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# Nupjautų Periodinių Sekų Dalumas
# Divisibility of Truncated Periodic Sequences

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## Abstract

A finite set of primes $S$ is called unavoidable with respect to integer $b > 1$ if and only if for every $\xi > 0$ the sequence of integers $\lfloor \xi b^k \rfloor$ contains infinitely many elements divisible by primes from $S$. It is known that unavoidable sets of primes exist when $b = 2, 3, 4, 6$ and it does not exist for an integer $b > 1$ such that $b - 1$ is square free. If there is no unavoidable set of primes for $b$, for every finite set of primes $S$ we call its falsifier a real $\xi_S > 1$ such that the sequence $\lfloor \xi_S b^k \rfloor$ has only finitely many elements divisible by primes from $S$. In this paper I classify all $b > 1$ whose all falsifiers have two ultimately repeating digits. In particular, I show that if $b > 3$ is prime, then there is no unavoidable set of primes for $b$. Based on this thesis, I co-authored an article which was published in the International Journal of Number Theory [12].

## Santrauka

Baigtinė pirminių skaičių aibė $S$ yra vadinama neišvengiama sveikojo skaičiaus $b > 1$ atžvilgiu tada ir tik tada, kai visiems realiems $\xi > 0$ sveikųjų skaičių seka $\lfloor \xi b^k \rfloor$ turi be galo daug narių, dalių iš aibės $S$ elementų. Yra žinoma, jog neišvengiamos pirminių skaičių aibės egzistuoja, kai $b = 2, 3, 4, 6$, ir neegzistuoja, kai $b-1$ yra bekvadratis. Jeigu $b$ atžvilgiu nėra neišvengiamos pirminių skaičių aibės $S$, tuomet mes $S$ falsifikatorių vadiname realų $\xi_S > 0$ tokį, kad seka $\lfloor \xi_S b^k \rfloor$ turi tik baigtinį skaičių narių dalių iš aibės $S$ elementų. Šiame straipsnyje aš klasifikuoju visus sveikus $b > 1$, kurių falsifikatoriai turi galiausiai du pasikartojančius skaitmenis, taip pat įrodau, jog kai $b > 3$ yra pirminis tuomet $b$ neturi neišvengiamos pirminių aibės $S$. Pagal šį baigiamąjį darbą buvo išleistas straipsnis moksliniame žurnale “International Journal of Number Theory” [12].

# 1 Introduction[1]

Consider the sequence of fractional parts $\{\xi b^k\}$ in $k \in \mathbb{N}_0$ where $\xi, b \in \mathbb{R}$ are such that $b > 1$ and $\xi \neq 0$. Sequences of this kind have been studied for a long time. One of the earliest results is from Hardy and Littlewood [19] which states that the sequence $\{\xi \lambda_k\}$ in $k \in \mathbb{N}_0$ is everywhere dense on $(0, 1)$ for almost all $\xi$ where $\lambda_k$ is an increasing sequence of integers.

In particular, if we consider $\lambda_k = b^k$ for an integer $b > 1$ and for $k \in \mathbb{N}_0$, we obtain that the sequence $\{\xi b^k\}$ is everywhere dense for almost all $\xi$. However, this obviously fails when $\xi$ is rational as then $\xi$ will have ultimately repeating digits in base $b$.[2] Mercifully, the set of rationals is of measure zero as it is countable.

Later this result was improved by Weyl [31]. Weyl's result is more general, but the particular case that will be important for us is that for any real $b > 1$ and almost all real $\xi$ the sequence $\{\xi b^k\}$ is everywhere dense.

Historically, the case $\xi = 1$ has received particular attention and has been studied by Pisot [25, 26], Vijayarhagavan [28, 29, 30], and Salem [27], among others. Although a lot is known about this special case, a lot remains open. For example although Koksma proved that the sequence $\{b^k\}$ is uniformly distributed in $(0, 1)$ for almost all $b > 1$ [21], Pisot and Salem proved that for a Salem number $b$, the sequence $\{b^k\}$ is everywhere dense on the unit interval but not uniformly distributed[27]. In general it remains open for which $b > 1$ the sequence $\{b^k\}$ is uniformly distributed, for which it is everywhere dense, and for which it is neither. Of particular interest is an old conjecture of Pisot and Vijayarhagavan which states that $\{b^k\}$ is everywhere dense on $(0, 1)$ when $b > 1$ is a non-integer rational [23]. For some work on this and similar problems see [5, 8, 10] and see [4] for more information on Pisot and Salem numbers.

In the other direction, if we instead fix $b$ we have some interesting conjectures as well. Of particular interest is that of Mahler [22]. Mahler studied the distribution of $\{\xi (3/2)^k\}$ and conjectured that there are no real $\xi \neq 0$ (which Mahler called Z-numbers) such that the sequence $\{\xi (3/2)^k\}$ all falls into the interval $[0, 1/2)$. Mahler's conjecture remains open, although see [1, 9, 13, 16, 32, 33] for some work in this direction.

On the other hand, surprisingly little is known on the truncated integer

[1] Based on this thesis, I have co-authored an article in [12].
Pagal šitą darbą išleistas straipsnis žurnale [12].

[2] See [15, p. 13] for a proof when $b$ is prime. The proof is identical for all $b$.

sequence $\lfloor \xi b^k \rfloor$. A conjecture of Whiteman on integer parts (see problem E19 in [18]) states that the sequence $\lfloor b^k \rfloor$ contains infinitely many primes for $b > 1$ a non-integer rational. Due to a remarkable paper by Baker and Harman [3] we know that for almost all real $b > 1$ the sequence $\lfloor b^k \rfloor$ contains infinitely many primes. Unfortunately, as is the common wisdom with metric results of this kind, important exceptions are bound to happen. Baker and Harman in the same paper show that although $\lfloor b^k \rfloor$ has infinitely many primes for almost all $b > 1$, the number of $b$ for which $\lfloor b^k \rfloor$ has only finitely many primes is in fact uncountable. See also [2, 24] for the existence of $b > 1$ with $\lfloor b^k \rfloor$ prime infinitely often.

