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Nonlinear Elliptic Equation with Nonlocal Integral Boundary Condition Depending on Two Parameters

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Abstract. In this paper, the two-dimensional nonlinear elliptic equation with the boundary integral condition depending on two parameters is solved by finite difference method. The main aim of this paper is to investigate the conditions under those all eigenvalues of corresponding difference eigenvalue problem are positive. For this purpose, we investigate the spectrum structure of one-dimensional difference eigenvalue problem with integral condition. In particular, conditions of the existence and some properties of negative eigenvalue are analyzed in details.

Keywords: nonlinear elliptic equation, nonlocal boundary condition, difference eigenvalue problem, M-matrix, real eigenvalues of nonsymmetric matrix.

AMS Subject Classification: 65M06; 65M12; 65N25.

1 Introduction

Problems with nonlocal conditions are widely used in mathematical modeling of various processes and phenomenon of the real world. The survey of the first papers on the applications of problems with nonlocal conditions could be found in [12, 22, 33]. It is worth to admit that the investigation of mathematical models with nonlocal conditions has great influence on the theory of modern numerical methods. As an example, could be taken investigation of the spectrum of nonsymmetrical matrices. Taking separately, the following inverse eigenvalue problem is important enough: what limitations should satisfy the parameters (or functions) of nonlocal conditions of the differential problem that the spectrum of a nonsymmetrical matrix of difference problem should possess the property which would be specified in advance. For example, all the eigenvalues would be real and positive, or all the eigenvalues would possess the property $Re\lambda > 0$ and so on. The similar problems were investigated in many papers [6, 10, 16, 17, 30].

In the present paper, we investigate the boundary value problem for nonlinear elliptic equation with nonlocal integral condition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = f(x, y, u), \quad (x, y) \in \Omega = \{0 < x < 1,\ 0 < y < 1\}, \tag{1.1}$$

$$u(x, 0) = \mu_1(x), \quad u(x, 1) = \mu_2(x), \quad u(0, y) = \mu_3(y), \tag{1.2}$$

$$u(1, y) = \gamma \int_\xi^1 u(x, y)dx + \mu_4(y), \tag{1.3}$$

where ξ and γ are real parameters and $\xi \in [0, 1)$, $\gamma \in (-\infty, \infty)$.

The finite difference or other numerical methods were applied for the solution of elliptic equations with various type nonlocal conditions. To investigate the numerical methods (error estimation, convergence, etc.) various approaches were used. One of them is an investigation of the structure of spectrum. It can be said, that eigenvalue problems of differential or difference equations with nonlocal conditions form separate quite important branch in the modern numerical analysis.

The values of solution of two-dimensional problem (1.1)–(1.3) in one coordinate direction are conected by nonlocal condition (1.3). This is quite often and characteristic formulation of nonlocal condition for elliptic equation in two- or multi-dimensional case [1, 2, 3, 4, 5, 13, 14, 18].

In paper [3] the existence and uniqueness of the solution for a multi-dimensional elliptic equation with integral conditions:

$$\int_0^{\xi_1} u(x, y)dy = 0, \quad \int_{\xi_2}^1 u(x, y)dy = 0, \quad x = (x_1, x_2, \dots, x_n),$$

were investigated. In these nonlocal conditions, as in condition (1.3), the interval of integration is less than interval of definition of the solution by one of the variables. In paper [5] the convergence of finite difference method for two-dimensional elliptic equation with nonlocal condition was proved.

In paper [1] boundary value problem for the multidimensional elliptic equation with one integral condition was investigated. In [18] the difference problem for Poisson equation with multipoint nonlocal condition

$$u(1,y)=\sum_{i=1}^{m}\alpha_i u(\xi_i,y)+\eta(y)$$

instead of integral condition was investigated. In [13] Laplace equation with the multipoint nonlocal condition was solved by the difference method of fourth order accuracy.

Situation when in nonlocal condition the interval of integration is less than the interval of definition of the solution is characteristic not only for elliptic equations. Such formulation for nonlocal conditions naturally occurs for parabolic equations too, as models of real processes. In [7], when formulating the problem for the one-dimensional heat equation, the following nonlocal condition was used:

$$E(t)=\int_0^{x(t)} u(x,t)dx,$$

in which $E(t)$ and $x(t)$ are known functions.

