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First-Order Integer-Valued Autoregressive Model with Bell-Touchard Innovations

Master's thesis

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Abstract

The integer-valued autoregressive process is commonly used to model time-series data. One of the challenges for such models is dealing with overdispersion. In this paper a new integer-valued autoregressive model, denoted as BT-INAR(1) is introduced, which can be used for data that exhibits overdispersion. The model consists of binomial thinning operator and Bell-Touchard innovations. A number of BT-INAR(1) model's properties are derived, such as mean, variance, covariance, autocorrelation, and joint probability function. The model's parameters are estimated using Yule-Walker and Conditional Maximum Likelihood methods. Parameter estimates are compared via Monte Carlo simulation. Finally, the model is applied to two real time-series count data sets, and the model's performance is evaluated.

Keywords: INAR(1), Bell, thinning operator, Bell-Touchard, BT-INAR(1), overdispersion.

Santrauka

Laiko eilučių duomenims modeliuoti dažniausiai naudojamas sveikųjų skaičių autoregresinis procesas. Vienas iš iššūkių su kuriais šie modeliai susiduria yra perteklinė dispersija. Šiame darbe pristatomas naujas sveikųjų skaičių autoregresinis modelis, žymimas kaip BT-INAR(1), kuris gali būti naudojamas duomenims, kurie turi perteklinę dispersiją. Modelį sudaro dvinaris retinimo operatorius ir Bell-Touchard inovacijos. Darbe yra išvedama keletas BT-INAR(1) modelio statistinių parametrų, tokių kaip vidurkis, dispersija, kovariacija, autokoreliacija ir jungtinė tikimybių funkcija. Modelio parametrai yra įvertinti naudojantis Yule-Walker ir Didžiausio tikėtinumo metodais. Modelio parametrų įverčiai yra palyginami naudojantis Monte Carlo simuliaciją. Galiausiai, modelis yra pritaikomas dviems sveikųjų skaičių laiko eilučių duomenų rinkiniams bei ant šių duomenų yra įvertinamas modelio veikimas.

Raktiniai žodžiai: INAR(1), Bell, retinimo operatorius, Bell-Touchard, BT-INAR(1), perteklinė dispersija.

Notation

Some notations used in the paper:

- α is parameter used in INAR(1) process.
- β and θ are parameters used in BT-INAR(1) model.
- ACF denotes autocorrelation function.
- $\mathbb{V}\text{ar}(\cdot)$ denotes variance.
- $\mathbb{E}(\cdot)$ denotes mean.
- $Cov(\cdot, \cdot)$ denotes covariance.

Contents

1 Introduction

There are many time-series counting processes that were developed and studied in recent years. These models are being used in various fields such as finance, economics, medicine, or insurance. The processes play a crucial role in not only explaining the past but also in forecasting the future. Therefore, in this thesis, I have decided to make a modification to the integer-valued autoregressive process and to apply it to count time-series data.

There are many integer valued time series models. Cox [7] proposed that integer valued time series models could be divided into two groups: observation-driven and parameter-driven models. The main difference between the two mentioned is that observation-driven model's parameters are deterministic functions of lagged dependent variables as well as contemporaneous and lagged exogenous variables. Whereas, parameters of parameter-driven models vary over time as dynamic processes with idiosyncratic innovations [11]. The main focus in this thesis will go to observation-driven first order integer-valued autoregressive process INAR(1). INAR model can be obtained by replacing multiplication in conventional ARMA models by thinning operator. This ensures integer discreteness of the process [19].

The INAR(1) process was introduced by Mckenzie in 1985 [14], and Al-Osh and Alzaid [1] which became the basis for numerous researches in count time-series field. The process is created based on a binomial thinning operator which was introduced by Steutel and van Harn [17].

The classical INAR(1) process is described bellow:

$$X_t = \alpha \circ X_{t-1} + \epsilon_t, \ t = 0, 1, 2, ...,$$

where X_t is a non-negative integer valued random variable and $\alpha \in (0, 1)$ is a constant. The binomial thinning operator $\circ$ is defined as $\alpha \circ X_t = \sum_{i=1}^{X} p_i$, where p_i is a sequence of independent identically distributed Bernoulli random variables with $P(p_i = 1) = 1 - P(p_i = 0) = \alpha$. Whereas, ϵ_t is a sequence of independent identically distributed discrete random variables, with mean μ_ϵ and finite variance σ_ϵ^2. One more noticeable characteristic is that ϵ_t is independent of p_i and X_{t-k} for $k \geq 1$ [8].

As described above first order integer valued autoregressive process is defined by two major components - thinning operator and innovations. Therefore, there are two most common approaches to how one can modify INAR(1) model: either by changing the thinning operator or by modifying the innovation assumptions.

There are quite many researches about different options for thinning operator and innovations for INAR(1) process. Therefore, based on the literature I have decided to introduce a new distribution for innovations and construct a completely new INAR(1) model with the binomial thinning operator and Bell-Touchard innovations which will be called BT-INAR(1) model. The main contribution is that such a model was never developed and applied to real count time-series data. The advantage of Bell-Touchard distribution is that it can be used for data with overdispersion. What is more, the Bell-Touchard distribution has a rather simple probability mass function to work with.

The main goal of this paper is to construct a new model for count time-series data with Bell-Touchard innovations that could deal with data that have overdispersion. To achieve that the paper will have a short summary of the literature to get an overview about INAR(1) process and will be divided into two main parts - theoretical and empirical.

In the theoretical part, the main characteristics and definitions of the BT-INAR(1) process are derived and proved. Firstly, based on the literature the theoretical part will focus on introducing the Bell-Touchard distribution and the binomial thinning operator to derive the needed information for the BT-INAR(1) model. Then the BT-INAR(1) model will be introduced. The BT-INAR(1) model main characteristics will be derived and proofs of the characteristics will be provided in this thesis. Those are mean, variance, conditional mean, conditional variance, covariance, and autocorrelation function. Secondly, parameter estimation will be performed. The estimation methods are based on Yule-Walker equations and the Conditional Maximum Likelihood function. Yule-Walker estimators for the BT-INAR(1) process will be calculated based on the method of moments. In addition, we will also use the sample mean, sample variance, and sample autocorrelation function in order to get estimates with Yule-Walker method. Conditional Maximum Likelihood estimator does not have an exact expression for the parameters, therefore, numerical estimator with `optim` function in R will be used to calculate them in the empirical part. The main reason for that is that Yule-Walker equations has a closed-form solution. Meaning, that the method has specific expressions.

In the empirical part based on the derived definitions and characteristics, the new model will be simulated. Firstly, Bell-Touchard and the binomial thinning operator will be generated. Then based on the latter obtained functions we will create a function that could generate random integer numbers based on the BT-INAR(1) model. Based on this function and using Monte-Carlo simulation parameter estimates will be calculated with Yule-Walker and Conditional Maximum Likelihood methods. In the last part, the model is applied to the real count time-series data.

2 Literature review

In literature there are numerous researches about different options for thinning operator and innovations for INAR(1) process.

Weiß (2018) [20] has provided a summary of several possible options for thinning operator. He provides summaries for thinning operators such as random coefficient thinning, iterated thinning and quasi-binomial thinning. Zhengwei and Fukang [12] focused on a specific thinning operator and provided a new extension of thinning-based integer-valued autoregressive model. Authors introduced a new thinning operator named extended binomial, which is a general case of the binomial thinning operator. Compared to the base binomial thinning operator the new one has two parameters and is more flexible in modelling. Nastic, Ristic and Janjic [15] focused on binomial and negative binomial thinning operators and derived new thinning operator called mixed thinning operator. It was created as a probability mixture of the mentioned thinning operators. In other researches authors chooses a very similar approach and tries to derive the best integer-valued autoregressive model for count time-series data by improving the thinning operator. For example, Mansour and Alamatsaz [9] defined two generalizations of binomial thinning operator by replacing Bernoulli random variables with zero-inflated Bernoulli and inflated parameter Bernoulli.

Other authors analysed a different approach for improving the INAR(1) model. They tend to achieve the best INAR(1) model by modifying the innovations. The most common choice for innovations is Poisson distribution. Poisson's distribution has mean and variance equal to each other. However, integer empirical data examples are usually overdispersed. Therefore, Poisson distribution is not always a good choice for modelling purposes. Overdispersion is not the only problem for INAR processes. Also, models encounter issues with data that is underdispersed or equidispersed. Bourguignon, Rodrigues and Santos-Neto [4] focused on the possible solution for the three observed issues for INAR processes. They propossed two new INAR(1) models with double Poisson and generalized Poisson innovations. Models are called INARDP(1) and INARGP(1) respectively. Their motivation is that the two mentioned distributions are more appropriate, because they have more then one type of dispersion or the probability mass function of distribution and moments of innovations are not so complicated. Huang and Fukang [8] provides a rather simple approach by introducing Bell distribution for innovations. The advantage of Bell distribution is that it has only one parameter, have infinite divisibility and have a rather simple probability mass function [6]. What is more, the BL-INAR(1) is suitable for data with overdispersion. Therefore, in this thesis the new BT-INAR(1) model is introduced where for innovations Bell-Touchard distribution is being used. The Bell-Touchard is very close distribution to Bell and both of them are introduced in the later sections with more details.

3 Bell distribution and Binomial thinning operator

In this section a select number of characteristics of Bell distribution and binomial thinning operator are presented.

3.1 Bell distribution

Before introducing Bell-Touchard distribution it is important to have a look at Bell distribution at first.

Definition 1. A discrete random variable Z with values $\mathbb{N}_0 = 0, 1, 2, ...$ has a Bell distribution with parameter θ and is denoted as $Z \sim Bell(\theta)$, if its probability function is given by

$$P(Z = z) = \frac{\theta^z \mathrm{e}^{-\mathrm{e}^\theta + 1} B_z(\theta)}{z!}, z \in \mathbb{N}_0, \tag{1}$$

where B_z is the Bell number which has the following form

$$B_z(\theta) = \mathrm{e}^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{2}$$

The Bell number B_z is the n-th moment of the Poisson distribution with parameter equal to 1 [8]. From the mean and variance one can notice that the Bell distribution is overdispersed.

$$\mathbb{E}(Z) = \theta \mathrm{e}^\theta; \mathbb{V}\mathrm{ar}(Z) = \theta(1 + \theta)\mathrm{e}^\theta,$$

then $\mathbb{V}\mathrm{ar}(Z)/\mathbb{E}(Z) = 1 + \theta > 1$. This characteristic means that the Bell distribution can be applied to count time series data that has overdispersion. We make the same assumption for Bell-Touchard distribution and it is proved in the following sections.

3.2 Binomial thinning operator

In this section basic properties of binomial thinning operator are listed. The random variable $\alpha \circ X := Y_i$ is called binomial thinning operator with counting series Y_i if X is a discrete random variable. Counting series Y_i is identically distributed binary random variables which are independent from X. Therefore, binomial thinning operator can only get integer values from 0 to X [20]. Below are listed conditional mean and conditional variance of binomial thinning operator which are necessary for following sections.

Conditional mean:

$$\mathbb{E}(\alpha \circ X) = \mathbb{E}(\mathbb{E}(\alpha \circ X | X)) = \mathbb{E}(\alpha X) = \alpha \mu, \tag{3}$$

where $\mu = \mathbb{E}(X)$.

