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APPLICATION OF THE EULER CHARACTERISTIC EQUATION TO THE STUDY OF SPECTRUM CURVES

OILERIO CHARAKTERISTINĖS LYGTIES TAIKYMAS TIRIANT SPEKTRINES KREIVES

Master's thesis

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Glossary of Notations

SLP	–	Sturm–Liouville Problem
dSLP	–	discrete Sturm–Liouville Problem
BC	–	Boundary Condition
NBC	–	Nonlocal Boundary Condition
NC	–	Nonlocal Condition
FDS	–	Finite Difference Scheme
$\mathbb{C}_q$	–	$\{q \in \mathbb{C} : -\pi/2 < \arg q \leqslant \pi/2 \text{ or } q = 0\}$
$\mathbb{R}_q$	–	$\{q \in \mathbb{C}_q : \lambda = (\pi q)^2 \in \mathbb{R}\}$
$\gamma_c(q)$	–	Complex Characteristic Function, Complex CF
$\mathcal{D}_\xi$	–	Domain of CF $:= \{q \in \mathbb{C}_q : \mathrm{Im}\ \gamma_c(q) = 0\}$
EP	–	Eigenvalue Point
CE	–	Constant Eigenvalue
●	–	Zero Point
●, CP	–	Critical Point
○, BP	–	Branch Point
○, BEP	–	Branch Eigenvalue Point
○, RP	–	Ramification Point
●, CEP	–	Constant Eigenvalue Point
○, PP	–	Pole Point
◎	–	Pole Point of the second order
$\mathcal{C}_\xi$	–	set of all CEPs
$\mathcal{P}_\xi$	–	set of all PPs
$\mathcal{Z}$	–	set of all zeros
$\mathcal{K}_\xi$	–	set of all Critical Points
$\mathcal{N}_\xi$	–	Spectrum Domain $:= \mathcal{D}_\xi \cup \mathcal{C}_\xi$
$\mathcal{N}_l$	–	Spectrum Curve
n_{ce}	–	number of nonregular Spectrum Curves
n_{nce}	–	number of regular Spectrum Curves
n_p	–	number of Pole Points (including order)
n_∞	–	order of Pole Point at $q = \infty$
n_{cr}	–	number of Critical Points in $\mathbb{R}_q^h$
n_{cr}^{+}	–	number of Critical Points in $\mathbb{C}_q^{h\pm}$
n_c	–	number of parts of Spectrum Curves between two Critical Points in $\mathbb{C}_q^h$

Abstract

Sturm–Liouville problem with nonlocal boundary conditions arise in many scientific fields such as chemistry, physics, or biology. There could be found some references to graph theory in a discrete Sturm–Liouville problem, especially in investigation of spectrum curves. In this master thesis, relations between discrete Sturm–Liouville problem with nonlocal boundary conditions characteristics (poles, critical points, spectrum curves) and graphs characteristics (vertices, edges and faces) were found.

Keywords: Sturm–Liouville problem; spectrum curves; nonlocal boundary conditions; graphs

Santrauka

Šturmo ir Liuvilio uždavinys su nelokaliosiomis kraštinėmis sąlygomis iškyla daugelyje mokslo šakų, tokiose kaip chemija, fizika ar biologija. Diskretizavus šį uždavinį bei išnagrinėjus spektrines kreives, galima įžvelgti grafų teorijos motyvų. Šiame magistro baigiamajame darbe pristatomos sąsajos tarp diskrečiojo Šturmo ir Liuvilio uždavinio su nelokaliosiomis kraštinėmis sąlygomis (poliai, kritiniai taškai ir spektrinės kreivės) bei grafų charakteristikų (viršūnės, briaunos ir veidai).

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys; spektrinės kreivės; nelokaliosios kraštinės sąlygos; grafai

Introduction

In differential equations theory, Sturm–Liouville problem has not been yet fully investigated although it is important in applied mathematics where such problems appear very frequently. One of the examples is quantum mechanics, the one-dimentional time-independent Schrödinger equation which is a Sturm–Liouville problem.

In general, the investigation of nonlocal boundary conditions' problems have first been started by J.R. Cannon in 1963 [6] and L.I. Kamynin in 1964 [15], later by A.V. Bitsadze and A.A. Samarskii in 1969 [3]. J.R. Cannon and L.I. Kamynin research area was the parabolic problems, they formulated new problems with Boundary Conditions (BC) [21] which are now called *nonlocal*, while A.V. Bitsadze and A.A. Samarskii's focus was on the general elliptic equation with nonlocal boundary conditions. A.V. Bitsadze and A.A. Samarskii have proved the existence and uniqueness of solution of such problem [4]. Currently some one-dimensional NBCs

$$u(0) = \gamma_0 u(\xi_0) \quad \text{or} \quad u(1) = \gamma_1 u(\xi_1)$$

are named after the latter researchers and called *Bitsadze–Samarskii type* NBCs.

Other than the fathers of NBC problems, there were many scientists who investigated various types of nonlocal conditions [21]. Two-point NBCs have started to be investigated by N.I. Ionkin, E.I. Moiseev in 1979 [12] and later in 2000 [13]. N.I. Ionkin himself considered a parabolic problem with an integral condition [11]

$$u(0,t) = \nu(t), \quad \int\limits_0^1 u(1,t)dt = \mu(t).$$

These and many other nonlocal problems were a rich research topicality for many scientists. Thus, in 2011, a special issue of 27 articles for nonlocal Boundary Conditions was published in the journal *Boundary Value Problems* [8]. This article deals with Boundary Value Problems (BVPs) with Nonlocal Conditions (NCs) for many types of equations: discrete, parabolic, ordinary and many other, as well as existence, multiplicity, asymptotical behaviour and approximation of solutions by multiple methods.

Although many world-wide scientists have built a good base investigations for nonlocal problems, Lithuanian mathematicians have given a great input in the field of differential and numerical problems with NBCs. Prof. M. Sapagovas was the pioneer in the study of such problems. In such topic, the first publications were about the investigation of a mercury droplet in electric contact given the droplet volume [21]. Later on, many other topics were researched like difference scheme for two-dimensional elliptic problem with an integral condition. In 1982–1985, M. Sapagovas doctoral student R. Čiegis was also involved in these topics as his study area was elliptic and parabolic problems with integral and Bitsadze–Samarskii type NBCs and finite-difference schemes of them [7].

Throughout the time the focus of differential problem has changed to eigenvalues studies. All type of eigenvalue questions are mostly applicable in real world problems, such as designing car stereo systems, where it helps to reproduce the vibration of the car due to music.

However, the investigation of the spectrum of Sturm–Liouville problem (SLP) with nonlocal boundary conditions is fairly new area. SLP is important for investigating existence and uniqueness of the solutions for classical stationary problems. These considered to be very complicated since such problems with NBCs are not self-adjoin and spectrum for such problems may be negative or complex [21]. Many scientists have spent time on studying SLP with various NBCs. N.I. Ionkin and E.A. Valikova have studied SLP with the following NBCs [14]:

$$u(0) = u(1), \quad u_x(1) = 0.$$

A.V. Goolin, N.I. Ionkin and V.A. Morozova have dealt with the stability, with respect to initial data of difference schemes that approximate the heat–conduction equation with contant coefficients and nonlocal boundary conditions [9]

$$u(0,t) = 0, \quad u_x(0,t) = u_x(1,t).$$

In other publications, A.V. Goolin, V.A. Morozova and N.S. Udovichenko have investigated parabolic problems with two parameters in NBCs:

$$u(0,t) = \alpha u(1,t), \quad u_x(1,t) = \gamma u_x(0,t).$$

Similar results for NBCs have been found by A.V. Goolin and A.Y. Mokin with

$$u(0,t) = 0, \quad u_x(1,t) = u_x(0,t) + \alpha u(1,t).$$

The existence of the solutions of the nonlinear SLP with integral BC was analized by Z. Yang [28]. The problem was formulated with the following integral BCs:

$$\begin{aligned} &-(au')' + bu = g(t)f(t,u), \quad t \in (0,1) \\ &(\cos\gamma_0)\, u(0) - (\sin\gamma_0)\, u'(0) = \int_0^1 u(\tau)\, d\alpha(\tau) \\ &(\cos\gamma_1)\, u(1) - (\sin\gamma_1)\, u'(1) = \int_0^1 u(\tau)\, d\beta(\tau) \end{aligned}$$

where $a \in C^1([0,1],(0,\infty))$ and $b \in C([0,1],[0,\infty))$, $f \in C([0,1] \times \mathbb{R},\mathbb{R})$ and $g \in C((0,1) \times [0,\infty)) \cap L(0,1)$, $\int_0^1 g(t)dt > 0$, α and β right continuous on $[0,1)$ and left continuous at $t = 1$, and nondecreasing on $[0,1]$, with $\alpha(0) = \beta(0) = 0$; $\gamma_0 \in [0,\pi/2]$ and $\gamma_1 \in [0,\pi/2]$; $\int_0^1 u(\tau)d\alpha(\tau)$ and $\int_0^1 u(\tau)d\beta(\tau)$ donate the Riemann–Stieltjes integrals of u with respect to α and β, respectively. The existence of nontrivial solution for the formulated problem was proved using the topological degree arguments and cone theory.

Nonetheless, Lithuanian mathematicians have given an insightful input to SLP with NBCs investigations. M. Sapagovas with co-authors were the first ones to begin to investigate eigenvalues for Bitsadze–Samarskii type

$$u(0) = 0, \quad u(1) = \gamma u(\xi), \quad 0 < \xi < 1,$$

and integral type

$$u(0) = \gamma_0 \int_0^1 \alpha_0(x)u(x)\,\mathrm{d}x, \quad u(1) = \gamma_1 \int_0^1 \alpha_1(x)u(x)\,\mathrm{d}x.$$

NBCs. They have showed that there exists eigenvalues which do not depend on parameters γ_0 and γ_1 in BCs, they also showed that complex eigenvalues may exist. It is worth mentioning that M. Sapagovas with co-authors have investigated the spectrum of discrete SLP, too. Such results were helpful to prove stability of Finite-Difference Schemes (FDS) for nonstationary problems and convergence of iterative methods.

There is many more Lithuanian mathematicians who investigated SLP with Nonlocal Boundary Conditions [21]. A. Štikonas and O. Štikoniene have described another approach of investigating spectrum of SLP with Nonlocal Boundary Conditions using characteristic function method [22]. Other than that, during the last decades, A. Štikonas together with his doctoral students S. Pečiulytė, A. Skučaitė and K. Bingelė [1, 2, 18, 20] have studied the spectrum of Sturm–Liouville problem with various boundary conditions.

The topicality of this master thesis has come from the question whether there could be a connection between different mathematical branches. The results gave some insights to look at the differential problem, especially spectrum, through the graphs theory. This lead to new findings including proofs of previous propositions [26, 27].

In this master thesis the relations between discreet Sturm–Liouville problem with non-local boundary conditions and graphs theory are presented.

Publications and conferences

The results were printed in the *Proceedings of the Lithuanian Mathematical Society*:

1. Jonas Vitkauskas, Artūras Štikonas, *Relations between spectrum curves of discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory.* Ser. A, Vol. 61, 2020.

2. Jonas Vitkauskas, Artūras Štikonas, *Relations between spectrum curves of discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory. II.* Ser. A, Vol. 62, 2021.

The results were also presented in 3 conferences:

1. J. Vitkauskas, A. Štikonas. Relations between spectrum curves and discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory. *Lithuanian Mathematical Society LXI conference.* Šiauliai, Lithuania, December 2020. Section: Differential Equations and Numerical Methods.

2. J. Vitkauskas. Relations between spectrum curves of discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory. *9th Young Mathematicians of Lithuania conference.* Vilnius, Lithuania, December 2020. In addition to the presentation, participants of the conference had voted it to be the most interesting short presentation.

3. J. Vitkauskas, A. Štikonas. Relations between spectrum curves and discrete Sturm–Liouville problem with nonlocal boundary conditions and graph theory. II. *Lithuanian Mathematical Society LXII conference.* Vilnius, Lithuania, June 2021. Section: Differential Equations and Numerical Methods.

1. Sturm–Liouville problem

In this section we describe previous results about the Sturm–Liouville Problem (SLP) with various Nonlocal Boundary Conditions (NBCs) [1, 2, 19, 20, 22].

1.1. Problem formulation

Let us consider Sturm–Liouville Problem (SLP)

$$-u'' = \lambda u, \qquad 0 < t < 1, \tag{1.1}$$

with one classical (Dirichlet or Neumann) Boundary Condition (BC)

$$u(0) = 0, \tag{1.2_d}$$
$$u'(0) = 0 \tag{1.2_n}$$

and one two-point Nonlocal Boundary Condition (NBC)

$$u(1) = \gamma u'(\xi), \tag{1.3_1}$$
$$u(1) = \gamma u(\xi) \tag{1.3_2}$$

or integral NBC

$$u(1) = \gamma \int_{\xi_1}^{\xi_2} u(t)\mathrm{d}t, \tag{1.4}$$

here $\lambda \in \mathbb{C}$, $\xi \in [0; 1]$ for (1.3_1)–(1.3_2) and $\boldsymbol{\xi} \in S_{\boldsymbol{\xi}} := \{(\xi_1,\xi_2) \in [0; 1]^2 : 0 \leqslant \xi_1 < \xi_2 \leqslant 1\}$ for (1.4) [1, 20].

1.2. Characteristic function method in the case of dSLP

We introduce a uniform grid and we use notation $\bar{\omega}^h = \{t_j = jh, j = 0, \dots, n; nh = 1\}$ for $2 < n \in \mathbb{N}$, and $\mathbb{N}^h := (0,n) \cap \mathbb{N}$, $\overline{\mathbb{N}}^h = \mathbb{N}^h \cup \{0,n\}$. For ($1.3_1$)–($1.3_2$), we make an assumption that ξ is located on the grid, i.e., $\xi = mh = m/n$, $0 < m < n$. We denote $\mathbb{N}_o = \{k \in \mathbb{N} \colon k - \text{odd}\}$, $\mathbb{N}_e = \{k \in \mathbb{N} \colon k - \text{even}\}$.

Let us consider a dSLP (an approximation by Finite–Difference Scheme (FDS)) [19]

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = \lambda U_i, \qquad i = 1, \dots, n-1, \tag{1.5}$$

$\lambda \in \mathbb{C}$, with classical discrete Dirichlet or Neumann Boundary Condition (BC)

$$U_0 = 0, \tag{1.6_d}$$
$$U_1 = U_0 \tag{1.6_n}$$

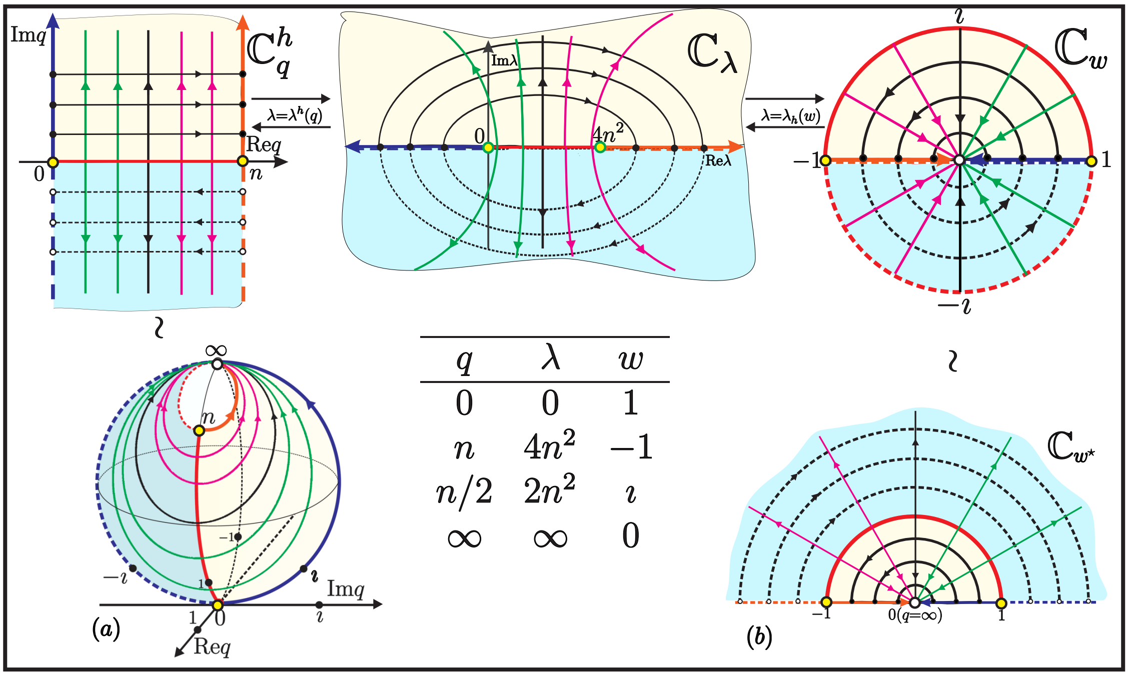


Figure 1. Bijective mappings: $\lambda = (4/h^2)\sin^2(\pi qh/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_q^h$; $\lambda = (2/h^2)(1-(w-w^{-1})/2)$ between $\mathbb{C}_\lambda$ and $\mathbb{C}_w$, ○ – Branch Point $\mathbb{C}_\lambda$, ○ – Ramification Point $\mathbb{C}_q$; (a) $\overline{\mathbb{C}}_q^h$ on Riemann sphere; (b) domain $\mathbb{C}_{w^\star}$ on the upper half-plane [2]

and Nonlocal Boundary Condition (NBC):

$$U_n = \gamma \frac{U_{m+1} - U_{m-1}}{2h}, \qquad (1.7_1)$$

$$U_n = \gamma U_m. \qquad (1.7_2)$$

So, we have four cases of two-point BCs: a) (1.6_d)–(1.7_1), b) (1.6_d)–(1.7_2), c) (1.6_n)–(1.7_1), d) (1.6_n)–(1.7_2). We denote $\varkappa = 0$ for (1.7_2) NBC and $\varkappa = 1$ for (1.7_1) NBC.