For two-dimensional parabolic equation also could be formulated condition of the same type [8].

The structure of spectrum for one-dimensional eigenvalue problem with nonlocal condition in the form (1.3) was investigated in paper [24]. The structure of spectrum of corresponding difference eigenvalue problem was considered in [6]. Analogous investigations with different forms of nonlocal conditions were performed in [19, 23, 25, 30] (see also the review article [33]). The structure of spectrum for ordinary differential equation of more general form with nonlocal condition (1.3) was investigated in [34].

The examination of structure of the spectrum for parabolic equations is one of the effective methods proving the stability of difference scheme with nonlocal conditions [9, 17, 19, 20, 21, 27].

The eigenvalue problem for two-dimensional Poisson equation with various types of nonlocal conditions was investigated in many papers (see, for example, [15] and references therein). The structure of the spectrum for two-dimensional elliptic equation with nonlocal condition in the form (1.3) when $\xi=0$ was considered in [35]. In papers [10,11,20,28,32] the structure of spectrum for two-dimensional problem in connection with the theory of M-matrices was applied for theoretical investigation (convergence of finite difference method, stability of difference schemes, convergence of iterative methods) of difference schemes with nonlocal condition (1.3) when $\xi=0$.

Difference method of fourth order of accuracy for two-dimensional Laplace equation with nonlocal condition

$$u(x,0)=\alpha\int_{\xi}^{b} u(x,y)dy+\mu(x),\quad 0<x<a,\quad 0<y<b$$

has been considered in [14].

The main aim of the present paper is to investigate the limitation of the parameters γ and ξ of the problem (1.1)–(1.3) assuming that all the eigenvalues of corresponding difference problem would be real and positive.

The structure of the paper is as follows. In Section 2, following the methodology provided in papers [6, 24], we investigated conditions of existence of negative eigenvalue of the corresponding one-dimensional problem and its properties. In Section 3, we applied these results for two-dimensional eigenvalue problem. In the last Section 4, the conclusions and generalizations are provided.

2 Spectrum structure of one-dimensional difference eigenvalue problem

We consider one-dimensional eigenvalue problem with nonlocal condition

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad x \in (0,1), \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \gamma \int_{\xi}^{1} u(x)dx, \tag{2.2}$$

where $\xi \in [0,1)$ and $\gamma \in (-\infty,\infty)$ are real numbers. We write down the corresponding difference eigenvalue problem

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{2.3}$$

$$u_0 = 0, \quad u_N = h\gamma\left(\frac{u_m + u_N}{2} + \sum_{i=m+1}^{N-1} u_i\right), \tag{2.4}$$

where $Nh = 1$, $\xi = mh$.

The specificity of nonlocal problem (2.1)–(2.2) is such that an interval of integration is only a part of an interval $[0,1]$.

We examine structure of spectrum of the problem (2.3)–(2.4). As it was mentioned in an Introduction, structure of spectrum of differential and difference problems (2.1)–(2.2) and (2.3)–(2.4) was considered in [6, 24]. Not to repeat the theoretical research done in [6, 24] we complement them by quantitative conclusions about dynamics of negative eigenvalue varying parameters γ, ξ and h.

In [24] the following main properties of the spectrum of problem (2.1)–(2.2) are proved.

Proposition 1. *[24] All eigenvalues of problem* (2.1)–(2.2) *with real* γ *and* ξ *are real.*

Proposition 2. *[24] For differential problem* (2.1)–(2.2) *the eigenvalue* $\lambda = 0$ *exists if and only if* $\gamma = 2/(1-\xi^2)$.

Proposition 3. *[24] For* $\gamma > 2/(1-\xi^2)$ *one negative eigenvalue of the problem* (2.1)–(2.2) *exists, and for* $\gamma < 2/(1-\xi^2)$ *there aren't negative eigenvalues.*

In [6] these statements are generalized for difference problem (2.3)–(2.4).

We examine in more detail conditions of existence of negative eigenvalue for the difference eigenvalue problem (2.3)–(2.4). According to the Propositions 2 and 3, all eigenvalues of the problem (2.3)–(2.4) are positive in the case $\gamma \leq 0$. So, we investigate in the present paper only the case $\gamma > 0$.

