the proof of Theorem 3.1. We observe that the boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Therefore, $P_{\underline{\alpha}}(G_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. This shows that the set G_ε is a continuity set of the measure $P_{\underline{\alpha}}$ for all but at most countably many $\varepsilon > 0$. Therefore, using Theorem 3.3, $1°$ and $3°$ of Lemma 3.5, and inequality (3.6), we obtain that the limit

$$\lim_{T\to\infty} P_{T,\underline{\alpha}}(G_\varepsilon) = P_{\underline{\alpha}}(G_\varepsilon) > 0$$

exists for all but at most countably many $\varepsilon > 0$. Thus, the definitions of $P_{T,\underline{\alpha}}$ and G_ε prove the theorem. □

Chapter 4

Joint discrete approximation of analytic functions by Hurwitz zeta-functions

Let $H(D)$ be the space of analytic functions on the strip $D = \{\sigma + it \in \mathbb{C} : 1/2 < \sigma < 1\}$. In Chapter 4, it is proved that there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H(D)$ such that every collection of the functions $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ is approximated by discrete shifts $(\zeta(s + ikh_1, \alpha_1), \dots, \zeta(s + ikh_r, \alpha_r))$, $h_j > 0$, $j = 1, \dots, r$, $k \in \mathbb{N} \cup \{0\}$, of Hurwitz zeta-functions with arbitrary parameters $\alpha_1, \dots, \alpha_r$.

4.1 Statements of the theorems

The aim of this chapter is a discrete version of Theorems 3.1 and 3.2. For brevity, let $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\underline{h} = (h_1, \dots, h_r)$.

Theorem 4.1. *Suppose that the numbers $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$ and positive numbers h_j, $j = 1, \dots, r$, are arbitrary. Then there exists a closed non-empty set $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ such that, for every compact sets $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\underline{\alpha},\underline{h}}$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j, \alpha_j) - f_j(s)| < \varepsilon \Big\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

It will be proved that the set $F_{\underline{\alpha},\underline{h}}$ is the support of a certain $H^r(D)$-valued random element.

4.2 Probabilistic results

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_{N,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}) \in A \right\},$$

where

$$\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}) = (\zeta(s + ikh_1, \alpha_1), \dots, \zeta(s + ikh_r, \alpha_r)).$$

In this section, we deal with weak convergence of $P_{N,\underline{\alpha},\underline{h}}$ as $N \to \infty$.

We start with definition of one probability space. Define

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore,

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, again is a compact topological Abelian group. Thus, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by $\omega_j(m)$ the mth component of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$, $m \in \mathbb{N}$. Characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m),$$

where the sign “$*$” shows that only a finite number of integers k_{jm} are distinct from zero. Therefore, putting $\underline{k} = \{k_{jm} : k_{jm} \in \mathbb{Z},\ m \in \mathbb{N}_0\}$, $j =, \dots, r$, we have that the Fourier transform $g(\underline{k}_1, \dots, \underline{k}_r)$ of a probability measure μ

on $(\Omega^r, \mathcal{B}(\Omega^r))$ is given by

$$g(\underline{k}_1, \ldots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m) \right) \mathrm{d}\mu. \tag{4.1}$$

Define two collections

$$A(\underline{\alpha}, \underline{h}) = \left\{ (\underline{k}_1, \ldots, \underline{k}_r) : \exp\left\{ -i \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}^{*} k_{jm} \log(m+\alpha_j) \right\} = 1 \right\}$$

and

$$B(\underline{\alpha}, \underline{h}) = \left\{ (\underline{k}_1, \ldots, \underline{k}_r) : \exp\left\{ -i \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}^{*} k_{jm} \log(m+\alpha_j) \right\} \neq 1 \right\}.$$

Let $Q_{\underline{\alpha},\underline{h}}$ be the probability measure on $(\Omega^r, \mathcal{B}(\Omega^r))$ having the Fourier transform

$$g_{\underline{\alpha},\underline{h}}(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in A(\underline{\alpha}, \underline{h}), \\ 0 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \in B(\underline{\alpha}, \underline{h}). \end{cases}$$

For $A \in \mathcal{B}(\Omega^r)$, define

$$\begin{aligned} Q_{N,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \Big(\Big((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0 \Big), \ \ldots, \\ \Big((m+\alpha_r)^{-ikh_r} : m \in \mathbb{N}_0 \Big) \Big) \in A \Big\}. \end{aligned}$$

Lemma 4.1. *$Q_{N,\underline{\alpha},\underline{h}}$ converges weakly to the measure $Q_{\underline{\alpha},\underline{h}}$ as $N \to \infty$.*

Proof. In view of (4.1), the Fourier transform $g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \ldots, \underline{k}_r)$ of $Q_{N,\underline{\alpha},\underline{h}}$ is given by

$$\begin{aligned} g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1, \ldots, \underline{k}_r) &= \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m) \right) \mathrm{d}Q_{N,\underline{\alpha},\underline{h}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha_j)^{-ikh_j k_{jm}} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{ -ik \sum_{j=1}^{r} h_j \sum_{m\in\mathbb{N}_0}^{*} k_{jm} \log(m+\alpha_j) \right\}. \end{aligned}$$

Thus, $g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = 1$ for $(\underline{k}_1,\dots,\underline{k}_r) \in A(\underline{\alpha},\underline{h})$. If $(\underline{k}_1,\dots,\underline{k}_r) \in B(\underline{\alpha},\underline{h})$, then by the sum formula of geometric progression, we have

$$g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = \frac{1-\exp\left\{-i(N+1)\sum\limits_{j=1}^{r} h_j \sum\limits_{m\in\mathbb{N}_0}^{*} k_{jm}\log(m+\alpha_j)\right\}}{(N+1)\left(1-\exp\left\{-i\sum\limits_{j=1}^{r} h_j \sum\limits_{m\in\mathbb{N}_0}^{*} k_{jm}\log(m+\alpha_j)\right\}\right)}.$$

Therefore,

$$\lim_{N\to\infty} g_{N,\underline{\alpha},\underline{h}}(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in A(\underline{\alpha},\underline{h}), \\ 0 & \text{if } (\underline{k}_1,\dots,\underline{k}_r) \in B(\underline{\alpha},\underline{h}), \end{cases}$$

This together with a continuity theorem for probability measures on compact groups proves the lemma. □

Now, let $\theta > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$v_n(m,\alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\theta}\right\}, \quad j = 1,\dots,r.$$

Define

$$\underline{\zeta}_n(s,\underline{\alpha}) = \left(\zeta_n(s,\alpha_1),\dots,\zeta_n(s,\alpha_r)\right),$$

where

$$\zeta_n(s,\alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r.$$

In view of the definition $v_n(m,\alpha_j)$, the latter Dirichlet series are absolutely convergent for $\sigma > \frac{1}{2}$. For $A \in \mathcal{B}(H^r(D))$, define

$$V_{N,n,\underline{\alpha},\underline{h}}(A) = \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \underline{\zeta}_n(s+ik\underline{h},\underline{\alpha}) \in A\right\}.$$

To obtain the weak convergence for $V_{N,n,\underline{\alpha},\underline{h}}$ as $N \to \infty$, introduce the mapping $u_{n,\underline{\alpha}} : \Omega^r \to H^r(D)$ given by

$$u_{n,\underline{\alpha}}(\omega) = \underline{\zeta}_n(s,\underline{\alpha},\omega), \quad \omega = (\omega_1,\dots,\omega_r) \in \Omega^r,$$

where

$$\underline{\zeta}_n(s, \underline{\alpha}, \omega) = (\zeta_n(s, \alpha_1, \omega_1), \ldots, \zeta_n(s, \alpha_r, \omega_r))$$

with

$$\zeta_n(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r.$$

Obviously, the latter series also are absolutely convergent for $\sigma > \frac{1}{2}$. Therefore, the mapping u_n is continuous, hence, it is $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Thus, the measure $Q_{\underline{\alpha},\underline{h}}$ defines the unique probability measure $V_{\underline{\alpha},\underline{h}}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ by the formula

$$V_{\underline{\alpha},\underline{h}}(A) = Q_{\underline{\alpha},\underline{h}}(u_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(D)).$$

Moreover, the definitions of $V_{N,n,\underline{\alpha},\underline{h}}$ and $Q_{N,\underline{\alpha},\underline{h}}$ imply the equality

$$V_{N,n,\underline{\alpha},\underline{h}}(A) = Q_{N,\underline{\alpha},\underline{h}}(u_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(D)).$$

All these remarks together with Lemma 4.1 and Lemma 2.2 lead to the following limit lemma.

Lemma 4.2. *$V_{N,n,\underline{\alpha},\underline{h}}$ converges weakly to $V_{n,\underline{\alpha},\underline{h}}$ as $N \to \infty$.*

To obtain a limit theorem for $P_{N,\underline{\alpha},\underline{h}}$, we need the estimation a distance between $\underline{\zeta}_n(s, \underline{\alpha})$ and $\underline{\zeta}(s, \underline{\alpha})$. Let $g_1, g_2 \in H(D)$. Recall that

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a certain sequence of compact subsets of the strip D, is a metric on $H(D)$ inducing its topology of uniform convergence on compacta. Let $\underline{g}_1 = (g_{11}, \ldots, g_{1r})$, $\underline{g}_2 = (g_{21}, \ldots, g_{2r}) \in H^r(D)$. Then

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j})$$

is a metric on $H^r(D)$ that induces the product topology.

Let θ be the same parameter as in definition of $v_n(m, \alpha_j)$, and

$$l_n(s, \alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) (n + \alpha)^{-s},$$

where $\Gamma(s)$ is the Euler gamma-function. Then the following integral representation is known [28].

Lemma 4.3. *For $s \in D$,*

$$\zeta(s,\alpha) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha) l_n(z,\alpha)\frac{\mathrm{d}z}{z}.$$

We will use some mean square results of discrete type. For the proof of them Gallagher lemma (Lemma 2.5) is useful.

Lemma 4.4. *Suppose that $0 < \alpha \leqslant 1$, $\frac{1}{2} < \sigma < 1$ and $h > 0$ are fixed numbers. Then, for every $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh + it, \alpha)|^2 \ll_{\alpha,\sigma,h} N(1+|t|).$$

Proof. It is well known that

$$\int_0^T |\zeta(\sigma+it,\alpha)|^2 \ll_{\alpha,\sigma} T$$

and

$$\int_0^T \left|\zeta'(\sigma+it,\alpha)\right|^2 \ll_{\alpha,\sigma} T.$$

Therefore, an application of Lemma 2.5 with $\delta = h$ gives the estimate of the lemma. □

The next lemma is very important for the proof of weak convergence for $P_{N,\underline{\alpha},\underline{h}}$.

Lemma 4.5. *For arbitrary $0 < \alpha_j \leqslant 1$ and $h_j > 0$, $j = 1, \dots, r$,*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} \underline{\rho}\left(\underline{\zeta}(s+ik\underline{h},\underline{\alpha}), \underline{\zeta}_n(s+ik\underline{h},\underline{\alpha})\right) = 0.$$

Proof. The definition of the metric $\underline{\rho}$ implies that it suffices to show the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} \rho\left(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)\right) = 0$$

for arbitrary $0 < \alpha \leqslant 1$ and $h > 0$. On the other hand, the latter equality is implied by

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)|=0 \tag{4.2}$$

for every compact subset $K\subset D$.