Remark 1. *(1.7_1)–(1.7_2) are called first and second two-point NBCs respectively. By nature, (1.7_1) is three-point NBC but is approximated from two-point (1.3_1) NBC.*

The function $\lambda^h\colon \mathbb{C}_z \to \mathbb{C}_\lambda$,

$$\lambda^h(z) := \frac{4}{h^2}\sin^2\frac{\pi zh}{2} = \frac{2}{h^2}(1-\cos(\pi zh)),$$

is an entire function which inverse is multivalued. In $\mathbb{C}_\lambda$, function $\lambda^h(z)$ has two Branch Points (BP) of the second order at $\lambda = 0$ and $\lambda = 4n^2$, also a logarithmic BP at $\lambda = \infty$. In $\mathbb{C}_z$, respectively, points $q = 0$ and $q = n$ are called Ramification Points (RP). The latter function λ^h could be made inverse single-valued if we take the subset interval $[0, 4n^2]$. Now let us consider a bijection [2].

$$\lambda = \lambda^h(q) := \frac{4}{h^2}\sin\frac{\pi qh}{2} \qquad (1.8)$$

between $\mathbb{C}_\lambda := \mathbb{C}$ and $\mathbb{C}_q^h$, $\mathbb{C}_q^h := \mathbb{R}_q^h \cup \mathbb{C}_q^{h+} \cup \mathbb{C}_q^{h-}$, $\mathbb{R}_q^h := \mathbb{R}_y^- \cup \{0\} \cup \mathbb{R}_x^h \cup \{n\} \cup \mathbb{R}_y^{h+}$, $\mathbb{R}_y^- := \{q = \imath y : y > 0\}$, $\mathbb{R}_x^h := \{q = x : 0 < x < n\}$, $\mathbb{R}_y^{h+} := \{q = n + \imath y : y > 0\}$, $\mathbb{C}_q^{h+} := \{q = x + \imath y : 0 < x < n, y > 0\}$, $\mathbb{C}_q^{h-} := \{q = x + \imath y : 0 < x < n, y < 0\}$. The domain of *Riemann sphere* is noted as $\overline{\mathbb{C}}_q^h = \mathbb{C}_q^h \cup \{\infty\}$. Then for any eigenvalue $\lambda \in \mathbb{C}_\lambda$ there exists the Eigenvalue Point (EP) $q \in \mathbb{C}_q^h$. Let us donate $\mathring{\mathbb{C}}_q^h := \mathbb{C}_q^h \setminus \{0,n\}$ the relative complement of RP points in $\mathbb{C}_q^h$ and $\mathring{\mathbb{R}}_q^h := \mathbb{R}_y^- \cup \mathbb{R}_x^h \cup \mathbb{R}_y^{h+}$. It follows that $\lambda < 0$ for $q \in \mathbb{R}_y^{h-}$; $0 < \lambda < 4/h^2$ for $q \in \mathbb{R}_x^h$; $\lambda > 4/h^2$ for $q \in \mathbb{R}_y^{h+}$. So, if $q \in \mathbb{R}_q^h := \mathring{\mathbb{R}}_q^h \cup \{0,n\}$, then corresponding eigenvalue is real. If EP is RP, then we call this point *Branch Eigenvalue Point (BEP)*.

Let $\lambda_h(w) : \mathbb{C}_\lambda \to \mathbb{C}_w$, $\lambda = \lambda_h(w) := 2/h^2(1 - (w - w^{-1})/2)$, be a bijection, where $\mathbb{C}_w := \{w \in \mathbb{C} : |w| \leqslant 1, w \neq 0\}$ (see Figure 1). The set $\mathbb{C}_w$ is good to investigate eigenvalues in the neighbourhood $\lambda \to \infty$ $(w \to 0)$. The bijection $\lambda = \lambda_h(w)$ maps $\lambda < 0$ to the interval $w \in (0,1)$; $0 < \lambda < 4/h^2$ to the upper unit semicircle; $\lambda > 4/h^2$ to the interval $(-1,0)$. Complex λ points correspond to the points w, $\operatorname{Im} w \neq 0$, inside the unit circle (see Figure 1). The points $w = \pm 1$ are RPs in $\mathbb{C}_w$ of the function $\lambda_h(w)$ and correspond to two BPs of the second order at $\lambda = 0$ and $\lambda = 4n^2$ in the set $\mathbb{C}_\lambda$. The function $w = e^{\imath\pi hq}$ maps $\mathbb{C}_q^{h+}$ to the upper unit semidisk in $\mathbb{C}_w$ or $\mathbb{C}_{w^\star}$ and $\mathbb{C}_q^{h-}$ to the outer part of the unit semidisk in $\mathbb{C}_q^{h-}$. All the corresponding points in different domains are shown in the table (see Figure 1).

The general solution U_j for a discrete equation (1.5) is equal to:

$$U_j = C_1 \frac{\sin(\pi q t_j)}{(1 - hq)\pi q} + C_2 \cos(\pi q t_j), \qquad j \in \overline{\mathbb{N}}^h. \tag{1.9}$$

With a classical Dirichlet BC $U_0 = 0$ we get $C_1 = 0$, so equation (1.9) has a solvable form

$$U_j = \begin{cases} C t_j & \text{for } q = 0, \\ C(-1)^j t_j & \text{for } q = n, \\ C \sin(\pi q t_j) & \text{(general case)}. \end{cases} \tag{1.10}$$

Similarly with Neumann BC $U_1 = U_0$ the equations are as follows:

$$U_j = \begin{cases} C & \text{for } q = 0, \\ C(-1)^{j+1}(2t_j/h - 1) & \text{for } q = n, \\ C \dfrac{\cos(\pi q(t_j - h/2))}{\cos(\pi q h/2)} & \text{(general case)}. \end{cases} \tag{1.11}$$

Now, by substituting (1.10) to both first and second NBCs (1.7_1)–(1.7_2) we get the following expressions [1]:

1. $q = 0$ $(\lambda = 0)$. Nontrivial $(C \neq 0)$ solution (eigenfunction) for $\lambda = 0$ exists if

$$\gamma = 1, \tag{1.12_1}$$

$$\gamma = 1/\xi. \tag{1.12_2}$$

2. $q = n$ $(\lambda = 4/h^2)$. Nontrivial solution (eigenfunction) for $\lambda = 4/h^2$ exists if

$$\gamma = (-1)^{n-m-1}, \tag{1.13$_1$}$$

$$\gamma = (-1)^{n-m}/\xi. \tag{1.13$_2$}$$

3. For the general case, the equations are as follows:

$$\frac{\sin(\pi q)}{\pi q} \cdot \frac{\pi q h}{\sin(\pi q h)} = \gamma \cos(\xi \pi q), \tag{1.14$_1$}$$

$$\frac{\sin(\pi q)}{\pi q} \cdot \frac{1}{1 - hq} = \gamma \frac{\sin(\xi \pi q)}{\pi q} \cdot \frac{1}{1 - hq}. \tag{1.14$_2$}$$

Similarly, by substituting (1.11) to both first and second NBCs (1.7$_1$)–(1.7$_2$) we get the following expressions [1]:

1. $q = 0$ $(\lambda = 0)$. Nontrivial $(C \neq 0)$ solution (eigenfunction) for $\lambda = 0$ exists if

$$C = C\gamma \cdot 0 = 0, \tag{1.15$_1$}$$

$$C = C\gamma. \tag{1.15$_2$}$$

2. $q = n$ $(\lambda = 4/h^2)$. Nontrivial solution (eigenfunction) for $\lambda = 4/h^2$ exists if

$$\gamma = (-1)^{n-m-1} \frac{2n-1}{2n}, \tag{1.16$_1$}$$

$$\gamma = (-1)^{n-m} \frac{2n-1}{2m-1}. \tag{1.16$_2$}$$

3. For the general case, the equations are as follows:

$$-\frac{\cos(\pi q(1 - h/2))}{\cos(\pi q h/2)} = \gamma \sin(\pi q(\xi - h/2)) \cdot \frac{\sin(\pi q h/2)}{h/2}, \tag{1.17$_1$}$$

$$\frac{\cos(\pi q(1 - h/2))}{\cos(\pi q h/2)} = \gamma \frac{\cos(\pi q(\xi - h/2))}{\cos(\pi q h/2)}. \tag{1.17$_2$}$$

1.2.1. Special points of Characteristic functions. Dirichlet BC

In this subsection the definitions and formulas of Characteristic Functions (see [22]) will be presented for the Dirichlet BC (1.6$_d$) and two-point NBCs (1.7$_1$)–(1.7$_2$) [1].

Let us consider the following functions:

$$Z^h(z) := Z(z) \cdot \frac{\pi z h}{\sin(\pi z h)}, \qquad Z(z) := \frac{\sin(\pi z)}{\pi z}; \tag{1.18$_1$}$$

$$Z^h(z) := Z(z) \cdot \frac{1}{\pi z (hz - 1)}, \qquad Z(z) := \sin(\pi z); \tag{1.18$_2$}$$

$$P_\xi^h(z) = P_\xi(z) := \cos(\xi\pi z); \tag{1.19$_1$}$$

$$P_\xi^h(z) = P_\xi(z) \cdot \frac{1}{\pi z(hz-1)}, \qquad P_\xi(z) := \sin(\xi\pi z). \tag{1.19$_2$}$$

The zeros of functions (1.18$_1$)–(1.18$_2$), $Z^h(q)$, $q \in \mathbb{C}_q^h$, is the set

$$\hat{\mathcal{Z}} := \mathbb{N}^h = \{1, \dots, n-1\}. \tag{1.20}$$

The zeros of functions (1.19$_1$)–(1.19$_2$), $P_\xi^h(q)$, $q \in \mathbb{C}_q^h$, are the sets

$$\overline{\mathcal{Z}}_\xi := \{p_k = (k-1/2)/\xi, \quad k = \overline{1,m}\}, \tag{1.21$_1$}$$

$$\overline{\mathcal{Z}}_\xi := \{p_k = k/\xi, \quad k = \overline{1,m-1}\}. \tag{1.21$_2$}$$

The zeros for the first two-point NBC always exists, moreover, there will always be one $1/2 < p_1 \leqslant n/2$ and $k_{\max} = m = n/(2p_1)$. For the second two-point NBC zeros of the function $P_\xi^h(q)$ do not exist for $m = 1$. Also, $1/2 < p_1 \leqslant n/2$ for $m > 1$, $k_{\max} = m - 1 = n/p_1 - 1$.

The equations (1.14) can be written in the form

$$Z^h(q) = \gamma P_\xi^h(q), \quad q \in \mathbb{C}_q^h. \tag{1.22}$$

Constant Eigenvalues. The Constant Eigenvalue (CE) is defined as the eigenvalue that does not depend on parameter γ. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q$. CEPs are roots of the system

$$Z^h(q) = 0, \quad P_\xi^h(q) = 0. \tag{1.23}$$

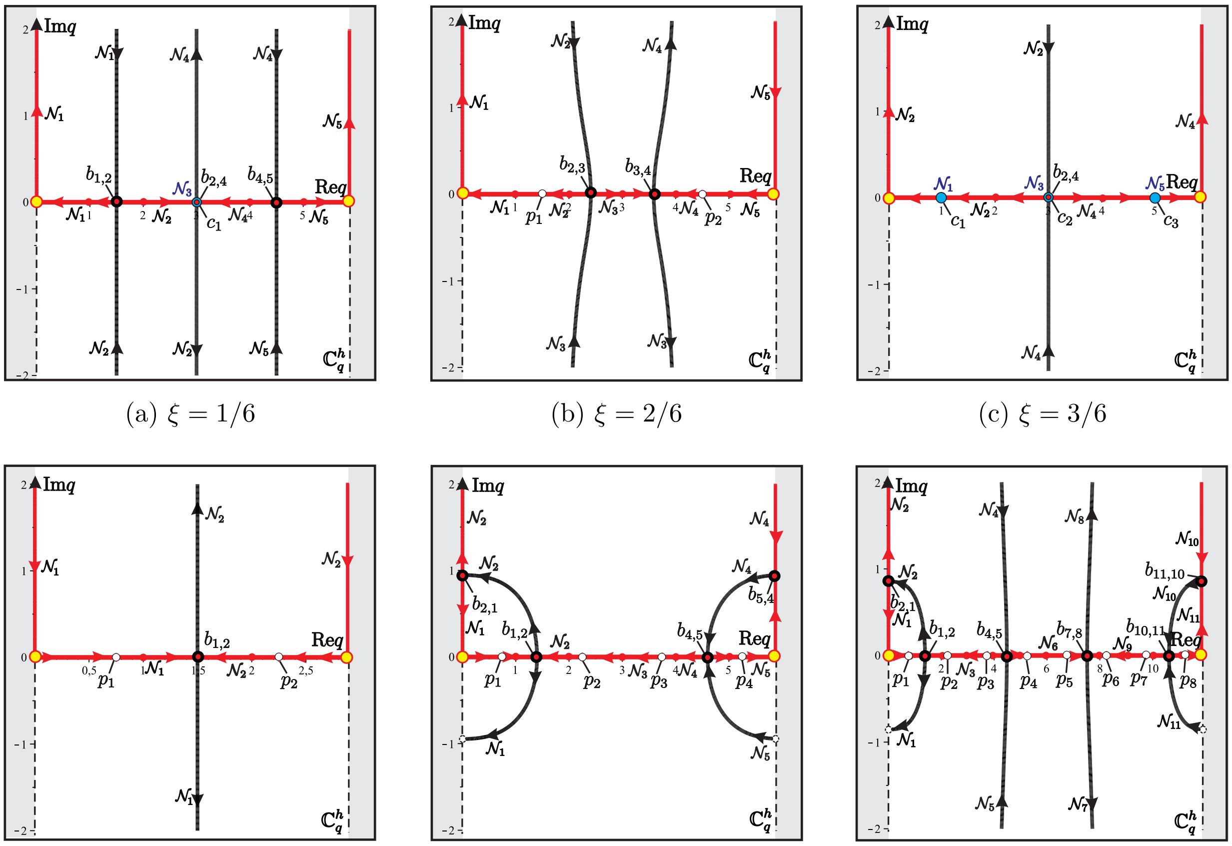
Remark 2. *Such systems to find Constant Eigenvalues were studied in S. Pečiulytė doctoral dissertation [18].*

For further lemmas the notations $\xi = m/n = M/N$, $\gcd(n,m) = K$, $\gcd(N,M) = 1$ are used.

Lemma 3 ([1])**.** *For dSPL* (1.5) *with Dirichlet BC (1.6$_d$) and first two-point NBC (1.7$_1$) Constant Eigenvalues exist only for $\xi = M/N \in (0,1)$, $M \in \mathbb{N}_o$, $N \in \mathbb{N}_e$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s), c_s := (s-1/2)N, s = \overline{1,K}$.*

Lemma 4 ([1])**.** *For dSPL* (1.5) *with Dirichlet BC (1.6$_d$) and second two-point NBC (1.7$_2$) Constant Eigenvalues exist only for $\xi = M/N \in (0,1)$, $M, N \in \mathbb{N}, K > 1$, values and those eigenvalues are equal to $\lambda_s = \lambda^h(c_s), c_s := Ns, s = \overline{1,K-1}$.*

Remark 5. *Note that for dSPL* (1.5) *with Dirichlet BC (1.6$_d$) and second two-point NBC (1.7$_2$) when $K = 1$ CEPs do not exist since for $m = n-1$, $\gcd(n,n-1) = 1$.*



(a) $\xi = 1/6$ (b) $\xi = 2/6$ (c) $\xi = 3/6$

(d) $\xi = 2/3$ (e) $\xi = 4/6$ (f) $\xi = 8/12$

Figure 2. Spectrum Curves (Dirichlet and first two-point BCs) [1]

Thus, taking lemmas 3–4 into consideration, expressions for CEPs $c_s = z_{l_s} = p_{k_s} = N(s-1/2)$ are as follows:

$$k_s = M(s-1/2)+1/2, \qquad l_s = N(s-1/2), \qquad s = \overline{1,K}, \tag{1.24$_1$}$$

$$k_s = Ms, \qquad l_s = Ns, \qquad s = \overline{1,K-1}. \tag{1.24$_2$}$$

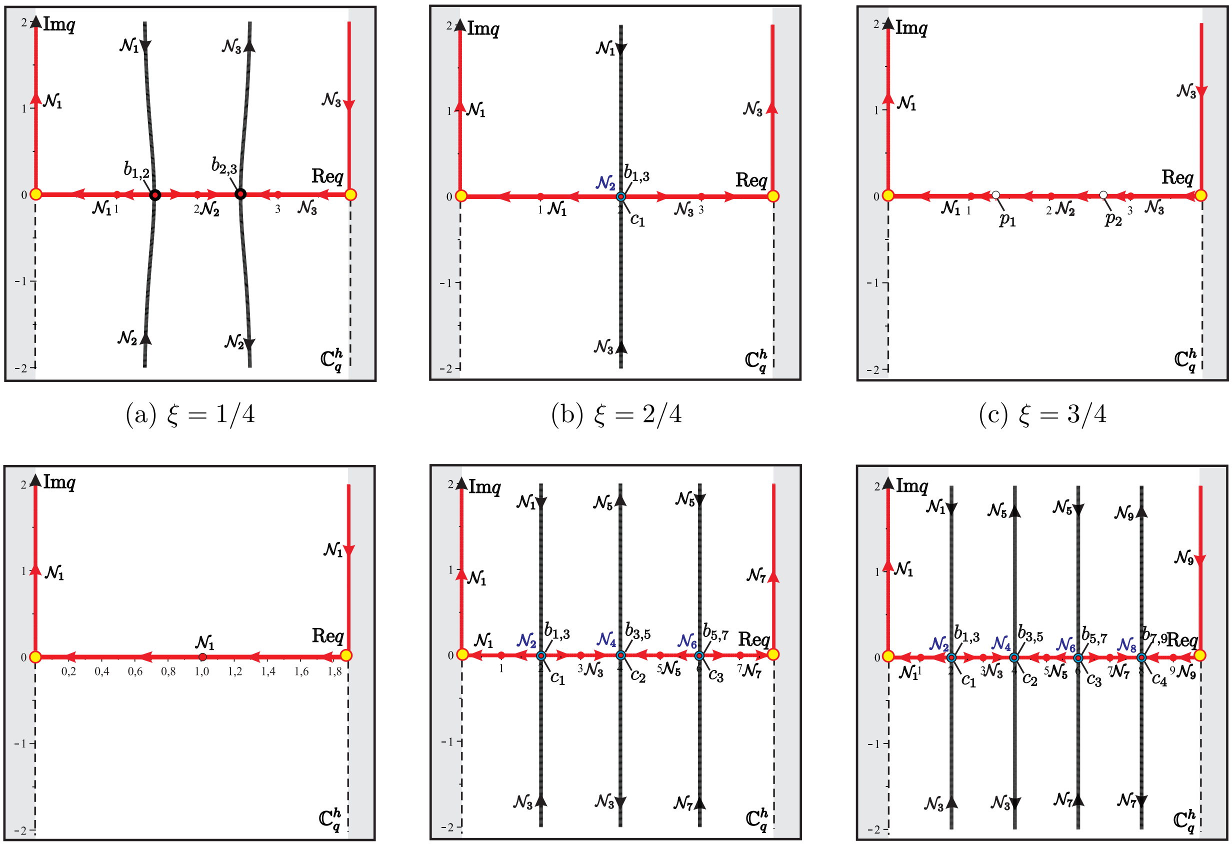
The notation $\mathcal{C}_\xi$ is used for the set of all CEPs.