Let us say that the following hypothesis is true.

HYPOTHESIS 1. Mesh size of the grid h is sufficiently small with respect to parameter γ: $h < 2/\gamma$.

When $\lambda < 0$, then

$$1 - \lambda h^2/2 > 1.$$

Consequently, in the Equation (2.3) instead of eigenvalue $\lambda < 0$ other parameter $\beta > 0$ could be introduced by one-to-one dependence:

$$1 - \lambda h^2/2 = \cosh(\beta h).$$

From there it follows

$$\lambda = -\frac{4}{h^2} \sinh^2\left(\frac{\beta h}{2}\right). \tag{2.5}$$

Now the general solution of the Equation (2.3) could be search in the form:

$$u_i = C_1 \cosh(i\beta h) + C_2 \sinh(i\beta h). \tag{2.6}$$

Requiring that nontrivial solution (eigenvector) (2.6) satisfying not only the Equation (2.3) but also conditions (2.4) should exist, we get the following expression:

$$\sinh(\beta) = \frac{\gamma h}{2} \frac{\cosh(\beta) - \cosh(\beta\xi)}{\tanh\left(\frac{\beta h}{2}\right)}. \tag{2.7}$$

Hence, we can formulate the following preliminary conclusion.

Proposition 4. *For difference eigenvalue problem* (2.3)–(2.4) *the unique negative eigenvalue exists, if and only if the unique root* $\beta_0 > 0$ *of the Equation* (2.7) *exists.*

Remark 1. In this paper, we use real parameter $\beta > 0$ to define negative eigenvalue. In papers [6, 24], the universal complex parameter q ($Re q > 0$) suitable for expression of whichever eigenvalue $\lambda \neq 0$, was used. When $\lambda < 0$, relation between β from present article and q from papers [6,24] is the following: $q = i\beta$ in [24] and $\pi q = i\beta$ in [6], where i - imaginary unit.

In that way the problem on the existence of negative eigenvalue (2.3)–(2.4) is reduced to more simple problem on the existence of the root $\beta_0 > 0$ of the Equation (2.7).The existence and uniqueness of this solution we present graphically, receiving the quantitative information needed in this way.

With such an aim we rewrite the Equation (2.7) in another form

$$\frac{2}{\gamma h} \tanh\left(\frac{\beta h}{2}\right) = \frac{\cosh(\beta) - \cosh(\beta\xi)}{\sinh(\beta)}. \tag{2.8}$$

Now we define functions $f_1(\beta)$ and $f_2(\beta)$ in the interval $\beta \in (0, \infty)$:

$$f_1(\beta) = \frac{\cosh(\beta) - \cosh(\beta\xi)}{\sinh(\beta)}, \quad f_2(\beta) = \frac{2}{\gamma h} \tanh\left(\frac{\beta h}{2}\right).$$

It is evident, that the unique solution β_0 for the Equation (2.7) exists if and only if graphics of functions $f_1(\beta)$ and $f_2(\beta)$ would have the only crossing point. Both of functions also depend on the parameters ξ or γ, h .

The behavior of the functions $f_1(\beta)$ and $f_2(\beta)$ and their derivatives in the limiting points of the interval $(0, \infty)$ is directly calculated

$$\lim_{\beta\to 0} f_1(\beta) = 0, \quad \lim_{\beta\to\infty} f_1(\beta) = 1, \quad \lim_{\beta\to 0} f_2(\beta) = 0, \quad \lim_{\beta\to\infty} f_2(\beta) = \frac{2}{\gamma h}, \qquad (2.9)$$

$$\lim_{\beta\to 0} f_1'(\beta)=\frac{1-\xi^2}{2}, \quad \lim_{\beta\to\infty} f_1'(\beta)=0, \quad \lim_{\beta\to 0} f_2'(\beta)=\frac{1}{\gamma}, \quad \lim_{\beta\to\infty} f_2'(\beta)=0. \qquad (2.10)$$

Furthermore, functions $f_1(\beta)$ and $f_2(\beta)$ are monotonically increasing functions in whole the interval $\beta \in (0, \infty)$. Graphics of the functions $f_1(\beta)$ and $f_2(\beta)$ with several different values of parameters γ and ξ are presented in Figures 1 and 2.