Thus, let $K\subset D$ be an arbitrary compact set. There exists $\varepsilon>0$ such that all points of the set K lie in the strip $\{s\in\mathbb{C}:\frac{1}{2}+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon\}$. Let $s=\sigma+it\in K$, and

$$\theta_1=\sigma-\frac{1}{2}-\varepsilon>0.$$

Then, in view of Lemma 4.3 and the residue theorem,

$$\zeta_n(s,\alpha)-\zeta(s,\alpha)=\frac{1}{2\pi i}\int\limits_{-\theta_1-i\infty}^{-\theta_1+i\infty}\zeta(s+z,\alpha)l_n(z,\alpha)\frac{\mathrm{d}z}{z}+R_n(s,\alpha),$$

where

$$R_n(s,\alpha)=\operatorname*{Res}_{z=1}\zeta(s+z,\alpha)l_n(z,\alpha)\frac{1}{z}=\frac{l_n(1-s,\alpha)}{1-s}.$$

Hence, for $s\in K$,

$$\begin{aligned}&\zeta_n(s+ikh,\alpha)-\zeta(s+ikh,\alpha)\\&\quad\ll\int_{-\infty}^{\infty}\left|\zeta(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha)\right|\sup_{s\in K}\left|\frac{l_n(\frac{1}{2}+\varepsilon-s+i\tau,\alpha)}{\frac{1}{2}+\varepsilon-s+i\tau}\right|\mathrm{d}\tau\\&\qquad+\sup_{s\in K}|R_n(s+ikh,\alpha)|.\end{aligned}$$

Therefore,

$$\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)|\ll I_1+I_2, \tag{4.3}$$

where

$$\begin{aligned}I_1=\int_{-\infty}^{\infty}&\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|\zeta(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha)\right|\right)\\&\times\sup_{s\in K}\left|\frac{l_n(\frac{1}{2}+\varepsilon-s+i\tau,\alpha)}{\frac{1}{2}+\varepsilon-s+i\tau}\right|\mathrm{d}\tau\end{aligned}$$

and

$$I_2 = \frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|R_n(s+ikh,\alpha)|.$$

The crucial role in the estimation of $l_n(s,\alpha)$ is played by the gamma-function. It is well known that there exists $c>0$ such that, uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}. \tag{4.4}$$

This estimate leads, for $\sigma+it\in K$, to

$$\begin{aligned}\frac{l_n(\frac{1}{2}+\varepsilon-\sigma-it+i\tau,\alpha)}{\frac{1}{2}+\varepsilon-\sigma-it+i\tau} &\ll \frac{(n+\alpha)^{\frac{1}{2}+\varepsilon-\sigma}}{\theta}\exp\{-(c/\theta)|\tau-t|\}\\ &\ll_{\theta,K} (n+\alpha)^{-\varepsilon}\exp\{-(c/\theta)|\tau|\}.\end{aligned}$$

Therefore, in view of Lemma 4.4,

$$\begin{aligned}I_1 \ll_{\theta,K} (n+\alpha)^{-\varepsilon}\int_{-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=0}^{N}\left|\zeta\Big(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\Big)\right|^2\right)^{\frac{1}{2}}\\ \times\exp\left\{-\Big(\frac{c}{\theta}\Big)|\tau|\right\}\mathrm{d}\tau \ll_{\theta,K,\varepsilon,h} (n+\alpha)^{-\varepsilon}.\end{aligned} \tag{4.5}$$

By estimate (4.4) again, we find that, for $s\in K$,

$$\begin{aligned}\frac{l_n(1-s-ikh,\alpha)}{1-s-ikh} &\ll_{\theta} (n+\alpha)^{1-\sigma}\exp\left\{-\Big(\frac{s}{\theta}\Big)|kh-t|\right\}\\ &\ll_{\theta,K} (n+\alpha)^{\frac{1}{2}-2\varepsilon}\exp\left\{-\Big(\frac{ch}{\theta}\Big)k\right\}.\end{aligned}$$

Therefore,

$$I_2 \ll_{\theta,K} (n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{1}{N}\sum_{k=0}^{N}\exp\left\{-\Big(\frac{ch}{\theta}\Big)k\right\} \ll_{\theta,K,h} (n+\alpha)^{\frac{1}{2}-2\varepsilon}\frac{\log N}{N}.$$

This, together with (4.5) and (4.3) proves the lemma. □

Now, we define the marginal measures of $V_{n,\underline{\alpha},\underline{h}}$. For $A\in\mathcal{B}(\Omega_j)$ define

$$Q_{N,\alpha_j,h_j}(A) = \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \Big((m+\alpha_j)^{-ikh_j} : m\in\mathbb{N}_0\Big)\in A\right\},\quad j=1,\dots,r.$$

Then by Lemma 1 of [45], Q_{N,α_j,h_j} converges weakly to a certain probability measure Q_{α_j,h_j} on $(\Omega_j, \mathcal{B}(\Omega_j))$ as $N \to \infty$, $j = 1, \dots, r$. Let the mapping $u_{n,\alpha_j} : \Omega_j \to H(D)$ be given by

$$u_{n,\alpha_j}(\omega_j) = \zeta_n(s, \alpha_j, \omega_j).$$

Define

$$V_{n,\alpha_j,h_j}(A) = Q_{\alpha_j,h_j} u_{n,\alpha_j}^{-1}(A) = Q_{\alpha_j,h_j}(u_{n,\alpha_j}^{-1} A), \qquad A \in \mathcal{B}(H(D)),\ j = 1, \dots, r.$$

Then in [45, Lemma 4], the following statement has been obtained.

Lemma 4.6. *For all $0 < \alpha_j \leqslant 1$ and $h_j > 0$, $j = 1, \dots, r$, the family of probability measures $\{V_{n,\alpha_j,h_j} : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that*

$$V_{n,\alpha_j,h_j}(K_j) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

We apply Lemma 4.6 for the family of probability measures $\{V_{n,\underline{\alpha},\underline{h}} : n \in \mathbb{N}\}$.

Lemma 4.7. *The family $\{V_{n,\underline{\alpha},\underline{h}} : n \in \mathbb{N}\}$ is tight.*

Proof. Let $\varepsilon > 0$ be an arbitrary number. By Lemma 4.6, there exists compact sets $K_1, \dots, K_r \subset H(D)$ such that

$$V_{n,\alpha_j,h_j}(K_j) > 1 - \frac{\varepsilon}{r} \tag{4.6}$$

for all $n \in \mathbb{N}$. Let $K = K_1 \times \cdots \times K_r$. Then K is a compact set in $H^r(D)$. Denoting

$$(H(D) \backslash K_j)_r = (\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \backslash K_j) \times H(D) \times \cdots \times H(D)),$$

by (4.6), we have

$$V_{n,\underline{\alpha},\underline{h}}(H^r(D)\backslash K) = V_{n,\underline{\alpha},\underline{h}}\left(\bigcup_{j=1}^{r}(H(D)\setminus K_j)\right)_r \leqslant \sum_{j=1}^{r} V_{n,\alpha_j,h_j}(H(D)\backslash K_j)$$
$$\leqslant \varepsilon$$

for all $n \in \mathbb{N}$. Thus,

$$V_{n,\underline{\alpha},\underline{h}}(K) \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$. □

Now we are in position to prove a limit theorem for $P_{N,\underline{\alpha},\underline{h}}$.

Theorem 4.3. *On $(H^r(D), \mathcal{B}(H^r(D)))$, there exists a probability measure $P_{\underline{\alpha},\underline{h}}$ such that $P_{N,\underline{\alpha},\underline{h}}$ converges weakly to $P_{\underline{\alpha},\underline{h}}$ as $N \to \infty$.*

Proof. Let ξ_N be a random variable defined on a certain probability space with measure μ and having the distribution

$$\mu\{\xi_N = k\} = \frac{1}{N+1}, \quad k = 0, 1, \ldots, N.$$

On the mentioned probability space, define the $H^r(D)$-valued random elements

$$X_{N,n,\underline{\alpha},\underline{h}} = X_{N,n,\underline{\alpha},\underline{h}}(s) = \zeta_n(s + i\xi_N \underline{h}, \underline{\alpha})$$

and

$$X_{N,\underline{\alpha},\underline{h}} = X_{N,\underline{\alpha},\underline{h}}(s) = \zeta(s + i\xi_N \underline{h}, \underline{\alpha}).$$

Moreover, let $Y_{n,\underline{\alpha},\underline{h}}$ be the $H^r(D)$-valued random element having the distribution $V_{n,\underline{\alpha},\underline{h}}$. Then, in view of Lemma 4.2,

$$X_{N,n,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} Y_{n,\underline{\alpha},\underline{h}}, \tag{4.7}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

By the Prokhorov theorem, see, for example, [7], every tight family of probability measures is relatively compact. Thus, in view of Lemma 4.7, the family $\{V_{n,\underline{\alpha},\underline{h}}\}$ is relatively compact. Therefore, there exists a subsequence $\{V_{n_l,\underline{\alpha},\underline{h}}\}$ weakly convergent to a certain probability measure $P_{\underline{\alpha},\underline{h}}$ as $l \to \infty$. Hence,

$$Y_{n_l,\underline{\alpha},\underline{h}} \xrightarrow[l\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}}. \tag{4.8}$$

Moreover, Lemma 4.5 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\left\{\underline{\rho}\left(X_{N,\underline{\alpha},\underline{h}}, X_{N,n,\underline{\alpha},\underline{h}}\right) \geqslant \varepsilon\right\}$$
$$\leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \underline{\rho}\left(\underline{\zeta}(s+ik\underline{h},\underline{\alpha}), \underline{\zeta}_n(s+ik\underline{h},\underline{\alpha})\right) = 0.$$

This, (4.7) and (4.8) together with Theorem 4.2 of [7] show that

$$X_{N,\underline{\alpha},\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\alpha},\underline{h}}.$$

Since the latter relation is equivalent to weak convergence of $P_{N,\underline{\alpha},\underline{h}}$ to $P_{\underline{\alpha},\underline{h}}$ as $N \to \infty$, the theorem is proved. □

4.3 Proof of approximation

Denote by $F_{\underline{\alpha},\underline{h}}$ the support of the limit measure $P_{\underline{\alpha},\underline{h}}$ in Theorem 4.3. Thus $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$ is a minimal closed set such that $P_{\underline{\alpha},\underline{h}}(F_{\underline{\alpha},\underline{h}}) = 1$. The set $F_{\underline{\alpha},\underline{h}}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighbourhood G of $\underline{g}$, the equality $P_{\underline{\alpha},\underline{h}}(G) > 1$ is satisfied. Obviously, $F_{\underline{\alpha},\underline{h}} \neq \varnothing$.

Proof. (Proof of Theorem 4.1).

1. Let $(f_1(s), \dots, f_r(s)) \in F_{\underline{\alpha},\underline{h}}$. Define the set

$$G_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\right\}.$$

Then G_ε is an open neighbourhood of an element of the support of the measure $P_{\underline{\alpha},\underline{h}}$, therefore $P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0$. Hence, by Theorem 4.3 and Lemma 3.5 (1°, 2°) in terms of open sets,

$$\liminf_{N\to\infty} P_{N,\underline{\alpha},\underline{h}}(G_\varepsilon) \geqslant P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0.$$

This, the definitions of $P_{N,\underline{\alpha},\underline{h}}$ and G_ε prove the first assertion of the theorem.

2. The boundary of the set G_ε lies in the set

$$\left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon\right\}.$$

Therefore, these boundaries do not intersect for different ε. Hence, the set

G_ε is a continuity set of the measure $P_{\underline{\alpha},\underline{h}}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 4.3 together with Lemma 3.5 (1°, 3°) in terms of continuity sets implies that

$$\lim_{N\to\infty} P_{N,\underline{\alpha},\underline{h}}(G_\varepsilon) = P_{\underline{\alpha},\underline{h}}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$, and the second assertion of the theorem is proved. □

Chapter 5

Approximation of analytic functions by the periodic Hurwitz zeta-functions

It is known that the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$ with transcendental or rational α is universal, i.e., its shifts $\zeta(s+i\tau,\alpha;\mathfrak{a})$ approximate all analytic functions defined in the strip $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$. In this chapter, it is proved that, for all $0<\alpha\leqslant 1$ and $\mathfrak{a}$, there exists a non-empty closed set $F_{\alpha,\mathfrak{a}}$ of analytic functions on D such that every function $f\in F_{\alpha,\mathfrak{a}}$ can be approximated by shifts $\zeta(s+i\tau,\alpha;\mathfrak{a})$.

5.1 Statements of the theorems

The function $\zeta(s,\alpha;\mathfrak{a})$ is also connected with the Lerch zeta-function $L(\lambda, \alpha, s)$, $\lambda\in\mathbb{R}$, which, for $\sigma>1$, is given by the series

$$L(\lambda,\alpha,s)=\sum_{m=0}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+\alpha)^s}.$$

For rational λ, the function $L(\lambda,\alpha,s)$ becomes the periodic Hurwitz zeta-function. Thus, the function $\zeta(s,\alpha;\mathfrak{a})$ is an extension of the Lerch zeta-function with rational parameter λ.

The function $\zeta(s,\alpha;\mathfrak{a})$ depends on the parameter α and periodic sequence $\mathfrak{a}$. The parameter α can be transcendental, rational or algebraic irrational. The case of transcendental α is the simplest one, and the universality of $\zeta(s,\alpha;\mathfrak{a})$

in this case has been proved in [22]. The universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with rational α has been studied in [46].

The universality of $\zeta(s, \alpha; \mathfrak{a})$ with algebraic irrational parameter α remains an open problem until our days. Therefore, in this chapter, we propose the following "approximation" to the universality of the function $\zeta(s, \alpha; \mathfrak{a})$. We state the theorems for all parameters α and sequences $\mathfrak{a}$ because they are a bit different from universality theorems with transcendental or rational parameter. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta.

Theorem 5.1. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that, for every compact subset $K \subset D$, $f(s) \in F_{\alpha,\mathfrak{a}}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

The positivity of a lower density of the set of shifts $\zeta(s + i\tau, \alpha; \mathfrak{a})$ can be replaced by the positivity of the density of that set. More precisely, we have the following modification of Theorem 5.1.

Theorem 5.2. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that, for every compact subset $K \subset D$ and $f(s) \in F_{\alpha,\mathfrak{a}}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Unfortunately, the set $F_{\alpha,\mathfrak{a}}$ is not explicitly given. We will prove, that this set is the support of a certain probability measure.

Theorems 5.1 and 5.2 can be generalized for some compositions. We give one example.