The situation is not much better when we consider the question if the sequence $\lfloor \xi b^k \rfloor$ contains infinitely many composite numbers. Due to the aforementioned result of Weyl [31] we know that the sequence $\{\xi b^k\}$ is everywhere dense on $(0, 1)$ for almost all $b > 1$ and almost all $\xi$. In particular, if for a given $\xi \in \mathbb{R}$ the sequence $\{(\xi/2)b^k\}$ is everywhere dense on the unit interval, it follows that $\{(\xi/2)b^k\} < 1/2$ infinitely often, and hence the sequence $\lfloor \xi b^k \rfloor$ is even (and therefore composite) infinitely often. Hence, we have the following lemma:

**Lemma 1.** *For all $b > 1$ and almost all $\xi \in \mathbb{R}$ the sequence $\lfloor \xi b^k \rfloor$ with $k \in \mathbb{N}_0$ is composite infinitely often.*

This then leads us to the following conjecture:

**Conjecture 1.** *Let $\xi, b \in \mathbb{R}$ be real such that $b > 1$ and $\xi \neq 0$. Then the sequence $\lfloor \xi b^k \rfloor$ contains infinitely many composite numbers.*

Unfortunately, not much is known with this conjecture. For fixed $\xi = 1$ the first result of this kind is by Forman and Shapiro [17] which states that the sequences $\lfloor (3/2)^k \rfloor$ and $\lfloor (4/3)^k \rfloor$ are composite infinitely often for $k \in \mathbb{N}_0$. Later the set of $b > 1$ for which conjecture 1 holds for fixed $\xi = 1$ was expanded to when $b > 1$ is a quadratic unit [6] and more generally a Pisot or Salem number [7].

If we do not fix $\xi \in \mathbb{R}$ then even less is known. Dubickas and Novikas in [14] observed that Forman and Shapiro's method also works for $\lfloor \xi (3/2)^k \rfloor$ and $\lfloor \xi (4/3)^k \rfloor$ where $\xi > 0$. In this paper Dubickas and Novikas establish that the sequence $\lfloor \xi b^k \rfloor$ has infinitely many composite numbers where $\xi > 0$, $k \in \mathbb{N}_0$ and $b \in \{2, 3, 4, 5, 6, 3/2, 4/3, 5/4\}$. These are in fact the only explicit numbers $b$ for which this result is known.

It should be noted now as to how we know that these numbers are composite. With Weyl's result we saw that the sequence $\lfloor \xi b^k \rfloor$ is composite for almost all $\xi \in \mathbb{R}$ because infinitely many elements of the sequence are even. And in [14] for each $b \in \{2, 3, 4, 6, 3/2, 4/3, 5/4\}$ there is a finite set of primes $S = S(b)$ was found such that for all $\xi > 0$ there are infinitely many elements in $\lfloor \xi b^k \rfloor$ that are divisible by some element of $S$. This then justifies the following definition:

**Definition 1.** *We say that a finite set of primes $S$ is **unavoidable** with respect to a real $b > 1$ if for every real $\xi > 0$ the sequence $\lfloor \xi b^k \rfloor$ has infinitely many member divisible by some element of $S$.*

It should be noted here that in [14] the number 5 **does not** have an unavoidable set of primes. Later Dubickas observed that if for an integer $b > 1$ the number $b - 1$ is not squarefree then $b$ does not have an unavoidable set of primes [11]. In the same article Dubickas conjectured that $b = 2, 3, 4, 6$ are the only integers $b > 1$ that have an unavoidable set of primes:

**Conjecture 2** (Dubickas)**.** *Let $b > 1$ be an integer with $b \neq 2, 3, 4, 6$. Then there is no unavoidable set of primes $S$ with respect to $b$.*

In this paper I generalise the result obtained by Dubickas in [11] by proving that the above conjecture 2 holds for when $b > 3$ is prime. The results obtained are more general and also establish conjecture 2 for certain other classes of integers, including, $b = 2^{2k+1} - 1$ for $k \in \mathbb{N}$.

In the next section we introduce some basic terminology and concepts. Following that we prove a theorem of Dubickas from [11] in a different way that will more naturally generalise to the case when $b > 3$ is prime. In section 3.2 we then establish the main result which we use in the final section to prove that when $b > 3$ is prime then $b$ does not have an unavoidable set of primes.

It should be noted here that based on this thesis I have an article published in the International Journal of Number Theory [12]. The main results are mostly the same, corollary 10 corresponds to proposition 4.1 in [12] and corollary 11 corresponds to theorem 1.2 in [12]. These are the main results in [12]. A few key differences between the articles: theorem 2.1 in [12] is slightly weaker than the corresponding theorem 9 here, as the slightly stronger premises (A1), (A2) in [12] were sufficient to establish the main results of [12]. The weaker premises B1-B5 are used here since in this article I am more focused on proving necessary and sufficient conditions for using the same method as in [12], as opposed to immediately establishing the main results of [12].

In addition, most proofs and results are the same, but some of them are expressed differently in [12] than in here as to fit the premises (A1),(A2) in that article. In particular, the proofs of theorem 2.1 in [12] and that of lemma 7 here are essentially the same, as well as the discussion in section 3.1 here and the discussion in section 2 of [12]. The proofs of corollaries 11, 10 in this thesis and the discussion in chapter 4 in [12] also remains essentially the same.

# 2 Basic Concepts and Theorems

In this section we introduce basic terminology, concepts, and results we shall use in the subsequent section.

First, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{P}$ is the set of prime numbers.

An integer $b \in \mathbb{N}$ is **squareful** if it is divisible by $c^2$ for some integer $c > 1$, otherwise it is called **squarefree**.

The **radical** of a positive integer is the product of all the primes that divide it.

$$\operatorname{rad} b = \prod_{p|b, p \in \mathbb{P}} p$$

From the above definitions it is clear that $b$ is squarefree if and only if $b = \operatorname{rad} b$, otherwise $\operatorname{rad} b < b$. In the following chapters we shall use the Chinese remainder theorem extensively. The statement of the theorem is given for completion. See [20, p. 34] for more details.