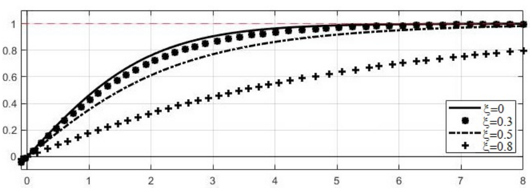

Figure 1. Graphics of the function $f_1(\beta)$ for various ξ.

In Figure 1 the values of function $f_1(\beta)$ with all values of the parameters ξ approach to 1, as $\beta \to \infty$. We note, as larger the parameter ξ value, as lower the graphic of function $f_1(\beta)$ in the coordinate plane is.

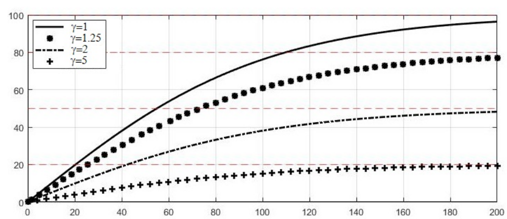

Figure 2. Graphics of the function $f_2(\beta)$ for various γ; $h = 0.02$.

Similary, in Figure 2 the graphic of function $f_2(\beta)$ is lower in the coordinate plane as value of the parameter γ is larger.

Few following conclusions follow from the properties of functions $f_1(\beta)$ and $f_2(\beta)$ which we demonstrate in Figures 3–10. In Figure 3, the illustration of statement from Proposition 3, i.e., if $\gamma \le 2/(1-\xi^2)$, then the negative eigenvalue for difference eigenvalue problem (2.3)–(2.4) does not exist, is presented. When ξ is fixed (in Figure 3, $\xi = 0.5$), then graphic of function $f_1(\beta)$ with the values of parameter γ less than $\frac{2}{1-\xi^2} = \frac{8}{3}$, has no crossing point with the graphic of function $f_2(\beta)$. Equation (2.8) has no roots, $\lambda < 0$ does not exist.

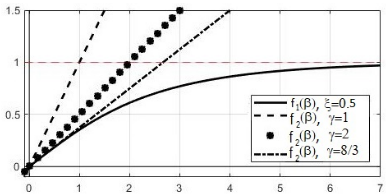

Figure 3. Graphics of the function $f_1(\beta)$ for $\xi = 0.5$ and function $f_2(\beta)$ for $h = 0.02$ and variuos γ. The root β_0 of Equation (2.8) does not exist.

In Figure 4, the illustration of another statement from Proposition 3 is presented. Namely, if $\gamma > \frac{2}{1-\xi^2}$ then one negative eigenvalue exists. Indeed, when $\xi = 0.5$ and $\gamma > \frac{2}{1-\xi^2} = \frac{8}{3}$, then all the graphics of function $f_2(\beta)$ have only one crossing point with the graphic of function $f_1(\beta)$.

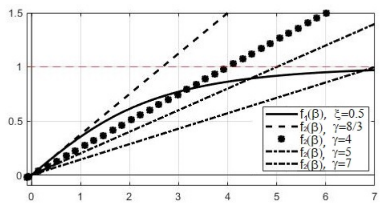

Figure 4. Graphics of the function $f_1(\beta)$ for $\xi = 0.5$ and function $f_2(\beta)$ for $h = 0.02$ and variuos γ. The root β_0 of Equation (2.8) exist.

The following proposition presented in Figures 5 and 6, could be interpreted not only as the illustration of the theory but also as the complementation of the theory. Here the situation is demonstrated when $h > \frac{2}{\gamma}$ (or $h\gamma > 2$), i.e., the Hypothesis 1 is not fulfilled.

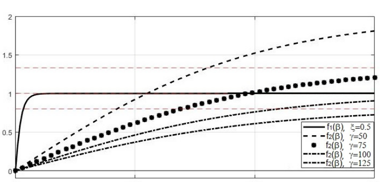

Figure 5. Graphics of the function $f_1(\beta)$ for $\xi = 0.5$ and function $f_2(\beta)$ for $h = 0.02$ and various γ. The existence of the root β_0 of Equation (2.8) depend of γ.