Theorem 5.3. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that if $\Phi : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty, then, for* *every compact subset $K \subset D$, $f(s) \in \Phi(F_{\alpha,\mathfrak{a}})$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi(\zeta(s+i\tau,\alpha;\mathfrak{a})) - f(s)| < \varepsilon\right\} > 0.$$

The next theorem is an analogue of Theorem 5.2 for the composition $\Phi(\zeta(s+i\tau,\alpha;\mathfrak{a}))$.

Theorem 5.4. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then there exists a non-empty closed set $F_{\alpha,\mathfrak{a}} \subset H(D)$ such that if $\Phi : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty, then, for every compact subset $K \subset D$ and $f(s) \in \Phi(F_{\alpha,\mathfrak{a}})$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi(\zeta(s+i\tau,\alpha;\mathfrak{a})) - f(s)| < \varepsilon\right\} > 0.$$

exists for all but at most countably many $\varepsilon > 0$.

For the proof of above theorems, we apply a probabilistic approach based on limit theorems for weakly convergent probability measures in the space $H(D)$. These limit theorems will be proved in the next section.

5.2 Limit theorems

In this section, we will prove a limit theorem for

$$P_{T,\alpha,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau,\alpha;\mathfrak{a}) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$.

Theorem 5.5. *Suppose that the parameter α, $0 < \alpha \leqslant 1$, and the periodic sequence $\mathfrak{a}$ are arbitrary. Then, on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{\alpha,\mathfrak{a}}$ such that $P_{T,\alpha,\mathfrak{a}}$ converges weakly to $P_{\alpha,\mathfrak{a}}$ as $T \to \infty$.*

First, we will prove a limit theorem for probability measures on $(\Omega, \mathcal{B}(\Omega))$, where

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

and $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all non-negative integers m. Since the unit circle is a compact set, by the Tikhonov theorem, the infinite-dimensional torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Denote by $\omega(m)$, $m \in \mathbb{N}_0$, the m-th component of an element $\omega \in \Omega$. Clearly, $\big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big)$ is an element of Ω. For $A \in (\Omega, \mathcal{B}(\Omega))$, define

$$Q_{T,\alpha}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big) \in A\right\}.$$

In the sequel, we assume that the parameter α, $0 < \alpha \leqslant 1$, and the sequence $\mathfrak{a}$ are arbitrary.

Lemma 5.1. *On* $(\Omega, \mathcal{B}(\Omega))$, *there exists a probability measure* Q_α *such that* $Q_{T,\alpha}$ *converges weakly to* Q_α *as* $T \to \infty$

Proof. The dual group of Ω is isomorphic to

$$\bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}$. Therefore, the characters χ of the group Ω are of the form

$$\chi(m) = \prod_{m=0}^{\infty} \omega^{k_m}(m), \quad \omega \in \Omega,$$

where only a finite number of integers k_m are distinct from zero. Hence, the Fourier transform $g_{T,\alpha,\mathfrak{a}}(\underline{k})$, $\underline{k} = (k_m \in \mathbb{Z} : m \in \mathbb{N}_0)$, of $Q_{T,\alpha}$ is defined by

$$g_{T,\alpha}(\underline{k}) = \int_\Omega \left(\prod_{m=0}^{\infty} \omega^{k_m(m)}\right) \mathrm{d}Q_{T,\alpha}, \tag{5.1}$$

where only a finite number of integers k_m are distinct from zero. We consider $g_{T,\alpha}$ as $T \to \infty$. Define two collections

$$\underline{k}_{1,\alpha} = \left\{\underline{k} = \{k_m \in \mathbb{Z}\} : \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) = 0\right\}$$

and

$$\underline{k}_{2,\alpha} = \left\{\underline{k} = \{k_m \in \mathbb{Z}\} : \sum_{m=0}^{\infty}{}' k_m \log(m+\alpha) \neq 0\right\},$$

where the sign “ ′ ” means that only a finite number of integers k_m are distinct from zero. In view of (5.1) and the definition of $Q_{T,\alpha}$,

$$\begin{aligned} g_{T,\alpha}(\underline{k}) &= \frac{1}{T}\int_0^T \left(\prod_{m=0}^{\infty}{}'(m+\alpha)^{-i\tau k_m}\right) \mathrm{d}\tau \\ &= \frac{1}{T}\int_0^T \exp\left\{-i\tau \sum_{m=0}^{\infty}{}'k_m \log(m+\alpha)\right\} \mathrm{d}\tau. \end{aligned}$$

Obviously,

$$g_{T,\alpha}(\underline{k}) = 1 \tag{5.2}$$

for $\underline{k} \in \{\underline{k}_{1,\alpha}\}$. If $\underline{k} \in \{\underline{k}_{2,\alpha}\}$, then after integration we have that

$$g_{T,\alpha}(\hat{\underline{k}}_{0,\alpha}) = \frac{1-\exp\left\{-iT \sum\limits_{m=0}^{\infty}{}'k_m \log(m+\alpha)\right\}}{iT \sum\limits_{m=0}^{\infty}{}'k_m \log(m+\alpha)}.$$

Therefore, in this case,

$$\lim_{T\to\infty} g_{T,\alpha}(\underline{k}) = 0.$$

This and (5.2) show that

$$\lim_{T\to\infty} g_{T,\alpha}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} \in \{\underline{k}_{1,\alpha}\}, \\ 0 & \text{if} \quad \underline{k} \in \{\underline{k}_{2,\alpha}\}. \end{cases}$$

Therefore, we obtained that the Fourier transform of $Q_{T,\alpha}$, as $T \to \infty$, converges to a continuous function in the discrete topology. Thus, by a continuity theorem for probability measures on compact groups, we deduce that $Q_{T,\alpha}$, as $T \to \infty$, converges weakly to a probability measure Q_α on $(\Omega, \mathcal{B}(\Omega))$ given by the Fourier transform

$$g_\alpha(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} \in \{\underline{k}_{1,\alpha}\}, \\ 0 & \text{if} \quad \underline{k} \in \{\underline{k}_{2,\alpha}\}. \end{cases}$$

□

Lemma 5.1 is applied for proving a limit theorem for absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number. Define

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\theta}\right\}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},$$

and

$$\zeta_n(s,\alpha,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then it is known [23] that the latter series is absolutely convergent for $\sigma > \frac{1}{2}$. For $A \in \mathcal{B}(H(D))$, let

$$P_{T,n,\alpha,\mathfrak{a}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta_n(s+i\tau,\alpha,\mathfrak{a}) \in A\right\}.$$

Lemma 5.2. *On $(H(D),\mathcal{B}(H(D)))$, there exists a probability measure $V_{n,\alpha,\mathfrak{a}}$ such that $P_{T,n,\alpha,\mathfrak{a}}$ converges weakly to $V_{n,\alpha,\mathfrak{a}}$ as $T \to \infty$.*

Proof. Consider the function $u_{n,\alpha,\mathfrak{a}} : \Omega \to H(D)$ given by

$$u_{n,\alpha,\mathfrak{a}}(\omega) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}, \quad \omega \in \Omega.$$

Since $|\omega(m)| = 1$, the latter series, as the series for $\zeta_n(s,\alpha,\mathfrak{a})$, is absolutely convergent for $\sigma > \frac{1}{2}$. Hence, the function $u_{n,\alpha,\mathfrak{a}}$ is continuous, therefore, it is $(\mathcal{B}(\Omega),\mathcal{B}(H(D)))$-measurable. Thus, the limit measure Q_α of Lemma 5.1 induces on $(H(D),\mathcal{B}(H(D)))$ the unique probability measure $V_{n,\alpha,\mathfrak{a}} \overset{def}{=} Q_\alpha u_{n,\alpha,\mathfrak{a}}^{-1}$, where

$$Q_\alpha u_{n,\alpha,\mathfrak{a}}^{-1}(A) = Q_\alpha(u_{n,\alpha,\mathfrak{a}}^{-1}A), \quad A \in \mathcal{B}(H(D)).$$

Clearly,

$$u_{n,\alpha,\mathfrak{a}}\left((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\right) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^{s+i\tau}}.$$

Therefore,

$$\begin{aligned} P_{T,n,\alpha,\mathfrak{a}}(A) &= \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \left((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\right) \in u_{n,\alpha,\lambda}^{-1}A\right\} \\ &= Q_{T,\alpha}\left(u_{n,\alpha,\mathfrak{a}}^{-1}A\right) = Q_{T,\alpha}u_{n,\alpha,\mathfrak{a}}^{-1}(A). \end{aligned}$$

This shows that $P_{T,n,\alpha,\mathfrak{a}} = Q_{T,\alpha}u_{n,\alpha,\mathfrak{a}}^{-1}$. Since the function $u_{n,\alpha,\mathfrak{a}}$ is continuous, and, by Lemma 5.1, $Q_{T,\alpha}$, as $T \to \infty$, converges weakly to Q_α, we obtain by Lemma 2.2 that $P_{T,n,\alpha,\mathfrak{a}}$ converges weakly to $V_{n,\alpha,\mathfrak{a}} = Q_\alpha u_{n,\alpha,\mathfrak{a}}^{-1}$ as $T \to \infty$. □

Now, we will approximate in the mean the function $\zeta(s,\alpha;\mathfrak{a})$ by $\zeta_n(s,\alpha;\mathfrak{a})$.

Denote by ρ the metric in $H(D)$ inducing the topology of uniform convergence on compacta.

Lemma 5.3. *The equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\left(\zeta(s+i\tau,\alpha;\mathfrak{a}),\zeta_n(s+i\tau,\alpha;\mathfrak{a})\right)\,\mathrm{d}\tau=0$$

holds.

Proof. Let the number θ be from the definition of the function $v_n(m,\alpha)$. Then it is known [23] that, for $\sigma>\frac{1}{2}$,

$$\zeta_n(s,\alpha;\mathfrak{a})=\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z,\alpha;\mathfrak{a})l_n(z,\alpha)\frac{\mathrm{d}z}{z}, \tag{5.3}$$

where

$$l_n(s,\alpha)=\frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)(n+\alpha)^s,$$

and $\Gamma(s)$ denote Euler gamma-function. For an arbitrary $\beta>0$, the residue theorem and (5.3) give

$$\begin{aligned}\zeta_n(s,\alpha;\mathfrak{a})-\zeta(s,\alpha;\mathfrak{a})=&\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\zeta(s+z,\alpha;\mathfrak{a})l_n(z,\alpha)\frac{\mathrm{d}z}{z}\\&+\operatorname*{Res}_{z=1-s}\frac{l_n(z,\alpha)}{z}\zeta(s+z,\alpha;\mathfrak{a}).\end{aligned} \tag{5.4}$$

Since

$$\operatorname*{Res}_{s=1}\zeta(s,\alpha;\mathfrak{a})=\frac{1}{q}\sum_{l=0}^{q-1}a_l\stackrel{def}{=}r,$$

we have

$$\operatorname*{Res}_{z=1-s}\frac{l_n(z,\alpha)}{z}\zeta(s+z,\alpha;\mathfrak{a})=r\frac{l_n(1-s,\alpha)}{1-s}. \tag{5.5}$$

By the definition,

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|},\quad g_1,g_2\in H(D),$$

where $\{K_l:l\in\mathbb{N}\}\subset D$ is a sequence of compact sets such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then K lies in some K_l. Therefore, it suffices to prove that, for every compact set K,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - \zeta_n(s+i\tau,\alpha;\mathfrak{a})| \, \mathrm{d}\tau = 0. \quad (5.6)$$

Thus, let K be a fixed compact subset of D. Suppose that $\varepsilon > 0$ is such that $\frac{1}{2} + 2\varepsilon \leqslant \mathrm{Re} s \leqslant 1 - \varepsilon$ for any point $s \in K$. Denote by $s = \sigma + iv$ the points of K. Moreover, let

$$\beta = \sigma - \varepsilon - \frac{1}{2} \quad \text{and} \quad \theta = \frac{1}{2} + \varepsilon.$$

Then, from (5.4) and (5.5) we find

$$\begin{aligned} &|\zeta(s+i\tau,\alpha;\mathfrak{a}) - \zeta_n(s+i\tau,\alpha;\mathfrak{a})| \\ &\quad \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |\zeta(s+i\tau-\beta+it,\alpha;\mathfrak{a})| \frac{|l_n(-\beta+it,\alpha)|}{|-\beta+it|} \, \mathrm{d}t \\ &\qquad + |r| \frac{|l_n(1-s-i\tau,\alpha)|}{|1-s-i\tau|}. \end{aligned}$$

Recall that $s = \sigma + iv$. We take t in place of $t + v$ in the above integral. This shift gives

$$\begin{aligned} &|\zeta(s+i\tau,\alpha;\mathfrak{a}) - \zeta_n(s+i\tau,\alpha;\mathfrak{a})| \\ &\quad \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau),\alpha;\mathfrak{a}\right)\right| \frac{|l_n(1/2+\varepsilon-s+it,\alpha)|}{|1/2+\varepsilon-s+it|} \, \mathrm{d}t \\ &\qquad + |r| \frac{|l_n(1-s-i\tau,\alpha)|}{|1-s-i\tau|}. \end{aligned}$$