**Theorem 2** (Chinese Remainder Theorem)**.** *Let* $a = a_1 \dots a_n$ *for* $a_i \in \mathbb{N}$ *and let* $\gcd(a_i, a_j) = 1$ *for* $i \neq j$*. Let* $t_1, \dots, t_n$ *be integers. The set of equations:*

$$x \equiv t_1 \mod a_1 \quad \dots \quad x \equiv t_n \mod a_n$$

*Always has a solution, and any two solutions differ by a factor of* $a$*. In particular, there is a solution* $0 < x \leq a$*.*

The set of congruence classes mod $a$, $\mathbb{Z}/\mathbb{Z}a$ for $a \in \mathbb{N}$ is always a ring and it is a field if $a$ is prime. In the latter case the set of non-zero remainders $\mathbb{Z}/\mathbb{Z}p - \{0\}$ is the set of all units in the ring, and as such form a multiplicative group of order $p - 1$. For more details see chapters 2 and 3 in [20].

Even though $x \in \mathbb{Z}/\mathbb{Z}a$ is an equivalence class of integers we often conflate it with the integer $\{0, 1, \dots, a-1\} \cap x$ as $x$ will always have a unique entry in this set and the whole equivalence class can be retrieved from such an integer. Such practice is standard.

An integer $x \in \mathbb{Z}$ coprime with $a \in \mathbb{N}$ is a **quadratic residue** mod $a$ if there exists $t \in \mathbb{Z}$ such that

$$x \equiv t^2 \mod a$$

Otherwise, $x$ is a **quadratic non-residue** mod $a$. It is a known fact that if $p > 2$ is prime then there are as many quadratic residues mod $p$ as there are quadratic non-residues mod $p$. For more details see chapter 5 in [20].

Next, we shall look at some terminology particular to the topic of this paper. First, here we reiterate the definition of an unavoidable set of primes:

**Definition 2.** *We say that a finite set of primes $S$ is **unavoidable** with respect to an integer $b > 1$ if for every real $\xi > 0$ the sequence $\lfloor \xi b^k \rfloor$ in $k \in \mathbb{N}_0$ has infinitely many member divisible by some element of $S$.*

*Otherwise, we say that $S$ is **avoidable** with respect to $b$. In this case, for a given $S$ we call its **falsifier** a real number $\xi > 0$ such that the sequence $\lfloor \xi b^k \rfloor$ in $k \in \mathbb{N}_0$ has only finitely many elements divisible by members of the set $S$.*

Notice that it is sufficient to consider the case $\xi > 1$ as the sequence $\lfloor \xi b^k \rfloor$ has infinitely many composite numbers if and only if the augmented sequence $\lfloor \xi b^{k+N} \rfloor$ has infinitely many composite numbers for all $N \in \mathbb{Z}$. If we take $N$ to be large enough we can guarantee that $\xi b^N > 1$ as $b > 1$. Hence, for all $\xi > 0$ there exists $\xi' = \xi b^N > 1$ such that the sequence $\lfloor \xi b^k \rfloor$ contains infinitely many composite numbers if and only if the augmented sequence $\lfloor \xi' b^k \rfloor$ contains infinitely many composite numbers. As such, $b > 1$ has an unavoidable set of primes $S$ if and only if for all $\xi > 1$ the sequence $\lfloor \xi b^k \rfloor$ has infinitely many numbers divisible by primes from $S$.

Likewise, if $S$ has a falsifier $\xi > 0$ it must also have a falsifier $\xi' > 1$ by the above considerations. Therefore, henceforth whenever we talk about a sequence $\lfloor \xi b^k \rfloor$ we imply that $\xi > 1$ is real, $b > 1$ is an integer and the sequence is in $k \in \mathbb{N}_0$, unless otherwise stated.

As mentioned earlier if $\xi > 1$ is rational then $\xi$ is ultimately repeating in base $b$ (see [15, p. 13]). This gives rise to the following definition:

**Definition 3.** *Let $b > 1$ be an integer. We say that $b$ has an **n digits solution** if for every finite set of primes $S$ there exists a rational falsifier $\xi$ such that in base $b$ the number $\xi$ has exactly $n$ ultimately repeating digits.*

Finally, in order to represent $\xi$ in base $b$ in a unique way we require that $\xi$ does not ultimately consist of the digit $b-1$ repeated indefinitely [15, p. 12–13]. For future use we note this condition here:

**Proposition 3.** *Let $\xi \in \mathbb{R}$ then when represented in base $b > 1$, $\xi$ does not ultimately consist of the digit $b-1$ repeated infinitely.*

# 3 Main Results

## 3.1 One Digit Solution

It has already been shown in [11] that Conjecture 2 holds when $b-1$ is squareful. In this section I give an alternative proof of this claim and connect it to a single digit solution. Indeed, the claim we shall proof is that $b$ has a single digit solution if and only if $b-1$ is squareful.

For future use we denote the condition here and proceed to the proof:

$$b - 1 \text{ is squareful} \tag{A1}$$

**Lemma 4.** *Let $b > 1$ be an integer with $b-1$ squareful. Then $b$ has a single digit solution.*

*Proof.* Let $S$ be a finite set of primes and define $B = \text{rad}(b-1)$. We shall find a falsifier for $S$ of the form $\xi = X_0, \overline{(B)}$. As $b-1$ is squareful, $B = \text{rad}(b-1) < b-1$ and $\xi$ is of the right form as in proposition 3.

For every $p \in S$ we will find a remainder $x_p$ such that if $X_0 \equiv x_p \mod p$ then for every $k \in \mathbb{N}_0$ the sequence $\lfloor \xi b^k \rfloor$ is not divisible by $p$. Then by the Chinese remainder theorem we know that there exists a natural $X_0 \in \mathbb{N}$ such that $X_0 \equiv x_p \mod p$ for all $p \in S$ as distinct primes are obviously coprime. This then finishes our construction for $\xi$.

First, if $p \mid b-1$ then $p \mid \text{rad}(b-1) = B$, as such we have that:

$$\lfloor \xi b^k \rfloor = X_0 b^k + B b^{k-1} + B b^{k-2} + \ldots + Bb + B \equiv X_0 (b - 1 + 1)^k \equiv X_0 \mod p$$

Thus we may choose $x_p = 1$ for such $p \in S$.