In Figure 5, the grid step h is fixed, and parameter γ of function $f_2(\beta)$ changes. When $h = 0.02$ and $\gamma \geq 100$, then $h\gamma \geq 2$. In this case, graphics of the functions $f_1(\beta)$ and $f_2(\beta)$ do not cross each other, although the condition $\gamma > \frac{2}{1-\xi^2}$ is fulfilled. Equation (2.8) has no the root, negative eigenvalue does not exist.

Similarly in Figure 6, when $\gamma = 50$ and $h \geq 0.04$, the Hypothesis 1 is not fulfilled. Equation (2.8) has no root. So, the inequality $\gamma > \frac{2}{1-\xi^2}$ is only necessary, but not sufficient condition for the existence of the root of Equation (2.8). The value of h influences on that, too. Such situation, that when $\xi = 0$, was observed in [27]. When $\xi \neq 0$, as far as it is known for authors, concrete limitations for the step h were not considered.

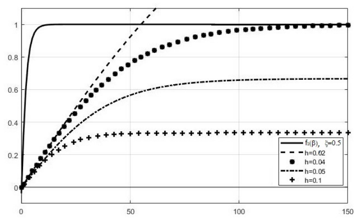

Figure 6. Graphics of the function $f_1(\beta)$ for $\xi = 0.5$ and function $f_2(\beta)$ for $\gamma = 50$ and vanish h. The existence of the root β_0 of Equation (2.8) depend on h.

So, according to the results presented in Figures 5 and 6, it is possible to specify the formulation of Proposition 3 for difference problem (2.3)–(2.4) on the structure of spectrum.

Proposition 5. *For difference eigenvalue problem* (2.3)–(2.4)*, the negative eigenvalue exists if and only if both conditions*

$$2/(1-\xi^2) < \gamma < 2/h$$

are fulfilled.

Now, we analyze how the root of Equation (2.8) vary depending on change of parameters γ and ξ. With this aim we rewrite this equation in the different form:

$$\gamma = f(\beta, \xi), \tag{2.11}$$

where

$$f(\beta, \xi) = \frac{2}{h} \tanh\left(\frac{\beta h}{2}\right) \frac{\sinh(\beta)}{\cosh(\beta) - \cosh(\beta\xi)}.$$

Proposition 6. *[6, 24] Function $f(\beta, \xi)$ is monotonically increasing function of variable β in whole the interval $\beta \in (0, \infty)$.*

Proposition 7. *Function $f(\beta, \xi)$ is monotonically increasing function of variable ξ in whole the interval $\xi \in [0, 1)$.*

This proposition follows from the inequality

$$\frac{\partial f}{\partial \xi} = \frac{2}{h} \tanh\left(\frac{\beta}{2}\right) \frac{\xi \sinh(\beta) \sinh(\beta\xi)}{(\cosh(\beta) - \cosh(\beta\xi))^2} > 0.$$

We denote by β_0^i the root of Equation (2.11), when the values of the parameters γ and ξ are γ^i, ξ^i, $i = 1, 2$.

Proposition 8. *Suppose, β_0^1 and β_0^2 are two solutions of Equation* (2.11)*, when ξ_1 and ξ_2 are different values and value of γ is fixed:*

$$\gamma = f(\beta_0^1, \xi_1), \quad \gamma = f(\beta_0^2, \xi_2). \tag{2.12}$$

If $\xi_2 > \xi_1$, then $\beta_0^2 < \beta_0^1$.

Proof. Suppose the contrary, i.e., it follows from conditions (2.12) and $\xi_2 > \xi_1$, that $\beta_0^2 \geq \beta_0^1$. Function $f(\beta, \xi)$ is monotonically increasing function of both variables β and ξ. Consequently, it follows from inequalities $\beta_0^2 \geq \beta_0^1$ and $\xi_2 > \xi_1$, that

$$f(\beta_0^2, \xi_2) > f(\beta_0^1, \xi_1).$$

We get contradiction with the conditions (2.12). So, $\beta_0^2 < \beta_0^1$ (see Figures 7–8). □

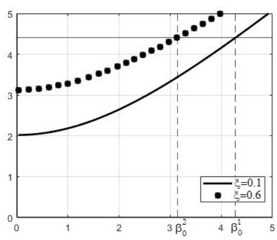

Figure 7. Grapfics of the function $f(\beta, \xi)$ for $\xi = 0.1$ and $\xi = 0.6$. $h = 0.02$; $\gamma = 3.44$. If $\xi_2 > \xi_1$, then $\beta_0^2 < \beta_0^1$.