Complex Characteristic Function. Lets consider *Complex Characteristic Function* $\gamma_c : \mathbb{C}_q^h \to \mathbb{C}$:

$$\gamma_c(q) = \gamma_c(q;\xi) := \frac{Z^h(q)}{P_\xi^h(q)}, \qquad q \in \mathbb{C}_q^h, \tag{1.25}$$

where Z^h and P_ξ^h are functions (1.18)–(1.19). Nonconstant Eigenvalue Points (NEPs) depend on the parameter γ, thus all NEPs are γ-points of the (1.25) meromorphic function [22]. If γ is fixed, then the γ-point is the root of the equation $\gamma_c(q) = \gamma$. Thus, NEPs are defined as $\lambda = (\pi q(\gamma))^2$.

Such approaches to investigate NEPs using characteristic function method are published in S. Pečiulytė doctoral dissertation [18], A. Štikonas and O. Štikonienė publication "Characteristic functions for Sturm–Liouville Problems with nonlocal Boundary Conditions" [22].



(a) $\xi = 1/4$ (b) $\xi = 2/4$ (c) $\xi = 3/4$

(d) $\xi = 1/2$ (e) $\xi = 4/8$ (f) $\xi = 5/10$

Figure 3. Spectrum Curves (Dirichlet and second two-point BCs) [1]

A *Complex-Real Characteristic Function* (CF) is the restriction of the Complex CF on subset $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q : \operatorname{Im}\gamma_c(q) = 0\} \subset \mathbb{C}_q$ and $\gamma = \gamma_c|_{\mathcal{N}} : \mathcal{N} \to \mathbb{R}$. in general case, a subset $\mathcal{N}$ is a union of curves. A complex-real characteristic function describes complex eigenvalue points (and complex eigenvalues) for real γ [22].

For dSPL (1.5) with Dirichlet BC (1.6_d) and two-point NBCs (1.7) we have meromorphic functions (Complex CF)

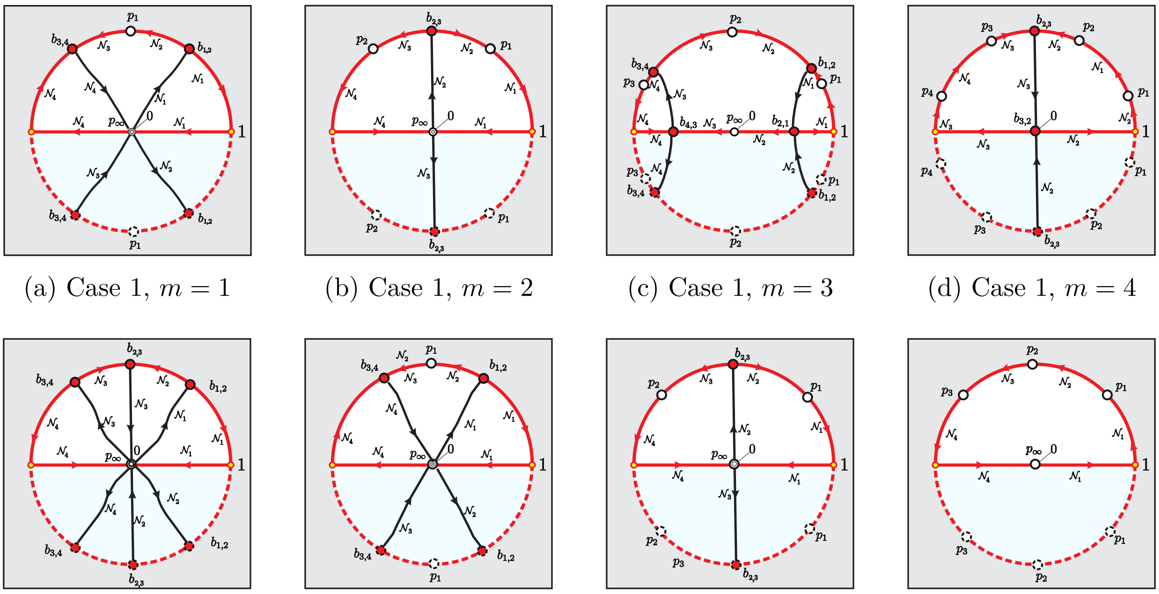
$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} \cdot \frac{\pi q h}{\sin(\pi q h)} = \frac{\sin(\pi q)}{\pi q \cos(\xi \pi q)} \cdot \frac{\pi q h}{\sin(\pi q h)}, \tag{1.26_1}$$

$$\gamma_c(q) := \frac{Z(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\sin(\xi \pi q)}. \tag{1.26_2}$$

All zeros and poles of meromorphic function $\gamma_c(q)$ lie in $(0,n) = \mathbb{R}^h_x$. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$, and the set of zeros for Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. Moreover, all these sets are finite.

Remark 6 ([1]). *If the point $q \in \mathcal{C}_\xi$, then it is Removable Singularity Point of Complex CF.*

Remark 7 ([1]). *The domain of Spectrum Curves, $\mathcal{N}_\xi$, for dSPL (1.5) with Dirichlet BC (1.6_d) and two-point NBCs (1.7_1)–(1.7_2) are symmetrical with respect to vertical line $x = n/2$. Figures 2 and 3 represent such behaviour of Spectrum Curves.*



(a) Case 1, $m = 1$ (b) Case 1, $m = 2$ (c) Case 1, $m = 3$ (d) Case 1, $m = 4$

(e) Case 2, $m = 1$ (f) Case 2, $m = 2$ (g) Case 2, $m = 3$ (h) Case 2, $m = 4$

Figure 4. Spectrum Curves in $\mathbb{C}_w$ (dSLP Dirichlet BC) for $n = 5$ [1]

Critical Points. For the dSLP (1.5)–(1.7) with two-point NBCs there are three types of Critical Points: the first, the second and the third order. If $\gamma_c'(b) = 0$, $b \in \mathbb{C}$, then b is called a *Critical Point* (CP) of the function γ_c [1]. It is said that CPs are saddle points of Complex CF. In general, if the function γ_c at the point $b \in \mathbb{C}_q$ satisfy conditions $\gamma_c'(b) = 0$, $\ldots$, $\gamma_c^{(k)} = 0$, $\gamma_c^{(k+1)} = 0$, then the point b is a CP of k-order.

The point at infinity $q = \infty \notin \mathbb{C}_q^h$. It is important to show the behaviour of Complex CF at the infinity point $q = \infty \notin \mathbb{C}_q^h$. In the domain $\mathbb{C}_w$, this point corresponds to $w = 0$ and the expression for Complex CF can be rewritten in the following manner:

$$\gamma_c(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi} + e^{-\imath\pi q\xi}} \cdot \frac{2h}{e^{\imath\pi qh} - e^{-\imath\pi qh}}, \qquad (1.27_1)$$

$$\gamma_c(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi} + e^{-\imath\pi q\xi}}. \qquad (1.27_2)$$

Now, it is more convenient to investigate Complex CF in the neighbourhood $w = 0$ ($w = e^{\imath\pi qh}$):

$$\gamma_c(w) = \frac{w^n - w^{-n}}{w^m + w^{-m}} \cdot \frac{2h}{w - w^{-1}} = \frac{2h}{w^{n-m-1}}\left(1 + w^2 + \mathcal{O}\left(w^4\right)\right), \qquad (1.28_1)$$

$$\gamma_c(w) = \frac{w^n - w^{-n}}{w^m + w^{-m}} = \frac{1}{w^{n-m}}\left(1 + \mathcal{O}\left(w^{2m}\right)\right). \qquad (1.28_2)$$

Functions $1/(w^{n-m-1})$ and $1/(w^{n-m})$ conclude the behaviour of Complex CF at the point $w = 0 \notin \mathbb{C}_w^h$ (see Figure 4).

Lemma 8 ([1]). *Complex CF at the point $q = \infty$ for dSPL (1.5) with Dirichlet BC (1.6_d) and the first two-point NBC (1.7_1) has Pole Point of the $n - m - 1$-order in case $m < n - 1$.*

Complex CF at the point $q = \infty$ for the same problem has Removable Singularity Point of Complex CF in case $m = n - 1$ and $\gamma_c(\infty) = 2h$. This point is Critical Point of the first order, too.

Lemma 9 ([1]). *Complex CF at the point $q = \infty$ for dSPL (1.5) with Dirichlet BC (1.6$_d$) and the second two-point NBC (1.7$_2$) has Pole Point of the $n - m$-order.*

Ramification Points. There is two Ramification Points (RP) in $\mathbb{C}_q^h$: point $q = 0$ and $q = n$. Ramification points are important in investigation of a dSLP because the plane folds on by those points (see Figure 1) [1].

$q = 0$ ($q = n$): Taylor series for $\gamma(q)$ at RP $q = 0$ is

$$\gamma(q) = 1 + \frac{\pi^2}{6n^2}\left(3m^2 - n^2 + 1\right) q^2 + \mathcal{O}\left(q^4\right), \tag{1.29$_1$}$$

$$\gamma(q) = \frac{n}{m} + \frac{\pi^2}{6mn}\left(m^2 - n^2\right) q^2 + \mathcal{O}\left(q^4\right). \tag{1.29$_2$}$$

Case 1: first two-point BC. In case 1, if $n^2 \neq 3m^2 + 1$, then a coefficient of the second term is nonzero and $q = 0$ is a CP of the first order in $\mathbb{C}_q^h$ but this point is not CP in $\mathbb{C}_\lambda$. Moreover, by symmetry, point $q = n$ has the same properties.

Also, if Pell's equation $n^2 = 3m^2 + 1$, $m,n \in \mathbb{N}$, $m + 1 < n$, is satisfied, then the RP will be a CP. Examples of such solutions are the pairs of (m,n): (4,7), (15,26), (56,97) etc. If $n^2 < 3m^2 + 1$ then there exists a negative CP ($b = \imath y, y > 0$) and large positive CP ($b = n + \imath y, y > 0$). If $n^2 > 3m^2 + 1$ then there is no such CP [1].

Case 2: second two-point BC. In case 2, a coefficient of the second term is negative since $m < n$, so $q = 0$ is the CP of the first order in $\mathbb{C}_q^h$ but BP $\lambda = 0$ is not a CP. Also, by symmetry, $q = n$ is the CP and $\lambda = 4n^2$ is not a CP.

Dirichlet BC is a special case in our investigated problems because the graphs of Spectrum Curves are symmetric by the vertical line $q = n/2$. Thus, if there is a CP in $q = 0$, there will be one in $q = n$ as well.

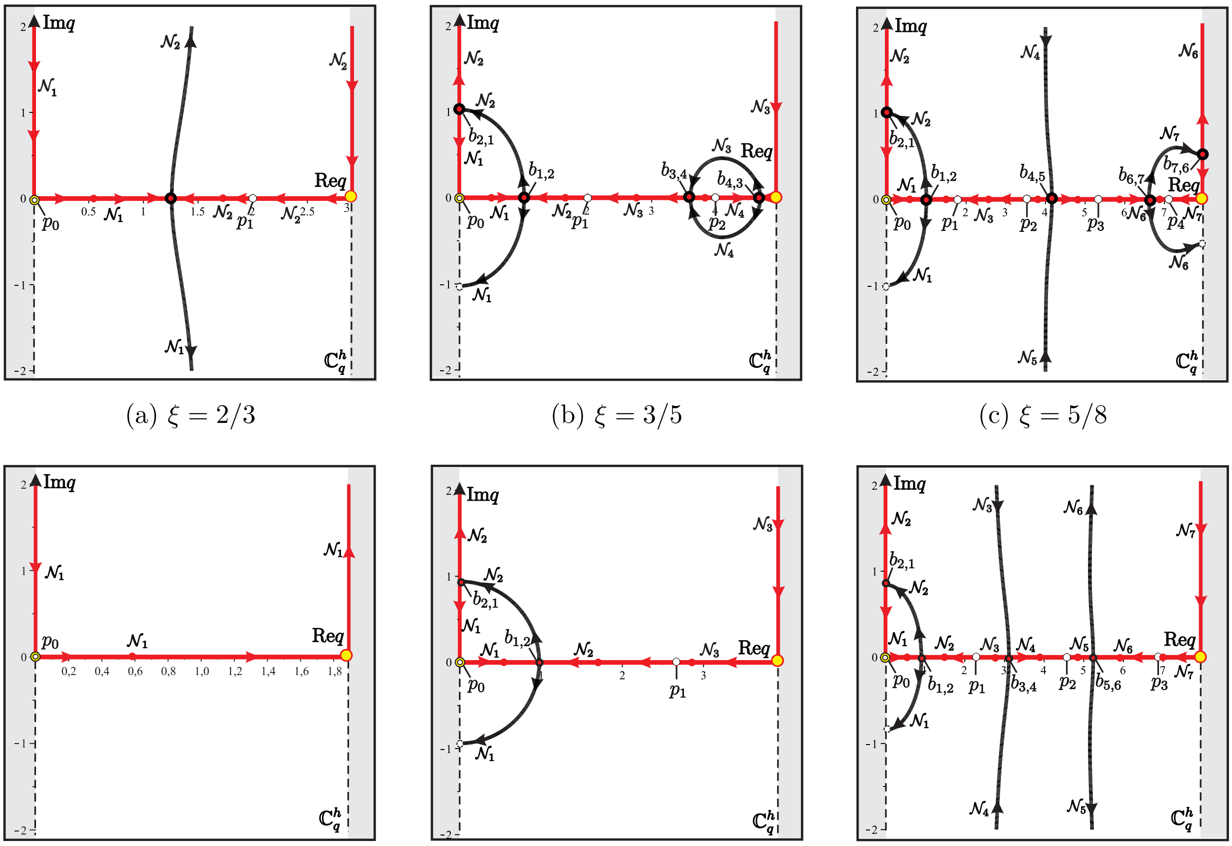
1.2.2. Special points of Characteristic functions. Neumann BC

As in subsection 1.2.1, dSLP with Neumann BC (1.6$_n$) and two-point NBCs (1.7) were investigated by K. Bingelė [1]. Let's consider functions

$$Z^h(z) := \frac{\cos(\pi z(1 - h/2))}{\cos(\pi z h/2)}; \tag{1.30}$$

$$P_\xi^h(z) := -\sin(\pi z(\xi - h/2)) \cdot \frac{\sin(\pi z h/2)}{h/2}, \tag{1.31$_1$}$$

$$P_\xi^h(z) := \frac{\cos(\pi z(\xi - h/2))}{\cos(\pi z h/2)}. \tag{1.31$_2$}$$



(a) $\xi = 2/3$ (b) $\xi = 3/5$ (c) $\xi = 5/8$

(d) $\xi = 1/2$ (e) $\xi = 2/4$ (f) $\xi = 4/8$

Figure 5. Spectrum Curves (Neumann and first two-point BCs) [1]

Zeros of the function $Z^h(z)$ (1.30) is the set

$$\hat{\mathcal{Z}} := \left\{ z_l = \frac{l-1/2}{1-h/2} = \frac{2l-1}{2n-1}n, \quad l = \overline{1,n-1} \right\}. \tag{1.32}$$

All zeros are simple. Zeros of the functions $P_\xi^h(q)$ (1.31) are the sets

$$\overline{\mathcal{Z}}_\xi := \left\{ p_k = \frac{k}{\xi - h/2} = \frac{2k}{2m-1}n, \quad k = \overline{0,m-1} \right\}, \tag{1.32$_1$}$$

$$\overline{\mathcal{Z}}_\xi := \left\{ p_k = \frac{k-1/2}{\xi - h/2} = \frac{2k-1}{2m-1}n, \quad k = \overline{1,m-1} \right\}. \tag{1.32$_2$}$$

Remark 10 ([1])**.** *All zeros p_k, $k = \overline{1,m-1}$ are simple. Zero p_0 for the first two-point NBC (1.7$_1$) is of the second order. If $m = 1$, then $\overline{\mathcal{Z}}_\xi = \{0\}$ If $m = 0$ for the second two-point NBC (1.7$_2$), then $\overline{\mathcal{Z}}_\xi = \varnothing$.*

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q^h$. CEP are roots of the system (1.23).