From this, we obtain

$$\frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - \zeta_n(s+i\tau,\alpha;\mathfrak{a})| \, \mathrm{d}\tau \leqslant I_1 + I_2, \quad (5.7)$$

where

$$\begin{aligned} I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} &\left(\frac{1}{T} \int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau),\alpha;\mathfrak{a}\right)\right| \mathrm{d}\tau \right) \\ &\times \sup_{s\in K} \frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it,\alpha\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|} \, \mathrm{d}t \end{aligned}$$

and

$$I_2 = |r| \frac{1}{T} \int_0^T \sup_{s \in K} \frac{|l_n(1-s-i\tau, \alpha)|}{|1-s-i\tau|} \, \mathrm{d}\tau.$$

By the definition of $l_n(s, \alpha)$ and the estimate

$$\Gamma(\sigma + it) \ll \mathrm{e}^{-c|t|}, \quad c > 0,$$

uniform for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, we find

$$\begin{aligned} \frac{\left|l_n\left(\frac{1}{2}+\varepsilon-s+it, \alpha\right)\right|}{\left|\frac{1}{2}+\varepsilon-s+it\right|} &\ll \frac{(n+\alpha)^{-\varepsilon}}{\theta} \exp\left\{-\frac{c}{\theta}|t-v|\right\} \\ &\ll_{\theta,K} (n+\alpha)^{-\varepsilon} \exp\left\{-\frac{c}{\theta}|t|\right\}. \end{aligned} \tag{5.8}$$

It is well known that, for

$$\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+it, \alpha\right)\right|^2 \mathrm{d}t \ll T.$$

Hence, in view of (5.1)

$$\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+it, \alpha; \mathfrak{a}\right)\right|^2 \mathrm{d}t \ll T.$$

Therefore,

$$\begin{aligned} &\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau), \alpha; \mathfrak{a}\right)\right| \mathrm{d}\tau \\ &\qquad \ll \left(T \int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i(t+\tau), \alpha; \mathfrak{a}\right)\right|^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \ll T(1+|t|). \end{aligned}$$

This and (5.8) imply the estimate

$$I_1 \ll_{\theta,K} (n+\alpha)^{-\varepsilon} \int_{-\infty}^{\infty} (1+|t|) \exp\left\{-\frac{c}{\theta}|t|\right\} \mathrm{d}t \ll_{\theta,K} (n+\alpha)^{-\varepsilon}. \tag{5.9}$$

Similarly as above, we obtain that

$$\sup_{s \in K} \frac{|l_n(1-s-i\tau, \alpha)|}{|1-s-i\tau|} \ll_{\theta,K} (n+\alpha)^{1-\sigma} \exp\left\{-\frac{c}{\theta}|\tau|\right\}.$$

Hence,

$$I_2 \ll_{\theta,K} |r| \frac{(n+\alpha)^{1-\sigma}}{T}. \tag{5.10}$$

Now, (5.7), (5.9) and (5.10) imply the equality (5.6). □

Now, we will examine the sequence $\{V_{n,\alpha,\mathfrak{a}} : n \in \mathbb{N}\}$, where $V_{n,\alpha,\mathfrak{a}}$ is the limit measure in Lemma 5.2.

Lemma 5.4. *The sequence $\{V_{n,\alpha,\mathfrak{a}} : n \in \mathbb{N}\}$ is relatively compact, i.e., every subsequence $\{V_{n_k,\alpha,\mathfrak{a}}\} \subset \{V_{n,\alpha,\mathfrak{a}}\}$ contains an another subsequence weakly convergent to a certain probability measure on $(H(D), \mathcal{B}(H(D)))$.*

Proof. In virtue of the Prokhorov theorem [7, Theorem 6.1], it is sufficient to prove that the sequence $\{V_{n,\alpha,\mathfrak{a}}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$V_{n,\alpha,\mathfrak{a}}(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Let ξ be a random variable on a certain probability space with measure $\mathbb{P}$ and uniformly distributed on $[0, 1]$, and let $Y_{n,\alpha,\mathfrak{a}} = Y_{n,\alpha,\mathfrak{a}}(s)$ be the $H(D)$-valued random element whose distribution is $V_{n,\alpha,\mathfrak{a}}$. Moreover, let

$$X_{T,n,\alpha,\mathfrak{a}} = X_{T,n,\alpha,\mathfrak{a}}(s) = \zeta_n(s + iT\xi, \alpha; \mathfrak{a}).$$

Then the assertion of Lemma 5.2 is equivalent to the relation

$$X_{T,n,\alpha,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} Y_{n,\alpha,\mathfrak{a}}. \tag{5.11}$$

We have mentioned that the series for $\zeta_n(s, \alpha; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$. Hence, by properties of absolutely convergent series, for fixed $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma + it, \alpha; \mathfrak{a})|^2 \, dt &= \sum_{m=0}^{\infty} \frac{|a_m|^2 \, v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \\ &\leqslant \sum_{m=0}^{\infty} \frac{|a_m|^2}{(m+\alpha)^{2\sigma}} \leqslant C_{\alpha,\mathfrak{a}} < \infty. \end{aligned} \tag{5.12}$$

Let K_l be a compact set from the definition of the metric ρ. Then (1.2) and the

Cauchy integral formula imply

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha;\mathfrak{a})| \,\mathrm{d}\tau \leqslant R_{l,\alpha,\mathfrak{a}} < \infty.$$

Now, taking $M = M_{l,\alpha,\mathfrak{a}}(\varepsilon) = R_{l,\alpha,\mathfrak{a}} 2^l \varepsilon^{-1}$, where $\varepsilon > 0$ is an arbitrary fixed number, we find that

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n,\alpha,\mathfrak{a}}(s)| > M\right) \\
&\qquad = \limsup_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau\in[0,T] : \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha;\mathfrak{a})| > M\right\} \\
&\qquad \leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{TM}\int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha;\mathfrak{a})| \,\mathrm{d}\tau \leqslant \frac{\varepsilon}{2^l}
\end{aligned} \tag{5.13}$$

for all $l \in \mathbb{N}$. The set

$$K = K(\varepsilon) = \left\{g\in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_{l,\alpha,\mathfrak{a}}(\varepsilon),\ l\in\mathbb{N}\right\}$$

is compact in the space $H(D)$, and, by (5.11) and (5.13),

$$\mathbb{P}\left(Y_{n,\alpha,\mathfrak{a}} \in K\right) \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$. Hence,

$$V_{n,\alpha,\mathfrak{a}}(K) \geqslant 1-\varepsilon$$

for all $n \in \mathbb{N}$. □

Now, we are in position to prove Theorem 5.5.

Proof of Theorem 5.5. In view of Lemma 5.4, there exists a subsequence $\{V_{n_k,\alpha,\mathfrak{a}}\} \subset \{V_{n,\alpha,\mathfrak{a}}\}$ weakly convergent to a certain probability measure $P_{\alpha,\mathfrak{a}}$ on $(H(D), \mathcal{B}(H(D)))$ as $k\to\infty$. Thus, we have the relation

$$Y_{n_k,\alpha,\mathfrak{a}} \xrightarrow[k\to\infty]{\mathcal{D}} P_{\alpha,\mathfrak{a}}. \tag{5.14}$$

Define

$$X_{T,\alpha,\mathfrak{a}} = X_{T,\alpha,\mathfrak{a}}(s) = \zeta(s+iT\xi,\alpha;\mathfrak{a}).$$

Then an application of Lemma 5.3 gives, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho\left(X_{T,\alpha,\mathfrak{a}}, X_{T,n,\alpha,\mathfrak{a}}\right) \geqslant \varepsilon\right) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \mathrm{meas}\left\{\tau \in [0,T] : \rho\left(\zeta(s+i\tau,\alpha;\mathfrak{a}), \zeta_n(s+i\tau,\alpha;\mathfrak{a})\right) \geqslant \varepsilon\right\} \\
&\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho\left(\zeta(s+i\tau,\alpha;\mathfrak{a}), \zeta_n(s+i\tau,\alpha;\mathfrak{a})\right) \, \mathrm{d}\tau = 0.
\end{aligned}$$

This equality, and relations (5.11) and (5.14) show that the hypotheses of Theorem 4.2 of [7] are satisfied. Hence,

$$X_{T,\alpha,\mathfrak{a}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\alpha,\mathfrak{a}},$$

or equivalently, $P_{T,\alpha,\mathfrak{a}}$ converges weakly to $P_{\alpha,\mathfrak{a}}$ as $T \to \infty$.

Corollary 5.1. *Suppose that $\Phi : H(D) \to H(D)$ is a continuous operator, and, for $A \in \mathcal{B}(H(D))$,*

$$P_{T,\Phi,\alpha,\mathfrak{a}}(A) = \frac{1}{T} \mathrm{meas}\left\{\tau \in [0,T] : \Phi(\zeta(s+i\tau,\alpha;\mathfrak{a})) \in A\right\}.$$

Then $P_{T,\Phi,\alpha,\mathfrak{a}}$ converges weakly to $\Phi^{-1}P_{\alpha,\mathfrak{a}}$ as $T \to \infty$.

Proof. It follows from the definitions of $P_{T,\Phi,\alpha,\mathfrak{a}}$ and $P_{T,\alpha,\mathfrak{a}}$ that $P_{T,\Phi,\alpha,\mathfrak{a}} = P_{T,\alpha,\mathfrak{a}}\Phi^{-1}$. Thus, the corollary is a result of Theorem 5.5, the continuity of Φ and Theorem 5.1 of [7]. □

5.3 Proof of the main theorems

Proof of Theorem 5.1. Let $F_{\alpha,\mathfrak{a}}$ be the support of the limit measure $P_{\alpha,\mathfrak{a}}$ in Theorem 5.5. By the definition of a support, $F_{\alpha,\mathfrak{a}}$ is a minimal closed set of the space $H(D)$ such that $P_{\alpha,\mathfrak{a}}\left(F_{\alpha,\mathfrak{a}}\right) = 1$. Thus, $F_{\alpha,\mathfrak{a}} \neq \varnothing$, and consists of all elements $g \in H(D)$ such that, for every open neighbourhood G of g, the inequality $P_{\alpha,\mathfrak{a}}(G) > 0$ is satisfied.

For $f \in F_{\alpha,\mathfrak{a}}$, define the set

$$G_\varepsilon = \left\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\right\}.$$

Then G_ε is an open neighbourhood of the element f of the support of the measure $P_{\alpha,\mathfrak{a}}$. Therefore $P_{\alpha,\mathfrak{a}}(G_\varepsilon) > 0$. Now, using the equivalent of weak convergence of probability measures in terms of open sets [7, Theorem 2.1], we deduce from Theorem 5.5 the inequality

$$\liminf_{T\to\infty} P_{T,\alpha,\mathfrak{a}}(G_\varepsilon) \geqslant P_{\alpha,\mathfrak{a}}(G_\varepsilon) > 0.$$

This and the definitions of $P_{T,\alpha,\mathfrak{a}}$ and G_ε prove the theorem.

Proof of Theorem 5.2. We preserve the notation of the set G_ε. The boundary ∂G_ε of G_ε lies in the set $\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\}$. Hence, we have that $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \varnothing$ for $\varepsilon_1 \neq \varepsilon_2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. This shows that the set G_ε is a continuity set of the measure $P_{\alpha,\mathfrak{a}}$, i.e., $P_{\alpha,\mathfrak{a}}(\partial G_\varepsilon) = 0$, for all but at most countably many $\varepsilon > 0$. Applying the equivalent of weak convergence of probability measures in terms of continuity sets [7, Theorem 2.1], we deduce from Theorem 5.5 the inequality

$$\lim_{T\to\infty} P_{T,\alpha,\mathfrak{a}}(G_\varepsilon) = P_{\alpha,\mathfrak{a}}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$, and the definitions of $P_{T,\alpha,\mathfrak{a}}$ and G_ε prove the theorem.

Proof of Theorem 5.3. By Corollary 5.1, $P_{T,\Phi,\alpha,\mathfrak{a}}$ converges weakly to $P_{\alpha,\mathfrak{a}}\Phi^{-1}$ as $T \to \infty$. We will show that the support of the measure $P_{\alpha,\mathfrak{a}}\Phi^{-1}$ contains the closure of the set $\Phi(F_{\alpha,\mathfrak{a}})$.

We suppose that, for every open set $\hat{G} \neq \varnothing$, the set $\left(\Phi^{-1}\hat{G}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty. Let g be an arbitrary element of the set $\Phi(F_{\alpha,\mathfrak{a}})$, and G be its any open neighbourhood. Then, the set $\Phi^{-1}G$ is open, and contains an element of the set $F_{\alpha,\mathfrak{a}}$. However, $F_{\alpha,\mathfrak{a}}$ is the support of the measure $P_{\alpha,\mathfrak{a}}$. Therefore, $P_{\alpha,\mathfrak{a}}\left(\Phi^{-1}G\right) > 0$, and

$$P_{\alpha,\mathfrak{a}}\Phi^{-1}(G) = P_{\alpha,\mathfrak{a}}\left(\Phi^{-1}G\right) > 0.$$

Hence, the support of the measure $P_{\alpha,\mathfrak{a}}\Phi^{-1}$ contains the set $\Phi(F_{\alpha,\mathfrak{a}})$, and, as a closed set, contains the closure of $\Phi(F_{\alpha,\mathfrak{a}})$.