Conditional variance:

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\alpha \circ X) &= \mathbb{V}\mathrm{ar}(\mathbb{E}(\alpha \circ X|X)) + \mathbb{E}(\mathbb{V}\mathrm{ar}(\alpha \circ X|X)) \\ &= \mathbb{V}\mathrm{ar}(\alpha X) + \mathbb{E}(\alpha(1-\alpha)X) = \alpha^2\sigma^2 + \alpha(1-\alpha)\mu\end{aligned} \tag{4}$$

4 Bell-Touchard distribution

In this section Bell-Touchard distribution [5] with the main characteristics is presented.

4.1 Introduction to Bell-Touchard distribution

The two parameter discrete distribution is derived based on the expansion presented by Touchard [18] and Bell ([3], [2]):

$$\exp[y(\mathrm{e}^X - 1)] = \sum_{n=0}^{\infty} \frac{T_n(y)}{n!} X^n, X, y \in \mathbb{R} \tag{5}$$

where the coefficients $T_n()$ are the Touchard polynomials, and are defined by

$$T_n(\theta) = \mathrm{e}^{-\theta} \sum_{k=0}^{\infty} \frac{k^n \theta^k}{k!} \tag{6}$$

The Touchard polynomial could be understood as, $T_0(\theta) = 1, T_1(\theta) = \theta, T_2(\theta) = \theta^2 + \theta, T_3(\theta) = \theta^3 + 3\theta^2 + \theta, T_4(\theta) = \theta^4 + 6\theta^3 + 7\theta^2 + \theta$, and so on.

Interesting characteristic is that if $\theta = 0$ then Touchard polynomial is equal to Bell number, which was introduced in the previous section.

From the formula of Touchard Polynomial and from the Figure 1 below one can notice that if $n \to \infty$ then $T_n(\theta) \to \infty$. This characteristic holds no matter what θ value is chosen.

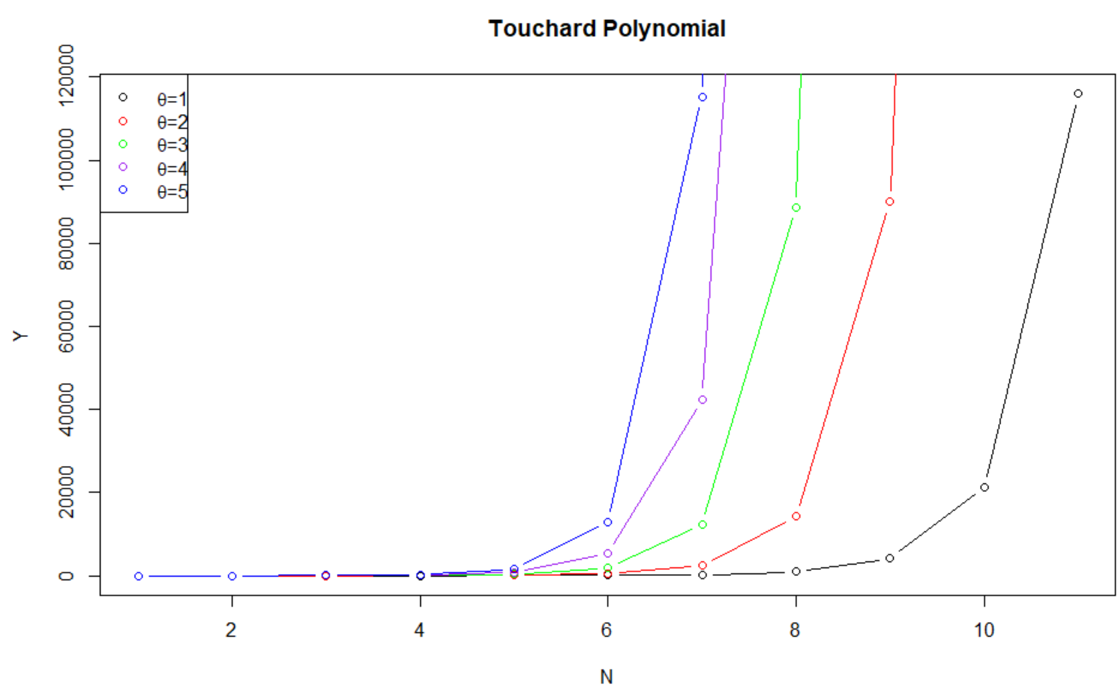

Figure 1: The Touchard Polynomial when $\theta \in \{1, 2, 3, 4, 5\}$.

4.2 Main characteristics of Bell-Touchard distribution

With the use of formulas (5) and (6) one can define the Bell-Touchard distribution.

Definition 2. A discrete random variable Y has a BT distribution if its probability mas function is given by

$$P(Y=y)=\frac{\beta^{y}\mathrm{e}^{\theta(1-\mathrm{e}^{\beta})}T_{y}(\theta)}{y!} \tag{7}$$

where $\beta>0, \theta>0$, and $T_k()$ are the Touchard polynomials in (6).

One can notice that BT distribution consists of two non-negative parameters β and θ. What is more, the probability mass function (7) does not have a complicated functional form. Therefore, it is relatively easy to deal with.

If the Y has Bell-Touchard distribution then mean and variance can be derived by using the probability generating function:

$$\mathbb{E}(Y)=\theta\beta\mathrm{e}^{\beta}, \mathbb{V}\mathrm{ar}(Y)=\theta(1+\beta)\beta\mathrm{e}^{\beta} \tag{8}$$

From mean and variance one can notice that Bell-Touchard distribution is overdispersed, meaning, that in certain situations it is suitable for count data with overdispersion. This can be seen by finding the index of dispersion: $\mathbb{V}\mathrm{ar}(Y)/\mathbb{E}(Y)=1+\beta>1$. Additionally, the index of dispersion depends only on parameter β. If β increases then the dispersion of Bell-Touchard distribution increases.

In the Figure 2 one can see different behaviour of Bell-Touchard's probability mass function with different combinations of parameters β and θ. To make sure that our function generates results that are expected in Appendix A we provided Bell-Touchard probability mass function and density plot of Bell-Touchard function.

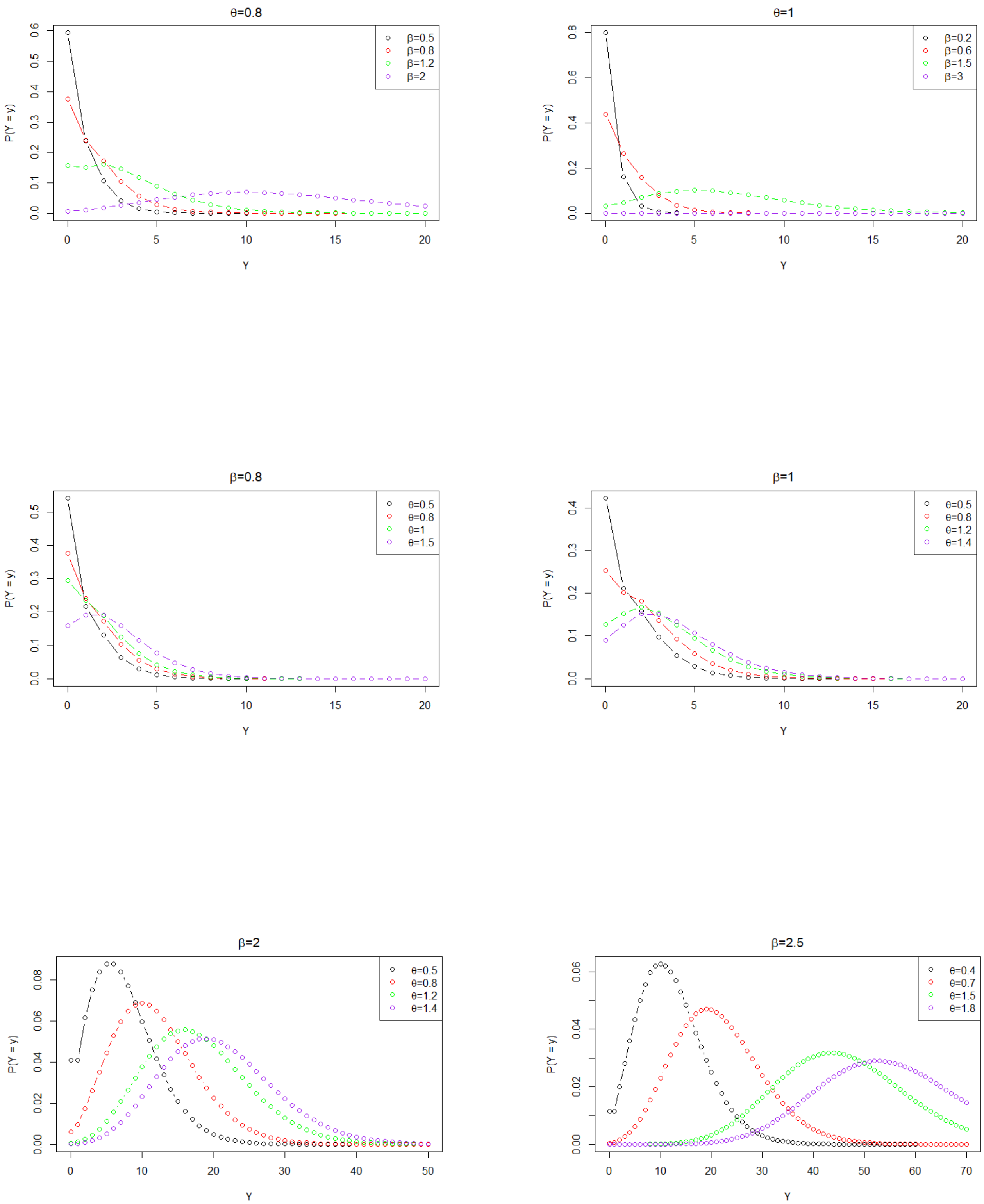

Figure 2: Bell-Touchard probability mass function with different variations of parameters β and θ.

5 BT-INAR(1) model

In this section we introduce a new INAR(1) process with Bell-Touchard innovations and derive a number of statistical properties of this model. The new model will be named as BT-INAR(1).

Definition 2. Let ϵ_t be a random variable with Bell-Touchard innovations, then X_t is defined by the following equation (9) is called a BT-INAR(1) process;

$$\begin{cases} X_t = \alpha \circ X_{t-1} + \epsilon_t, t \in \mathbb{N} \\ \epsilon_t \sim BT(\beta, \theta), \end{cases} \tag{9}$$

where $0 < \alpha < 1$, $\theta > 0$, $\beta > 0$.