The equations (1.17) are rewritten in the form (1.22). For further lemmas some notations are used: if $\xi = m/n \in \mathbb{Q}$ and $\gcd(2n-1, 2m-1) = K$, here $N = ((2n-1)/K + 1)/2$, $M = (2m-1)/K + 1$, then $\gcd(2N-1, 2M-1) = 1$ and $K \in \mathbb{N}_o$.

Lemma 11 ([1]). *For dSPL (1.5) with Neumann BC (1.6$_n$) and first two-point NBC (1.7$_1$) Constant Eigenvalues do not exist.*

Lemma 12 ([1]). *For dSPL (1.5) with Neumann BC (1.6$_n$) and second two-point NBC (1.7$_2$) Constant Eigenvalues exist only for $\xi = m/n \in (0,1)$, $K > 1$, values and those eigenvalues are are equal to $\lambda_s = \lambda^h(c_s)$, $c_s := n/K \cdot (2s-1)$, $s = \overline{1,(K-1)/2}$.*

Remark 13 ([1]). *For the second two-point NBC (1.7$_2$) and $K = 1$ CEPs do not exist. For example, such situation is for $m = 0,1,n-4,n-2,n-1$ $(n > 3)$ because for such m we have* $\gcd(2n-1,2m-1) = 1$.

Finally, from lemma 12 we get expressions for CEPs $c_s = z_{l_s} = p_{k_s} = n/K \cdot (2s-1) = ((N-1/2)K + 1/2)/K(2s-1)$, $s = \overline{1,(K-1)/2}$:

$$l_s = \frac{(2N-1)(2s-1)+1}{2}, \quad k_s = \frac{(2M-1)(2s-1)+1}{2}. \tag{1.34}$$

Complex Characteristic Function. Let's consider *Complex Characteristic Function* $\gamma_c : \mathbb{C}_q^h \to \mathbb{C}$ (1.25) where Z^h and P_ξ^h are functions (1.30)–(1.31).

For dSPL (1.5) with Neumann BC (1.6$_n$) and two-point NBCs (1.7) we have meromorphic functions (Complex CF)

$$\gamma_c(q) := -\frac{\cos(\pi q(1-h/2))}{\sin(\pi q(\xi - h/2))} \cdot \frac{h}{\sin(\pi q h)}, \tag{1.35$_1$}$$

$$\gamma_c(q) := \frac{\cos(\pi q(1-h/2))}{\cos(\pi q(\xi - h/2))}. \tag{1.35$_2$}$$

In Case 2 all zeroes and poles of meromorphic function $\gamma_c(q)$ lie in $(0,n) = \mathbb{R}_x^h$. In Case 1 we have additional pole of the second order in $q = 0$. A set of PPs for Complex CF is $\mathcal{P}_\xi := \overline{\mathcal{Z}}_\xi \setminus \hat{\mathcal{Z}} = \overline{\mathcal{Z}}_\xi \setminus \mathcal{C}_\xi$. The set of zeros for this Complex CF is $\mathcal{Z}_\xi := \hat{\mathcal{Z}} \setminus \overline{\mathcal{Z}}_\xi = \hat{\mathcal{Z}} \setminus \mathcal{C}_\xi$. All these sets are finite.

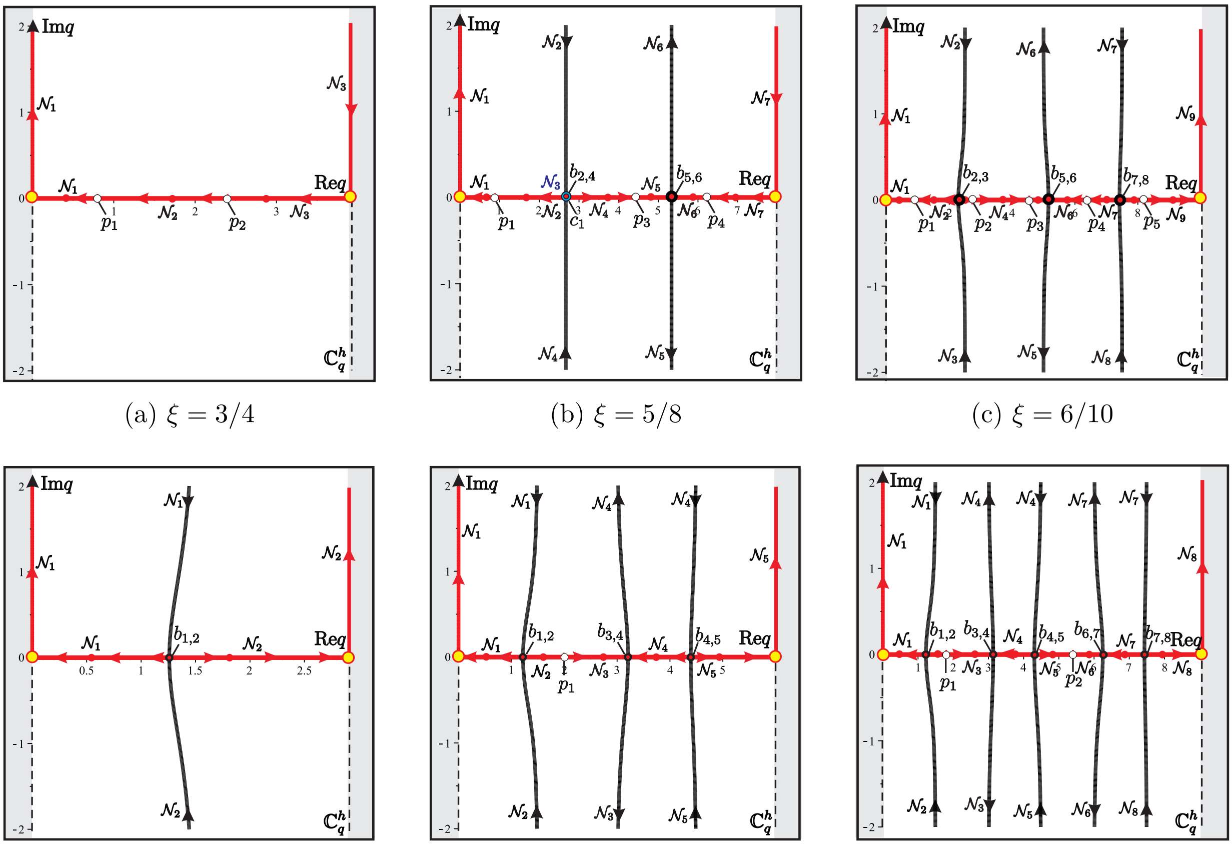
The point at infinity $q = \infty \notin \mathbb{C}_q^h$. As in subsection 1.2.1, point $q = \infty \notin \mathbb{C}_q^h$ with Neumann BC with two-point NBCs also have been investigated by K. Bingelė [1]. Now, it is more convenient to investigate Complex CF in the neighbourhood $w = 0$ $(w = e^{\imath\pi q h})$:

$$\gamma_c(q) = \frac{2h}{w^{n-m-1}} \left(1 + w^{\min(m,2)} + \mathcal{O}\left(w^2\right)\right), \tag{1.36$_1$}$$

$$\gamma_c(q) = \frac{1}{w^{n-m}} \left(1 + \mathcal{O}\left(w^{2m-1}\right)\right). \tag{1.36$_2$}$$

Functions $1/(w^{n-m-1})$ and $1/(w^{n-m})$ conclude the behaviour of Complex CF at the point $w = 0 \notin \mathbb{C}_w^h$.

Lemma 14 ([1]). *Complex CF at the point $q = \infty$ for dSPL (1.5) with Neumann BC (1.6$_n$) and the first two-point NBC (1.7$_1$) has Pole Point of the $n-m-1$-order in case $m < n-1$.*



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 1/3$ (e) $\xi = 2/6$ (f) $\xi = 3/9$

Figure 6. Spectrum Curves (Neumann and second two-point BCs) [1]

Complex CF at the point $q = \infty$ for the same problem has Removable Singularity Point of Complex CF in case $m = n - 1$ and $\gamma_c(\infty) = 2h$. This point is Critical Point of the first order, too.

Lemma 15 ([1]). *Complex CF at the point $q = \infty$ for dSPL (1.5) with Neumann BC (1.6_n) and the second two-point NBC (1.7_2) has Pole Point of the $n - m$-order.*

Ramification Points. Similarly as with Dirichlet BC, there are two Ramification Points (RPs) in $\mathbb{C}_q^h$: point $q = 0$ and $q = n$.

***q* = 0:** Taylor series for $\gamma(q)$ at RP $q = 0$ is

$$\gamma(q) = -\frac{2n}{2m-1}\pi^{-2}q^{-2} + \frac{6n^2 - 2m^2 - 6n + 2m - 1}{6n(2m-1)} + \mathcal{O}\left(q^2\right); \qquad (1.37_1)$$

$$\gamma(q) = 1 - \frac{\pi^2}{2n^2}(n-m)(n+m-1)q^2 + \mathcal{O}\left(q^4\right). \qquad (1.37_2)$$

Case 1: first two-point BC. In case 1, RP $q = 0$ is a PP of the second order in $\mathbb{C}_q^h$ and BP $\lambda = 0$ is a PP of the first order [1]. It is clear from the Taylor series that $q = 0$ will always be a PP.

Case 2: second two-point BC. In case 2, RP $q = 0$ is a PP of the first order in $\mathbb{C}_q^h$ and BP $\lambda = 0$ is not a CP in $\mathbb{C}_\lambda$.

$\boldsymbol{q = n}$**:** Taylor series for $\gamma(q)$ at RP $q = n$ is

$$\begin{aligned}\gamma(q) =&(-1)^{n-m-1}\frac{2n-1}{2n}\\ &+(-1)^{n-m-1}\frac{\pi^2(2n-1)}{24n^3}\left(6m^2-2n^2-6m+2n+3\right)(q-n)^2\\ &+\mathcal{O}\left((q-n)^4\right); \end{aligned} \tag{1.38$_1$}$$

$$\begin{aligned}\gamma(q) =&(-1)^{n-m}\frac{2n-1}{2m-1}\\ &+(-1)^{n-m+1}\frac{\pi^2(2n-1)}{6n^2(2m-1)}(n-m)(n+m-1)(q-n)^2\\ &+\mathcal{O}\left((q-n)^4\right). \end{aligned} \tag{1.38$_2$}$$

In both cases there will not be CPs.

Case 1: first two-point BC. In case 1, for $m + 1 < n$ there will be one negative CP $b_- \in \mathbb{R}_q^{h-}$. Moreover, for $6m^2 - 2n^2 + 6m - 2n + 3 > 0$ and $m + 1 < n$ there will be one positive CP $b_+ \in \mathbb{R}_q^{h+}$.

Case 2: second two-point BC. In case 2, there is no positive or negative CPs in RP.

1.2.3. Special points of Characteristic functions. Dirichlet and integral BC

For integral BC we assume that ξ_1 and ξ_2 are located on the grid $\bar{\omega}^h$, i.e., $\xi_1 = m_1h = m_1/n$, $\xi_2 = m_2h = m_2/n$, $\boldsymbol{m} \in S_{\boldsymbol{\xi}}^h := \{(m_1,m_2) : 0 \leqslant m_1 < m_2 \leqslant n, m_1,m_2 \in \overline{\mathbb{N}}^h\}$. Let $\boldsymbol{\xi} = \boldsymbol{m}/n = (m_1/n,m_2/n)$, $\xi = \xi_1/\xi_2 = m_1/m_2$, $\xi_+ = \xi_1 + \xi_2 = m_+/n$, $\xi_- = \xi_2 - \xi_1 = m_-/n$, here $m_+ := m_1 + m_2$, $m_- = m_2 - m_1$.

Real grid function on $\overline{\omega}^h$ is a space H. For such functions we can use a notation of inner product

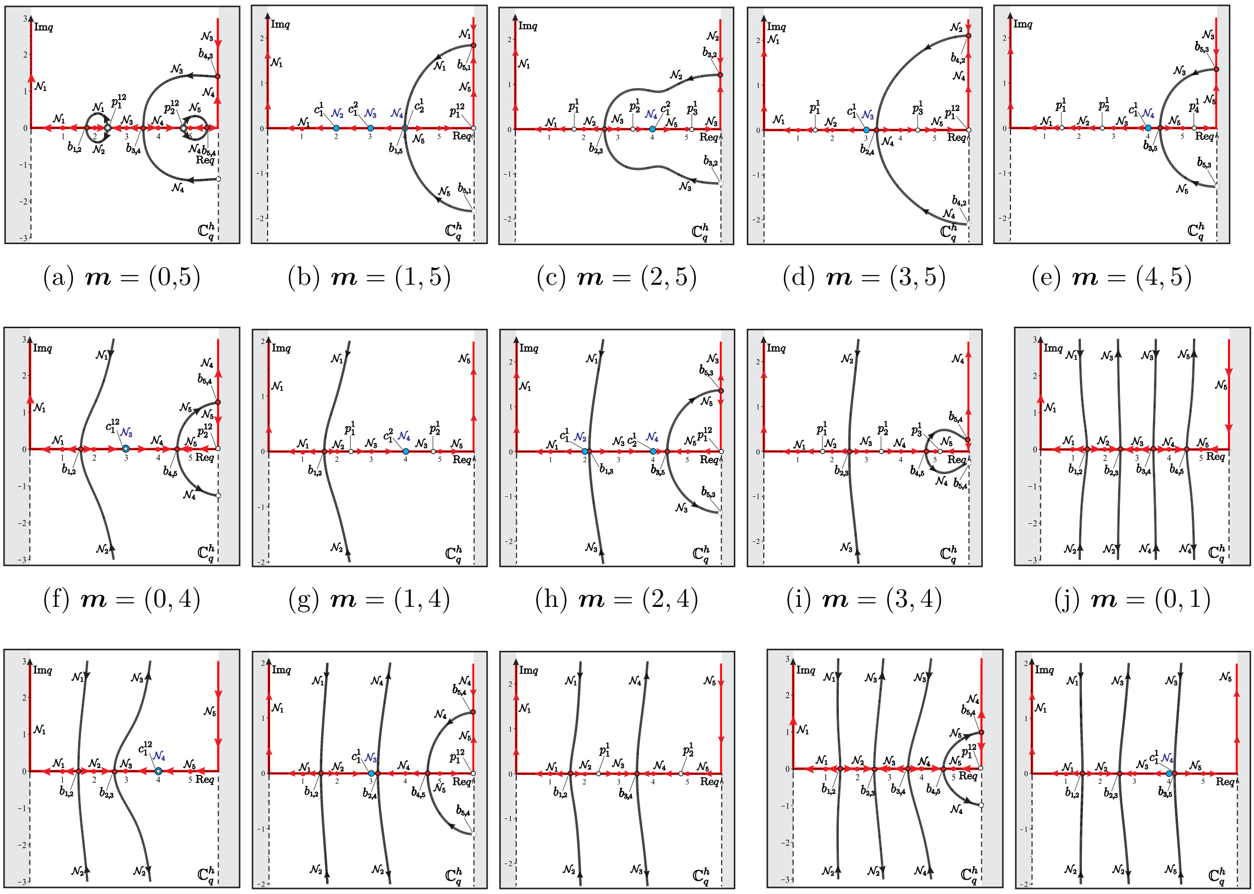
$$[U,V] := \sum_{j=0}^{n} U_jV_j.$$

Then the integral NBC could be approximated by trapezoid formula [2]:

$$\int_a^b u\,dt \approx u_{m_1}\frac{h}{2} + \sum_{i=m_1+1}^{m_2-1} u_ih + u_{m_2}\frac{h}{2} = [\chi_{[a,b]},u],$$

where

$$\chi_{[a,b],j} = \chi_{[a,b]}(t_j) = \begin{cases} 0, & \text{for } t_j < a \text{ or } t_j > b,\\ \frac{h}{2}, & \text{for } t_j = a \text{ or } t_j = b, \qquad t_j \in \bar{\omega}^h,\\ h, & \text{for } a < t_j < b, \end{cases}$$



(a) $\boldsymbol{m} = (0{,}5)$ (b) $\boldsymbol{m} = (1,5)$ (c) $\boldsymbol{m} = (2,5)$ (d) $\boldsymbol{m} = (3,5)$ (e) $\boldsymbol{m} = (4,5)$

(f) $\boldsymbol{m} = (0,4)$ (g) $\boldsymbol{m} = (1,4)$ (h) $\boldsymbol{m} = (2,4)$ (i) $\boldsymbol{m} = (3,4)$ (j) $\boldsymbol{m} = (0,1)$

(k) $\boldsymbol{m} = (0,3)$ (l) $\boldsymbol{m} = (1,3)$ (m) $\boldsymbol{m} = (2,3)$ (n) $\boldsymbol{m} = (0,2)$ (o) $\boldsymbol{m} = (1,2)$

Figure 7. Spectrum Curves for $n = 6$ (integral and Dirichlet BCs) [20].