Now, suppose that, for every polynomial $p = p(s)$, the set $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty. We will show that then, for every open set $G \subset H(D)$, the set $\left(\Phi^{-1}G\right) \cap F_{\alpha,\mathfrak{a}}$ is non-empty. It is well known that the approximation in the space $H(D)$ reduces to that on compact sets with connected complements, i.e., in the definition of the metric ρ we may take the sets K_l with connected complements. Let$K \subset D$ be a compact set with connected complement, and

$g \in G$. Then, by the Mergelyan theorem [50], for every $\varepsilon > 0$ there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}.$$

Then, by the above remark, we have that $\rho(g, p) < \varepsilon$ provided the set K is well chosen. Thus, if ε is small enough, then $p \in G$. Therefore, $\Phi^{-1}\{p\} \subset \Phi^{-1}G$, and the set $\left(\Phi^{-1}G\right) \cap F_{\alpha,\mathfrak{a}} \neq \varnothing$ if $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}} \neq \varnothing$.

For $f \in \Phi(F_{\alpha,\mathfrak{a}})$, define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then G_ε is an open neighbourhood of the element f of the support of the measure $P_{\alpha,\mathfrak{a}}\Phi^{-1}$. Thus $P_{\alpha,\mathfrak{a}}\Phi^{-1}(G_\varepsilon) > 0$. Hence, by Corollary 5.1,

$$\liminf_{T \to \infty} P_{T,\Phi,\alpha,\mathfrak{a}}(G_\varepsilon) \geqslant P_{\alpha,\mathfrak{a}}\Phi^{-1}(G_\varepsilon) > 0.$$

This and the definitions of $P_{T,\Phi,\alpha,\mathfrak{a}}$ and G_ε give the assertion of the theorem.

Proof of Theorem 5.4. We repeat with evident changes the arguments of the proof of Theorem 5.2.

Chapter 6

Conclusions

The results of the dissertation lead us to the following conclusions.

1. The approximation of a wide class of analytic functions by the shifts $\zeta(s + i\gamma_k h, F)$ of zeta-functions of cusp forms, where γ_k is a sequence of imaginary parts of non-trivial zeros of the Riemann zeta function and $h > 0$, is valid.

2. The theorem on joint continuous approximation of a certain set of analytic functions by shifts of Hurwitz zeta-functions $\zeta(s + i\tau, \alpha_1), ...,$ $\zeta(s + i\tau, \alpha_r)$ with arbitrary parameters $\alpha_1 ... \alpha_r$ is valid.

3. The theorem on joint discrete approximation of a certain set of analytic functions by discrete shifts of Hurwitz zeta-functions $\zeta(s+ikh_1, \alpha_1), ...,$ $\zeta(s + ikh_r, \alpha_r)$ with arbitrary parameters $\alpha_1 ... \alpha_r$ and $h_j > 0$ is valid.

4. The theorem on approximation of analytic functions by shifts of periodic Hurwitz zeta-functions $\zeta(s + i\tau, \alpha; \mathfrak{a})$ is valid.

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Santrauka (Summary in Lithuanian)

Tyrimo objektas

Disertacijos tyrimo objektas - kai kurių dzeta funkcijų aproksimavimo savybės. Prisiminkime, kad dzeta funkcijos kurioje nors kompleksinėje $\sigma > \sigma_0$ pusplokštumėje yra apibrėžiamos Dirichlė eilute

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

su koeficientais $a_m \in \mathbb{C}$, turinčiais vienokią ar kitokią aritmetinę prasmę. Dzeta funkcijų "motina" yra laikoma Rymano dzeta funkcija $\zeta(s)$, $s = \sigma + it$, kuri pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

Funkcija $\zeta(s)$ yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus tašką $s = 1$, kuris yra paprastasis polius su reziduumu 1. Funkcija $\zeta(s)$ naudojama ne tik analizinės skaičių teorijos uždaviniuose, bet ir kituose matematikos srityse. Be to, Rymano dzeta funkcija rado pritaikymus fizikoje (kvantinė elektrodinamika), statistikoje, astronomijoje (astrofizikoje)ir netgi muzikos teorijoje (pavyzdžiui, [47], [13], [14], [1]). Taip pat, $\zeta(s)$ yra naudinga pirminių skaičių pasiskirstymo nagrinėjimui. Pirminių skaičių skaičiaus asimptotinė formulė

$$\sum_{p \leqslant x} 1 \sim \int_2^x \frac{\mathrm{du}}{\log u}, \quad x \to \infty, \tag{1.1}$$

buvo įrodyta remiantis faktu, kad $\zeta(s) \neq 0$, kai $\sigma \geqslant 1$. Be kita ko, funkcija turi ir geras aproksimavimo savybes, plati analizinių funkcijų klasė gali būti aproksimuojama norimu tikslumu šios funkcijos postūmiais $\zeta(s+i\tau), \tau \in \mathbb{R}$.

Disertacijoje nagrinėjame funkcijos $\zeta(s)$ apibendrinimą, Hurvico tipo dzeta funkcijas. Tegul $0 < \alpha \leqslant 1$ yra fiksuotas parametras. Hurvico dzeta funkcija $\zeta(s, \alpha)$ pusplokštumėje $\sigma > 1$ yra apibrėžiama Dirichlė eilute

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

ir taip pat, kaip ir $\zeta(s)$, yra analiziškai pratęsiama į visą kompleksinę plokštumą, išskyrus paprastąjį polių taške $s = 1$ su reziduumu 1. Funkcijos $\zeta(s, \alpha)$ analizinės savybės priklauso ir nuo parametro α aritmetikos. Nesunku pastebėti, jog $\zeta(s, 1) = \zeta(s)$ ir $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$. Tačiau funkcijos $\zeta(s)$ ir $\zeta(s, \alpha)$ turi skirtingas analizines savybes. Šie skirtumai paaiškinami tuo, kad funkcija $\zeta(s)$ turi Oilerio sandaugą pagal pirminius skaičius, o Hurvico dzeta funkcija $\zeta(s, \alpha)$ bendruoju atveju tokios sandaugos neturi

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \ \sigma > 1. \tag{1.2}$$

Disertacijoje yra nagrinėjamas ir funkcijos $\zeta(s, \alpha)$ apibendrinimas - periodinė Hurvico dzeta funkcija. Tegul $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ yra periodinė kompleksinių skaičių seka su minimaliuoju periodu $q \in \mathbb{N}$. Periodinė Hurvico dzeta funkcija $\zeta(s, \alpha; \mathfrak{a})$ pusplokštumėje $\sigma > 1$, yra apibrėžiama Dirichlė eilute

$$\zeta(s, \alpha, \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

Iš sekos $\mathfrak{a}$ periodiškumo išplaukia, kad pusplokštumėje $\sigma > 1$

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{l+\alpha}{q}\right).$$

Iš šios lygybės gauname, kad periodinė Hurvico dzeta funkcija yra analizinė visoje kompleksinėje plokštumoje, išskyrus paprastąjį polių taške $s = 1$ su reziduumu

$$\hat{a} \stackrel{def}{=} \frac{1}{q} \sum_{l=0}^{q-1} a_l.$$

Jeigu $\hat{a} = 0$, tai $\zeta(s, \alpha; \mathfrak{a})$ yra sveikoji funkcija.

Dar vienas disertacijos tyrimo objektas yra parabolinių formų dzeta funkcijos. Tegul

$$SL(2, \mathbb{Z}) \stackrel{def}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

yra pilnoji modulinė grupė. Jei funkcija $F(z)$ yra analizinė pusplokštumėje $\mathfrak{Im} z > 0$ su visais elementais

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

ir su kuriuo nors $\kappa \in 2\mathbb{N}$ tenkina funkcinę lygybę

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z), \tag{1.3}$$

tai $F(z)$ vadinama svorio κ moduline forma grupės $SL(2, \mathbb{Z})$ atžvilgiu. Aišku, kad $F(z)$ yra periodinė funkcija su periodu 1, todėl turi Furjė skleidinį

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) \mathrm{e}^{2\pi i m z}.$$

Jei $c(m) = 0$ su visais $m \leqslant 0$, tada $F(z)$ yra vadinama svorio κ paraboline forma pilnos modulinės grupės atžvilgiu. Kai $\sigma > \frac{\kappa+1}{2}$, apibėžiame

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

Tada funkcija $\zeta(s, F)$ turi analizinį pratęsimą į visą kompleksinę plokštumą ir yra vadinama parabolinės formos $F(z)$ dzeta funkcija.

Tegul $v \in \mathbb{N}$ ir

$$\Gamma_0(v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 (\mathrm{mod}\ v) \right\}$$

pilnos modulinės grupės pogrupis. Jis yra vadinamas lygio v Heckės pogrupiu. Jei $F(Z)$ tenkina (1.3) lygybę su visais $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(v)$, tada parabolinė forma $F(z)$ yra vadinama svorio κ ir lygio v paraboline forma. Pastarųjų parabolinių formų dzeta funkcijos buvo svarbios paskutiniosios Ferma teoremos įrodymui, kad diofantinė lygtis

$$x^n + y^n = z^n$$

neturi netrivialiųjų sprendinių, kai $n \geqslant 3$. Tegul E yra eliptinė kreivė virš $\mathbb{Q}$, duota Vejerštraso lygtimi

$$y^2 = x^3 + ax + b, a, b \in \mathbb{Z},$$

su diskriminantu $D = -16(4a^3 + 27b^2)$. Pažymėkime E_p kreivės E redukcija moduliu p, kuri yra kreivė virš baigtinio kūno $\mathbb{F}_p$. Kreivės E_p taškų skaičių žymime $|E(\mathbb{F}_p)|$, o $\lambda(p)$ apibrėžiame lygybe

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p).$$

Tuomet kreivės $\mathbb{E}$ dzeta funkcija $\zeta_{\mathbb{E}}(s)$, kai $\sigma > \frac{3}{2}$, apibrėžiama sandauga

$$\zeta_{\mathbb{E}}(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1},$$

ir gali būti analiziškai pratęsiama į visą kompleksinę plokštumą.

Japonų matematikai G. Šimura (Shimura) ir Y. Tanijama (Taniyama) iškėlė hipotezę, jog parabolinės formos su tam tikru svoriu ir lygiu dzeta funkcija sutampa su elipsinės kreivės dzeta funkcija. Vėliau paaiškėjo, kad iš Šimuros-Tanijamos hipotezės išplaukia paskutinioji Ferma teorema. Galiausiai, A. Vailsas (Wiles) [67] įrodė dalinį Šimuros-Tanijamos hipotezės atvejį ir gavo paskutiniosios Ferma teoremos įrodymą.

Tarkime, kad $F(z)$ yra svorio κ parabolinė forma pilnosios modulinės grupės atžvilgiu. Papildomai reikalaujame, kad $F(z)$ būtų Heckės-eigenforma, t.y., kad $F(z)$ būtų tikrinė funkcija visų Heckės operatorių

$$T_m f(z) = m^{\kappa-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^\kappa} \sum_{b(\mathrm{mod} d)} f\left(\frac{az+b}{d}\right), m \in \mathbb{N},$$

atžvilgiu. Tada $F(z)$ forma gali būtų normuojama, todėl galime laikyti, jog $c(1) = 1$. Šiuo atveju, dzeta funkcija $\zeta(s, F)$, kai $\sigma > \frac{\kappa+1}{2}$, turi Oilerio sandaugą pagal pirminius skaičius

$$\zeta(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

čia $\alpha(p)$ ir $\beta(p)$ yra kompleksiniai jungtiniai skaičiai ir $\alpha(p)+\beta(p)=c(p)$.

Dauguma dzeta funkcijų tenkina funkcines lygtis. Pavyzdžiui, Rymano dzeta funkcijai turime tokią funkcinę lygtį

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

čia $\Gamma(s)$ yra Oilerio gama funkcija. Iš šios lygties išplaukia, kad $\zeta(-2k)=0$, kai $k\in\mathbb{N}$. Skaičiai $s=-2k$ vadinami funkcijos $\zeta(s)$ trivialiaisiais nuliais. Be to, $\zeta(s)$ funkcija turi be galo daug netrivialiųjų kompleksinių nulių $\rho=\beta+i\gamma$, gulinčių juostoje $\{s\in\mathbb{C}:0<\sigma<1\}$. Tegul $\{\gamma_l:\gamma_l>0,\ l\in\mathbb{N}\}$ yra funkcijos $\zeta(s)$ netrivialiųjų nulių teigiamų menamųjų dalių seka.