Based on equation (8) we know that the mean and variance of ϵ_t is finite for reasonable values of θ and β. Therefore the process X_t described in (9) is a stationary Markov chain [10] with transition probabilities that can be described as follows:

$$\begin{aligned} P_{ij} =& P(X_t = i | X_{t-1} = j) = P(\alpha \circ X_{t-1} + \epsilon_t | X_{t-1} = j) \\ =& \sum_{m=0}^{min(i,j)} P(\alpha \circ X_{t-1} = m | X_{t-1} = j) P(\epsilon_t = i - m) \\ =& \sum_{m=0}^{min(i,j)} \begin{pmatrix} j \\ m \end{pmatrix} \alpha^m (1-\alpha)^{j-m} \frac{\beta^{i-m} \mathrm{e}^{\theta(1-\mathrm{e}^{\beta})} T_{i-m}(\theta)}{(i-m)!} \end{aligned} \tag{10}$$

Using transition probabilities one can obtain joint probability function:

$$\begin{aligned} f(i_1, i_2, ..., i_T) =& P(X_1 = i_1, X_2 = i_2, ..., X_T = i_T) \\ =& P(X_1 = i_1) P(X_2 = i_2 | X_1 = i_1) .... P(X_T = i_T | X_{T-1} = i_{T-1}) \\ =& \prod_{k=2}^{T-1} \left[\sum_{m=0}^{min(i_k, i_{k-1})} \begin{pmatrix} i_k \\ m \end{pmatrix} \alpha^m (1-\alpha)^{i_k - m} P(\epsilon_{k-1} = i_{k-1} - m) \right] \end{aligned} \tag{11}$$

Further, below we describe the main properties of the new BT-INAR(1) model. Such as mean, variance, conditional mean, conditional variance, covariance and autocorrelation function.

Definition 3. Let X_t be a BT-INAR(1) process, described in (9). Then X_t has the following properties:

3.1 $\mathbb{E}(X_t | X_{t-1}) = \alpha X_{t-1} + \theta \beta \mathrm{e}^{\beta}$;

3.2 $\mathbb{V}\mathrm{ar}(X_t | X_{t-1}) = \alpha(1-\alpha) + \theta(1+\beta)\beta \mathrm{e}^{\beta}$;

3.3 $\mu_X = \mathbb{E}(X_t) = \frac{\theta \beta \mathrm{e}^{\beta}}{1-\alpha}$;

3.4 $\sigma_X^2 = \mathbb{V}\mathrm{ar}(X_t) = \frac{\theta \beta \mathrm{e}^{\beta}(\alpha+1+\beta)}{1-\alpha^2}$;

3.5 $\gamma_k = Cov(X_k, X_{k+1}) = \alpha^k \sigma_X^2$, $k \geq 1$;

3.6 $\rho_k = Corr(X_k, X_{k+1}) = \alpha^k,\ k \geq 1.$

Proof of 3.1. With the use of thinning operator and ϵ_t characteristics listed in previous sections below we provide proof for conditional mean:

$$\begin{aligned}\mathbb{E}(X_t|X_{t-1}) &= \mathbb{E}(\alpha \circ X_{t-1} + \epsilon_t|\epsilon_{t-1})\\ &= \mathbb{E}(\alpha \circ X_{t-1}|X_{t-1}) + \mathbb{E}(\epsilon_t|X_{t-1}) = \mathbb{E}(\alpha \circ X_{t-1}) + \mathbb{E}(\epsilon_t|X_{t-1})\\ &= \alpha X_{t-1} + \theta\beta \mathrm{e}^{\beta}.\end{aligned}$$

□

Proof of 3.2. With the use of thinning operator and ϵ_t characteristics listed in previous sections below we provide proof for conditional variance:

$$\begin{aligned}\mathbb{V}\mathrm{ar}(X_t|X_{t-1}) &= \mathbb{V}\mathrm{ar}(\alpha \circ X_{t-1} + \epsilon_t|\epsilon_{t-1})\\ &= \alpha(1-\alpha)X_{t-1} + \sigma_\epsilon^2 = \alpha(1-\alpha)X_{t-1} + \theta(1+\beta)\beta\mathrm{e}^{\beta}.\end{aligned}$$

□

Proof of 3.3. With the use of thinning operator, conditional mean and ϵ_t characteristics below we provide proof for mean:

$$\begin{aligned}\mu_X &= \mathbb{E}(\mathbb{E}(X_t|X_{t-1})) = \mathbb{E}(\mathbb{E}(\alpha \circ X_{t-1} + \epsilon_t|X_{t-1}))\\ &= \mathbb{E}(\alpha X_{t-1} + \epsilon_t) = \mathbb{E}(\alpha(\alpha X_{t-2} + \epsilon_{t-1} + \epsilon_t)\\ &= \mathbb{E}(\alpha^2 X_{t-2} + \alpha\epsilon_{t-1} + \epsilon_t) = \mu_\epsilon(1 + \alpha + \alpha^2 + ... + \alpha^n)\\ &= \mu_\epsilon \sum_{j=0}^{\infty} \alpha^j = \frac{\theta\beta\mathrm{e}^{\beta}}{1-\alpha}.\end{aligned}$$

□

Proof of 3.4. With the use of thinning operator, conditional mean, conditional variance, mean and ϵ_t characteristics below we provide proof for variance:

$$\begin{aligned}\sigma_X^2 &= \mathbb{V}\mathrm{ar}(\mathbb{E}(X|X_{t-1})) + \mathbb{E}(\mathbb{V}\mathrm{ar}(X_t|X_{t-1})\\ &= \mathbb{V}\mathrm{ar}(\alpha X_{t-1} + \mu_\epsilon) + \mathbb{E}(\alpha(1-\alpha)X_{t-1} + \sigma_\epsilon^2) = \alpha^2\mathbb{V}\mathrm{ar}(X_{t-1}) + \alpha(1-\alpha)\mathbb{E}(X_{t-1}) + \sigma^2\epsilon\\ &= \alpha^2\sigma_X^2 + \alpha(1-\alpha)\mu_X + \sigma_\epsilon^2 = \frac{\theta\beta\mathrm{e}^{\beta}(\alpha+1+\beta)}{1-\alpha^2}.\end{aligned}$$

□

Proof of 3.5 and 3.6. Al-Osh and Alzaid [1] expressed that the first-order INAR process is second-oder stationary process no matter what innovation's distribution is for all $0 \leq \alpha < 1$, with $\rho(X_t, X_{t-k}) = \alpha^k$, which only depends on k. So the covariance and correlation is described as follows:

$$\begin{aligned}\gamma_k &= Cov(X_k, X_{k+1}) = \alpha^k\sigma^2\\ \rho_k &= Corr(X_k, X_{k+1}) = \alpha^k.\end{aligned}$$

□

6 Parameter Estimation

We have three unknown parameters - (θ, α, β). Therefore, we need to estimate them. The estimation methods are based on Yule-Walker equations and Conditional Maximum Likelihood function.

6.1 Yule-Walker Estimation

In this section we will calculate Yule-Walker estimators, which are based on the method of moments, for the BT-INAR(1) model's parameters - (θ, α, β). In addition, to obtain the Yule-Walker estimators we will use sample mean, sample variance and sample autocorrelation function.

Let X_t is a process from Definition 2 where $t = 1, 2, ..., T$.

From the BT-INAR(1) characteristics from Definition 3 we know that $\rho_k = \alpha^k$. Therefore, we can derive the Yule-Walker estimate of α:

$$\hat{\alpha}_{YW} = \hat{\rho}(1) = \frac{\sum_{t=1}^{T-1}(X_t - \bar{X})(X_{t-1} - \bar{X})}{\sum_{t=1}^{T}(X_t - \bar{X})^2}. \tag{12}$$

To obtain $\hat{\beta}_{YW}$ we use sample variance and we have that $\hat{\sigma}^2_{YW} = \bar{\sigma}^2_X$. From this and Def.3 (3.4) property we get that the Yule-Walker estimate for $\hat{\beta}$ is as follows:

$$\hat{\beta}_{YW} = \frac{\bar{\sigma}^2_X(1 + \hat{\alpha}_{YW})}{\bar{\mu}_X} - 1 - \alpha_{YW}. \tag{13}$$

Then we have that $\hat{\mu}_{YW} = \bar{X}$, where $\bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$. By using this characteristic we can find Yule-Walker estimator for θ parameter.

We know that $\mu = \bar{X} = \frac{\theta\beta e^{\beta}}{1-\alpha}$ from Def.3 (3.3). From that we get a straightforward equation and we can obtain the Yule-Walker estimate for $\hat{\theta}_{YW}$:

$$\hat{\theta}_{YW} = \frac{\bar{X}(1 - \hat{\alpha}_{YW})}{\hat{\beta}_{YW} e^{\hat{\beta}_{YW}}}. \tag{14}$$

Below we provide proof for eq. (13):

Proof.

$$
\begin{aligned}
\hat{\sigma}^2_{YW} = \bar{\sigma}^2_X &= \frac{\hat{\theta}_{YW}\hat{\beta}_{YW}\mathrm{e}^{\hat{\beta}_{YW}}(\hat{\alpha}_{YW} + 1 + \hat{\beta}_{YW})}{1 - \hat{\alpha}^2_{YW}} \\
&= (\frac{\hat{\theta}_{YW}\hat{\beta}_{YW}\mathrm{e}^{\hat{\beta}_{YW}}}{1 - \hat{\alpha}})(\frac{\hat{\alpha}_{YW} + 1 + \hat{\beta}_{YW}}{1 + \hat{\alpha}}) \\
&= \bar{\mu}_X \frac{\hat{\alpha}_{YW} + 1 + \hat{\beta}_{YW}}{1 + \hat{\alpha}} 1 + \hat{\alpha} \\
&\Rightarrow \hat{\beta}_{YW} = \frac{\bar{\sigma}^2_X(1 + \hat{\alpha}_{YW})}{\bar{\mu}_X} - 1 - \alpha_{YW}.
\end{aligned}
$$

□

6.2 Conditional Maximum Likelihood Estimation

In this section we will describe how to calculate conditional maximum estimates for the parameters of BT-INAR(1) process. We omit first lag from the calculations as there is no past to predict from, assuming $t \geq 1$.

By using the joint probability function (11) we can obtain the likelihood function as follows:

$$
\begin{aligned}
f(X_2, ..., X_T) &= \prod_{t=2}^{T-1} P(X_{t+1} = x_{t+1} | X_t = x_t) \\
&= \prod_{t=2}^{T-1} \left[\sum_{m=0}^{\min(X_t, X_{t+1})} \binom{X_t}{m} \alpha^m (1-\alpha)^{X_t - m} P(\epsilon_{t+1} = X_{t+1} - m) \right].
\end{aligned} \tag{15}
$$

Conditional log-likelihood function:

$$
L(\alpha, \theta, \beta) = \sum_{t=2}^{T-1} \log P(X_{t+1} = X_{t+1} | X_t = x_t) \to \max_{\alpha, \theta, \beta} \tag{16}
$$

The CML estimates of the parameters (α, θ, β) are the values $(\hat{\alpha}_{CML}, \hat{\theta}_{CML}, \hat{\beta}_{CML})$ obtained by maximizing the conditional log likelihood function $L(\alpha, \theta, \beta)$. This will be achieved with optim function in R.

7 Monte Carlo Simulation

Monte Carlo Simulation estimates for BT-INAR(1) model parameters are calculated using Yule-Walker and Conditional Maximum Likelihood methods. To evaluate the results it was decided to calculate the empirical mean of the estimates, the mean squared errors, draw box-plots and density plots.

The empirical mean of the estimate could show how close to the actual parameter value simulated estimates were. The empirical mean takes into account all observed parameter values during the different simulations. The mean squared errors (MSE) measures the average squared difference between the estimated values and the actual values:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(\phi_i - \hat{\phi}_i)^2, \tag{17}$$

where ϕ_i is actual value and $\hat{\phi}_i$ is estimated parameter value. The lower the MSE the lower the difference between estimated and actual parameters.