$a,b \in \overline{\omega}^h$ and $a < b$. To simplify, integral NBC could be expressed as follows [20]:

$$U_n = \gamma h \left(\frac{U_{m_1} + U_{m_2}}{2} + \sum_{k=m_1+1}^{m_2-1} U_k \right). \tag{1.39}$$

Thus, we are considerig a dSLP approximation by the FDS (1.5) with a classical Dirichlet BC (1.6_d) and an integral NBC (1.39).

Now, let us substitute expressions (1.10) into the integral NBC (1.39):

1. $q = 0$ ($\lambda = 0$). Nontrivial ($C \neq 0$) solution (eigenfunction) for $\lambda = 0$ exists if

$$1 = \frac{\gamma\left(\xi_2^2 - \xi_1^2\right)}{2}. \tag{1.40}$$

   So, $\lambda = 0$ exists if and only if $\gamma = 2n^2/\left(m_2^2 - m_1^2\right)$ and $q = 0$ is BEP for all ξ_1, ξ_2 values. Note that if $\gamma = 2/\left(\xi_2^2 - \xi_1^2\right)$ then $\lambda = 0$.

2. $q = n$ ($\lambda = 4/h^2$). Nontrivial solution (eigenfunction) for $\lambda = 4/h^2$ exists if

$$1 = \frac{\gamma h^2 (-1)^n \left((-1)^{m_2} - (-1)^{m_1}\right)}{4}. \tag{1.41}$$

So, the eigenvalue $\lambda = 4/h^2$ ($q = n$) exists if and only if

$$m_2 - m_1 \in \mathbb{N}_o \quad \text{and} \quad \gamma = \frac{2(-1)^{n-m_2}}{h^2}, \tag{1.42}$$

and $q = n$ is BEP for $m_2 - m_1 \in \mathbb{N}_o$. If $m_2 - m_1 \in \mathbb{N}_e$, the eigenvalue $\lambda = 4/h^2$ does not exist ($\gamma = \infty$). In this case, we have a pole point at RP.

3. For the general case, the equations are as follows:

$$\frac{\sin(\pi q)}{\pi q} \cdot \frac{\sin(\pi q h/2)}{\pi q h \cos(\pi q h/2)} = \gamma \frac{\sin(\pi q(\xi_2 - \xi_1)/2) \sin(\pi q(\xi_2 + \xi_1)/2)}{\pi^2 q^2}. \tag{1.43}$$

Note that this equation is valid with limit cases for $q = 0$ and $q = n$, too.

Similarly as with two-point BCs, by using (1.6_d) and (1.11) we get an equation:

$$Z^h(q) = \gamma P^h_{\boldsymbol{\xi}}(q), \quad q \in \mathbb{C}^h_q, \tag{1.44}$$

where functions $Z^h(q)$ and $P^h_{\boldsymbol{\xi}}(q)$ are as follows:

$$Z^h(q) = \frac{\sin(\pi q)}{\pi q} \cdot \frac{\sin(\pi q h/2)}{\pi q h \cos(\pi q h/2)}, \quad P^h_{\boldsymbol{\xi}}(q) = \frac{\sin(\pi q(\xi_2 - \xi_1)/2)}{\pi q} \cdot \frac{\sin(\pi q(\xi_2 + \xi_1)/2)}{\pi q}.$$

To make it easier to investigate zeros of functions $Z^h(q)$ and $P^h_{\boldsymbol{\xi}}(q)$, we will break them into parts:

$$Z^h(z) := Z(z) \cdot \frac{\sin(\pi z h/2)}{\pi z h \cos(\pi z h/2)}, \qquad Z(z) := \frac{\sin(\pi z)}{\pi z}; \tag{1.45_1}$$

$$P_{\boldsymbol{\xi}}(z) := 2P^1_{\boldsymbol{\xi}}(z) P^2_{\boldsymbol{\xi}}(z),$$

$$P^1_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_1 + \xi_2)/2)}{\pi z}, \quad P^2_{\boldsymbol{\xi}}(z) := \frac{\sin(\pi z(\xi_2 - \xi_1)/2)}{\pi z}, \quad z \in \mathbb{C}. \tag{1.45_2}$$

All zeros of these functions in the domain $\mathbb{C}^h_q$ are simple and positive. It is not hard to derive that the zeros of the functions $Z(q)$, $Z^h(q)$ (1.45_1) coincide with the EPs in the classical case $\gamma = 0$ meaning that a set of zeros for function $Z^h(q)$, $\mathbb{C}^h_q$, is $\mathcal{Z} := \mathbb{N}^h = \{1, \ldots, n-1\}$.

Zeros of the functions $P^1_{\boldsymbol{\xi}}$ and $P^2_{\boldsymbol{\xi}}$ are respectively the sets

$$\overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} = \left\{ p^1_k = \frac{2nk}{m_+}, \quad k = 1, \ldots, \left\lfloor \frac{n}{p^1_1} \right\rfloor \right\}, \qquad \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \left\{ p^2_l = \frac{2nl}{m_-}, \quad l = 1, \ldots, \left\lfloor \frac{n}{p^2_1} \right\rfloor \right\}. \tag{1.46}$$

where $\lfloor\rfloor$ is the floor function.

Remark 16 ([2]). *If $\boldsymbol{m} = (0{,}1)$, then there are no zeros of the functions $P^1_{\boldsymbol{\xi}}$ and $P^2_{\boldsymbol{\xi}}$ in $\mathbb{C}^h_q$, i.e., $\overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} = \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \varnothing$. If $m_2 - 1 = m_1 > 0$, then there exists $p^1_1 \in \mathbb{C}^h_q$ and $\overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} = \varnothing$. If $m_2 - m_1 > 1$, then both sets are nonempty.*

Lemma 17 ([2]). *A set $\mathcal{Z}^P_{\boldsymbol{\xi}} = \mathcal{Z}^1_{\boldsymbol{\xi}} + \mathcal{Z}^2_{\boldsymbol{\xi}} + \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ describes all zeros of $P_{\boldsymbol{\xi}}(z)$ where $\mathcal{Z}^1_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ and $\mathcal{Z}^2_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}} \setminus \mathcal{Z}^{12}_{\boldsymbol{\xi}}$ are two families of the first-order zeros of $P^1_{\boldsymbol{\xi}}(z)$, $P^2_{\boldsymbol{\xi}}(z)$, respectively, and $\mathcal{Z}^{12}_{\boldsymbol{\xi}} := \overline{\mathcal{Z}}^1_{\boldsymbol{\xi}} \cap \overline{\mathcal{Z}}^2_{\boldsymbol{\xi}}$ is a family of the second-order zeros*

$$\mathcal{Z}^{12}_{\boldsymbol{\xi}} = \left\{ p^{12}_s = \frac{2ns}{\gcd(m_+; m_-)}, \quad s = 1, \ldots, \left\lfloor \frac{n}{p^{12}_1} \right\rfloor \right\}.$$

Remark 18 ([2]). *If $m_1 = 0$, then all zeros of P_ξ are of the second order and $\mathcal{Z}_\xi = \mathcal{Z}_\xi^{12}$. The point $q = n$ is zero of P_ξ^1 if and only if $m_2 + m_1 \in \mathbb{N}_e$ and $q = n$ is zero of P_ξ^2 if and only if $m_2 - m_1 \in \mathbb{N}_e$. So, if $q = n$ is zero of P_ξ, then it is of the second order and belongs to $\mathcal{Z}_\xi^{12}$.*

Constant Eigenvalues. For any CE $\lambda \in \mathbb{C}_\lambda$ there exists the *Constant Eigenvalue Point* (CEP) $q \in \mathbb{C}_q^h$. CEP are roots of the system

$$Z^h(q) = 0, \quad P_\xi^h(q) = 0. \tag{1.47}$$

In the case of dSLP (1.5) with Dirichlet BC (1.6_d) and integral NBC (1.39) CEPs belong to two sets, $Z(q) = 0$ and $P_\xi^1(q)$ or $Z(q) = 0$ and $P_\xi^2(q)$ respectively

$$\overline{\mathcal{C}}_\xi^1 = \mathcal{Z} \cap \overline{\mathcal{Z}}_\xi^1 = \left\{ c_k^1 = \frac{2nk}{\gcd(2n; m_+)}, \quad k = 1, \ldots, \left\lfloor \frac{n-1}{c_1^1} \right\rfloor \right\},$$
$$\overline{\mathcal{C}}_\xi^2 = \mathcal{Z} \cap \overline{\mathcal{Z}}_\xi^2 = \left\{ c_l^2 = \frac{2nl}{\gcd(2n; m_-)}, \quad l = 1, \ldots, \left\lfloor \frac{n-1}{c_1^2} \right\rfloor \right\}.$$

Similarly as with two-point NBCs, out of the sets with the integral NBC, a numerous lemmas and remarks were expressed by A. Skučaitė and other authors [2, 20].

Lemma 19 ([2]). *A set $\mathcal{C}_\xi^P = \mathcal{C}_\xi^1 + \mathcal{C}_\xi^2 + \mathcal{C}_\xi^{12}$ describes all Constant Eigenvalue Points, where $\mathcal{C}_\xi^1 := \overline{\mathcal{C}}_\xi^1 \setminus \mathcal{C}_\xi^{12} = \mathcal{Z} \cap \mathcal{Z}_\xi^1$, $\mathcal{C}_\xi^2 := \overline{\mathcal{C}}_\xi^2 \setminus \mathcal{C}_\xi^{12} = \mathcal{Z} \cap \mathcal{Z}_\xi^2$ and*

$$\mathcal{C}_\xi^{12} = \overline{\mathcal{C}}_\xi^1 \cap \overline{\mathcal{C}}_\xi^2 = \mathcal{Z} \cap \mathcal{Z}_\xi^{12} = \left\{ c_s^{12} = \frac{2ns}{gcd(2n; m_+, m_-)}, \quad s = 1, \ldots, \left\lfloor \frac{n-1}{c_1^{12}} \right\rfloor \right\}.$$

Remark 20 ([2]). *If $m_2 + m_1 = n$, then $c_1^1 = 2$. So, CEPs exist for all $n \geqslant 3$.*

If $q \notin \mathbb{N}^h$ ($Z^h(q) \neq 0$) and q satisfies equation $P_\xi = 0$ then equality (1.44) is not valid for all γ and such point q is a Pole Point (PP).

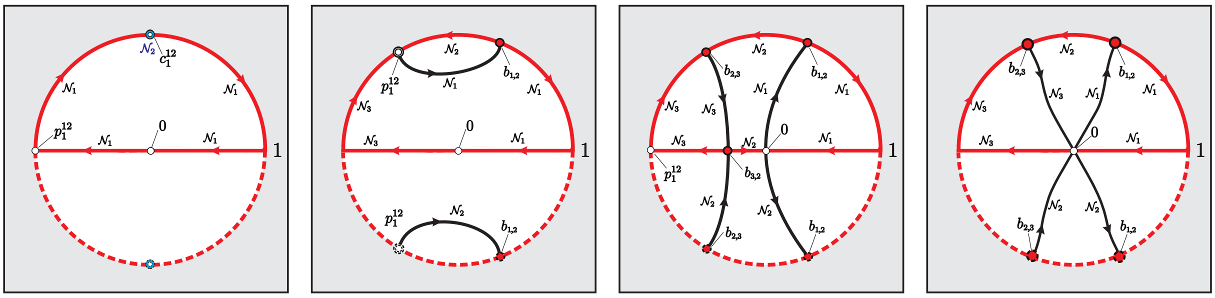
Lemma 21 ([2]). *A set $\mathcal{P}_\xi^P = \mathcal{P}_\xi^1 + \mathcal{P}_\xi^2 + \mathcal{P}_\xi^{12}$ describes all Pole Points of Complex CF, where $\mathcal{P}_\xi^1 := \mathcal{Z}_\xi^1 \setminus \mathcal{Z} = \mathcal{Z}_\xi^1 \setminus \mathcal{C}_\xi^1$, $\mathcal{P}_\xi^2 := \mathcal{Z}_\xi^2 \setminus \mathcal{Z} = \mathcal{Z}_\xi^2 \setminus \mathcal{C}_\xi^2$, $\mathcal{P}_\xi^{12} := \mathcal{Z}_\xi^{12} \setminus \mathcal{Z} = \mathcal{Z}_\xi^{12} \setminus \mathcal{C}_\xi^{12}$.*

Remark 22 ([2]). *If $q \in \mathcal{C}_\xi^{12}$, then we have PP of Complex CF and CEP at this point. If $q \in \mathcal{P}_\xi^{12}$, then we have PP of the second order.*

Remark 23 ([2]). *If the point $q \in \mathcal{C}_\xi^1$ or $q \in \mathcal{C}_\xi^2$, then it is Removable Singularity Point of Complex CF.*

Complex Characteristic Function. Let us consider the meromorphic function $\gamma_c = \gamma(q)$ on $\mathbb{C}_q^h$ as Complex CF and donate this function as $\gamma(q)$, $q \in \mathbb{C}_q^h$:

$$\gamma(q) = \frac{Z^h(q)}{P_\xi(q)} = \frac{\sin(\pi q)}{\sin(\pi q(\xi_2 + \xi_1/2) \sin(\pi q(\xi_2 - \xi_1)/2)} \cdot \frac{\tan(\pi q h/2)}{h}. \tag{1.48}$$



(a) $\boldsymbol{m} = (0{,}4)$ (b) $\boldsymbol{m} = (0{,}3)$ (c) $\boldsymbol{m} = (0{,}2)$ (d) $\boldsymbol{m} = (0{,}1)$

Figure 8. Bijective mappings in domain $\mathbb{C}_w^h$ for different $\boldsymbol{m}$ values ($n = 4$) [20]

The point at infinity $q = \infty \notin \mathbb{C}_q^h$. We rewrite equality (1.48) in the following form:

$$\gamma(q) = \frac{e^{\imath\pi q} - e^{-\imath\pi q}}{e^{\imath\pi q\xi_1} + e^{-\imath\pi q\xi_1} - e^{\imath\pi q\xi_2} - e^{-\imath\pi q\xi_2}} \cdot \frac{1 - e^{\imath\pi qh}}{1 + e^{\imath\pi qh}} \cdot \frac{2}{h}. \tag{1.49}$$

Now, similarly as with two-point NBCs, it is more convenient to investigate Complex CF in the neighbourhood $w = 0$ ($w = e^{\imath\pi qh}$):

$$\begin{aligned} \gamma(w) &= \frac{w^n - w^{-n}}{w^{m_1} + w^{-m_2} - w^{m_2} - w^{-m_2}} \cdot \frac{1 - w}{1 + w} \cdot \frac{2}{h} \\ &= \frac{w^{-n}}{w^{-m_2}} \cdot \frac{\left(1 - w^{2n}\right)(1 - w)}{\left(1 - w^{m_2+m_1} - w^{m_2-m_1} + w^{2m_2}\right)(1 + w)} \cdot \frac{2}{h} \\ &= \frac{1}{w^{n-m_2}} \cdot \frac{2}{h} \cdot (1 + \mathcal{O}(w)), \quad w = e^{\imath\pi qh}. \end{aligned} \tag{1.50}$$

Function $1/(w^{n-m_2})$ conclude the behaviour of Complex CF at the point $w = 0 \notin \mathbb{C}_w^h$. The point $w = 0 \in \mathbb{C}_w^h$ is an isolated singularity point. Point $w = 0$ ins called a *removable singularity* point if $m_2 = n$ ($\lim_{w\to 0}\gamma(w) = 2/h$). In the case of $w = 0$ being a removable singularity point, the same Spectrum Curve enters and leaves this point.