Dzeta funkcija $\zeta(s,F)$ taip pat turi funkcinę lygtį

$$(2\pi)^{-s}\Gamma(s)\zeta(s,F)=(-1)^{\frac{\kappa}{2}}(2\pi)^{s-\kappa}\zeta(\kappa-s,F).$$

Disertacijoje plačią analizinių funkcijų klasę aproksimuojame postūmiais $\zeta(s+ih\gamma_k,F), h>0$. Be to, nagrinėjamas jungtinis aproksimavimas postūmių rinkiniais $\big(\zeta(s+i\tau,\alpha_1),...,\zeta(s+i\tau,\alpha_r)\big)$ ir $\big(\zeta(s+ikh,\alpha_1),...,\zeta(s+ikh,\alpha_r)\big)$ su bet kuriais $\alpha_1,...,\alpha_r$ parametrais. Paskutiniai analizinių funkcijų aproksimavimo rezultatai yra skirti postūmiams $\zeta(s+i\tau,\alpha;\mathfrak{a})$, su bet kuriuo α.

Tikslas ir uždaviniai

Disertacijos tikslas yra analizinių funkcijų aproksimavimas normuotų Heckės-eigen parabolinių formų dzeta funkcijų ir Hurvico tipo dzeta funkcijų postūmiais. Uždaviniai yra šie:

1. Plačios analizinių funkcijų klasės aproksimavimas normuotomis Heckės-eigen parabolinių formų dzeta funkcijų sąsukomis su netrivialiaisiais Rymano dzeta funkcijos nuliais.

2. Jungtinis analizinių funkcijų aproksimavimas Hurvico dzeta funkcijų postūmiais su bet kuriais parametrais.

3. Jungtinis diskretusis analizinių funkcijų aproksimavimas Hurvico dzeta funkcijų postūmiais su bet kuriais parametrais.

4. Analizinių funkcijų aproksimavimas periodinių Hurvico dzeta funkcijų postūmiais.

Aktualumas

Analizinių funkcijų aproksimavimo problemos yra vienos svarbiausių šiuolaikinės tiek teorinėje, tiek ir taikomojoje matematikoje. Pagal Mergeliano teoremą, kiekviena analizinė funkcija gali būti norimu tikslumu aproksimuojama polinomais kompaktinėse aibėse, turinčiuose jungųjį papildinį. Vėliau paaiškėjo, kad kai kurios dzeta funkcijos yra kur kas galingesnės negu polinomai, nes plati analizinių funkcijų klasė gali būti aproksimuojama vienos ir tos pačios dzeta funkcijos postūmiais. Šis aproksimavimas apibendrina Boro-Kuranto (Bohr-Courant) tankio rezultatus dzeta funkcijoms ir turi gilią matematinę prasmę. Todėl, mūsų nuomone, yra svarbu toliau vystyti analizinių funkcijų aproksimavimo teoriją dzeta funkcijoms, ieškoti naujų dzeta funkcijų ir jų postūmių su aproksimavimo savybėmis. Be to, aproksimavimo teorijos tyrimas yra viena iš Lietuvos analizinės skaičių teorijos mokyklos krypčių, todėl tai yra jaunų matematikų pareiga tęsti vyresnių kolegų darbus šioje srityje.

Metodai

Dzeta funkcijų aproksimavimo teoremų įrodymai remiasi Dirichlė eilučių teorija, Rymano dzeta funkcijos nulių pasiskirstymo teorija, Furjė analizės bei silpnojo tikimybinio mato konvergavimo teorija.

Naujumas

Visi disertacijoje pateikti rezultatai yra nauji. Rymano dzeta funkcijų netrivialūs nuliai yra pirmą kartą panaudoti analizinių funkcijų aproksimavime parabolinės formos dzeta funkcijų postūmiais. Analizinių funkcijų aproksimavimas Hurvico tipo dzeta funkcijų postūmiais su bet kuriais parametrais yra taip pat nauja kryptis analizinėje skaičių teorijoje.

Tyrimų istorija ir rezultatai

Dzeta funkcijos ir Dirichlė eilutės yra pagrindiniai analizinės skaičių teorijos įrankiai. Tegul pusplokštumėje $\sigma > \sigma_0$

$$A(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}. \tag{1.4}$$

Funkcijos $A(s)$ analizinės savybės leidžia gauti informaciją apie koeficientus a_m, visų pirma, apie jų vidurkį

$$M(x) = \sum_{m \leqslant x} a_m,$$

kai $x \to \infty$. Pavyzdžiui, pagal klasikinę Perono formulę, jeigu eilutė (1.4) absoliučiai konverguoja, kai $\sigma > 1, |a_m| \leqslant g(m)$ su teigiama monotoniškai didėjančia $g(m)$ ir, kai $\sigma \to 1 + 0$,

$$\sum_{m=1}^{\infty} \frac{|a_m|}{m^s} = O\big((\sigma - 1)^a\big), a > 0, \tag{1.5}$$

tada su kiekvienu $b_0 \geqslant b > 1, T \geqslant 1, x = n + \frac{1}{2}$, galioja formulė

$$M(x) = \frac{1}{2\pi} \int_{b-i\tau}^{b+i\tau} A(s) \frac{x^a}{s} ds + O\big(\frac{x^b}{T(b-1)^a}\big) + O\big(\frac{xg(2x)\log x}{T}\big).$$

Pastaroji formulė naudojama pirminių skaičių asimptotinio pasiskirstymo dėsniui gauti. Iš Oilerio sandaugos (1.2) dzeta funkcijai $\zeta(s)$ turime, kad

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}, \sigma > 1, \tag{1.6}$$

čia $\Lambda(m)$ yra Mangoldto funkcija,

$$\Lambda(m) = \begin{cases} \log p \text{ if } m = p^k, k \geqslant 1, \\ 0 \text{ kitu atveju.} \end{cases}$$

Iš (1.5), (1.6) ir lygybės išplaukia, jog

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + const,$$

čia ρ perbėga visus netrivialiusius $\zeta(s)$ nulius gaunama, kad

$$\sum_{m\in k}\Lambda(m) = x + o(x), x \to \infty,$$

ir tai įrodo (1.1) asimptotinę formulę.

Funkcija $\zeta(s, F)$ taip pat yra naudojama koeficientų $c(m)$ tyrimams. Pavyzdžiui, [57] darbe buvo gauta, jog

$$\sum_{m\leqslant x} c^2(m) = A_F x^{\kappa} + O(x^{\kappa-\frac{2}{5}}), x \to \infty,$$

su tam tikra konstanta $A_F > 0$. Svarbus yra ir P. Delinio (P. Deligne) rezultatas [10]

$$|c(m)|\leqslant d(m) m^{\frac{x-s}{2}},$$

čia $d(m)$ yra daliklių funkcija. Pati funkcija $\zeta(s, F)$ buvo taip pat nagrinėjama. Daug dėmesio buvo skiriama momentams

$$I_k(T,\sigma,F) = \int_0^T |\zeta(\sigma+it,F)|^{2k} dt.$$

Pirmą rezultatą šioje kryptyje gavo H. S. A. Poteris (H. S. A. Potter) [54]. Jis įrodė asimptotinę formulę

$$I_1(T,\sigma,F) \sim T\sum_{m=1}^{\infty}\frac{c^2(m)}{2^{2\sigma}}, \sigma > \frac{\kappa}{2}, T \to \infty,$$

ir [55] darbe gavo įvertį

$$I_1(T, \frac{\kappa}{2}, F) \ll T\log T.$$

A. Gudas (A. Good) patikslino [19] pastarąjį rezultatą iki

$$I_1(T, \frac{\kappa}{2}, F) \sim 2\kappa A_F T\log T,$$

ir [20]

$$I_1(T, \frac{\kappa}{2}, F) = \begin{cases} 2\kappa A_F T \log T + O(T) \text{ kai } \sigma = \frac{\kappa}{2}, \\ A_F(\sigma)T + O(T^{\kappa+1-2\sigma}) \text{ kai } \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, \\ A_F(\sigma)T + O((\log T)^2) \text{ kai } \sigma > \frac{\kappa+1}{2} \end{cases}$$

su tam tikra konstanta $A_F(\sigma)$. M. Jutila gavo įvertį [24]

$$I_3(T, \frac{\kappa}{2}, F) \ll T^{2+\varepsilon}, \quad \varepsilon > 0$$

Straipsnyje [31], buvo įrodyta, kad kai $k = \frac{1}{n}, n \in \mathbb{N}$,

$$I_k(T, \frac{\kappa}{2}, F) \ll T(\log T)^{k^2}.$$

Dzeta funkcijų taikymo analizinių funkcijų aproksimavimui idėja priklauso S. Voroninui. Darbe [66], jis gavo tokį svarbų rezultatą.

Teorema 1. *Tarkime, kad* $0 < r < \frac{1}{4}$, *o funkcija* $f(s)$ *yra tolydi ir nevirstanti nuliumi skritulyje* $|s| \leqslant r$, *bei analizinė to skritulio viduje. Tuomet kiekvieną* $\varepsilon > 0$ *atitinka toks* $\tau = \tau(\varepsilon) \in \mathbb{R}$, *su kuriuo yra teisinga nelygybė*

$$\max_{|s| \leqslant r} |\zeta(s + \frac{3}{4} + i\tau) - f(s)| < \varepsilon.$$

S. Voroninas pastarąją $\zeta(s)$ funkcijos savybę pavadino universalumu. Išties, 1 teorema turi universalumo savybę, nes plati analizinių funkcijų klasė yra aproksimuojama tos pačios dzeta funkcijos $\zeta(s)$ postūmiais. Voronino teoremą sustiprino ir išplėtojo įvairūs autoriai. Tegul $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Simboliu $\mathcal{K}$ žymėsime juostos D kompaktinių aibių, turinčių jungųjį papildinį, klasę, o $H_0(K)$, $K \in \mathcal{K}$, bus funkcijų, tolydžių ir nevirstančių nuliu aibėje K, bei analizinių aibės K viduje, klasę. Be to, $\text{meas}A$ tegul žymi mačios aibės $A \subset \mathbb{R}$ Lebego matą. Tuomet yra teisinga tokia Teoremos 1 modifikacija

Teorema 2. *Tarkime,* $K \in \mathcal{K}$ *ir* $f(s) \in H_0(K)$. *Tuomet su kiekvienu* $\varepsilon > 0$ *yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

2 teoremos nelygybė rodo, kad aibė postūmių $\zeta(s + i\tau)$, aproksimuojančių funkciją $f(s) \in H_0(K)$, yra begalinė, kadangi jos apatinis tankis yra griežtai

teigiamas. Kita vertus, jokios konkrečios reikšmės τ mes nežinome.

Darbuose [48] ir [35], buvo įrodyta, jog 2 teoremos apatinis tankis gali būti pakeistas tankiu. Taigi, yra teisingas toks 2 teoremos praplėtimas.

Teorema 3. *Tarkime, kad $K \in \mathcal{K}$ ir $f(s) \in H_0(K)$. Tuomet riba*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Sakykime, kad $H(D)$ yra analizinių juostoje D funkcijų erdvė su tolygaus konvergavimo kompaktinėse aibėse topologija. Kadangi erdvė $H(D)$ yra begaliniamatė, tai į Voronino universalumo teoremą galima žiūrėti kaip į Boro-Kuranto teoremos [8] begaliniamatį apibendrinimą.

Dzeta funkcijos $\zeta(s, F)$ universalumas buvo pradėtas nagrinėti [25] darbe, reikalaujant, kad egzistuotų toks $\eta > 0$, kad eilutė

$$\sum_{\substack{p \\ |c|p<\eta}} \frac{1}{p^\delta}, c_p = c(p)p^{\frac{1-\kappa}{2}},$$

konverguotų su $\delta > \frac{1}{2}$. Straipsnyje [27] šis reikalavimas buvo panaikintas. Tarkime, kad $D_F = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$. Simboliu $\mathcal{K}_F$ žymėsime juostos D_F kompaktinių aibių, turinčių jungųjį papildinį, klasę, o $H_{0F}(K)$ su $K \in \mathcal{K}_F$ - funkcijų, tolydžių ir nevirstančių nuliu aibėje K, bei analizinių aibės K viduje, klasę.

Teorema 4. *Tarkime, kad $K \in \mathcal{K}_F$ ir $f(s) \in H_{0F}(K)$. Tada su kiekvienu $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, F) - f(s)| < \varepsilon\right\} > 0.$$

4 teorema turi ir diskrečiąją versiją. Tegul $\#A$ žymi aibės A elementų skaičių, o N perbėga aibę $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Teorema 5. *Tarkime, kad $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$ ir $h > 0$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+ikh, F) - f(s)| < \varepsilon\right\} > 0.$$

Jei $\exp\{\frac{2\pi m}{h}\}$ yra iracionalusis skaičius su visais $m \in \mathbb{Z} \setminus \{0\}$, tai 5 teorema buvo gauta [29] darbe, o aukščiau formuluota versija įrodyta [38] straipsnyje.