On the other hand if $p \nmid b-1$ we may take

$$x_p \equiv -B(b-1)^{-1} \mod p$$

Since $p \nmid b-1$ then $p \nmid \text{rad}(b-1) = B$, and as such $p \nmid x_p$. We shall show that $\lfloor \xi b^k \rfloor \equiv x_p \mod p$ by induction. For this purpose, denote $X_k = \lfloor \xi b^k \rfloor$. Notice that $X_0 \equiv x_p \mod p$ by construction of $X_0$. Assume that $X_k \equiv x_p \mod p$. Then:

$$\begin{aligned} X_{k+1} &= bX_k + B \equiv bx_p + B \equiv x_p + (b-1)x_p + B \equiv \\ &\equiv x_p - B(b-1)^{-1}(b-1) + B \equiv x_p \mod p \end{aligned}$$

Therefore, by induction, for all $k \in \mathbb{N}_0$, $X_k \equiv x_p \not\equiv 0 \mod p$, as required. □

The proof in the other direction is slightly simpler:

**Lemma 5.** *Assume that $b > 1$ has a single digit solution. Then $b - 1$ is squareful.*

*Proof.* Let $S$ consist of all the prime divisors of $b - 1$. Then, clearly, $S$ is a finite set of primes. $S \neq \varnothing$ as $b > 2$ since 2 has an inevitable set of primes (see [14] for proof of this claim) and as such cannot have a single digit solution. Then let $\xi = \overline{X_0, (B)}$ be a single digit falsifier of $S$. Let $p \mid b - 1$ and thus $p \in S$, then

$$\begin{aligned} \left\lfloor \xi b^k \right\rfloor &= X_0 b^k + Bb^{k-1} + Bb^{k-2} + \ldots + B \equiv \\ &\equiv X_0 1^k + B1^{k-2} + B1^{k-2} + \ldots + B \equiv X_0 + kB \mod p \end{aligned}$$

Assume that there exists $p \mid b - 1$ such that $p \nmid B$. Let $k_0 \in \mathbb{N}$ be an integer such that $k_0 \equiv -X_0 B^{-1} \mod p$ and let $n \in \mathbb{N}_0$ be arbitrary, then

$$\left\lfloor \xi b^{(np+1)k_0} \right\rfloor \equiv X_0 + (np + 1)k_0 B \equiv X_0 + k_0 B \equiv X_0 - X_0 B^{-1} B \equiv 0 \mod p$$

Thus, the sequence $\lfloor \xi b^k \rfloor$ has infinitely many elements divisible by $p$. By contradiction, we have that if $p \mid b - 1$ then $p \mid B$. In particular, $\text{rad}(b - 1) \mid B$, and thus $\text{rad}(b - 1) \leq B$. By proposition 3, the relation $B < b-1$ holds. Hence, $\text{rad}(b-1) < b-1$, and therefore $b-1$ is squareful. □

We can put the two lemmas together into the following theorem:

**Theorem 6.** *$b > 1$ has a single digit solution if and only if $b-1$ is squareful.*

By giving an alternative, stronger proof of theorem 4 in [11] we have fully categorised the $b > 1$ with a single digit solution. Now we shall turn our attention to two digit solutions and categorise them. We will do this in the following subsection.

## 3.2 Two Digit Solution

Since a one digit solution is automatically also a two digit solution in what follows in light of theorem 6 we assume that $b-1$ is squarefree. Assuming, that $b-1$ is squarefree the requisite conditions for a two digits solution are as follows:

$$b \text{ is odd} \tag{B1}$$

In addition we require there to exist a positive integer $C \in \mathbb{N}$ satisfying the following conditions:

$$1 < C < b \tag{B2}$$
$$2 \nmid C \tag{B3}$$
$$\gcd(b, C(C-1)) = 1 \tag{B4}$$
$$\operatorname{rad}(b+1) \mid 2C \tag{B5}$$

First we shall prove that these conditions are sufficient for the existence of a two digits solution. Afterwards we prove the necessity of these conditions. We finish this chapter by discussing the conditions B1-B5.

**Lemma 7.** *Let $b > 1$ be an integer such that $b-1$ is squarefree. If the conditions B1-B5 hold then b has a two digit solution.*

*Proof.* For a given finite set of primes $S$ we will denote our falsifier as $\xi = \overline{X_0, (l_1 l_0)}$. Here $X_0$ will depend only on $S$ and $b$, and $l_1, l_0$ will depend on only $b$.

Denote:

$$B = (b-1)C \tag{1}$$

Since by B2, $1 < C < b$, then $b < B < b^2 - 1$. As such $B$ can be expressed in two digits in base $b$. Furthermore, since $B < b^2 - 1$, the number $\xi$ satisfies proposition 3 and is expressed appropriately.

It is easy to see that the two digits are

$$l_1 = C - 1 \quad l_0 = b - C \tag{2}$$

In which case

$$bl_1 + l_0 = b(C-1) + b - C = C(b-1) = B$$

By B2, $0 < l_1 < b$ and $0 < l_0 < b$ and therefore these are digits in base $b$. Furthermore, by uniqueness of expression in base $b$,[3] the numbers $l_1, l_0$ in (2) must precisely be the digits for $B$.

We will proceed to construct $\xi$ as follows. The choice for $l_0, l_1$ are as in (2). For each $p \in S$ we choose a remainder $x_p$, such that if $X_0 \equiv x_p \mod p$ then none of the elements $\lfloor \xi b^k \rfloor$ for $k \in \mathbb{N}_0$ are divisible by $p$. By the Chinese remainder theorem we are guaranteed that there exists $X_0 \in \mathbb{N}$ such that $X_0 \equiv x_p \mod p$ for each $p \in S$. This will be our choice for $X_0$. Although the Chinese remainder theorem generates multiple integers with the desired property, we do not care which positive integer we choose, so we may choose the smallest positive integer with the desired property.

For the purpose of choosing $x_p$ we define the following sequences for $k \in \mathbb{N}_0$:

$$X_k = \left\lfloor \xi b^{2k} \right\rfloor = X_0 b^{2k} + B\left(b^{2k-2} + b^{2k-4} + \ldots + b^2 + 1\right) \tag{3}$$

$$Y_k = \left\lfloor \xi b^{2k+1} \right\rfloor = X_0 b^{2k+1} + Bb\left(b^{2k-2} + b^{2k-4} + \ldots + b^2 + 1\right) + l_1 \tag{4}$$

Thus, we will be showing that for a given choice of $x_p$, the prime number $p$ divides no element from the sequences $X_k, Y_k$. We now proceed to find an appropriate choice for $x_p$ depending on the kind of $p \in S$ we have, classifying $p$ into the following five cases.