Box-plot and Density function will give us a visual look at the relationship between actual and estimated parameters.

Six different parameter variations were chosen. For each variation, it was decided to simulate 2500 replications when choosing two different sample sizes: 100 and 400. It is interesting to see the differences between having a relatively small sample (100) and a relatively big sample (400). So in total, there are 12 different calculations done, which are presented in the Table 1. The table shows different parameters (α, θ, β) variations with the empirical mean of the estimates as well as with the mean squared errors in parentheses. What is more, when generating the data an additional number of 'burn-in' observations were generated to initialize the process.

The parameters were chosen based on the ratio between BT-INAR(1) models variance and mean from Definition 3 - 3.3 and 3.4 equations. Table are ordered from highest to lowest ratio where variance is bigger than mean.

N	α_{YW}	θ_{YW}	β_{YW}	α_{CML}	θ_{CML}	β_{CML}
			$\alpha = 0.3,\ \theta = 0.8,\ \beta = 2$			
100	0.27551	1.47001	1.93931	0.28642	1.23127	1.94338
	(0.01)	(3.65395)	(0.335)	(0.00761)	(0.98499)	(0.25221)
400	0.29313	0.92997	1.98724	0.29523	0.925	1.98225
	(0.00245)	(0.22301)	(0.08549)	(0.00209)	(0.20121)	(0.07916)
			$\alpha = 0.3,\ \theta = 0.5,\ \beta = 1.6$			
100	0.27886	0.86668	1.55345	0.28738	0.77668	1.54246
	(0.00991)	(1.08703)	(0.25154)	(0.00666)	(0.49067)	(0.18945)
400	0.29356	0.57834	1.59052	0.29662	0.56299	1.58681
	(0.00247)	(0.08652)	(0.06958)	(0.00164)	(0.06303)	(0.05295)
			$\alpha = 0.6,\ \theta = 1,\ \beta = 1.6$			
100	0.5666	3.12286	1.48893	0.59125	1.58305	1.58272
	(0.00789)	(429.6266)	(0.4703)	(0.00406)	(1.50488)	(0.30858)
400	0.59032	1.38072	1.56206	0.59513	1.25528	1.5738
	(0.00176)	(1.15685)	(0.12511)	(0.00121)	(0.54597)	(0.09964)
			$\alpha = 0.5,\ \theta = 0.6,\ \beta = 1.4$			
100	0.46824	1.51704	1.31294	0.48442	1.06217	1.33385
	(0.00915)	(30.24141)	(0.34583)	(0.00555)	(1.01883)	(0.22421)
400	0.49242	0.75437	1.38578	0.49561	0.70751	1.38534
	(0.00205)	(0.25675)	(0.09233)	(0.00126)	(0.15203)	(0.06351)
			$\alpha = 0.5,\ \theta = 1.2,\ \beta = 0.5$			
100	0.4655	1.08763	0.45601	0.49045	1.6458	0.51247
	(0.00983)	(4984.828)	(0.14688)	(0.00499)	(1.20911)	(0.06448)
400	0.49118	2.00794	0.4869	0.4962	1.42664	0.49208
	(0.00233)	(60.94533)	(0.04093)	(0.00129)	(0.49859)	(0.01981)
			$\alpha = 0.2,\ \theta = 1.5,\ \beta = 0.2$			
100	0.1763	0.79045	0.18672	0.19892	1.80124	0.23821
	(0.0125)	(539.4529)	(0.05222)	(0.00729)	(2.72648)	(0.02591)
400	0.19295	0.83592	0.1975	0.19557	1.7607	0.20766
	(0.00337)	(5989.986)	(0.01399)	(0.00255)	(0.74877)	(0.0079)

Table 1: Empirical means and mean squared errors (in parentheses) of the estimates of the BT-INAR(1) model parameters.

From the Table 1 one can see that parameter estimates are more accurate when BT-INAR (1) model was generated on a bigger sample size. This is valid for all six parameter variations. Therefore, one can conclude that for newly developed BT-INAR(1) bigger count time-series data is necessary to achieve better results.

What is more, for some of the parameter variations Yule-Walker estimates of $\hat{\theta}$ have a quite large mean squared error as seen Table 1. This is mainly driven by several big outlier values.

What is more, not all parameter variations with bigger sample sizes have accurate parameter estimates. Parameter estimation accuracy decreases when the ratio between variance and mean decreases. For example, parameter estimates are more accurate when parameter values were chosen ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$) compared to ($\alpha = 0.2$, $\theta = 1.5$, $\beta = 0.2$). The latter parameter variation has a smaller ratio between variance and mean. The smaller ratio might cause higher outliers for the parameter estimates. This is observed when looking at the MSE. When the ratio between variance and mean decreases the MSE increases significantly. This can be noticed in the above-mentioned example as well. Therefore, MSE suggests that the empirical mean of the estimates might be biased due to some outliers in the calculation.

To have a better look let's compare box-plots of the Yule-Walker and Conditional Maximum Likelihood estimates when parameter values were chosen ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$) and ($\alpha = 0.2$, $\theta = 1.5$, $\beta = 0.2$) and sample size is 400.

Figure 3 shows box-plots when ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$) and Figure 4 shows box-plots when ($\alpha = 0.2$, $\theta = 1.5$, $\beta = 0.2$). When drawing box-plots the outliers are ommited.

The box plots show that with both parameter variations the median of the estimates are very close to the actual parameter value when outliers were removed. However, interquartile range is bigger when ($\alpha = 0.2$, $\theta = 1.5$, $\beta = 0.2$). This suggests that 50% of the estimates has a bigger volatility when comparing to calculation when parameter values were ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$). From this, we can conclude that the best estimates were calculated when the sample size is higher and the ratio between variance and mean is bigger.

Other parameter variations have a very similar pattern, therefore, more detailed explanations and graphs are omitted.

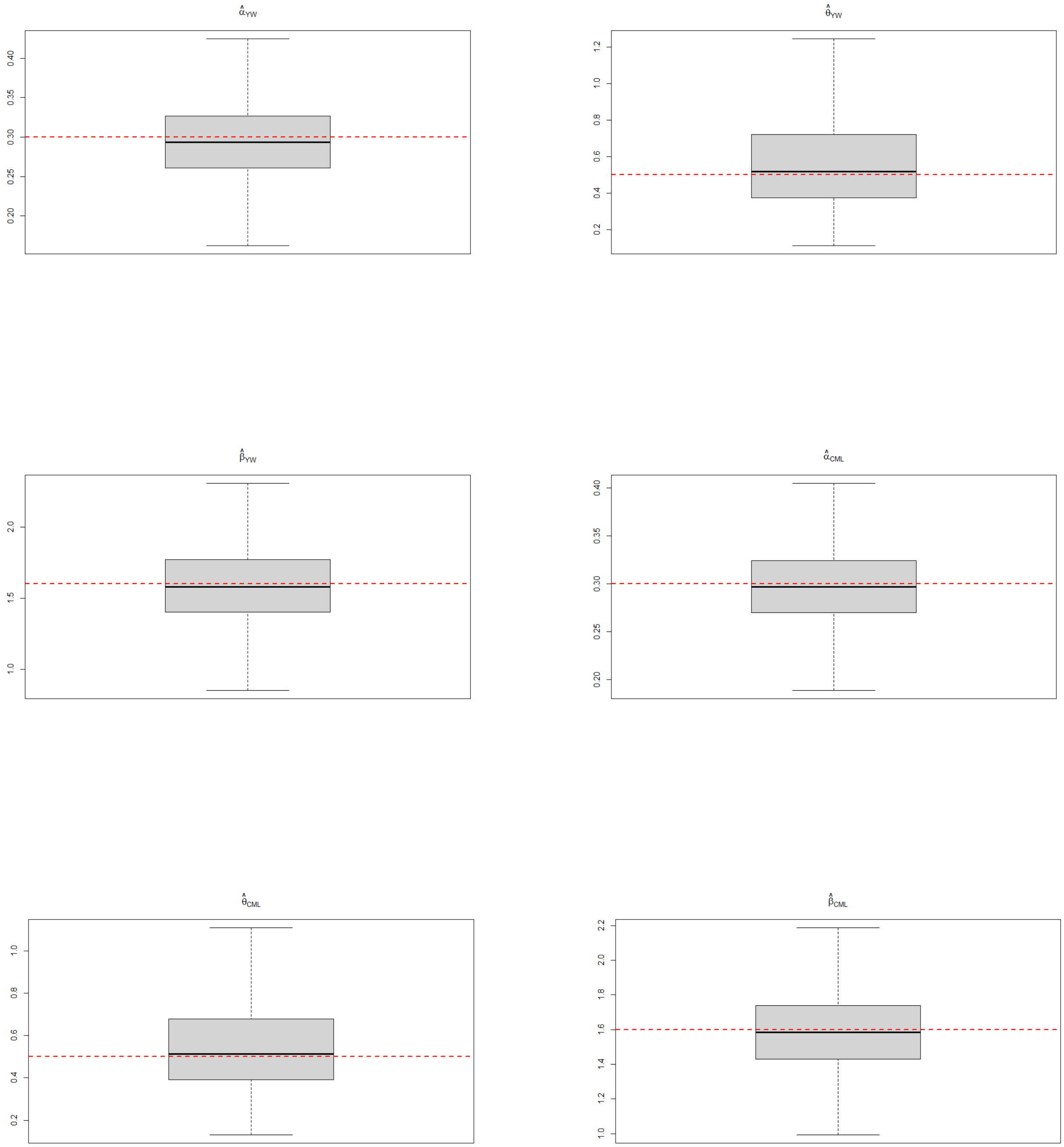

Figure 3: Yule-Walker and Conditional Maximum Likelihood estimates when $\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$.