Analizinių funkcijų aproksimavimui galima naudoti bendresnius postūmius $\zeta(s+i\varphi(\tau), F)$ vietoj $\zeta(s+i\tau, F)$. Tai ir buvo padaryta [65] darbe. Simboliu $U(T_0), T_0 > 0$, žymėsime funkcijų $\varphi(\tau)$, kurios tenkina šias sąlygas, klasę:

1° $\varphi(\tau)$ yra diferencijuojamos, teigiamos, didėjančios funkcijos, įgyjančios realias reikšmes intervale $[T_o, \infty)$.

2° Išvestinės $\varphi'(\tau)$ yra monotoniškos, tolydžios, teigiamos intervale $[T_o, \infty)$ ir $(\varphi'(\tau))^{-1} = o(\tau)$ kai $\tau \to \infty$.

3° Galioja įvertis

$$\varphi(2\tau) \max_{\tau \leqslant u \leqslant 2\tau} \frac{1}{\varphi'(u)} \ll \tau, \tau \to \infty.$$

Teorema 6. *Tarkime, kad $\varphi(\tau) \in U(T_0)$. Tegul $K \in \mathcal{K}_F$, $f(s) \in H_{0F}(K)$, Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T-T_0} meas \left\{ \tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\varphi(\tau), F) - f(s)| < \varepsilon \right\} > 0.$$

Be to, riba

$$\lim_{T\to\infty} \frac{1}{T-T_0} meas \left\{ \tau \in [T_0, T] : \sup_{s\in K} |\zeta(s+i\varphi(\tau), F) - f(s)| < \varepsilon \right\} > 0.$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

6 teorema turi diskrečią versiją. Jos formulavimui yra reikalinga tolygaus pasiskirstymo moduliu 1 sąvoka. Tegul $\{u\}$ yra trupmeninė dalis $u \in \mathbb{R}$, o χ_I yra aibės I indikatorius. Seka $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ yra vadinama tolygiai pasiskirstanti moduliu 1, jeigu su kiekvienu intervalu $I = [a, b) < [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = b - a.$$

Kaip ir 6 teoremoj , naudojame tam tikra funkcijų klasę. Tegul $k_0 \in \mathbb{N}$. Funkcija $\varphi(t) \in U(k_0)$, jei patenkintos šios sąlygos:

1° $\varphi(t)$ yra reali, didėjanti, teigiama funkcija intervale $[k_0 - \frac{1}{2}, \infty)$;

2° $\varphi(t)$ turi tolydžią išvestinę intervale $[k_0 - \frac{1}{2}, \infty)$, tenkinančią įvetrį

$$\varphi(2t)\big(\max_{t\leqslant u\leqslant 2t}\frac{1}{\varphi'(u)} + \max_{t\leqslant u\leqslant 2t}\varphi'(u)\big) \ll t, \quad t\to\infty;$$

3° Seka $\{a\varphi(k) : k\geqslant k_0\}$ su kiekvienu $a \in \mathbb{R}\setminus\{0\}$ yra tolygiai pasiskirsčiusi moduliu 1.

Pavyzdžiui, funkcija $\varphi(t) = t\log^\alpha(t)$ su $0 < \alpha < 1$ yra klasės $U(2)$ elementas.

Straipsnyje [40] pateiktas toks tvirtinimas.

Teorema 7. *Tarkime, kad $\varphi(t) \in U(k_0)$. Tegul $K \in \mathcal{K}_F$ ir $f(s) \in H_{0F}(K)$. Tuomet kiekvienam $\varepsilon > 0$*

$$\liminf_{N\to\infty}\frac{1}{N-k_0+1}\#\Big\{k_0\leqslant k\leqslant N : \sup_{s\in K}|\zeta(s+i\varphi(k),F) - f(s)| < \varepsilon\Big\} > 0.$$

Be to, riba

$$\lim_{N\to\infty}\frac{1}{N-k_0+1}\#\Big\{k_0\leqslant k\leqslant N : \sup_{s\in K}|\zeta(s+i\varphi(k),F) - f(s)| < \varepsilon\Big\} > 0$$

egzistuoja visiems $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Tegul $0 < \gamma_1 < \gamma_2 < ...\leqslant\gamma_k\leqslant\gamma_{k+1}...$ yra Rymano dzeta funkcijos netrivialių nulių menamų dalių seka. Yra žinomos universalumo teoremos su dzeta funkcijų postūmiais, apimančios seką $\{\gamma_k\}$. Pirmas šios krypties rezultatas buvo gautas [17] darbe. Netrivialių nulių pasiskirstymas yra mįslingas, mes neturime pakankamai informacijos apie seką $\{\gamma_k\}$. Straipsnyje [17] buvo panaudota prielaida, kad kai $c > 0$,

$$\sum_{\substack{\gamma_k\leqslant T \\ |\gamma_k-\gamma_l|<\frac{c}{\log T}}}\sum_{\gamma_l\leqslant T} 1 \ll T\log T, T\to\infty. \tag{1.7}$$

Pastarasis įvertis išplaukia iš gerai žinomos Montgomerio porų koreliacijos hipotezės [52], kad su $\alpha < \beta$

$$\sum_{\substack{\gamma_k\leqslant T \\ \frac{2\pi\alpha}{\log T}<\gamma_k-\gamma_l<\frac{2\pi\beta}{\log T}}}\sum_{\gamma_l\leqslant T} 1 \sim \Bigg(\int\limits_\alpha^\beta\bigg(1-\Big(\frac{\sin\pi u}{\pi u}\Big)^2\bigg)du+\delta(\alpha,\beta)\Bigg)\frac{T\log T}{2\pi},\ T\to\infty$$

čia

$$\delta(\alpha,\beta) = \begin{cases} 1 \text{ jei } 0 \in [\alpha,\beta], \\ 0 \text{ kitu atveju.} \end{cases}$$

Taigi, naudojant (1.7) įverti darbe [17] buvo įrodyta, jog postūmiai $\zeta(s+ih\gamma_k)$, su kiekvienu $h > 0$ aproksimuoja klasės $H_0(K), K \in \mathcal{K}$ funkcijas. Darbe [15] tokie patys rezultatai buvo gauti naudojant Rymano hipotezę ir momentų įverčius iš [16]. Darbas [44] yra skirtas analizinių funkcijų aproksimavimui Hurvico dzeta funkcijos postūmiais $\zeta(s+i\gamma_k h, \alpha)$ su tiesiškai nepriklausoma virš $\mathbb{Q}$ aibe $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$.

Antrasis disertacijos skyrius yra skirtas analizinių funkcijų aproksimavimui postūmiais $\zeta(s+i\gamma_k h, F)$. Pagrindinis skyriaus rezultatas yra tokia teorema.

Teorema 2.1. *Tarkime, kad įvertis* (1.7) *yra teisingas. Tegul* $K \in \mathcal{K}_F$*,* $f(s) \in H_{0F}(K)$ *ir* $h > 0$*. Tuomet su kiekvienu* $\varepsilon > 0$

$$\liminf_{N\to\infty} \frac{1}{N}\#\left\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+i\gamma_k h, F) - f(s)| < \varepsilon\right\} > 0.$$

Be to, riba

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{1\leqslant k\leqslant N : \sup_{s\in K} |\zeta(s+i\gamma_k h, F) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$*, išskyrus ne daugiau negu skaičią reikšmių aibę.*

Teoremos 2.1 įrodymas remiasi mato

$$\frac{1}{N}\#\left\{1\leqslant k\leqslant N : \zeta(s+i\gamma_k h, F) \in A\right\}, A \in \mathcal{B}(H(D_F))$$

silpnuoju konvergavimu, kai $N \to \infty$. Čia $H(D_F)$ yra analizinių juostoje D_F funkcijų erdvė su tolygaus konvergavimo kompaktinėse aibėse topologija, o $\mathcal{B}(H(D_F))$ yra erdvės $H(D_F)$ Borelio σ kūnas.

Neseniai, profesorius Jornas Štoidingas (Joern Steuding) informavo, kad 2.1 teoremoje reikalavimo, kad būtų teisingas (1.7) įvertis, galima atsisakyti [61].

Skyriaus 2 rezultatai paskelbti [5] straipsnyje.

Likusieji disertacijos skyriai yra skirti analizinių funkcijų aproksimavimui Hurvico tipo dzeta funkcijų postūmiais. Simboliu $H(K)$, kai $K \in \mathcal{K}$, žymėsime funkcijų, tolydžių aibėje K ir analizinių aibės K viduje, klasę. Taigi, $H_0(K) \subset$

$H(K)$. Funkcijos $\zeta(s,\alpha)$ aproksimavimo savybės (arba universalumas) yra aprašytos toliau pateiktoje teoremoje. Primename, kad skaičius α vadinamas transcendenčiuoju, jeigu jis nėra jokio polinomo $p(s) \not\equiv 0$ su racionaliaisiais koeficientais šaknis, t.y., $p(\alpha) = 0$. Priešingu atveju, α yra vadinamas algebriniu.

Teorema 8. *Tarkime, kad parametras α yra transcendentusis arba racionalusis skaičius $\neq 1, \frac{1}{2}$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0. \quad (1.8)$$

Ši teorema buvo gauta skirtingais metodais B. Bagči (B.Bagchi) [2], S. M. Goneko (S.M.Gonek) [18], bei A. Laurinčiko ir R. Garunkščio [28] darbuose. Atvejo su racionaliuoju α įrodymas remiasi jungtiniu universalumu Dirichlė L-funkcijų, tuo tarpu, kai α yra transcendentusis, yra patogu taikyti tikimybinį metodą.

Funkcijų iš klasės $H(K)$ aproksimavimas (arba universalumas), kai α yra algebrinis iracionalusis skaičius, yra atvira problema. Šiuo atveju, yra žinomas toks [3] rezultatas. Tegul $H(D)$ žymi analizinių juostoje D funkcijų erdvę .

Teorema 9. *Tarkime, kad parametras α yra algebrinis iracionalusis. Tuomet egzistuoja toks uždaras, netuščias poaibis $F_\alpha \subset H(D)$, kad su kiekviena kompaktine aibe $K \subset D$, $f(s) \in F_\alpha$ ir $\varepsilon > 0$ galioja* (1.8) *nelygybė. Be to, riba*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Pastarojoje teoremoje aibė F_α nėra žinoma, yra įrodytas tik jos egzistavimas.

Labai svarbus ir sudėtingas universalumo rezultatas funkcijai $\zeta(s,\alpha)$ su algebriniu iracionaliuoju α buvo neseniai gautas darbe [62]. Buvo įrodyta, kad su visais algebriniais iracionaliaisiais α, išskyrus baigtinę jų aibę, funkcija $\zeta(s,\alpha)$ yra universali. Be to, nurodytas intervalas $[T, 2T]$, kuriame yra τ, tenkinantis (1.8), kai K yra skritulys.

Teorema 8 turi tam tikrą jungtinį apibendrinimą. Skaičiai $\alpha_1, ..., \alpha_r$ vadinami algebriškai nepriklausomais virš $\mathbb{Q}$, jei jie nėra jokio polinomo $p(s_1, ..., s_r) \not\equiv$

0 su racionaliaisiais koeficientais šaknys, t.y., $p(\alpha_1, ..., \alpha_r) = 0$. Tegul

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}.$$

Tuomet turime tokį 8 teoremos praplėtimą [32].

Teorema 10. *Tarkime, kad aibė $L(\alpha_1, \dots, \alpha_r)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H(K_j)$, $j = 1, \dots, r$. Tuomet su kiekvienu $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Nesunku pastebėti, jog iš parametrų $\alpha_1, ..., \alpha_r$ algebrinės nepriklausomybės išplaukia aibės $L(\alpha_1, \dots, \alpha_r)$ tiesiška nepriklausomybė.

Disertacijos 3 skyriuje yra gautas jungtinis 9 teoremos apibendrinimas.

Teorema 3.1. *Tarkime, kad skaičiai α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$, kad su visomis kompaktinėmis aibėmis $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ ir $\varepsilon > 0$ yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Teorema 3.1 turi modifikaciją tankio prasme.

Teorema 3.2. *Tarkime, kad skaičiai α_j, $0 < \alpha_j < 1$, $\alpha_j \neq \frac{1}{2}$, $j = 1, \dots, r$, yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$, kad su visomis kompaktinėmis aibėmis $K_1, \dots, K_r \subset D$, $(f_1, \dots, f_r) \in F_{\alpha_1,\dots,\alpha_r}$ riba*

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Teoremų 3.1 ir 3.2 įrodymui yra naudojamas tikimybinis metodas. Šios teoremas publikuotos [11] darbe.

Analizinių funkcijų aproksimavimas yra nagrinėjamas ir su diskrečiais Hurvico dzeta funkcijų postūmiais. Šiuo atveju postūmiuose $\zeta(s + ikh, \alpha)$ atsiranda papildomas parametras h. Straipsnyje [2] buvo gauta tokia diskrečioji teorema.

Teorema 11. *Tarkime, kad α yra racionalusis skaičius didesnis už 0, $\alpha < 1$, $\alpha \neq \frac{1}{2}$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > 0. \tag{1.9}$$

11 teoremos įrodymui yra naudojamas tikimybinis metodas ir funkcijos $\zeta(s,\alpha)$ išraiška Dirichlė L-funkcijomis. Kitu metodu ši teorema buvo įrodyta [59] darbe.