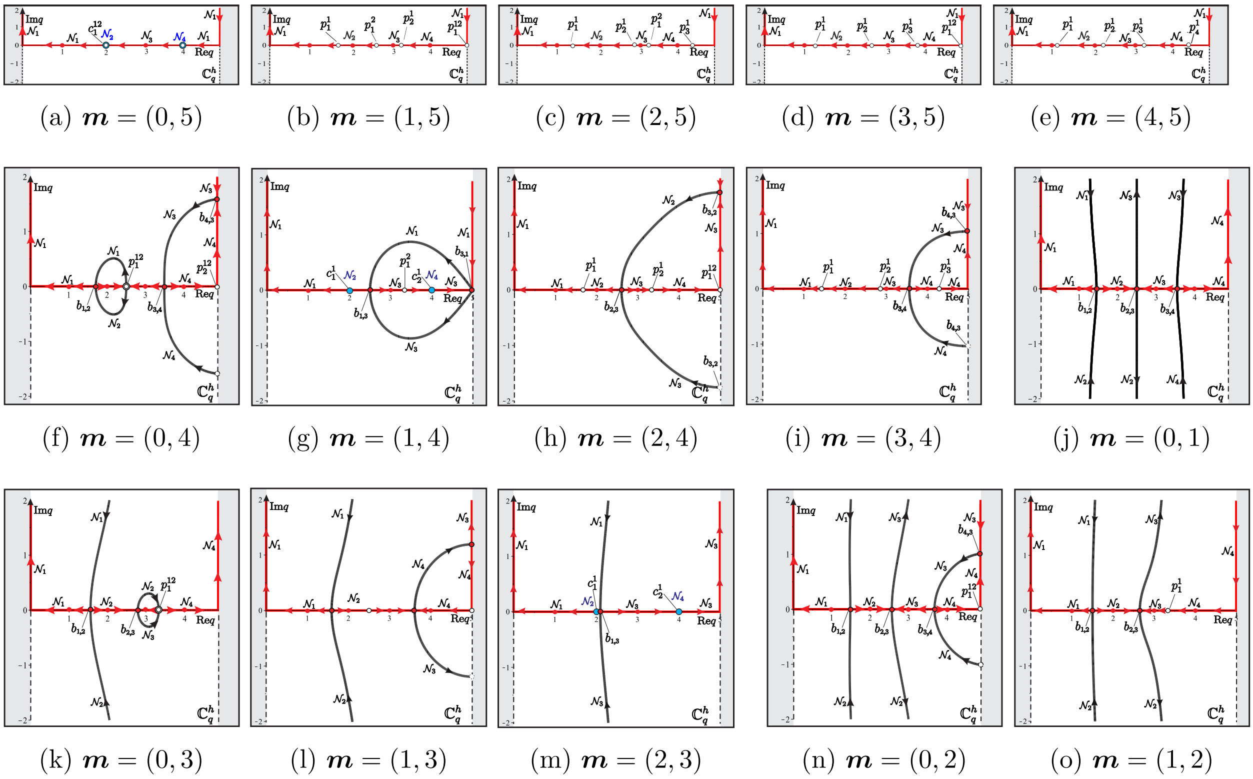
Point $w = 0$ is a Pole Point if $m_2 \leqslant n - 1$. Furthermore, the difference $n - m_2$ shows the order of the pole. If $n - m_2 = 1$, two real Spectrum Curves enter or leave the point $w = 0$. On the right hand side of the first order pole point $w = 0$ one spectrum curve ($\mathcal{N}_1 \subset (0{,}1)$) enters and on the left hand side of this point one spectrum curve ($\mathcal{N}_{n-1} \subset (-1{,}0)$) leaves this point. In case of $n - m_2 \geqslant 2$, there exist additional Spectrum Curves that enter and leave the point $w = 0$ [20].

Thus, the point $q = \infty \in \overline{\mathbb{C}_q^h}$ could be either a pole or removable singularity point [20] (see Figure 8).

Ramification Points. Similarly as with two-point BC, Ramification Points exist to integral boundary condition as well [2].

$q = 0$: Taylor series for CF at the point $q = 0$ is

$$\gamma(q) = \frac{2n^2}{m_2^2 - m_1^2} + \frac{\pi^2}{(m_2^2 - m_1^2)} \left(1 - 2n^2 + m_2^2 + m_1^2\right) q^2 + \mathcal{O}\left(q^4\right). \tag{1.51}$$



(a) $\boldsymbol{m} = (0, 5)$ (b) $\boldsymbol{m} = (1, 5)$ (c) $\boldsymbol{m} = (2, 5)$ (d) $\boldsymbol{m} = (3, 5)$ (e) $\boldsymbol{m} = (4, 5)$

(f) $\boldsymbol{m} = (0, 4)$ (g) $\boldsymbol{m} = (1, 4)$ (h) $\boldsymbol{m} = (2, 4)$ (i) $\boldsymbol{m} = (3, 4)$ (j) $\boldsymbol{m} = (0, 1)$

(k) $\boldsymbol{m} = (0, 3)$ (l) $\boldsymbol{m} = (1, 3)$ (m) $\boldsymbol{m} = (2, 3)$ (n) $\boldsymbol{m} = (0, 2)$ (o) $\boldsymbol{m} = (1, 2)$

Figure 9. Spectrum Curves for $n = 5$ (integral and Dirichlet BCs) [20]

Diophantine equation $1 - 2n^2 + m_2^2 + m_1^2 = 0$ does not have solutions such that $0 \leqslant m_1 < m_2 \leqslant n$, so there is no CPs in $q = 0$.

$\boldsymbol{q = n}$**:** If $m_- = m_2 - m_1 \in \mathbb{N}_o$, Taylor series for $\gamma(q)$ at RP $q = n$ is

$$\begin{aligned}\gamma(q) =& 2(-1)^{n-m_2} n^2 \\ &+ \frac{\pi^2 (-1)^{n-m_2}}{2} \left(\left(m_2^2 + m_1^2 \right) - \frac{2n^2+1}{3} \right) (q-n)^2 \\ &+ \frac{\pi^4 (-1)^{n-m_2}}{8n^2} \left(\frac{6n^4 + 10n^2 - 1}{45} - \frac{m_2^4 + m_1^4}{3} \right. \\ &+ \left. \left(\left(m_2^2 + m_1^2 \right) - \frac{2n^2+1}{3} \right) \left(m_2^2 + m_1^2 \right) \right) (q-n)^4 \\ &+ \mathcal{O}\left((q-n)^6 \right). \end{aligned} \tag{1.52}$$

If conditions

$$m_1^2 + m_2^2 = \frac{2n^2+1}{3}, \tag{1.53_1}$$

$$m_1^4 + m_2^4 = \frac{6n^4 + 10n^2 - 1}{15} \tag{1.53_2}$$

are valid, then the second term and the third term in (1.52) vanishes. It was proved by K. Bingelė, A. Bankauskienė and A. Štikonas [2] that the system (1.53) has no solution and both terms (the second and the third) in (1.52) can not vanish simultaneously for all values n,m_1,m_2. On the other hand, Diophantine equation (1.53_1) has solutions for $m_- = m_2 - m_1 \in \mathbb{N}_o$, so in this case $\gamma''(n) = 0$ In such cases, this point has properties of the second order CP in the domain $\mathbb{C}_q^h$ and the first order CP in the domain $\mathbb{C}_\lambda$. Thus, $q = n$ is the first order CP and the second term vanishes if the following equality holds [20]:

$$m_2^4 + m_1^4 = \frac{6n^4 + 10n^2 - 1}{15} + \left(3\left(m_2^2 + m_1^2\right) - 2n^2 - 1\right)\left(m_2^2 + m_1^2\right). \tag{1.54}$$

However, there is no such solutions to this equation and thus in the domain $\mathbb{C}_q^h$ CP at $q = n$ can be the first or the second order only. If (1.53_1) is not valid, then the point $q = n$ is a pole or the first order BP.

$\boldsymbol{q = 0:}$ In the case $m_- = m_2 - m_1 \in \mathbb{N}_e$, Taylor series for $\gamma(q)$ at BP $q = n$ is

$$\gamma(q) = \frac{8(-1)^{n-m_2+1}n^4}{\pi^2\left(m_2^2 - m_1^2\right)} \cdot \frac{1}{(q-n)^2} + \mathcal{O}(1). \tag{1.55}$$

Remark 24 ([2])**.** *If at RP $q = n$, we have pole, then this pole is of the second order. Indeed, if $p_l^2 = n$, then $m_2 - m_1 = 2l \in \mathbb{N}_e$ and $m_2 + m_1 = 2s \in \mathbb{N}_e$. So, $p_s^1 = n$. But at $\lambda = 4n^2 \in \mathbb{C}_\lambda$, we have pole of the first order.*

So, in case $m_- = m_2 - m_1 \in \mathbb{N}_e$ there will not be CPs in $q = n$, On the other hand, in this case there will always be a PP at $q = n$.

1.3. Spectrum Curves

For fixed parameter ξ in NBC Spectrum points define the *Spectrum Curves* [1, 2]. Each Spectrum Curve is the trajectory of a Spectrum point in domain $\mathbb{C}_q$. Spectrum Domain is the set $\mathcal{N}_{\boldsymbol{\xi}} = \mathcal{D}_{\boldsymbol{\xi}} \cup \mathcal{C}_{\boldsymbol{\xi}}$, where $\mathcal{D}_{\boldsymbol{\xi}}$ is the domain of CF $\{q \in \mathbb{C}_q : \operatorname{Im} \gamma_c(q) = 0\} \subset \mathbb{C}_q$, and the set $\mathcal{C}_{\boldsymbol{\xi}}$ describes CEPs. Function γ_c has real values in the set $\mathcal{D}_{\boldsymbol{\xi}}$ except PPs. $\mathcal{E}_{\boldsymbol{\xi}}(\gamma_0) := \gamma^{-1}(\gamma_0)$ is the set of all nonconstant eigenvalue points for $\gamma_0 \in \mathbb{R}$. Thus, $\mathcal{D}_{\boldsymbol{\xi}} = \bigcup_{\gamma \in \mathbb{R}} \mathcal{E}_{\boldsymbol{\xi}}(\gamma)$. If $q \in \mathcal{D}_{\boldsymbol{\xi}}$ and $\gamma_c'(q) \neq 0$ (q is not a critical point of CF), then $\mathcal{E}_{\boldsymbol{\xi}}(\gamma)$ is a smooth parametric curve $\mathcal{N} : \mathbb{R} \to \mathbb{C}_q^h$ locally and we can add arrow on this curve (arrows show the direction in which $\gamma \in \mathbb{R}$ is increasing). Such curves are called *regular Spectrum Curves* and they can be enumerated by classical case ($\gamma = 0$) depending on the BCs:

1. Dirichlet and two-point BCs: if $z_k = k \in \mathbb{N}$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k = k \in \mathbb{N}$.

2. Neumann and two-point BCs: if $z_k = (k - 1/2)/(1 - h/2)$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k$.

3. Dirichlet and integral BCs: if $z_k = k \in \mathbb{N}$ belongs to Spectrum Curve, then the index of this Spectrum Curve is k. So, $\mathcal{N}_k(0) = z_k = k \in \mathbb{N}$.

Moreover, two or more Spectrum Curves can intersect at the Critical Point because of the order of PP or CP. For example, if the order of a CP is 2, there will be two Spectrum Curves crossing this point and intersecting. At this point, Spectrum Curves change direction with the angle of size $\pi/(k+1)$ for the Critical Point of the kth order. The "right-hand rule" was used to determine this angle [1]. This means that in the intersection Spectrum Curve turns to the right.

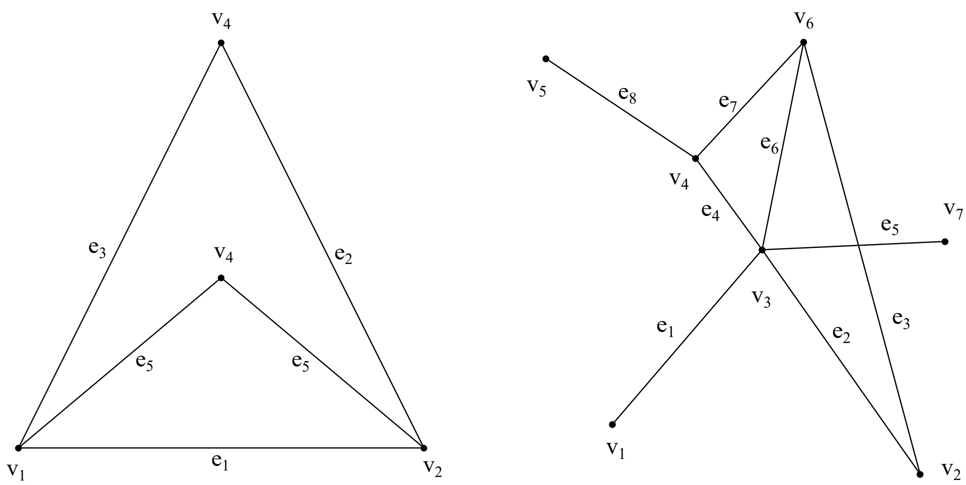
For the $\gamma \to \pm\infty$, Spectrum Curve $\mathcal{N}_k(\gamma)$ approaches a PP or the point ∞. The index of a CP is formed of indices of Spectrum Curves which intersect at the CP. If the CP is of the first order and real, then the left index coincides with the index of Spectrum Curve which is defined by the smaller real λ values, and the right index is defined by greater λ values.

For every CEP $c_j = j$ the *nonregular Spectrum Curve* $\mathcal{N}_j = \{c_j\}$ is defined. Nonregular Spectrum Curves can overlap with a point of a regular Spectrum Curve. Thus, $\mathcal{N}_\xi$ is a finite union of Spectrum Curves $\mathcal{N}_l$ where $l = \overline{1, n-1}$. So, the domain of CF could be expressed as $\mathcal{D}_\xi = \bigcup_{l \in \mathcal{Z}_\xi}$ for all ξ.

2. Graphs. Euler's characteristic. Digraphs

2.1. Main definitions

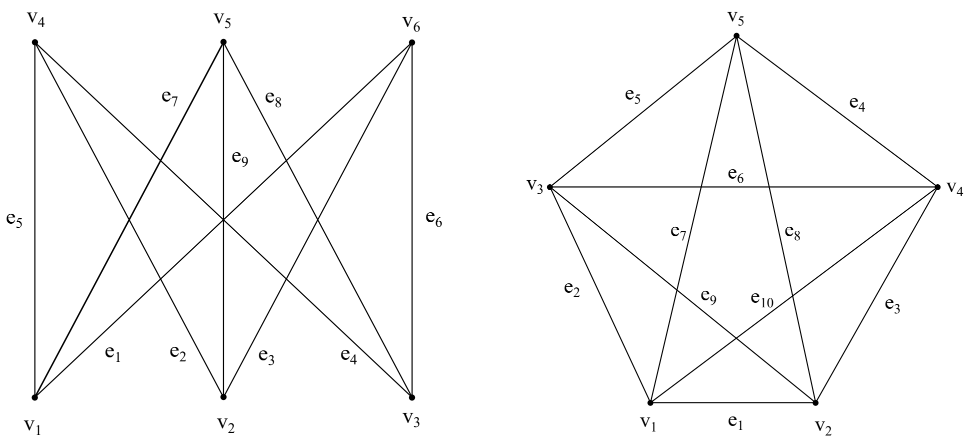
A graph is a pair of sets $G = (V,E)$ that consists of a nonempty set of vertices (nodes or points), $V = \{v_i, i = \overline{1,I}, I \in \mathbb{N}\}$, and a set of edges, $E = \{e_i, i = \overline{0,J}, J \in \mathbb{N}\}$. We say that $e_j := (v_{i_1}, v_{i_2}) =: v_{i_1}v_{i_2} = v_{i_2}v_{i_1} \in E$, $v_{i_1}, v_{i_2} \in V$, and v_{i_1} (or v_{i_2}) is the end of an edge e_j. The powers of sets V and E are $|V| = v$ and $|E| = e$.



(a) Planar graph K_4 (b) Planar graph G

Figure 10. Examples of planar graphs

To depict graphs in the plane, Jordan curves come in handy. A Jordan curve is a plane curve which is topologically equivalent to the unit circle, i.e., it is simple and closed. This means that Jordan curves neither touch each other nor intersect. A graph with Jordan curves is called *planar* (see Figure 10). A graph is said to be *nonplanar* if it cannot be drawn in a plane so that no edge cross. (see Figure 11)



(a) Nonplanar graph $K_{3,3}$ (b) Nonplanar graph K_5

Figure 11. Examples of nonplanar graphs

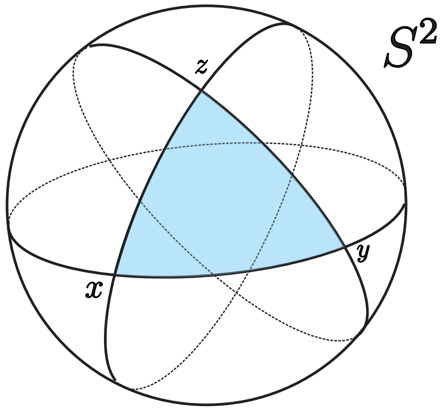


Figure 12. Triangulation of a sphere

The *faces* of a planar graph are the areas which are surrounded by edges. We denote f the number of such faces. A *tree* is an undirected, connected and acyclic graph while a *forest* is an undirected, disconnected, acyclic graph.

Theorem 25 ([17])**.** *A connected graph $G = (V,E)$ is a tree if and only if*

$$|E| = |V| - 1. \tag{2.1}$$

2.2. Euler's characteristics

One of the main theorems in graph theory is Euler's polyhedron formula.

Theorem 26 ([17])**.** *If G is a connected graph, n – number of vertices, m – number of edges l – number of faces, then*

$$n - m + l = 2. \tag{2.2}$$

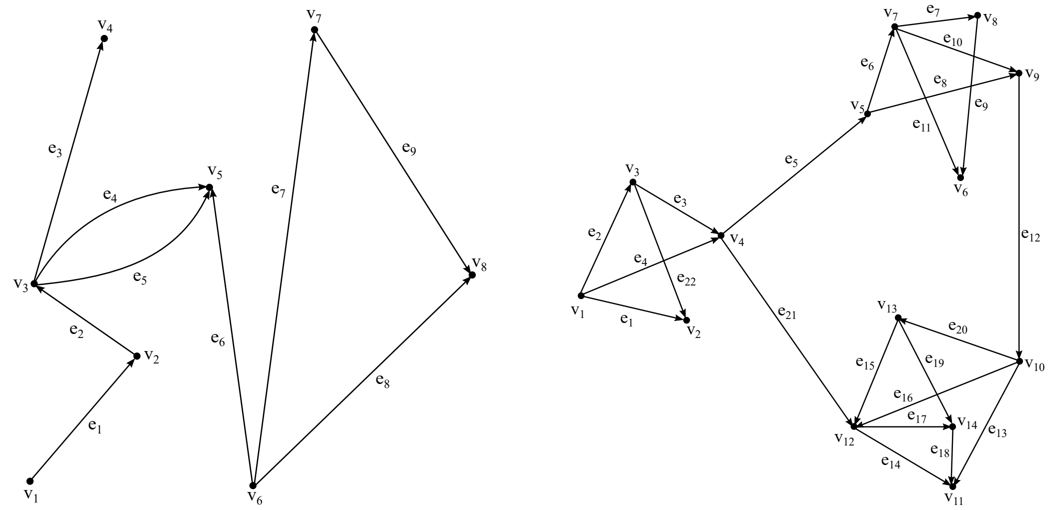
This theorem is one of many cases of Euler characteristic. *Euler characteristic* is a topological invariant, a number that describes a topological space's shape or structure regardless of the way it is bent. The common notation for Euler characteristic is donated by χ.