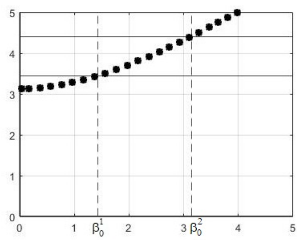

Figure 8. Grapfics of the function $f(\beta, \xi)$ for $\xi = 0.6$. If $\gamma_2 > \gamma_1$, then $\beta_0^2 > \beta_0^1$.

Proposition 9. *Suppose, β_0^1 and β_0^2 are two solutions of Equation* (2.11) *with different values of γ_1 and γ_2 and fixed value of ξ: $\gamma_1 = f(\beta_0^1, \xi)$, $\gamma_2 = f(\beta_0^2, \xi)$. If $\gamma_2 > \gamma_1$, then $\beta_0^2 > \beta_0^1$.*

Proof. Whereas $\gamma_2 > \gamma_1$, then $f(\beta_0^2, \xi) > f(\beta_0^1, \xi)$. It follows from the monotony of the function $f(\beta, \xi)$, that $\beta_0^2 > \beta_0^1$. □

From the definition of functions $f_1(\beta)$, $f_2(\beta)$ and $f(\beta, \xi)$ follows, that

$$f(\beta, \xi) = \frac{\gamma f_2(\beta)}{f_1(\beta)}.$$

So, accordingly with the expressions (2.9)–(2.10), we get the following limiting values of the function $f(\beta, \xi)$:

$$\lim_{\beta \to 0} f(\beta, \xi) = \lim_{\beta \to 0} \frac{\gamma f_2(\beta)}{f_1(\beta)} = \lim_{\beta \to 0} \frac{\gamma f_2'(\beta)}{f_1'(\beta)} = \frac{2}{1 - \xi^2}, \tag{2.13}$$

$$\lim_{\beta \to \infty} f(\beta, \xi) = \lim_{\beta \to \infty} \frac{\gamma f_2(\beta)}{f_1(\beta)} = \frac{2}{h}. \tag{2.14}$$

We take fixed values $\xi_1 \in [0,1)$ and $\beta_0 \in (0,\infty)$ and define

$$\gamma_1 = f(\beta_0, \xi_1). \tag{2.15}$$

It follows from (2.13) and (2.14), that

$$2/(1-\xi_1^2) < \gamma_1 < 2/h.$$

Equation (2.15) could be interpreted in a following way. $\beta_0 > 0$ is the root of Equation (2.11) with the values of parameters γ_1 and ξ_1.

On the other hand, it is possible to state, that formula (2.15) depicts the constructive algorithm how to obtain the value of the parameter γ such, that in the presence of fixed value ξ_1 the root of Equation (2.11) would be equal to given in advance value $\beta_0 > 0$. The conclusion follows from these considerations and Propositions 4 and 9.

Corollary 1. Suppose, the root of Equation (2.11) with the values of parameters γ_1 and ξ_1 is β_0, i.e., $\gamma_1 = f(\beta_0, \xi_1)$. Then equation $\gamma = f(\beta, \xi_1)$ with all values of γ, satisfying the condition

$$\frac{2}{1-\xi_1^2} < \gamma \le \gamma_1 = f(\beta_0, \xi_1)$$

possesses the root $\beta > 0$, belonging to the interval $(0, \beta_0)$.

We reformulate this conclusion for the eigenvalues (2.5) of difference eigenvalue problem.