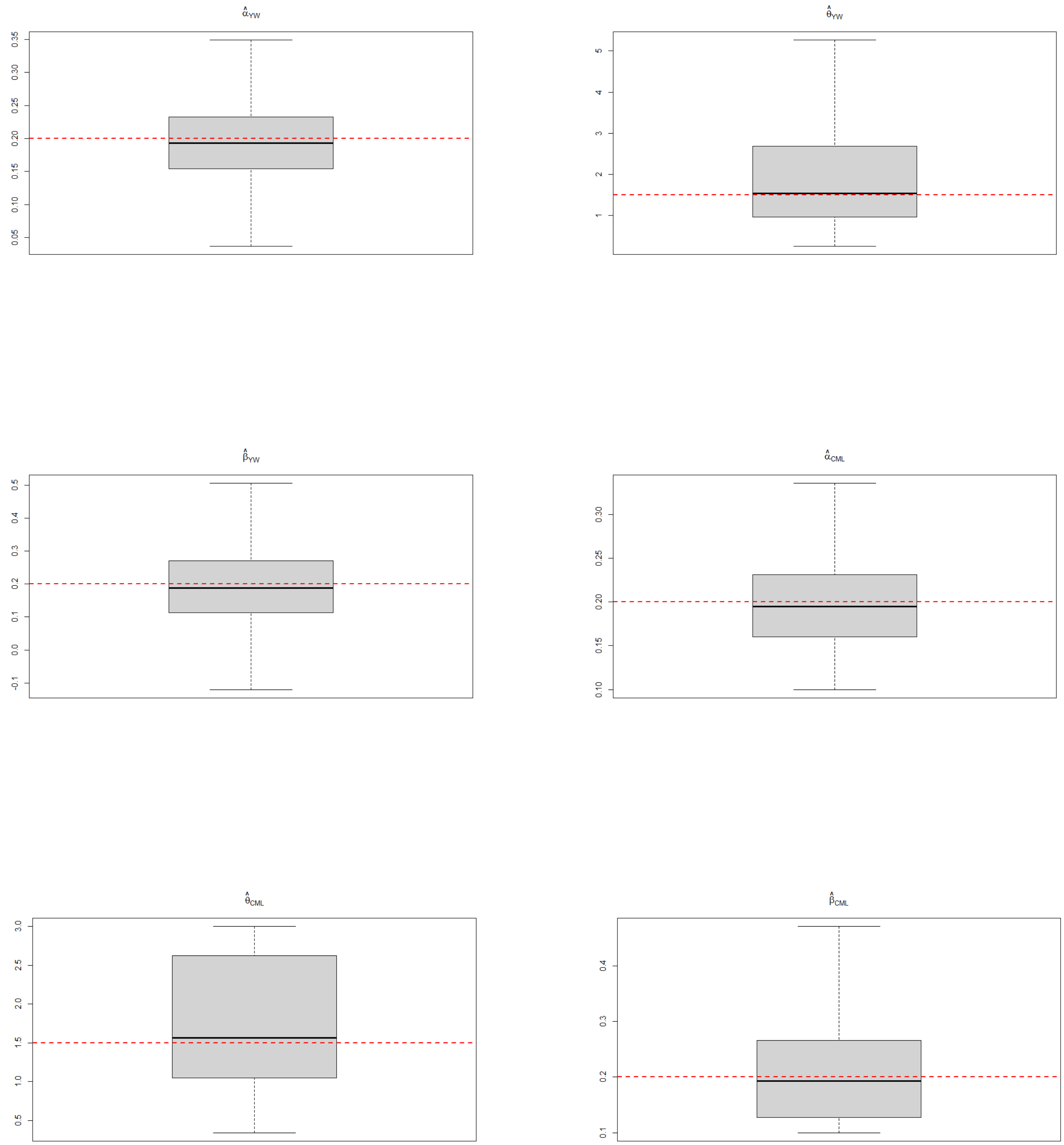

Figure 4: Yule-Walker and Conditional Maximum Likelihood estimates when $\alpha = 0.2$, $\theta = 1.5$, $\beta = 0.2$.

From the Monte Carlo analysis, it is hard to determine which method, Yule-Walker or Conditional Maximum Likelihood, provides better estimates for the BT-INAR(1) model parameters. To determine that density plots are used. For the graphs ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$) parameter variation is chosen as this calculation provided one of the most accurate results.

In the Figure 5 one can see density plots for parameter estimates with Yule-Walker and Conditional Maximum Likelihood methods when actual parameter values were - ($\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$).

When looking at the graphs it is easy to notice that density plots for all three parameters using both methods form bell shape. The vertical line refers to the actual parameter value. Peaks of all six density plots are not exactly on the vertical line (actual parameter value), but are close to it. This suggests that both methods calculate parameter estimates quite well, but not perfect. However, Conditional Maximum Likelihood's density plots for the estimates have higher heads and sharper tails. This implies that CML estimates are slightly better compared to Yule-Walker. But, CML has higher computational costs, therefore in the empirical part Yule-Walker method is being used to calculate parameter values for the real count time-series data sets.

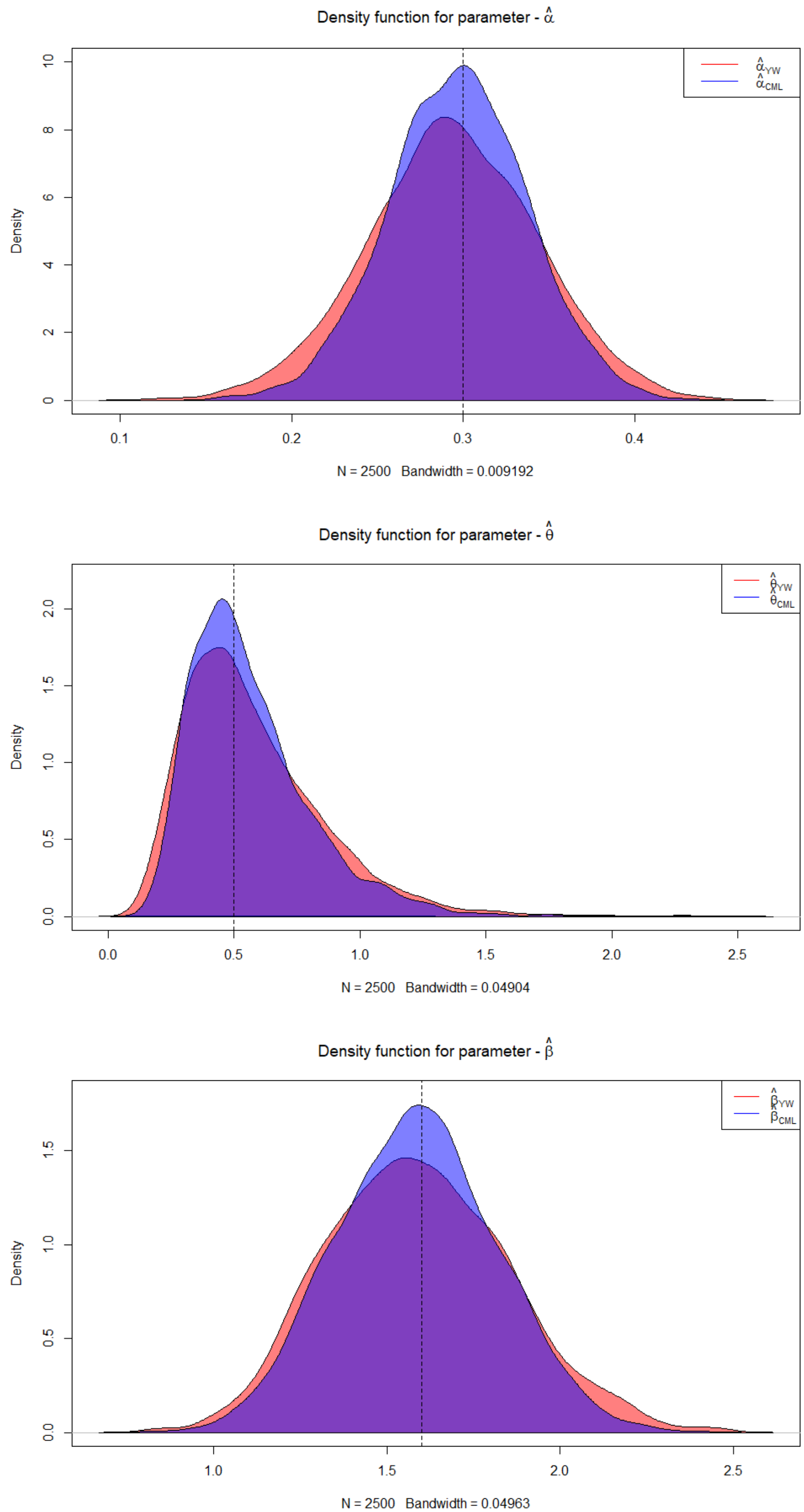

Figure 5: Yule-Walker and Conditional Maximum Likelihood estimates density plots when $\alpha = 0.3$, $\theta = 0.5$, $\beta = 1.6$.

From the Monte Carlo simulation one can make several conclusions:

1. Parameter estimation is more accurate when the sample size is bigger.

2. Parameter estimation is more accurate when the difference between variance and mean in the sample is higher.

3. Parameter estimation is quite good with both methods - Yule-Walker and Conditional Maximum Likelihood. However, due to simplicity in the computational part Yule-Walker method will be used in the empirical data example section.

8 Empirical Data Example

In this section BT-INAR(1) model is applied to actual count time-series data. Two different data sets are analysed.

8.1 Covid-19 data

The first data set consists of daily deaths due to Covid-19 in Lithuania from 2020-02-01 until 2021-12-21. The data set contains 690 observations which can be seen in Figure 6. From the graphical view, it is hard to notice any seasonality or trends in the data.

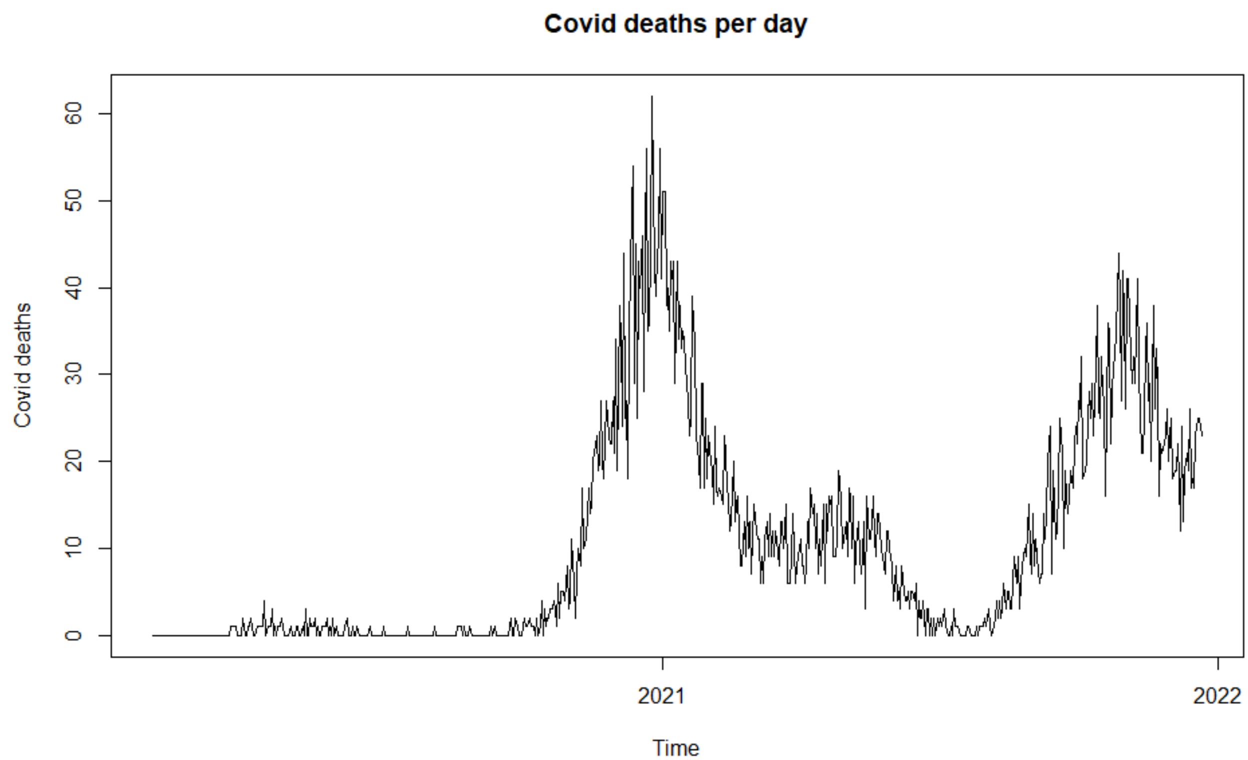

Figure 6: Number of deaths due to Covid-19 in Lithuania.