**Case 1:** $p \mid b$

By B4, $p \nmid C, C-1$. Thus we have the following, where $k > 0$ in the first case and $k \geq 0$ in the second one:

$$X_k = X_0 b^{2k} + B\left(b^{2k-2} + \ldots + b^2 + 1\right) \equiv B \equiv bl_1 + l_0 \equiv l_0 \equiv b - C \equiv$$
$$\equiv -C \not\equiv 0 \mod p$$
$$Y_k = X_0 b^{2k+1} + Bb\left(b^{2k-2} + \ldots + b^2 + 1\right) + l_1 \equiv l_1 \equiv C - 1 \not\equiv 0 \mod p$$

[3] See [15, p. 13]. The case proven in the reference is for prime $b$. Nothing changes if $b$ is composite.

Thus, we only require that $p \nmid X_0$, so we may choose $x_p \equiv 1 \mod p$.

**Case 2:** $p \mid b^2 - 1$

In this case $p \mid (b-1)(b+1)$. If $p \mid b-1$ then $p \mid B$ by (1). If $p \mid b+1$ then $p \mid 2C$ by B5, thus, either $p = 2$, or $p \mid C$. From the latter follows that $p \mid B = (b-1)C$. If, on the other hand, $p = 2$ then since $b$ is odd by B1, $b-1$ is even and therefore $p \mid B = (b-1)C$.

It is obvious that $X_0 \equiv X_0 \mod p$, and assume that $X_k \equiv X_0 \mod p$, then:

$$X_{k+1} = b^2 X_k + B = X_k + (b^2 - 1)X_k + B \equiv X_k \equiv X_0 \mod p$$

Thus, by induction, $X_k \equiv X_0 \mod p$ for all $k \in \mathbb{N}_0$. In particular, as long as $p \nmid X_0$ then $p \nmid X_k$ for all $k \in \mathbb{N}_0$. Also, for all $k \in \mathbb{N}_0$ we have that:

$$Y_k = bX_k + l_1 \equiv bX_0 + l_1 \mod p$$

Thus, we need to choose a remainder $x_p$ for $p$ such that $p \nmid x_p$ and $p \nmid bx_p + l_1$. If $p > 2$ then $p$ cannot divide both $b + l_1$ and $2b + l_1$. Otherwise, $p$ would have to divide their difference, that is $p \mid b$. But since $p \mid b^2 - 1$ and $\gcd(b, b^2 - 1) = 1$ this is impossible. Thus, if $p > 2$ we choose $x_p$ to be either 1 or 2 for whichever $p \nmid bx_p + l_1$.

On the other hand, if $p = 2$ then as $p \nmid x_p$ we must choose $x_p = 1$. For this purpose we need to check that $2 \nmid b + l_1$. By (2), $l_1 = C - 1$. Since by B3 the number $C$ is odd, it follows that $l_1$ is even. And since by B1 the number $b$ is odd, the number $b + l_1$ is also odd, as desired.

**Case 3:** $p \nmid b(b^2 - 1)$, $p \mid B$

By (3), $X_k \equiv X_0 b^{2k} \mod p$. Since $p \nmid b$, as long as $p \nmid X_0$ we have that $p \nmid X_k$ for every $k \in \mathbb{N}_0$. On the other hand, by (4), $Y_k \equiv X_0 b^{2k+1} + l_1 \mod p$. If $p \mid l_1$ then $Y_k \equiv X_0 b^{2k+1} \mod p$ and since $p \nmid b$, as long as $p \nmid X_0$ then $p \nmid Y_k$ for all $k \in \mathbb{N}_0$ as before. Therefore, let us assume that $p \nmid l_1$. Otherwise, we may take $x_p \equiv 1 \mod p$.

Assume that $x_p$ is such that $p \nmid x_p$ and assume that $p \mid Y_k$ for some $k \in \mathbb{N}_0$. Then since $p \nmid bx_p$ we have that

$$Y_k = x_p b^{2k+1} + l_1 \equiv 0 \mod p \iff x_p^2 b^{2k+2} + l_1 x_p b \equiv 0 \mod p$$

Since $p \nmid b(b^2-1)$ we have that $p > 2$ (as 2 must divide one of the three consecutive numbers $b-1, b, b+1$). And since $p > 2$ is a prime, therefore, the number $p-1 \geq 2$ is even. Hence, there exists a quadratic nonresidue $u \mod p$ (see section 2 above). Let $x_p \equiv -ul_1^{-1}b^{-1} \mod p$. Note that, by definition of $u$, we have that $p \nmid u$, and thus $p \nmid x_p$. Then we have:

$$0 \equiv b^{2k+2}x_p^2 - ul_1^{-1}b^{-1}l_1 b \equiv b^{2k+2}x_p^2 - u \mod p$$

That is

$$u \equiv \left(b^{k+1}x_p\right)^2 \mod p$$

This contradicts the definition of $u$. By contradiction then $p \nmid Y_k$ for all $k \in \mathbb{N}_0$ as long as $x_p \equiv -ul_1^{-1}b^{-1} \mod p$. As mentioned before in this case $p \nmid x_p$ and thus $p \nmid X_k$ for all $k \in \mathbb{N}_0$ as well.

**Case 4:** $p \nmid b(b^2-1)B$, $C \equiv b+1 \mod p$
In this case we have that

$$\begin{aligned} l_1 &= C - 1 \equiv b \mod p \\ B &= (b-1)C \equiv (b-1)(b+1) \equiv b^2 - 1 \mod p \end{aligned}$$

Then, by (3) we have

$$\begin{aligned} X_k &= X_0 b^{2k} + B\left(b^{2k-2} + b^{2k-4} + \ldots + b^2 + 1\right) \equiv \\ &\equiv X_0 b^{2k} + (b^2-1)\left(b^{2k-2} + \ldots + b^2 + 1\right) \equiv \\ &\equiv X_0 b^{2k} + b^{2k} - 1 \equiv (X_0+1)b^{2k} - 1 \mod p \end{aligned}$$

Thus, $X_k \equiv 0 \mod p$ if and only if $X_0 \equiv b^{-2k} - 1 \mod p$. Since $p \nmid b(b^2-1)$, we have that $p > 3$. In particular, $p$ is odd and $p-1$ is even. As such, as previously stated in section 2, there are as many quadratic residues as there are quadratic non-residues mod $p$. In particular, there are $(p-1)/2$ of each. Since $(b^{-2})^a = (b^{-a})^2$, the remainder $b^{-2}$ only generates quadratic residues mod $p$. Therefore, the order of $b^{-2}$ in the multiplicative group $\mathbb{Z}/\mathbb{Z}p - \{0\}$ is at most $(p-1)/2$.