In other words, the *Euler characteristic* χ of a subdivision of a surface is $\chi = v - e + f$. Since spectrum curves are on Riemann's sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$, we are interested in Euler's characteristic of a sphere S^2. *Euler's characteristics of a plane* [17] *of and a sphere* [10] *are* $\chi = 2$. The proof for Euler's characteristics of a sphere is derived using the triangulation techniques (see Figure 12). A *triangulation* is a subdivision of a planar object into triangles, and by extension the subdivision of a higher-dimension geometric object into simplices.

2.3. Digraphs

In graph theory, a *directed graph* or *digraph* is a graph that is made up of a set of points connected by arrows (edges with direction). We say that an arc $(v_i,v_j) \in G$ is drawn as an arrow from v_i to v_j. If a graph contains both arcs (v_i,v_j) and (v_j,v_i), these two arcs are distinct and thus the graph is called *not simple*. Other way around, *simple digraphs* have no loops and no multiple arrows with same source and target points (see Figure 13).

Let $G = (V, A)$ and $v \in V$. We donate by E_v^- the set of all arcs of the form $(v_j,v_i) \in G$, and by E_v^+ the set of all arcs of the form $(v_i,v_j) \in G$. The *indegree* of v, denoted $\deg^-(v)$, is the number of arcs in E_v^- and the *outdegree*, $\deg^+(v)$, is the number of arcs in E_v^+. If the vertices are $V = \{v_i,i = \overline{1,n}, n \in \mathbb{N}\}$, the degrees are donated $\deg^-(v_1)$, $\deg^-(v_2)$, ..., $\deg^-(v_n)$ and $\deg^+(v_1)$, $\deg^+(v_2)$, ..., $\deg^+(v_n)$.



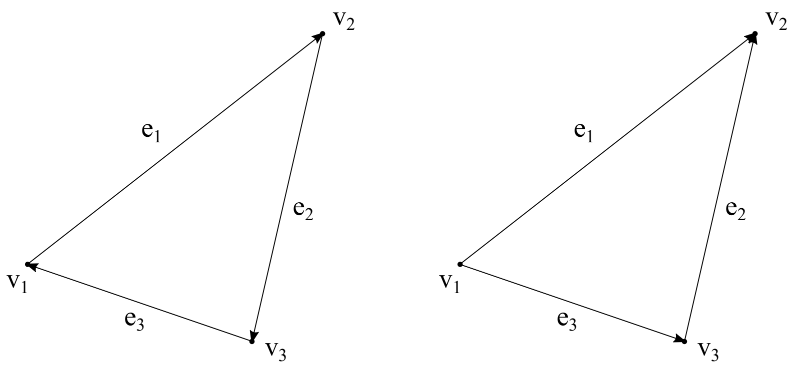
(a) Directed graph G (b) Directed graph G

Figure 13. Examples of directed graphs

The degree sum formula states that, for a digraph,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |A| = e. \tag{2.3}$$

If for every vertex $v \in V$, $\deg^+(v) = \deg^-(v)$, the graph is called a *balanced directed graph* (see Figure 14). The ordered pair is called weakly connected if an undirected path leads from v_1 to v_2 after replacing all of its directed arrows with undirected edges.



(a) Balanced digraph K_3 (b) Unbalanced digraph K_3

Figure 14. Examples balanced and unbalanced graphs

A *cycle digraph* is a digraph that consists of a single cycle (e.g. Figures 14a, 14b).

3. Relations between dSLP and graphs properties

This section is the main results of the master thesis. It is possible to define relations between properties of dSLP and graph theory. Poles or CPs refer to vertices of a certain graph and parts of Spectrum Curves could be interpreted as edges. In our case, we have a simple balanced weakly connected digraph [26, 27].

The results of this research (2 articles) were published in the journal "Proc. of the Lithuanian Mathematical Society" and presented in 3 conferences while studying master's degree (see *Publications and conferences* on page 7).

3.1. Properties of Spectrum Curves

The number of PPs, CPs, regular and nonregular Spectrum Curves, CEPs were found by both K. Bingelė [1] and A. Skučaitė [20].

There is $n-1$ Spectrum Curves for every $n \in \mathbb{N}$, $n \geqslant 2$. Nonregular Spectrum Curves are CEPs and belong to $\mathbb{R}^h_x = (0,n)$. For two-point NBCs the number of such Spectrum Curves is equal to

$$n_{ce} = K \text{ for } N \in \mathbb{N}_e, \qquad n_{ce} = 0 \text{ for } N \in \mathbb{N}_o, \tag{3.1$_a$}$$

$$n_{ce} = K - 1, \tag{3.1$_b$}$$

$$n_{ce} = 0, \tag{3.1$_c$}$$

$$n_{ce} = (K-1)/2, \tag{3.1$_d$}$$

here $\xi = m/n = M/N$, $K = \gcd(n,m)$, $\gcd(N,M) = 1$. For integral NBC the number of nonregular Spectrum Curves is

$$\begin{aligned} n_{ce} = &\left\lfloor \frac{n-1}{2n} \gcd(2n, m_1 + m_2) \right\rfloor + \left\lfloor \frac{n-1}{2n} \gcd(2n, m_2 - m_1) \right\rfloor \\ &- \left\lfloor \frac{n-1}{2n} \gcd(2n, m_1 + m_2, m_2 - m_1) \right\rfloor . \end{aligned} \tag{3.2}$$

In the classical case ($\gamma = 0$) we have $n-1$ eigenvalues. Thus, the number of regular Spectrum Curves is $n_{nce} = n - 1 - n_{ce}$.

Each regular Spectrum Curve begins at the PP ($\gamma = -\infty$) of CF and ends at the PP ($\gamma = +\infty$) of CF. Let n_{1p} be the number of PPs of te first order and n_{2p} – PPs of the second order. Then we have $n_p = n_{1p} + 2n_{2p}$. Then we have that the number of PPs of the first and the second order is $n_{1p} + n_{2p} = n_p - n_{2p}$. So, the set of PPs $\mathcal{P} := \{p_i, i = \overline{1, n_p - n_{2p}}\}$. For two-point NBCs $\mathcal{P} \subset \mathbb{R}^h_x \cup \{0\}$ and all PPs are of the first order (we write $\deg^+(p) = 1$, $p \in \mathcal{P}$, $n_{2p} = 0$). For integral NBC there could also be PPs of second order ($\deg^+(p) = 2$, $p \in \mathcal{P}$) in $\mathbb{R}^h_x$.

We donate $\varkappa = 1$ for the first two-point NBC (1.7$_1$) and $\varkappa = 0$ for the second two-point NBC (1.7$_2$). Then, for two-point NBCs, the PP at $q = \infty$ is of

$$n_\infty = n - m - \varkappa \tag{3.3}$$

order (see lemmas 8, 9, 14 and 15). For (1.7_1) NBC ($\varkappa = 1$) in the case $n = m+1$ the point $q = \infty$ is CP of the first order. So, $n_\infty^+ = \deg^+(\infty) = n_\infty + 2\varkappa\lfloor(m+1)/n\rfloor$. For integral NBC, the PP at $q = \infty$ is of

$$n_\infty = n - m_2 \tag{3.4}$$

order.

The poles of CF belong to $\mathbb{R}_x^h \cup \{0\} \cup \{n\} \cup \{\infty\}$ and $n_p + n_\infty = n_{nce}$. So, we have formula

$$n_p + n_{ce} = m - 1 + \varkappa \tag{3.5}$$

for two-point BCs and

$$n_p + n_{ce} = m_2 - 1 \tag{3.6}$$

for integral BC.

Remark 27. *Note that n_p describes the number of PPs including the order. For two-point NBCs all PPs are of the first order while integral NBC can have second order poles included. So, for two-point NBCs n_p is the number of PPs in $\mathbb{C}_q^h$.*

Two or more Spectrum Curves may intersect at CPs. We denote a set of CPs $\mathcal{B} := \{b_i, i = \overline{1, n_b}\}$, where n_b is the number of CPs in $\mathbb{C}_q^h$. The number of CPs at $\mathbb{R}_q^h$ and $\mathbb{C}_q^{h+}$ we denote as n_{cr} and n_{cr}^+, respectively. Note that the part of the spectrum domain in set $\mathbb{C}_q^{h+}$ is symmetric to the part in set $\mathbb{C}_q^{h-}$. So,

$$n_b = n_{cr} + 2n_{cr}^+. \tag{3.7}$$

If $b \in \mathcal{B}$ then $\deg^+(b)$ is one unit larger than the order of this CP.

Remark 28. *There is a not yet proved hypothesis that there exist CPs of dSLP with NBCs in $\mathbb{C}_q^{h+}$ (by symmetry in $\mathbb{C}_q^{h-}$, too). In fact, there was not yet found a rational ξ for two-point NBCs or ξ_1, ξ_2 for integral NBC such that a CP exists in $\mathbb{C}_q^{h\pm}$. Since in our theory it was not yet proved nor disproved, we consider to take a general case that such points exist. This problem was investigated in S. Pečiulytė doctoral dissertation [18].*

Remark 29. *In the figures presented in the first section, $n_{cr}^+ = 0$ for all BCs.*

Let us denote

$$\deg_r^+ := \sum_{b \in \mathcal{B} \cap \mathbb{R}_q^h} \deg^+(b), \quad \deg_c^+ := \sum_{b \in \mathcal{B} \cap \mathbb{C}_q^{h\pm}} \deg^+(b) = 2 \sum_{b \in \mathcal{B} \cap \mathbb{C}_q^{h+}} \deg^+(b).$$

Let n_c be the number of Spectrum Curves parts in $\mathbb{C}_q^{h+} \cup \{\infty\}$ between two CPs (including $q = \infty$ for $n = m+1$ for the first two-point BC, see Figure 4d) [26, 27].

3.2. Spectrum domain as a graph. Two-point BCs

We consider Spectrum domain as graph on sphere (Riemann sphere $\overline{\mathbb{C}}$) because $\mathbb{C}_q^h \cup \{\infty\} \sim S^2$. The poles and CPs of the CF are the vertices of this graph. The point ∞ is the PP or CP.

Lemma 30 ([26]). *The number of vertices is*

$$v = n_p + n_b + 1 = n_p + n_{cr} + 2n_{cr}^+ + 1. \tag{3.8}$$

Proof. For dSLP with two-point NBCs and either Dirichlet or Neumann BC the number of vertices consists of the total number of PPs and CPs in $\mathbb{C}_q^h$ and one point in infinity (either PP or CP). Thus, $v = n_p + n_b + 1$. Considering (3.7), $v = n_p + n_{cr} + 2n_{cr}^+ + 1$. □

Lemma 31 ([26]). *The number of edges is*

$$e = n_p + \deg_r^+ + \deg_c^+ + n_\infty + 2\varkappa\lfloor (m+1)/n \rfloor. \tag{3.9}$$

Proof. For dSLP with two-point NBCs and either Dirichlet or Neumann BC the number of edges consists of the total number of incoming and outgoing Spectrum Curves to PPs and CPs in $\mathbb{C}_q^h$ and one point in infinity (either PP or CP). This can be rewritten in the form of $e = \sum_{p\in\mathcal{P}} \deg^+(p) + \sum_{b\in\mathcal{B}} \deg^+(b) + \deg^+(\infty)$ (see (2.3)), where $\sum_{p\in\mathcal{P}} \deg^+(p)$ is the sum of degrees of outgoing Spectrum Curves of PPs, $\sum_{b\in\mathcal{B}} \deg^+(b)$ is the sum of degrees of outgoing Spectrum Curves of CPs and $\deg^+(\infty)$ is the degree of outgoing Spectrum Curves to the point at infinity.

Since the order of PPs in $\mathbb{R}_q^h$ is 1 for both two-point NBCs, $\sum_{p\in\mathcal{P}} \deg^+(p) = n_p$.

CPs, $b \in \mathbb{C}_q^h$, can be located on either $\mathbb{R}_q^h$ or $\mathbb{C}_q^{h\pm}$. The degree of outgoing Spectrum Curves of CPs on $\mathbb{R}_q^h$ is $\sum_{b\in\mathcal{B}\cap\mathbb{R}_q^h} \deg^+(b)$. Similarly, the degree of outgoing Spectrum Curves of CPs on $\mathbb{C}_q^{h\pm}$ is $\sum_{b\in\mathcal{B}\cap\mathbb{C}_q^{h\pm}} \deg^+(b)$. Moreover, CPs on the complex part of the map $\mathbb{C}_q^{h\pm}$ are all symmetrical to $\mathbb{R}_x^h$. So, we can express outgoing parts of Spectrum Curves to CPs as

$$\deg_r^+ := \sum_{b\in\mathcal{B}\cap\mathbb{R}_q^h} \deg^+(b), \quad \deg_c^+ := \sum_{b\in\mathcal{B}\cap\mathbb{C}_q^{h\pm}} \deg^+(b) = 2 \sum_{b\in\mathcal{B}\cap\mathbb{C}_q^{h+}} \deg^+(b).$$

Thus, $\sum_{b\in\mathcal{B}} \deg^+(b) = \deg_r^+ + \deg_c^+$.

The pole at $q = \infty$ is of $n_\infty = n - m - \varkappa$ order. For (1.7_1) BC ($\varkappa = 1$) in the case $n = m + 1$ the point $q = \infty$ is CP of the first order ($n_\infty = 0$). So, $\deg^+(\infty) = n_\infty^+ = n_\infty + 2\varkappa\lfloor (m+1)/n \rfloor$. □

Lemma 32 ([26]). *The number of faces is*

$$f = 2\left(n_\infty + \varkappa\lfloor (m+1)/n \rfloor - n_{cr}^+ + n_c\right). \tag{3.10}$$

Proof. For dSLP with two-point NBCs and either Dirichlet or Neumann BC the number of faces consists of the degree of the point at $q = \infty$ and critical points in $\mathbb{C}_q^h$. It is not hard to notice that the order of $q = \infty$, n_∞, shows the number of incoming and outgoing Spectrum Curves. Thus, for both two-point BCs with no CPs on $\mathbb{R}_q^h$ we express the number of faces by $2(n_\infty + \varkappa\lfloor (m+1)/n \rfloor)$. Moreover, if there is any CPs on $\mathbb{C}_q^h$, the number of faces will be increased with the relation of $2\,(n_c - n_{cr}^+)$ where n_c is the number of Spectrum Curves parts in $\mathbb{C}_q^{h+}$ between two CPs and n_{cr}^+ is the number of CPs in $\mathbb{C}_q^{h+}$. Combining both relations we get that $f = 2\,(n_\infty + \varkappa\lfloor (m+1)/n \rfloor - n_{cr}^+ + n_c)$. □

This lemma is valid for $n_c = n_{cr}^+ = 0$. Each part of spectrum curve between two CPs b_1, $b_2 \in \mathbb{R}_q^h$ increases the number of faces by one. So, this formula is valid for the case $n_{cr}^+ = 0$. Each additional CP $b \in \mathbb{C}_q^{h+}$ increases the number of faces by $2(\deg^+(b) - 1)$ and number parts of Spectrum Curves between this CP and other CPs by $2\deg^+(b)$.

Numbers of spectrum vertices, edges and faces, expressed by the formulas above, inserted to the Euler's characteristic's formula of sphere $v - e + f = 2$ (sphere's Euler characteristic $\lambda = 2$) give new relation.

Theorem 33 ([26]). *The Euler's characteristic's formula is equivalent to*

$$\sum_{b\in\mathcal{B}} \deg^+(b) = \deg_r^+ + \deg_c^+ = n_\infty + 2n_c + n_{cr} - 1. \tag{3.11}$$

Proof. By applying (3.8), (3.9) and (3.10) to Euler characteristic equation for sphere

$$v - e + f = 2$$

we get

$$\begin{aligned} & n_p + n_{cr} + 2n_{cr}^+ + 1 \\ & -\left(n_p + \deg_r^+ + \deg_c^+ + n_\infty + 2\varkappa\lfloor (m+1)/n \rfloor\right) \\ & +2\left(n_\infty + \varkappa\lfloor (m+1)/n \rfloor - n_{cr}^+ + n_c\right) = 2. \end{aligned}$$

By simplifying the equation we get the desired result. □

Remark 34. *Formula* (3.11) *was formulates as hypothesis in [1] in the case with no CPs in* $\mathbb{C}_q^{h\pm}$ *(*$\deg_c^+ = 0$*), all CPs are of the first order (*$\deg_r^+ = 2n_{cr}$*) and* $n_c = 0$ *in the case a), c). Then it can be rewritten as*

$$n_{cr} = 2n_c + n_\infty - 1 = 2n_c + n - m - \varkappa - 1.$$

Corollary 35 ([26]). *The number of edges is*

$$e = 2n_\infty + n_p + n_{cr} + 2n_c + 2\varkappa\lfloor (m+1)/n \rfloor - 1. \tag{3.12}$$

Formula (3.12) is derived from (3.9) and (3.11).

Remark 36 ([26]). *In the case $m+1<n$ the formulas (3.8), (3.10)–(3.12) are*

$$\begin{aligned} v &= m + n_{cr} + 2n_{cr}^{+} - n_{ce} + \varkappa, \\ e &= 2n - m - 2 + n_{cr} + 2n_c - n_{ce} - \varkappa, \\ f &= 2\left(n_{\infty} - n_{cr}^{+} + n_c\right), \\ \deg_r^{+} + \deg_c^{+} &= n - m + 2n_c + n_{cr} - \varkappa - 1, \end{aligned}$$

where n_{ce} is defined by (3.1).