Corollary 2. Suppose, Equation (2.11) with the parameters γ_1 and ξ_1 possesses the root $\beta_0 > 0$, i.e., the negative eigenvalue exists for the problem (2.3)–(2.4)

$$\lambda = -\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right).$$

Then with all the values of γ, satisfying condition

$$\frac{2}{1-\xi_1^2} < \gamma \le \gamma_1 = f(\beta_0, \xi_1),$$

negative eigenvalue belonging to the interval $\left[-\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right), 0\right)$ of the eigenvalue problem (2.3)–(2.4) exists.

This conclusion is the main result of Section 2 which will be needful for the investigation of two-dimensional eigenvalue problem.

3 Two-dimensional eigenvalue problem

We solve the differential problem (1.1)–(1.3) by finite difference method. We write down the corresponding difference problem in the following form:

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} = f_{ij}(u_{ij}), \quad i,j = \overline{1, N-1}, \tag{3.1}$$

$$u_{i0} = (\mu_1)_i, \quad u_{iN} = (\mu_2)_i, \quad u_{0j} = (\mu_3)_j, \tag{3.2}$$

$$u_{Nj} = h\gamma\left(\frac{u_{mj} + u_{Nj}}{2} + \sum_{i=m+1}^{N-1} u_{ij}\right) + (\mu_4)_j, \quad j = \overline{1, N-1}, \tag{3.3}$$

where

$$\delta_x^2 u_{ij} = \frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2}, \quad \delta_y^2 u_{ij} = \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2},$$

$hN = 1$, $\xi = mh$, N and m are integers.

If the solution of boundary value problem (1.1)–(1.3) is smooth enough, then the error of approximation of differential problem is $O(h^2)$.

The system of difference equations (3.1)–(3.3) we write down in the matrix form. With this aim we express from Equation (3.3) u_{Nj} by another unknowns

$$u_{Nj} = \sum_{i=1}^{N-1} \alpha_i u_{ij} + \frac{2\gamma h}{2-\gamma h}(\mu_4)_j, \tag{3.4}$$

where, depending on $\xi = mh$, in the case $1 < m < N-1$

$$\alpha_i = \begin{cases} 0, & \text{if } i = \overline{1, m-1}, \\ \gamma h/(2-\gamma h), & \text{if } i = m, \\ 2\gamma h/(2-\gamma h), & \text{if } i = \overline{m+1, N-1}. \end{cases}$$

In the case $\xi = 0$ ($m = 0$), formula (3.3) is reduced to

$$u_{Nj} = \frac{2\gamma h}{2-\gamma h}\left(\sum_{i=1}^{N-1} u_{ij} + \frac{(\mu_3)_j}{2} + (\mu_4)_j\right).$$

From here it is clear that it is possible to write down the equation in the form (3.4) only when $h \neq \frac{2}{\gamma}$. We remind that in Section 2 the certain condition was required also, defining the relation between h and γ ($h\gamma < 2$ according to presumption Hypothesis 1).

Putting down the expression (3.4) into Equations (3.1), when $i = N-1$, we rearrange the system of difference equations (3.1) to another equivalent form:

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} = f_{ij}(u_{ij}), \quad i = \overline{1, N-2}, \tag{3.5}$$

$$\frac{u_{N-2,j} - 2u_{N-1,j} + \sum\limits_{i=1}^{N-1} \alpha_i u_{ij}}{h^2} + \delta_y^2 u_{N-1,j} = f_{N-1,j}(u_{N-1,j}) - \frac{(\mu_4)_j}{h^2}, \tag{3.6}$$

where $j = \overline{1, N-1}$.

In this way, the system of difference equations (3.1), (3.3) with boundary conditions (3.2), in which there are $(N-1)N$ equations and the same number of unknowns is rearranged to the different form. In the new form there are two systems. These are the system (3.5), (3.6) with boundary conditions (3.2), in which there are $(N-1)^2$ of equations and the same number of unknowns, and another system (3.4) (more precisely, the explicit formulas).

Now we can at first solve the system (3.5), (3.6), (3.2). Indeed, in the system (3.5), (3.6), (3.2) there are no more nonlocal conditions and unknowns u_{Nj}, $j = \overline{1, N-1}$. So, we find values u_{ij}, $i, j = \overline{1, N-1}$. This system has an unique solution. After that we find u_{Nj}, $j = \overline{1, N-1}$ according to the formulas

(3.4). This approach is described in many papers in which spectrum structure of two-dimensional difference eigenvalue problems and iterative methods for systems of difference equations were investigated [10, 11, 28, 30, 35].