Thus, we must reject at most $(p-1)/2$ possible choice for $x_p$ in this case. Notice that $k = 0$ gives us $X_0 \equiv 0 \mod p$, therefore, we reject this option as well.

Now consider $Y_k$. By (4):

$$Y_k = X_0 b^{2k+1} + B\left(b^{2k-1} + b^{2k-3} + \ldots + b\right) + l_1 \equiv$$
$$\equiv X_0 b^{2k+1} + b(b^2 - 1)\left(b^{2k-2} + \ldots + b^2 + 1\right) + b \equiv$$
$$\equiv X_0 b^{2k+1} + b\left(b^{2k} - 1\right) + b \equiv X_0 b^{2k+1} + b^{2k+1} - b + b \equiv (X_0 + 1)b^{2k+1} \mod p$$

Since $p \nmid b$ it is clear that $p \mid Y_k$ if and only if $X_0 \equiv -1 \mod p$. Thus we must in addition reject at most one more possible remainder for the value of $x_p$.

Therefore, in total, we must reject at most $(p-1)/2 + 1 = (p+1)/2$ possible values for $x_p$. As $p > 3$ this leaves us with $p-(p+1)/2 = (p-1)/2 > 0$ possible values for $x_p$ to choose from. We then proceed to choose an arbitrary satisfactory $x_p$ for such a $p \in S$.

**Case 5:** $p \nmid b(b^2-1)B$, $C \not\equiv b+1 \mod p$

In this case, as $p \nmid (b^2-1)B$ we take the remainder:

$$x_p \equiv -B(b^2-1)^{-1} \mod p$$

Clearly, $p \nmid x_p$. First, we shall show that $X_k \equiv x_p \mod p$ by induction. Notice that $X_0 \equiv x_p \mod p$ by construction of $X_0$. Next, assume that $X_k \equiv x_p \mod p$ for some $k \in \mathbb{N}_0$. Then:

$$X_{k+1} = b^2 X_k + B \equiv b^2 x_p + B \equiv x_p + (b^2-1)x_p + B \equiv$$
$$\equiv x_p - (b^2-1)(b^2-1)^{-1}B + B \equiv x_p \mod p$$

Thus, $p \nmid X_k$. Next, assume that $p \mid Y_k$. Then since $X_k \equiv x_p \mod p$ and $Y_k = bX_k + l_1$, we have that $p \mid bx_p + l_1$ and, hence, $x_p \mid bx_p(b^2-1) + l_1(b^2-1)$. Since $x_p \equiv -B(b^2-1) \mod p$, we have that

$$0 \equiv x_p b(b^2-1) + l_1(b^2-1) \equiv -Bb(b^2-1)^{-1}(b^2-1) + l_1(b^2-1) \equiv$$
$$\equiv -Bb + l_1(b^2-1) \mod p$$

Now consider the following:

$$Bb - (b^2 - 1)l_1 \equiv 0 \mod p$$
$$(b-1)bC - (b^2 - 1)l_1 \equiv 0 \mod p \qquad \text{Since } B = (b-1)C \text{ by (1)}$$
$$(b-1)bC - (b^2 - 1)(C-1) \equiv 0 \mod p \qquad \text{Since } l_1 = C - 1 \text{ by (2)}$$
$$bC - (b+1)(C-1) \equiv 0 \mod p \qquad \text{Since } p \nmid b - 1$$
$$bC - bC + b - C + 1 \equiv 0 \mod p$$
$$b + 1 - C \equiv 0 \mod p$$
$$C \equiv b + 1 \mod p$$

This, however, contradicts the assumption in this case that $C \not\equiv b + 1 \mod p$. By contradiction then $p \nmid Y_k$ for all $k \in \mathbb{N}_0$. □

Now we go on to show that conditions B1-B5 are necessary for there to exist a two digit solution if $b - 1$ is squarefree.

**Lemma 8.** *Let $b > 1$ be an integer with $b-1$ squarefree. If $b$ has a two digit solution then the conditions B1-B5 hold.*

*Proof.* Let $S$ consist of all prime divisors of $b(b^2-1)$ and let $\xi = \overline{X_0, (l_1 l_0)}$ be a falsifier for $S$. Without losing generality, we may furthermore assume that none of the elements in the sequence $\lfloor \xi b^k \rfloor$ are divisible by primes from $S$. Indeed, if $\lfloor \xi b^K \rfloor$ is the last element of the sequence divisible from primes in $S$, if we take $\xi' = \xi b^{K+1}$ then no element in the sequence $\lfloor \xi' b^k \rfloor$ are divisible by primes in $S$.

As in the previous proof, we define $B = l_1 b + l_0$ and obtain the expressions (3) and (4) for $X_k$ and $Y_k$ respectively. Let $p \mid b^2 - 1$ then by (3) we have:

$$X_k = X_0 b^{2k} + B\left(b^{2k-2} + \ldots + b^2 + 1\right) \equiv X_0 1^{2k} + B\left(1^{2k-2} + \ldots + 1^2 + 1\right) \equiv$$
$$\equiv X_0 + kB \mod p$$

Assume that $p \nmid B$ and let $k_0 \in \mathbb{N}$ be such that $k_0 \equiv -X_0 B^{-1} \mod p$. Then taking $k = (np+1)k_0$ for arbitrary $n \in \mathbb{N}_0$, we have that

$$X_k \equiv X_0 + (np+1)k_0 B \equiv X_0 + k_0 B \equiv X_0 - X_0 B^{-1} B \equiv X_0 - X_0 \equiv 0 \mod p$$

Hence, there are infinitely many elements in the sequence $X_k$ divisible by $p$. This contradicts the choice of $\xi$. By contradiction we, therefore, have $p \mid B$.