Autocorrelation function graph in Figure 7 also shows that there is no seasonality in the data. What is more, some correlation is observed from the ACF graph. However, to verify that the correlation in the data is statistically significant we performed the Ljung-Box test which was introduced by Ljung and Box [13]. Test shows that the $p-value = 2.2 \cdot 10^{-16} < 0.05$. This suggests that, indeed, Covid-19 data have some correlation. This is one of the characteristics which implies that we could apply the BT-INAR(1) model to the Covid-19 data set, if the variance is greater than the mean.

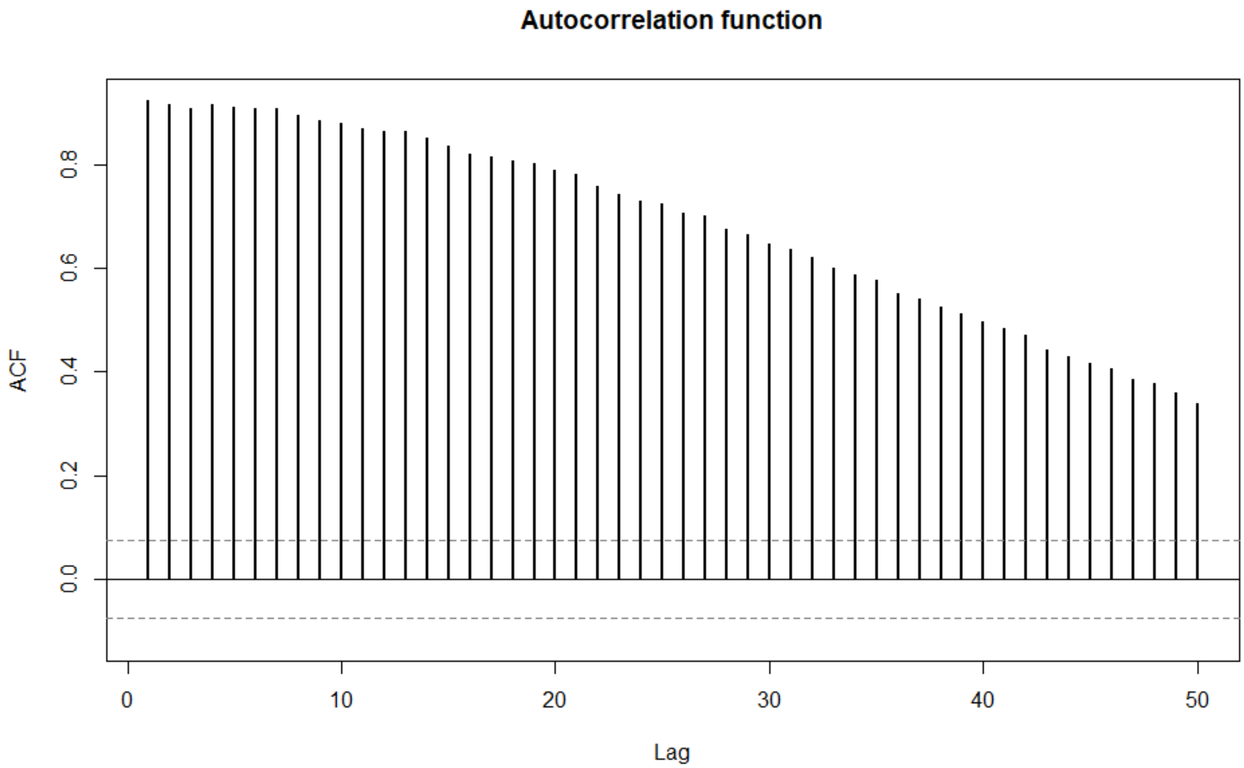

Figure 7: Autocorrelation function of Covid-19 data.

The BT-INAR(1) is suitable for modeling data that is overdispersed. The sample mean and sample variance are $\bar{X} = 10.43478$ and $S_X^2 = 160.3651$,respectively. One can see that the difference between the sample mean and sample variance appears to be significant. To verify whether this hypothesis holds we calculated the dispersion index $I_X = \frac{S_X^2}{\bar{X}} = 15.36832$. According to Schweer and Weiß [16] the critical value is equal to 1.1994. As we can see the dispersion index for Covid-19 data is much higher, thus we can conclude that the data is overdispersed and suitable for modeling with BT-INAR(1) model.

Using the Yule-Walker method parameter values were calculated. The estimated coefficient values are: $\hat{\alpha} = 0.92$, $\hat{\theta} = 2.98 \cdot 10^{-14}$ and $\hat{\beta} = 27.62$. The difference between $\hat{\theta}$ and $\hat{\beta}$ is quite large so it is interesting to see how the innovation distributions look with the above-mentioned parameter values. In Figure 8 one can see that due to very high difference and very low value of θ the probability $P(Y = 0)$ of the innovations density function is very close to 1 with the others remaining very close to 0. This idicates that the BT-INAR(1) model might not work on Covid-19 data as the parameter values are extreme.

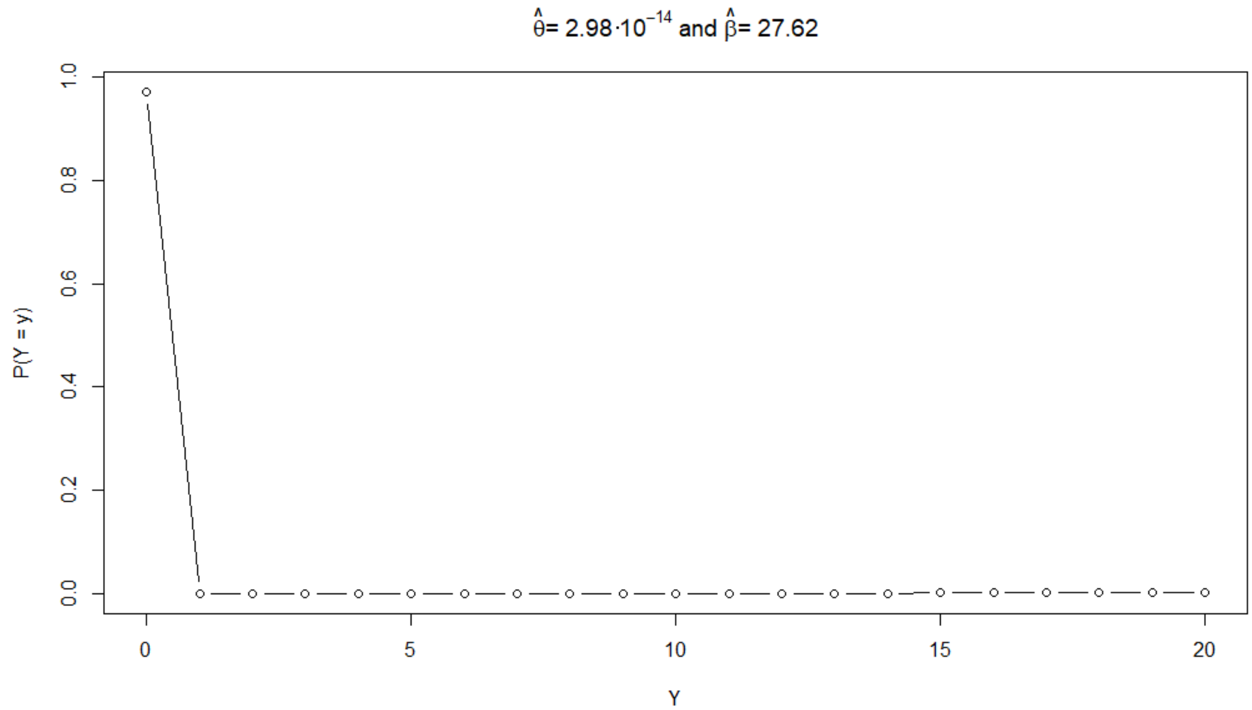

Figure 8: Density function of innovations when $\theta = 2.98 \cdot 10^{-14}$ and $\beta = 27.62$.

In order to check if the BT-INAR(1) model was appropriate for the Covid-19 data we performed residual analysis. The main goal is to check if the residuals are uncorrelated with a mean close to 0 and variance close to 1 as this will show that the model works. Standardized Pearson residuals approach described by Weiß [20] is used and the formula for the approach is as follows:

$$\epsilon_t = \frac{x_t - \mathbb{E}[X_t|x_{t-1}]}{\sqrt{\mathbb{V}\text{ar}[X_t|x_{t-1}]}}, \tag{18}$$

where $t = 2, ...T$.

The mean of the residuals is equal to 0.34, whereas, the variance is equal to 1.01. Nevertheless, that the variance satisfies the condition but from Figure 9 one can see that there is a significant correlation between the lags. This suggests that the BT-INAR(1) model is not suitable for Covid-19 data.

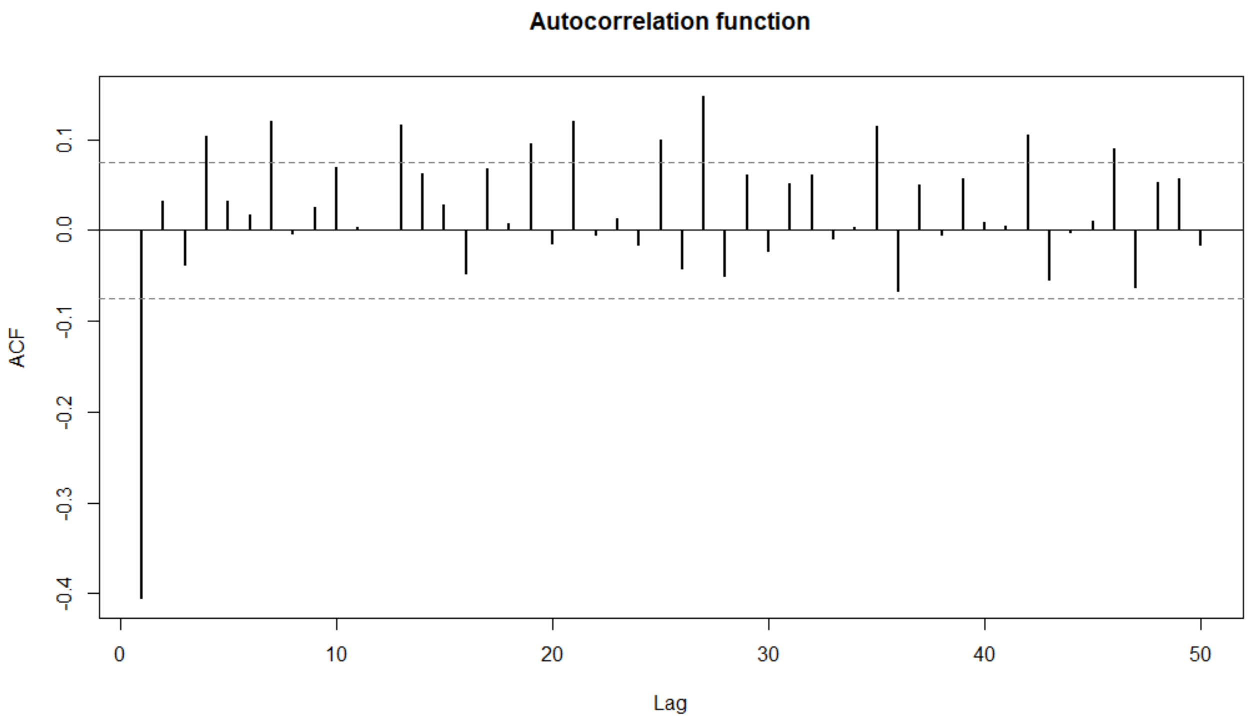

Figure 9: Autocorrelation function of Covid-19 data residuals.