In the case $m+1=n$ we have $n_{\infty}=1-\varkappa$ and $n_p=n-2+\delta$, where $\delta=1$ for the case c) and for $n\in\mathbb{N}_o$ in the case a), else $\delta=0$. So, for $m+1=n$ the following formulas

$$\begin{aligned} v &= n + n_{cr} + 2n_{cr}^{+} - 1 + \delta, \\ e &= n + n_{cr} + 2n_c - 1 + \delta, \\ f &= 2\left(1 - n_{cr}^{+} + n_c\right), \\ \deg_r^{+} + \deg_c^{+} &= 2n_c + n_{cr} - \varkappa \end{aligned}$$

are valid.

Remark 37 ([26]). *If $n_{cr}^{+}=0$ ($\deg_c^{+}=0$) then $\deg_r^{+}-n_{cr}=n_c+n-m-\varkappa>0$ shows that there exist CPs in $\mathbb{R}_q^h$ of the second or the higher order.*

Corollary 38. *If $\deg_r^{+}=n_{\infty}+2n_c+n_{cr}-1$, then there is no CPs in $\mathbb{C}_q^{h+}$, i.e. $n_{cr}^{+}=0$.*

Example 1. Let's take an example of dSLP with Dirichlet and first two-point ($\varkappa=1$) BCs and $\xi=8/12$ ($m=8$ and $n=12$)(see Figure 2f). The number of poles is defined by formulas (3.1_a) and (3.5), so we have

$$n_p = m - 1 + \varkappa - n_{ce} = 8 - 1 + 1 - 0 = 8.$$

From Figure 2f we have

$$n_c = 2, \qquad n_{cr} = 6, \qquad \deg_r^{+} = 12.$$

By Lemma 8, the point at infinity $p=\infty$ is a PP of order $n_{\infty}=n-m-1=12-8-1=3$. By Corollary 38 we have

$$12 = \deg_r^{+} = 3 + 2\cdot 2 + 6 - 1 = 12,$$

so, $\deg_c^{+}=0$ meaning $n_{cr}^{+}=0$. This means that there is no CPs in $\mathbb{C}_q^{h+}$, all CPs lie in $\mathbb{R}_q^h$.

Then we can derive vertices (v), edges (e) and faces (f):

$$\begin{aligned} v &= m + n_{cr} - n_{ce} + \varkappa = 8 + 6 + 1 = 15, \\ e &= 2n - m - 2 + n_{cr} + 2n_c - n_{ce} - \varkappa = 2\cdot 12 - 8 - 2 + 6 + 2\cdot 2 - 0 - 1 = 23, \\ f &= 2\left(n_{\infty} + n_c\right) + n_c = 2\left(3 + 2\right) = 10. \end{aligned}$$

Example 2. Let's take another example of dSLP with Neumann and second two-point ($\varkappa = 0$) BCs and $\xi = 3/4$ ($m = 3$ and $n = 4$)(see Figure 6a). The number of poles is

$$n_p = m - 2 + \delta = 4 - 2 + 0 = 2.$$

From Figure 6a we have

$$n_c = 0, \qquad n_{cr} = 0.$$

Since $n_{cr} = 0$, we have that $\deg_r^+ = 0$, too. So, by Corollary 38 we have $\deg_c^+ = 0$. Thus, there is no CPs in $\mathbb{C}_q^h$ at all.

The point at infinity $p = \infty$ is a PP of order $n_\infty = 1 - \varkappa = 1 - 0 = 1$. Now we can derive vertices (v), edges (e) and faces (f):

$$\begin{aligned} v &= n - 1 + \delta = 4 - 1 + 0 = 3, \\ e &= n - 1 + \delta = 4 - 1 + 0 = 3, \\ f &= 2. \end{aligned}$$

In this case we have a cycle digraph.

3.3. Spectrum domain as a graph. Integral BC

For integral BC, the point $q = \infty$ is a PP or a removable singularity point.

Lemma 39 ([27]). *The number of vertices is*

$$\begin{aligned} v &= n_p - n_{2p} + n_b + 1 - \lfloor m_2/n \rfloor \\ &= n_p - n_{2p} + n_{cr} + 2n_{cr}^+ + 1 - \lfloor m_2/n \rfloor \\ &= m_2 - n_{ce} - n_{2p} + n_{cr} + 2n_{cr}^+ - \lfloor m_2/n \rfloor. \end{aligned} \tag{3.13}$$

Proof. For dSLP with integral NBC and Dirichlet BC the number of vertices consists of the total number of PPs and CPs in $\mathbb{C}_q^h$ and one point in infinity (PP if $m_2 < n$, a Singularity point if $m_2 = n$). Thus, $v = n_p - n_{2p} + n_b + 1 - \lfloor m_2/n \rfloor$. Considering (3.7) and (3.6), $v = n_p - n_{2p} + n_{cr} + 2n_{cr}^+ + 1 - \lfloor m_2/n \rfloor = m_2 - n_{ce} - n_{2p} + n_{cr} + 2n_{cr}^+ - \lfloor m_2/n \rfloor$. □

Remark 40. *Note that in the proof of lemma 39 the total number of PPs in $\mathbb{C}_q^h$ is considered as $n_p - n_{2p}$ since n_p covers all orders of PPs (including second order poles) and n_{2p} gathers the number of second order PPs only.*

Lemma 41 ([27]). *The number of edges is*

$$e = n_p + \deg_r^+ + \deg_c^+ + n_\infty. \tag{3.14}$$

Proof. For dSLP with integral NBC and Dirichlet BC the number of edges consists of the total number of incoming and outgoing Spectrum Curves to PPs and CPs in $\mathbb{C}_q^h$ and one point in infinity (PP if $m_2 < n$, no point if $m_2 = n$). This can be rewritten in the form of $e = \sum_{p \in \mathcal{P}} \deg^+(p) + \sum_{b \in \mathcal{B}} \deg^+(b) + \deg^+(\infty)$ (see (2.3)), where $\sum_{p \in \mathcal{P}} \deg^+(p)$ is the degree of outgoing Spectrum Curves to the PPs, $\sum_{b \in \mathcal{B}} \deg^+(b)$ is the degree of outgoing Spectrum Curves to the CPs and $\deg^+(\infty)$ is the degree of outgoing Spectrum Curves to the point at infinity.

Since the order of PPs in $\mathbb{C}_q^h$ is either 1 or 2 for integral NBC, the number of outgoing Spectrum curves to PPs is $\sum_{p \in \mathcal{P}} \deg^+(p) = n_p$.

CPs, $b \in \mathbb{C}_q^h$, can be located on either $\mathbb{R}_q^h$ or $\mathbb{C}_q^{h\pm}$. The degree of outgoing Spectrum Curves of CPs on $\mathbb{R}_q^h$ is $\sum_{b \in \mathcal{B} \cap \mathbb{R}_q^h} \deg^+(b)$. Similarly, the degree of outgoing Spectrum Curves of CPs on $\mathbb{C}_q^{h\pm}$ is $\sum_{b \in \mathcal{B} \cap \mathbb{C}_q^{h\pm}} \deg^+(b)$. Moreover, CPs on the complex part of the map $\mathbb{C}_q^{h\pm}$ are all symmetrical to $\mathbb{R}_x^h$. So, we can express outgoing parts of Spectrum Curves to CPs as

$$\deg_r^+ := \sum_{b \in \mathcal{B} \cap \mathbb{R}_q^h} \deg^+(b), \quad \deg_c^+ := \sum_{b \in \mathcal{B} \cap \mathbb{C}_q^{h\pm}} \deg^+(b) = 2 \sum_{b \in \mathcal{B} \cap \mathbb{C}_q^{h+}} \deg^+(b).$$

Thus, $\sum_{b \in \mathcal{B}} \deg^+(b) = \deg_r^+ + \deg_c^+$.

The pole at $q = \infty$ is of $n_\infty = n - m_2$ order. In the case $n = m + 1$ there is a Singularity point at $q = \infty$ which is not counted in. So, $\deg^+(\infty) = n_\infty^+ = n_\infty$. □

Lemma 42 ([27]). *The number of faces is*

$$\begin{aligned} f &= 2\left(n_\infty + n_{2p} - n_{cr}^+ + n_c + \lfloor m_2/n \rfloor\right) \\ &= 2\left(n - m_2 + n_{2p} - n_{cr}^+ + n_c + \lfloor m_2/n \rfloor\right). \end{aligned} \tag{3.15}$$

Proof. For dSLP with integral NBC and Dirichlet BC the number of faces consists of the degree of the point at $q = \infty$, PPs of the second order and critical points in $\mathbb{C}_q^h$. It is not hard to notice that the order of $q = \infty$, n_∞, shows the number of incoming and outgoing Spectrum Curves. Thus, for integral BC with no CPs on $\mathbb{R}_q^h$ we express the number of faces by $2(n_\infty + \lfloor m_2/n \rfloor)$. Moreover, if there is any CPs or second order PPs on $\mathbb{C}_q^h$, the number of faces will be added with the relation of $2(n_{2p} - n_{cr}^+ + n_c)$ where n_c is the number of Spectrum Curves parts in $\mathbb{C}_q^{h+}$ between two CPs, n_{2p} is the number of PPs of the second order and n_{cr}^+ is the number of CPs on $\mathbb{C}_q^{h+}$. Combining both relations and considering (3.4) we get that $f = 2(n_\infty + n_{2p} - n_{cr}^+ + n_c + \lfloor m_2/n \rfloor) = 2(n - m_2 + n_{2p} - n_{cr}^+ + n_c + \lfloor m_2/n \rfloor)$. □

Theorem 43 ([27]). *The Euler's characteristic's formula is equivalent to*

$$\begin{aligned} \sum_{b \in \mathcal{B}} \deg^+(b) = \deg_r^+ + \deg_c^+ &= n_\infty + n_{2p} + 2n_c + n_{cr} - 1 + \lfloor m_2/n \rfloor \\ &= n - m_2 + n_{2p} + 2n_c + n_{cr} - 1 + \lfloor m_2/n \rfloor. \end{aligned} \tag{3.16}$$

Proof. By applying (3.13), (3.14) and (3.15) to Euler characteristic equation for sphere

$$v - e + f = 2$$

we get

$$\begin{aligned}&n_p - n_{2p} + n_{cr} + 2n_{cr}^{+} + 1 - \lfloor m_2/n \rfloor \\ &- \left(n_p + \deg_r^{+} + \deg_c^{+} + n_\infty \right) \\ &+2\left(n_\infty + n_{2p} - n_{cr}^{+} + n_c + \lfloor m_2/n \rfloor \right) = 2.\end{aligned}$$

By simplifying the equation we get the desired result. □

Remark 44. *Formula (3.16) was formulates as hypothesis in [20] in the case with no CPs in $\mathbb{C}_q^{h\pm}$ ($\deg_c^{+} = 0$), all CPs are of the first order ($\deg_r^{+} = 2n_{cr}$) and $n_c = 0$. Then it can be rewritten as*

$$n_{cr} = 2n_c + n_{2p} + n_\infty - 1 = 2n_c + n_{2p} + n - m_2 - 1.$$

Corollary 45 ([27])**.** *The number of edges is*

$$e = 2n_\infty + n_p + n_{2p} + n_{cr} + 2n_c - 1. \tag{3.17}$$

Formula (3.17) is derived from (3.14) and (3.16).

Remark 46 ([27])**.** *In the case $m_2 < n$ the formulas (3.14)–(3.17) are*

$$\begin{aligned}v &= m_2 - n_{2p} + n_{cr} + 2n_{cr}^{+} - n_{ce},\\ e &= 2n - m_2 + n_{2p} - 2 + n_{cr} + 2n_c - n_{ce},\\ f &= 2\left(n - m_2 + n_{2p} - n_{cr}^{+} + n_c\right),\\ \deg_r^{+} + \deg_c^{+} &= n - m_2 + n_{2p} + 2n_c + n_{cr} - 1,\end{aligned}$$

where n_{ce} is defined by (3.2).

In the case $m_2 = n$ we have $n_\infty = 0$, $n_p = n - 1 - n_{ce}$ and $n_{2p} = 0$. Note, that in this case there are no critical points, so $n_b = n_{cr} + 2n_{cr}^{+} = 0$. Thus, for $m_2 = n$ the following formulas

$$\begin{aligned}v &= n - 1 - n_{ce},\\ e &= n - 1 - n_{ce},\\ f &= 2,\\ \deg_r^{+} + \deg_c^{+} &= 0\end{aligned}$$

are valid. Note, that in this case we have cycle digraph.

Remark 47 ([27])**.** *If $n_{cr}^{+} = 0$ ($\deg_c^{+} = 0$) then $\deg_r^{+} - n_{cr} = n_c + n - m_2 + n_{2p} - 1 > n_{cr}$ shows that there exist CPs in $\mathbb{R}_q^h$ of the second or higher order.*

Corollary 48. *If $\deg_r^{+} = n_\infty + n_{2p} + 2n_c + n_{cr} - 1 + \lfloor m_2/n \rfloor$, then there is no CPs in $\mathbb{C}_q^{h+}$, i.e. $n_{cr}^{+} = 0$.*

Example 3. Let's take an example of dSLP with Dirichlet and integral BCs with $n = 6$ and $\boldsymbol{m} = (0{,}5)$ (see Figure 7a). The number of poles (including order, see remark 27) is defined by formulas (3.2) and (3.6), so we have

$$n_p = m_2 - 1 - n_{ce} = 5 - 1 - 0 = 4.$$

From Figure 7a we have

$$n_c = 1, \qquad n_{cr} = 4, \qquad n_{2p} = 2, \qquad \deg_r^+ = 8.$$

The point at infinity $p = \infty$ is a PP of order $n_\infty = n - m_2 = 6 - 5 = 1$. Now, by Corollary 48 we have

$$8 = \deg_r^+ = 1 + 2 + 2 \cdot 1 + 4 - 1 + 0 = 8,$$

so, $\deg_c^+ = 0$ meaning $n_{cr}^+ = 0$. This means that there is no CPs in $\mathbb{C}_q^{h+}$, all CPs lie in $\mathbb{R}_q^h$. Furthermore, we can derive vertices (v), edges (e) and faces (f):

$$\begin{aligned}
v &= m_2 - n_{2p} + n_{cr} - n_{ce} = 5 - 2 + 4 - 0 = 7,\\
e &= 2n - m_2 + n_{2p} - 2 + n_{cr} + n_c - n_{ce} = 2 \cdot 6 - 5 + 2 - 2 + 4 + 2 - 0 = 13,\\
f &= 2\,(n - m_2 + n_{2p} + n_c) = 2\,(6 - 5 + 2 + 1) = 8.
\end{aligned}$$

Example 4. Let's take another example of dSLP with Dirichlet and integral BCs with $n = 5$ and $\boldsymbol{m} = (0{,}5)$ (see Figure 9a). From the figure we have

$$n_c = 0, \qquad n_{cr} = 0, \qquad n_{2p} = 0.$$

The point at infinity $p = \infty$ is a removable singularity point. Since $n_{cr} = 0$, we have that $\deg_r^+ = 0$, too. So, by Corollary 48 we have $\deg_c^+ = 0$. Thus, there is no CPs in $\mathbb{C}_q^h$ at all. Now we can derive vertices (v), edges (e) and faces (f):

$$\begin{aligned}
v &= n - 1 - n_{ce} = 5 - 1 - 2 = 2,\\
e &= n - 1 - n_{ce} = 5 - 1 - 2 = 2,\\
f &= 2.
\end{aligned}$$

In this case we have a cycle digraph.

4. Conclusions

In this master's degree thesis the relations between graphs theory and discreet Sturm–Liouville problem with nonlocal boundary conditions were found. Two mathematical fields (differential equations and graphs theory) were joined to receive desired results.

First of all, theory of the Spectrum of a discrete Sturm–Liouville problem with various Nonlocal Boundary Conditions was studied. The main literature was K. Bingelė and A. Skučaitė doctoral dissertations about Sturm–Liouville problem with various Nonlocal Boundary Conditions

Secondly, lemmas for vertices, edges and faces were presented and proved using the theory of Spectrum Curves and digraphs. Nonetheless, theorems for parts of spectrum curves in $\mathbb{C}_q^h$ were formulated and proved using lemmas. Moreover, some corollaries and remarks were derived for specific cases, e.g. $m_2 < n$ for integral Boundary Condition. A couple of examples illustrate the theory which was derived in this master thesis.

The main results were theorems 33 (case of two-point NBCs) and 43 (case of integral NBC). These theorems are direct conclusion from Euler characteristic equation for sphere. Also, corollaries 35 and 45 are important results from the latter theorems as the number of graph edges can be expressed in terms of characteristic function parameters.

Moreover, some results about spectrum that were investigated by K. Bingelė and A. Skučaitė were proved using Euler's characteristic formula of sphere. In addition to this, more general cases were considered with the hypothesis that Critical Points in $\mathbb{C}_q^{h\pm}$ exist with either rational ξ_1, ξ_2 or ξ.

Finally, it was derived that there exist Critical Points in $\mathbb{R}_q^h$ of the second or higher order. Up until now it is a difficult problem to determine the number of Critical Points in $\mathbb{C}_q^h$, especially $\mathbb{C}_q^{h\pm}$, thus this result is a step ahead into the investigation of Spectrum Curves of a discrete Sturm–Liouville problem with various Nonlocal Boundary Conditions.

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