Since this hold for all $p \mid b^2 - 1$ it follows that $\mathrm{rad}(b^2 - 1) \mid B$ and therefore $\mathrm{rad}(b-1) \mid B$. However, by assumption, $b-1$ is squarefree. Therefore, $\mathrm{rad}(b-1) = b-1$, and hence $b-1 \mid B$. As such we can express $B$ as:

$$B = (b-1)C \tag{5}$$

Since $\mathrm{rad}(b+1) \mid \mathrm{rad}(b^2-1) \mid B$ and $\gcd(b-1, b+1) \le 2$ it follows that all prime divisors of $b+1$ must either be 2 or divide $C$. Hence, $\mathrm{rad}(b+1) \mid 2C$ and B5 holds.

Consider $p \mid b$. If $p$ divides one of the digits $l_0, l_1$ both of which repeat infinitely often then the sequence $\lfloor \xi b^k \rfloor$ would be infinitely often divisible by $p$ as $X_k \equiv l_0 \mod p$, and $Y_k \equiv l_1 \mod p$. By the choice of $\xi$ this would lead to a contradiction. As such $p \nmid l_0, l_1$.

In particular, $C > 0$ since otherwise $B = 0$, and hence $l_0, l_1 = 0$ and $p \mid l_0, l_1$. If $C = 1$ then $B = b - 1 < b$ and therefore $l_1 = 0$. Again, we would have that $p \mid l_1$ and obtain a contradiction. Thus, $C > 1$.

As $B$ has two digits it is clear that $B = (b-1)C \le b^2 - 1$. Dividing both sides by $b-1$ we obtain $C \le b+1$. If $C = b+1$ then $B = b^2 - 1$. As such, $l_0 = l_1 = b - 1$ and this contradicts the condition in proposition 3. On the other hand, if $C = b$ then obviously $l_0 = 0$, and as such $p \mid l_0$ and we yet again obtain a contradiction. Therefore, $C < b$.

Therefore, we have just show that $1 < C < b$, i.e. that B2 holds.

Now, define, as before:

$$l_1 = C - 1 \quad l_0 = b - C$$

Since $1 < C < b$ it follows that $0 < l_0, l_1 < b$ and hence these numbers are digits in base $b$. Furthermore:

$$bl_1 + l_0 = b(C-1) + b - C = bC - b + b - C = (b-1)C = B$$

By the uniqueness of expression of $B$ in base $b$ (see e.g. p. 13 in [15]) it follows that $l_0, l_1$ are the required digits.

Next, assume that $b$ is even. Then since $b$ must be coprime with both $l_0$ and $l_1$ it follows that $l_0, l_1$ are odd. Since $l_1 = C - 1$ is odd, therefore, $C$ is even. And since $l_0 = b - C$ is odd and $b$ is even, $C$ is odd. Since $C$ is both even and odd, we have a contradiction. Therefore, $b$ must be odd. This shows B1.

As mentioned previously, $\gcd(b, l_0 l_1) = 1$. Let $p \mid b$. Then $p \nmid l_1 = C - 1$ and also $l_0 = b - C \equiv -C \mod p$. Threfore, $p \nmid C$. To sum up, if $p \mid b$ then $p \nmid C(C-1)$. Equivalently, $\gcd(b, C(C-1)) = 1$. This shows B4.

Finally, both $l_0, l_1$ must be even. To see this, note that if $l_0$ is odd then:

$$X_{k+1} = bY_k + l_0 \equiv Y_k + 1 \mod 2$$

Therefore, one of $Y_k, X_{k+1}$ must be divisible by 2. In particular, there are infinitely many elements in the sequence $\lfloor \xi b^k \rfloor$ that are divisible by 2. Since $2 \mid b(b^2 - 1)$, we have that $2 \in S$ and we obtain a contradiction. Likewise, if $2 \mid l_1$ we have that

$$Y_k = bX_k + l_1 \equiv X_k + 1 \mod 2$$

and for all $k$, one of $X_k, Y_k$ must be even. Again, we obtain a contradiction.

Therefore, $l_0, l_1$ must be even. In particular, as $l_1 = C - 1$, $C$ must be odd and B3 holds. This then finishes the proof. □

To sum up, the two lemmas combine into the following theorem:

**Theorem 9.** *Let $b > 1$ be an integer with $b - 1$ squarefree. Then $b$ has a two digit solution if and only if the conditions B1-B5 hold.*

The conditions B1-B5 can be replaced by some other conditions. We can replace B5 with the condition that $\mathrm{rad}(b^2 - 1) \mid B$ since we have proved this in the course of lemma 8 and it is this fact we used in the proof of lemma 7.

Likewise, B3 can be replaced with the claim that $l_1$ is even, since we do not otherwise use the fact that $C$ is odd. Finally, B4 can be replaced by perhaps the more illuminating claim that $\gcd(b, l_1 l_2) = 1$ since we only use B4 to prove this condition.

In this light, we can see that B5 is a direct analogue to A1. Condition B4 just requires that the prime divisors of $b$ would not divide any of the repeating digits which is not a surprising condition, and finally, B3 is a reflection of the fact that the digits must have a different parity than $b$ or otherwise we would have infinitely many elements of the sequence $\lfloor \xi b^k \rfloor$ divisible by 2.

Condition B2 is not particularly interesting either. It is a simple conclusion of the requirements that $C = B/(b-1)$, $l_0, l_1 \neq 0$, and $B < b^2 - 1$. Thus, perhaps unexpectedly, the only surprising necessary and sufficient condition here is the requirement that $b$ is odd in B1.

## 4 Conclusions and Questions

Theorem 9 implies that a certain important class of numbers $b$ does not have an unavoidable set of primes $S$ since it has a two digit solution. We first show this for $b = 2^{2k+1} - 1$ with $k \in \mathbb{N}$ and then for $b > 3$ prime. We finish this section with a discussion on further potential research in this area.