8.2 Downloads data

The second data set consists of the daily number of downloads of a TeX editor for the period from June 2006 to February 2007 which can be seen in Figure 10. From the graph, it can be noticed that the daily number of downloads varies from 0 to 14. What is more, it is hard to observe any trends or seasonality.

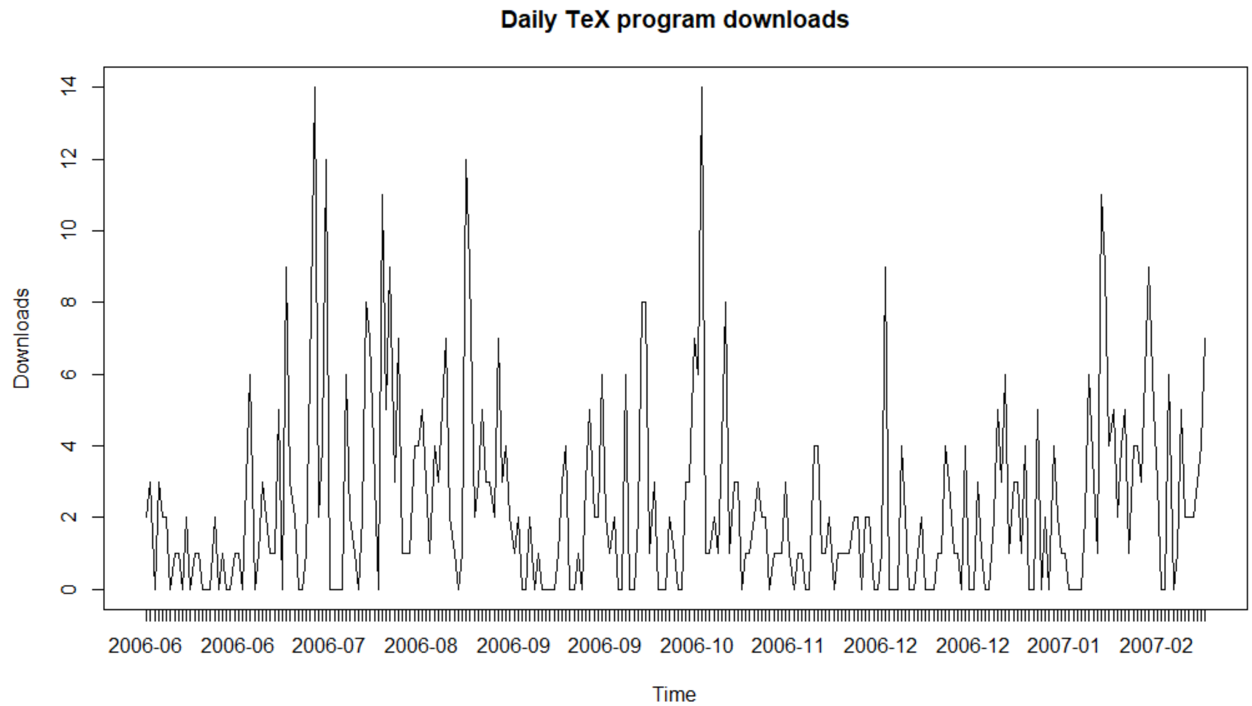

Figure 10: Number of daily downloads of TeX program.

To make sure that there is no seasonality we have plotted the autocorrelation function which can be seen in 11. From the ACF function, we failed to observe any seasonality as well. What is more, the autocorrelation function shows that there is some correlation in the data. To confirm that we performed the Ljung-Box test. Test shows that $p - value = 2.729 \cdot 10^{-5} < 0.05$. This suggests that downloads of the TeX program data set have some correlation. So the BT-INAR(1) model might be appropriate for this data.

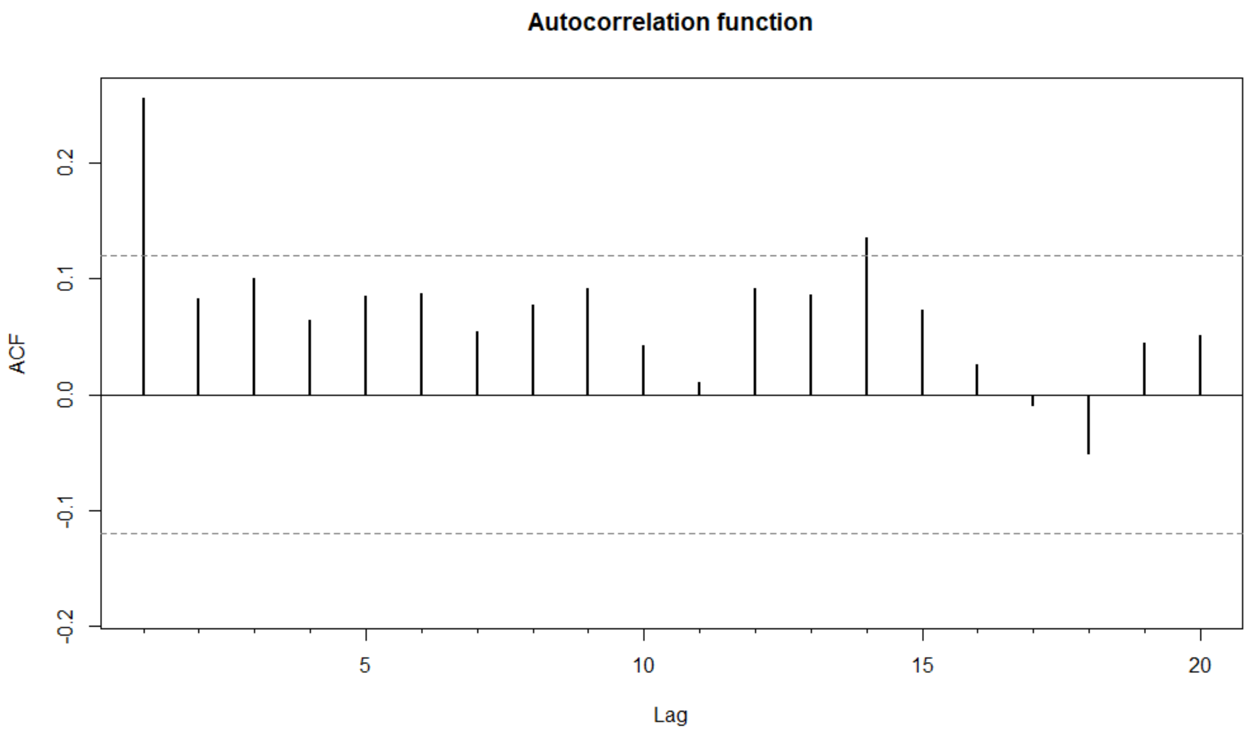

Figure 11: Autocorrelation function of the daily downloads of TeX program.

The downloads data is examined in the same way as Covid-19 data. The sample mean and sample variance are $\bar{X} = 2.368421$ and $S_X^2 = 7.282622$, respectively. Immediately we can see that the difference between the sample mean and sample variance is lower compared with what we had with the Covid-19 data. However, it is still significant as the dispersion index $I_X = \frac{S_X^2}{\bar{X}} = 3.074885$. The dispersion index is higher than the critical value 1.1994, therefore the data of downloads of the TeX program is overdispersed and can be used in the modeling with BT-INAR(1) process.

With the use of the Yule-Walker method parameter estimates were calculated. The estimated coefficients are: $\hat{\alpha} = 0.26$, $\hat{\theta} = 0.05$ and $\hat{\beta} = 2.61$. The estimated values are closer to the ones that were used in the Monte Carlo simulation. In Figure 12 one can notice that the probability $P(Y = 0)$ is the highest again ($P(Y = 0) > 0.5$) and then goes to 0 when Y increases.

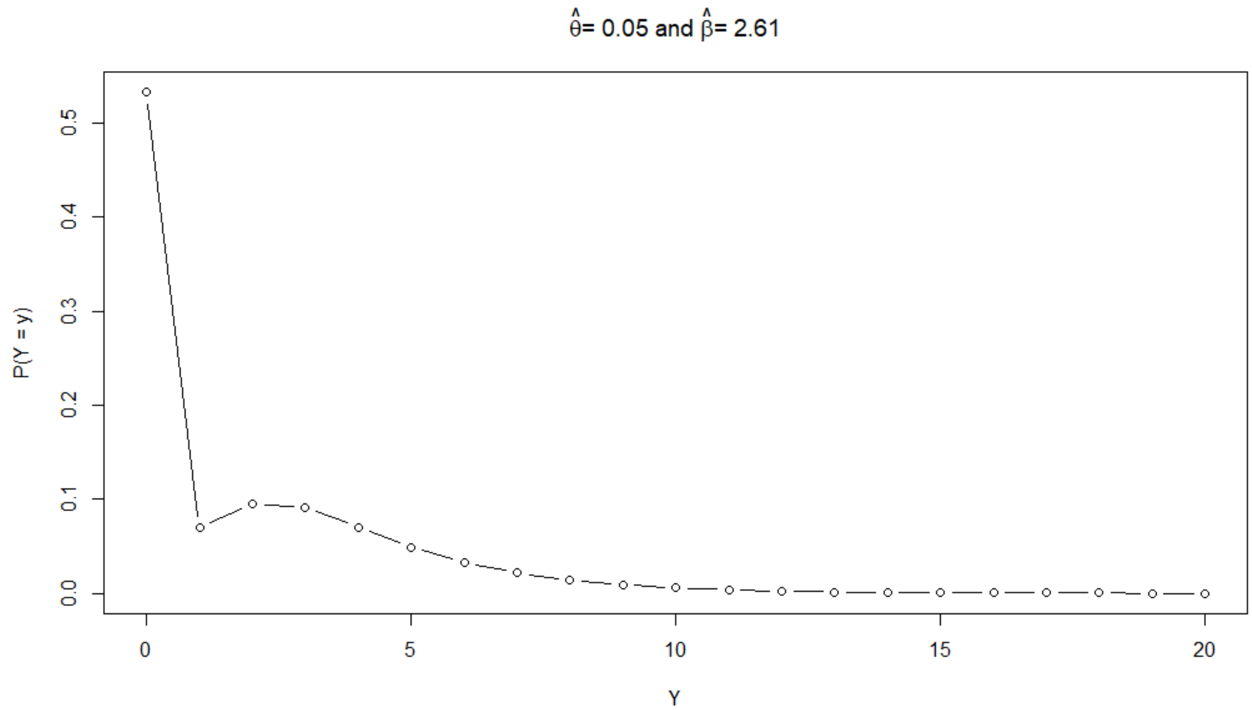

Figure 12: Density function of innovations when $\theta = 0.05$ and $\beta = 2.61$.