Diskretusis atvejis su transcendenčiuoju α yra sudėtingesnis negu su racionaliuoju α ir reikalauja sąryčio tarp h ir π.

Teorema 12. *[33]. Tarkime, kad skaičius $\exp\{\frac{2\pi}{h}\}$, $h > 0$, yra racionalusis, o α yra transcendentusis. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet galioja* (1.9) *nelygybė.*

Darbe [36], skaičiaus $\exp\{\frac{2\pi}{h}\}$ racionalumas buvo pakeistas silpnesne sąlyga. Tegul aibė

$$L(\alpha, h, \pi) = \left\{ (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.$$

Tuomet galioja toks 12 teoremos praplėtimas [36].

Teorema 13. *Tarkime, kad aibė $L(\alpha, h, \pi)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet galioja* (1.9) *nelygybė.*

Jeigu skaičiai α ir $\exp\{\frac{2\pi}{h}\}$ yra algebriškai nepriklausomi virš $\mathbb{Q}$, tada nesunku matyti, jog aibė $L(\alpha, h, \pi)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Taigi, pagal Nesterenko teoremą (π ir e^{π} yra algebriškai nepriklausomi), 13 teorema galioja su $\alpha = \frac{1}{\pi}$ ir racionaliuoju h.

Vietoje postūmių $\zeta(s+ikh,\alpha)$ galima naudoti ir apibendrintus diskrečius postūmius $\zeta(s+i\varphi(k),\alpha)$. Tegul $U(k_0)$ yra ta pati funkcijų klasė kaip ir Teoremoje 7. Sekanti teorema yra pateikta [39] straipsnyje.

Teorema 14. *Tarkime, kad aibė $L(\alpha)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$, o $\varphi(k) \in U(k_0)$. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N-k_0+1} \# \left\{ k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+i\varphi(k),\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Be to, riba

$$\lim_{N\to\infty} \frac{1}{N-k_0+1} \# \left\{ k_0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+i\varphi(k),\alpha) - f(s)| < \varepsilon \right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Teigiamiems $h_1, ..., h_r$, apibrėžkime aibę

$$L(\alpha_1, ..., \alpha_r; h_1, ..., h_r; 2\pi) = \big\{ \big(h_1 \log(m+\alpha_1) : m \in \mathbb{N}_0\big),$$

$$..., \big(h_r \log(m+\alpha_r) : m \in \mathbb{N}_0\big), 2\pi \big\}$$

Tuomet seka jungtinė diskrečioji teorema Hurvico dzeta funkcijai [37].

Teorema 15. *Tarkime, kad aibė $L(\alpha_1, ..., \alpha_r; h_1, ..., h_r; 2\pi)$ yra tiesiškai nepriklausoma virš $\mathbb{Q}$. Kai $j = 1, \ldots, r$, tegul $K_j \in \mathcal{K}$ ir $f_j(s) \in H(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Ketvirtame skyriuje yra pateiktos diskrečios Theoremų 3.1 ir 3.2 versijos. Tam tikra prasme šie rezultatai praplėčia 15 teoremą bet kuriems $\alpha_1, ...\alpha_r$.

Tegul, trumpumo dėlei, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ ir $\underline{h} = (h_1, \ldots, h_r)$.

Teorema 4.1. *Tarkime, kad skaičiai $0 < \alpha_j < 1$, ($\alpha_j \neq 1/2$) ir teigiami skaičiai h_j, $j = 1, \ldots, r$, yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$, kad su visomis kompaktinėmis aibėmis $K_1, \ldots, K_r \subset D$, $(f_1, \ldots, f_r) \in F_{\underline{\alpha},\underline{h}}$ ir $\varepsilon > 0$*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Kita teorema yra teoremos 3.2 diskreti versija.

Teorema 4.2. *Tarkime, kad skaičiai $0 < \alpha_j < 1$, ($\alpha_j \neq 1/2$) ir teigiami skaičiai h_j, $j = 1, \ldots, r$, yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\underline{\alpha},\underline{h}} \subset H^r(D)$, kad su visomis kompaktinėmis aibėmis $K_1, \ldots, K_r \subset D$, $(f_1, \ldots, f_r) \in F_{\underline{\alpha},\underline{h}}$ riba*

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ikh_j,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Teoremos 4.1 ir 4.2 yra publikuotos [6] straipsnyje.

Disertacijos penktas skyrius yra skirtas 9 teoremos apibendrinimui periodinei Hurvico dzeta funkcijai $\zeta(s, \alpha; \mathfrak{a})$.

Pirmąjį analizinių funkcijų aproksimavimo postūmiais $\zeta(s+i\tau, \alpha; \mathfrak{a})$ rezultatą su transcendenčiuoju parametru α gavo A. Javtokas ir A. Laurinčikas (2006).

Teorema 16. *Tarkime, kad parametras α yra transcendentusis. Tegul $K \in \mathcal{K}$ ir $f(s) \in H(K)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

A. Laurinčikas ir R. Macaitienė (2009) įrodė 16 teoremos diskrečiąją versiją.

Jungtinės universalumo teoremos periodinėms Hurvico dzeta funkcijoms buvo nagrinėjamos daugelyje darbų. Paminėkime [30]. Jungtiniu atveju, periodinių sekų $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}, j = 1, ..., r$ savybės turi svarbų vaidmenį. Tegul q yra sekos $\mathfrak{a}_j, r \leqslant q$ periodas. Apibrėžkime matricą

$$A = \begin{pmatrix} a_{01} & a_{02} & \dots & a_{0r} \\ a_{11} & a_{12} & \dots & a_{1r} \\ \dots & \dots & \dots & \dots \\ a_{q1} & a_{q2} & \dots & a_{qr} \end{pmatrix}.$$

Teorema 17. *Tarkime, kad α yra transcendentus skaičius ir $rank(A) = r$. Tegul $K_j \in \mathcal{K}$, $j = 1, ..., r$, o $f_j(s) \in H(K_j)$. Tuomet su kiekvienu $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha; \mathfrak{a}_j) - f_j(s)| < \varepsilon \right\} > 0.$$

17 teoremos apibendrinimai su skirtingais parametrais $\alpha_1, ...\alpha_r$ ir sekomis $\mathfrak{a}_j$ su skirtingais periodais buvo nagrinėjami [60] ir [43]. Šiuo atveju reikalaujama, kad skaičiai $\alpha_1, ...\alpha_r$ būtų algebriškai nepriklausomi virš $\mathbb{Q}$, o q matricoje A keičiamas periodų $q_1, ...q_r$ mažiausiu bendru kartotiniu.

Disertacijoje įrodytos tokios teoremos.

Teorema 5.1. *Tarkime, kad parametras α, $0 < \alpha \leqslant 1$, ir periodinė seka $\mathfrak{a}$ yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\alpha,\mathfrak{a}} \subset H(D)$, kad su* *kiekvienu kompaktiniu poaibiu $K \subset D$, $f(s) \in F_{\alpha,\mathfrak{a}}$ ir $\varepsilon > 0$, yra teisinga nelygybė*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Kita teorema yra Theoremos 5.1 versija tankio terminais.

Teorema 5.2. *Tarkinme, kad parametras α, $0 < \alpha \leqslant 1$, ir periodinė seka $\mathfrak{a}$ yra bet kokie. Tuomet egzistuoja tokia uždara netuščia aibė $F_{\alpha,\mathfrak{a}} \subset H(D)$, kad su kiekvienu kompaktiniu poaibiu $K \subset D$ ir $f(s) \in F_{\alpha,\mathfrak{a}}$ riba*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)| < \varepsilon\right\} > 0$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Teoremos 5.1 ir 5.2 yra apibendrintos kai kurioms kompozicijoms. Pirmos aproksimacijos kompozicijų postūmiais $F(\zeta(s+i\tau)), F : H(D) \to H(D)$ buvo gautos [34] darbe. Pavyzdžiui, postūmiai $\sin(\zeta(s+i\tau))$ aproksimuoja analizines funkcijas, juostoje D neįgyjančias 1 ir -1.

Disertacijoje išnagrinėti tokie kompozicijų universalumo atvejai.

Teorema 5.3. *Tarkime, kad paramentras α, $0 < \alpha \leqslant 1$, ir periodinė seka $\mathfrak{a}$ yra bet kokie. Tuomet egzistuoja tokia netuščia uždara aibė $F_{\alpha,\mathfrak{a}} \subset H(D)$, kad jei $\Phi : H(D) \to H(D)$ yra toks tolydus operatorius, kad su kiekvienu polinomu $p = p(s)$ aibė $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ yra netuščia, tada su kiekvienu kompaktiniu poaibiu $K \subset D$, $f(s) \in \Phi\left(F_{\alpha,\mathfrak{a}}\right)$ ir $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi\left(\zeta(s+i\tau,\alpha;\mathfrak{a})\right) - f(s)| < \varepsilon\right\} > 0.$$

Paskutinioji teorema yra teoremos 5.2 apibendrinimas kompozicijoms.

Teorema 5.4. *Tarkime, kad paramentras α, $0 < \alpha \leqslant 1$, ir periodinė seka $\mathfrak{a}$ yra bet kokie. Tuomet egzistuoja tokia netuščia uždara aibė $F_{\alpha,\mathfrak{a}} \subset H(D)$, kad jei $\Phi : H(D) \to H(D)$ yra toks tolydus operatorius, kad su kiekvienu polinomu $p = p(s)$ aibė $\left(\Phi^{-1}\{p\}\right) \cap F_{\alpha,\mathfrak{a}}$ yra netuščia, tada su kiekvienu kompaktiniu poaibiu $K \subset D$, $f(s) \in \Phi\left(F_{\alpha,\mathfrak{a}}\right)$ riba*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\Phi\left(\zeta(s+i\tau,\alpha;\mathfrak{a})\right) - f(s)| < \varepsilon\right\} > 0.$$

egzistuoja su visais $\varepsilon > 0$, išskyrus ne daugiau negu skaičią reikšmių aibę.

Šių teoremų įrodymui yra taikomas tikimybinis metodas. 5 skyriaus rezultatai yra paskelbti [12] straipsnyje.

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Publikacijos

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1. V. Franckevič, A. Laurinčikas, D. Šiaučiūnas, On joint value distribution of Hurwitz zeta-functions, *Chebysh. Sbr.* **19(3)** (2018), 219–230.

2. V. Franckevič, A. Laurinčikas, D. Šiaučiūnas, On approximation of analytic functions by periodic Hurwitz zeta-functions, *Math. Model. Anal.* **24(1)** (2019), 20–33.

3. A. Balčiūnas, V. Franckevič, V. Garbaliauskienė, R. Macaitienė, A. Rimkevičienė, Universality of zeta-functions of cusp forms and non-trivial zeros of the Riemann zeta-function, *Math. Model. Anal.* **26(1)** (2021), 82–93.

4. A. Balčiūnas, V. Garbaliauskienė, V. Lukšienė, R. Macaitienė, A. Rimkevičienė, Joint discrete approximation of analytic functions by Hurwitz zeta-functions, *Math. Model. Anal.* **27(1)** (2022), 88–100.

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1. V. Franckevič, A. Laurinčikas. On joint universality of Hurwitz zeta - functions, Abstracts of MMA2019, May 28–31, 2019, Talinn, Estonia, pp.
2. V. Franckevič. Universality of the periodic Hurwitz zeta - function. Abstracts of XVII International Conference «Algebra, Number Theory and Discrete Geometry: modern problems, applications and problems of history», September 23-28, 2019, Tula, Russia, pp.
3. V. Franckevič. Approximation of analytic functions by discrete shifts of periodic Hurwitz zeta - functions, Abstracts of MMA2021.

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Publications by the Author

1st publication

On joint value distribution of Hurwitz zeta-functions

V. Franckevič, A. Laurinčikas, D. Šiaučiūnas

Algebra, Number Theory and Discrete Geometry: modern problems, applications and problems of history, XVII International Conference, Tula, Chebysh. Sbr. ***19(3)*** *(2018), 219–230.*

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On approximation of analytic functions by periodic Hurwitz zeta-functions

V. Franckevič, A. Laurinčikas, D. Šiaučiūnas

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https://journals.vilniustech.lt/index.php/MMA/article/view/2331

3rd publication

Universality of zeta-functions of cusp forms and non-trivial zeros of the Riemann zeta-function

A. Balčiūnas, **V. Franckevič**, V. Garbaliauskienė, R. Macaitienė, A. Rimkevičienė

Mathematical Modelling and Analysis ***26(1)*** *(2021), 82–93.*

Link to the publication:
https://journals.vilniustech.lt/index.php/MMA/article/view/12447

4th publication

Joint discrete approximation of analytic functions by Hurwitz zeta-functions

A. Balčiūnas, V. Garbaliauskienė, **V. Lukšienė**, R. Macaitienė,
A. Rimkevičienė

Mathematical Modelling and Analysis ***27(1)*** *(2022), 88–100.*

Link to the publication:
https://journals.vilniustech.lt/index.php/MMA/article/view/15068

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