**Corollary 10.** *Let $b = 2^{2k+1} - 1$ for $k \in \mathbb{N}$. Then $b$ has a two digit solution.*

*Proof.* First it is clear that $b$ is odd and as such B1 holds. Next, we choose

$$C = 2^{2k+1} - 3$$

Clearly, $C$ is odd and so B3 holds. Also, $b+1 = 2^{2k+1}$ and so $\mathrm{rad}(b+1) = 2$. As such, $\mathrm{rad}(b+1) = 2 \mid 2C$ trivially. Hence, B5 holds as well.

It will take a little more work to verify B2 and B4. For B2 consider the following

$$C > 1 \iff 2^{2k+1} - 3 > 1 \iff 2^{2k+1} > 4 \iff 2k + 1 > 2 \iff k > 0$$

Since $k \in \mathbb{N}$, $k$ is a positive integer and therefore the last inequality holds. Hence, $C > 1$. Next, consider the following:

$$C < b \iff 2^{2k+1} - 3 < 2^{2k+1} - 1$$

That is, the inequality holds trivially. Therefore, $1 < C < b$ and hence B2 holds.

Finally, let $p \mid b = 2^{2k+1} - 1$. If $p \mid C = 2^{2k+1} - 3$ as well, then $p$ divides the difference of the two numbers. Hence, $p \mid 2$. Since $p$ is primes this implies that $p = 2$, but then we have that $2 = p \mid 2^{2k+1} - 1 = b$ which is clearly not the case. By contradiction then $p \nmid C$.

If we assume that $p \mid C - 1 = 2^{2k+1} - 4$ in addition to $p \mid b$, then we have that

$$p \mid b - (C - 1) = 3$$

As such, since $p$ is prime, we obtain that $p = 3$. However, this implies that

$$0 \equiv b = 2^{2k+1} - 1 \equiv 2 \cdot 4^k - 1 \equiv 2 \cdot 1 - 1 \equiv 1 \mod 3$$

By contradiction then $p \nmid C - 1$. In conclusion, B4 holds.

Hence, by theorem 9, $b$ has a two digit solution and, hence, there are no unavoidable sets of primes for $b = 2^{2k+1} - 1$. □

The previous corollary is also interesting due to the following result:

**Corollary 11.** *Let $q > 3$ be prime. Then there is a two digit solution for $q$.*

*Proof.* We now consider the following two cases depending on the form of $q$.

**Case 1: q is of the form $2^R - 1$**

Assume that $R = 2r$ is even, then

$$q = 2^{2r} - 1 = 4^r - 1 \equiv 1 - 1 \equiv 0 \mod 3$$

As $3 \mid q$ and $q$ is prime it follows that $q = 3$. However, by hypothesis, $q > 3$. By contradiction then $R$ is odd. Since, furthermore, $R > 1$ (for if $R = 1$ then $q = 1$) the preceding corollary 10 applies. Hence, there exists a two digit solution for $q$.

**Case 2: q is not of the form $2^R - 1$**

As $q > 3$ is prime condition B1 is automatic. Let $R$ be the highest power of 2 that divides $q + 1$, i.e. $R$ is the largest such that $2^R \mid q + 1$.

Assume that $q - 1$ is squarefree since, otherwise, the claim holds by theorem 6. Then $2 \mid q - 1$ but $4 \nmid q - 1$. As such, $q - 1 \equiv 2 \mod 4$ and hence $4 \mid q + 1$. That is, $R \geq 2$. We now choose $C$ to be as follows:

$$C = \frac{q+1}{2^R}(2^R - 1)$$

Let $p \mid \operatorname{rad}(q + 1)$. If $p \neq 2$ then $p \mid (q + 1)/2^R$, and as such $p \mid C \mid 2C$. If on the other hand $p = 2$ then $p \mid 2C$ trivially. Either way we obtain that if $p \mid \operatorname{rad}(q + 1)$ then $p \mid 2C$. Hence, $\operatorname{rad}(q + 1) \mid 2C$ and B5 holds.

By definition of $R$, $(q + 1)/2^R$ is odd. Since $2^R - 1$ is clearly odd then $C$ is odd and B3 holds. For B2 consider the following:

$$C > 1 \iff \frac{q+1}{2^R}(2^R - 1) > 1 \iff (q + 1)(2^R - 1) > 2^R \iff$$
$$\iff q2^R - q + 2^R - 1 > 2^R \iff q2^R > q + 1$$

Since $R \geq 2$ and $q > 3$ the last equation holds. Hence, $C > 1$. Now consider the other case:

$$C < q \iff \frac{q+1}{2^R}(2^R - 1) < q \iff (q+1)(2^R-1) < q2^R \iff$$
$$\iff q2^R + 2^R - q - 1 < q2^R \iff 2^R < q + 1$$

Since $2^R \mid q+1$ then $2^R \leq q+1$. Furthermore, since $q$ is not of the form $2^R - 1$ by hypothesis, it follows that $2^R < q+1$ and hence $C < q$. Therefore, $1 < C < q$ and B2 holds.

Finally, since $q$ is prime and $1 < C < q$ implies that $0 < C, C-1 < q$, we have that $q \nmid C(C-1)$. Therefore, in particular, $\gcd(q, C(C-1)) = 1$ and B4 holds.

Thus, by theorem 9, the number $q$ has a two digit solution. □

In light of this theorem, the smallest natural number for which conjecture 2 is open is 8. Since $8 - 1 = 7$ is squarefree theorem 6 does not apply, and since 8 is even theorem 9 does not apply. The smallest odd positive integer for which the conjecture is open is 15. This is because since $1 < C < 15$ is odd by conditions B2, B3, and coprime with 15 by B4, the remaining options for $C$ are $7, 11, 13$. However, for these choices for $C$, the number $C-1$ is not coprime with 15, and thus B4 cannot be maintained. A different approach is therefore required for $b = 8, 15$.

This investigation leads naturally to the following questions which remain open. First, one would need to categorise $n$ digit solutions for all $n \in \mathbb{N}$. Second, it remains open if for all $b > 1$ that do not have an unavoidable set of primes there exists $n \in \mathbb{N}$ such that $b$ has an $n$ digit solution. Third, it also remains in principle open if integers $b > 1$ other than $2, 3, 4, 6$ have an unavoidable set of primes. These questions require further investigation and at least some of them would need to be answered before we know if the numbers 8 and 15 have an unavoidable set of primes.

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