Residual analysis was performed for the downloads of the TeX program data as well. The mean of the residuals is 1.38, whereas, the variance is equal to 1.05. The mean is slightly higher than we expect, but the variance strongly satisfies the condition and is very close to 1. In addition, Figure 13 shows that there is no significant correlation of the residuals. This suggests that the BT-INAR(1) model is appropriate for modeling the downloads of the TeX program data.

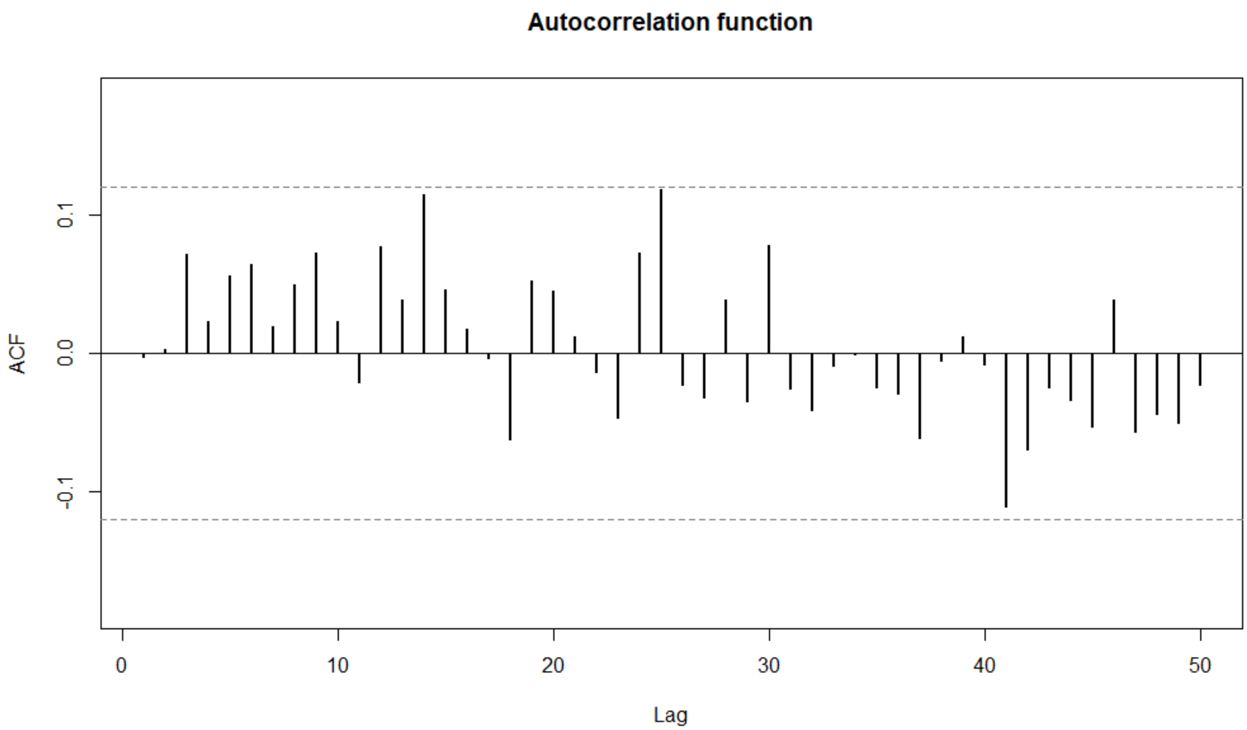

Figure 13: Autocorrelation function of downloads of the TeX program data residuals.

9 Conclusions

This thesis focuses on developing a new integer-valued autoregressive process that could be used for modeling count time-series data with overdispersion. From the literature review, one can notice that there were numerous different approaches on applying the INAR processes for counting time-series data with different problems. However, the most common approaches were by modifying thinning operator or innovations.

One of the main objectives of this thesis was to propose a new INAR(1) model that could deal with count time-series data that has overdispersion. Based on the literature review it was decided to use Bell-Touchard distribution for innovations. By the use of the new distribution for innovations a new BT-INAR(1) process was derived. A number of properties were obtained and proved, such as transition probabilities, mean, variance, conditional mean, conditional variance, covariance, and autocorrelation function. For the unknown parameters, Yule-Walker and Conditional Maximum Likelihood methods were used. What is more, the Monte-Carlo simulation was performed to compare Yule-Walker and Conditional Maximum likelihood estimators.

Another objective was to apply the new BT-INAR(1) process in practice. The BT-INAR(1) model was applied to two different data sets that are overdispersed. The first was the Covid-19 deaths per day in Lithuania from 2020 up until the end of 2022. The second data set was downloads of the TeX program from 2006 June to 2007 February. From the residual analysis it was found that the residuals of the Covid-19 data set have a correlation which implies that the BT-INAR(1) model is not suitable for the data. However, the residuals of the downloads of the TeX program are not correlated, which means that the BT-INAR(1) process can be used for modeling the data. Therefore, we found that BT-INAR(1) model is suitable for data that is overdispersed, but when the difference between the sample mean and the sample variance is not too big.

For future researches it would be interesting to to analyse the asymptotic properties of Yule-Walker and Conditional Maximum Likelihood estimates for BT-INAR(1) model. What is more, it could be beneficial to develop BT-INAR model on higher order lags. Finally, as there might be the case that the BT-INAR(1) model works differently on different ratios between mean and variance, so it would be interesting to find how the model is impacted by the mean-variance ratio in more detailed way.

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10 Appendix A

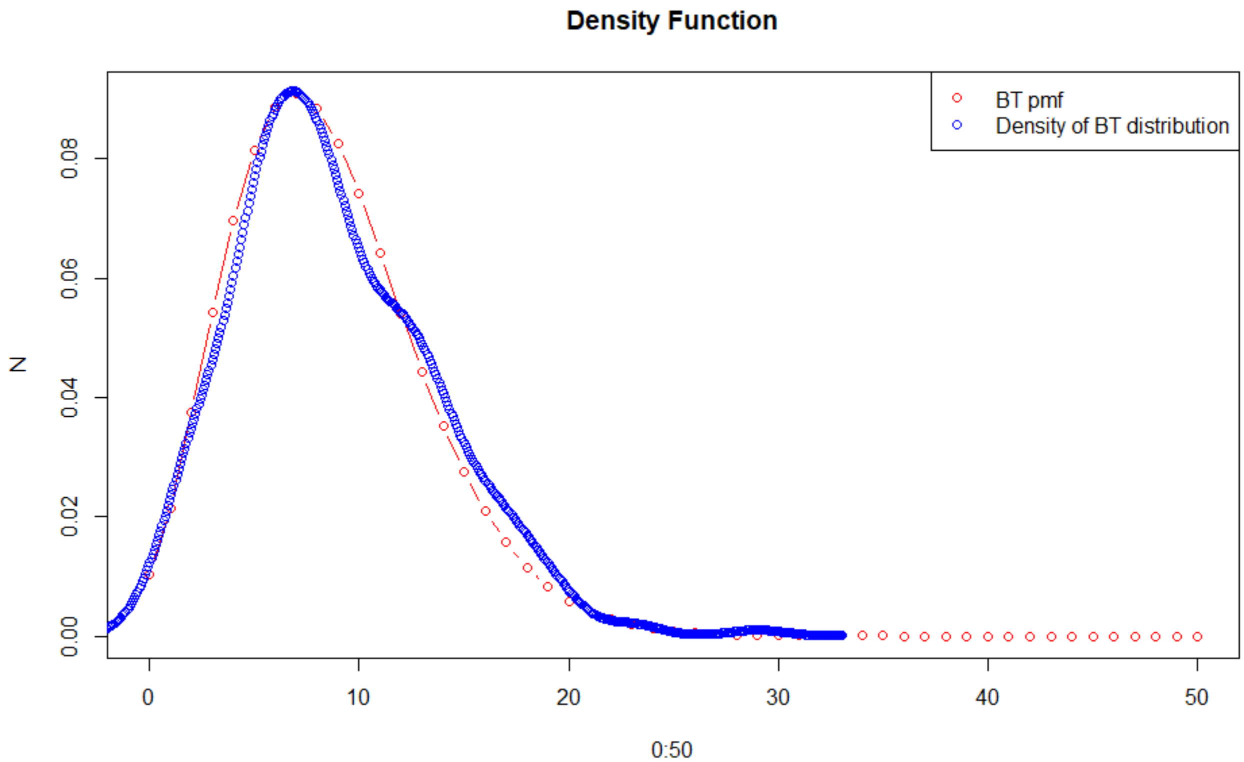

Figure 14: Bell-Touchard pmf and density plot of the Bell-Touchard function.

11 Appendix B: R code

In this section we are providing the key elements of the code that was used in this thesis.

Touchard polynomial function:

```
touchard_polynomial <- function(N, theta){

unlist(sapply(N, function(x) sum(theta^(0:x) * Stirling2(x, 0:x))))
}
```

Bell-Touchard probability mass function:

```
BT_pmf <- function(y, theta, alpha_e){

rez <- exp(theta * (1 - exp(alpha_e))) *
alpha_e^y * touchard_polynomial(N = y, theta = theta) / factorial(y)

rez[which(!is.finite(rez))] <- 0
```

```
return(rez)
}
```

Bell-Touchard innovations:

```
set.seed(123)

epsilon <- function(N,theta,alpha_e){

  BT <- BT_vec(N,theta,alpha_e)
  u <- runif(N, min = 0, max = 1)
  m <- c()
  for (i in 1:N){
    vec <- min(c(0:N)[BT >= u[i]])
    m <- c(m,vec)
  }
  return(m)

}
```

BT-INAR(1) series generation:

```
genr_series <- function(alpha, theta,alpha_e, N, burn_in = 200){
  epsil <- epsilon(N + burn_in, theta, alpha_e);
  Y <- rep(NA, burn_in + N)
  for(i in 1:(N+burn_in)){
    if(i == 1){
      Y[i] = epsil[i]
    }else{
      Y[i] = sum(rbern(Y[i-1], prob = alpha)) + epsil[i]
    }
  }
  rez <- Y[-c(1:burn_in)]
  return(rez =rez)

}
```

Transition probabilities and log-likelihood function:

```
tpbtinar <- function(k, l, alpha, theta, alpha_e){
  tp <- 0
  for(j in c(0:min(k,l))){
    tp <- tp + dbinom(j,l,alpha)*BT_pmf(k-j,theta, alpha_e)
  }
  tp
```

```
}


llbtinar <- function(par,X){
  alpha = par[1]
  theta = par[2]
  alpha_e = par[3]
  T <- length(X)
  value <- 0
  for(t in c(2:T)){
    value <- value - log(tpbtinar(k = X[t],
    l = X[t-1],alpha = alpha,theta = theta,alpha_e = alpha_e))

  }
  return(value)
}
```

Yule-Walker estimation:

```
 estim_YW <- function(X){

  alpha_YW <- acf(X, pl = FALSE)
  alpha_YW <- alpha_YW$acf[2]
  mean <- mean(X)
  var <- var(X)
  alpha_e_YW <- ((var*(1 + alpha_YW))/mean) - 1 - alpha_YW
  theta_YW <- (mean*(1 - alpha_YW)/(alpha_e_YW*exp(alpha_e_YW)))
  return(list(mean = mean,var = var, alpha_YW = alpha_YW,theta_YW
  =theta_YW,alpha_e_YW = alpha_e_YW  ))

}
```