Onward we write down the system (3.5), (3.6) in the matrix form

$$Au + f(u) = \varphi, \tag{3.7}$$

where A is the matrix of order $(N-1)^2$ and u, $f(u)$ and φ are the vectors of order $(N-1)^2$ (φ is a vector composed by the values μ_1, μ_2, μ_3, μ_4 of functions of boundary conditions). Matrix A could be writen as $A = \Lambda - C$, where $\Lambda = \Lambda_1 + \Lambda_2$ is a matrix corresponding to difference operator $-\delta_x^2 - \delta_y^2$ in the area Ω with Dirichlet type conditions. C is a matrix composed by coefficients α_i in Equations (3.6). More precisely, C is a block matrix

$$C = diag(C_1, C_1, \ldots, C_1)$$

where

$$C_1 = h^{-2} \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{N-1} \end{pmatrix}.$$

Number of blocks C_1 in matrix C is $N-1$, order of block C is also $N-1$.

We write down the eigenvalue problem of matrix A. With this aim firstly, we write down the difference eigenvalue problem, corresponding to differential eigenvalue problem for Laplace operator with homogeneous boundary conditions (1.2), (1.3)

$$\delta_x^2 u_{ij} + \delta_y^2 u_{ij} + \lambda u_{ij} = 0, \quad i, j = \overline{1, N-1}, \tag{3.8}$$

$$u_{Nj} = h\gamma\left(\frac{u_{mj} + u_{Nj}}{2} + \sum_{i=m+1}^{N-1} u_{ij}\right), \quad j = \overline{1, N-1}, \tag{3.9}$$

$$u_{i0} = u_{iN} = u_{0j} = 0. \tag{3.10}$$

We reduce this problem to another form analogously as we reduced the system of equations (3.1)–(3.3). With this aim we express u_{Nj} from Equation (3.9) and put it into Equation (3.8), where $i = N-1$. After these transformations we get matrix form

$$Au = \lambda u, \tag{3.11}$$

where A is the same matrix as in the system (3.7). So we get the following statement.

Proposition 10. *If Hypothesis 1 is fulfilled, then the eigenvalue problem* (3.11) *for matrix A as matrix of system of difference equations, is equivalent to two-dimensional eigenvalue problem* (3.8)–(3.10).

We refer some properties of matrix A.

Proposition 11. *If $h\gamma < 1$, then diagonal elements of matrix A are positive. Nondiagonal elements of matrix A are always nonpositive.*

Proof. From Equations (3.5) and (3.6) we get that diagonal elements of matrix A are $4h^{-2}$ or $h^{-2}(4-\alpha_i)$. We estimate diagonal element, when

$$\alpha_i = 2\gamma h/(2-\gamma h).$$

We get

$$\frac{4-\alpha_i}{h^2} = \left(4 - \frac{2h\gamma}{2-h\gamma}\right)\frac{1}{h^2} = \frac{8-6h\gamma}{h^2(2-h\gamma)} > \frac{1}{h^2},$$

when $h\gamma < 1$. Nondiagogal elements $A = \Lambda - C$ are composed by nondiagonal elements of matrices Λ and C, which are nonpositive. □

We investigate, with wich values of parameters γ and ξ all the eigenvalues of difference problem (3.8)–(3.10) are positive.

Using the Fourier method, we separate variables in problem (3.8)–(3.10) as $u_{ij} = v_i w_j$. In this way two-dimensional eigenvalue problem is reduced to two one-dimensional problems:

$$\begin{aligned} &\delta_x^2 v_i + \eta v_i = 0, \quad i = \overline{1, N-1} \\ &v_0 = 0, \quad v_N = h\gamma\left(\frac{v_m + v_{N-1}}{2} + \sum_{i=m+1}^{N-1} v_i\right), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\delta_y^2 w_j + \mu w_j = 0, \quad j = \overline{1, N-1}, \\ &w_0 = 0, \quad w_N = 0. \end{aligned} \tag{3.13}$$

For eigenvalues of problem (3.8)–(3.10) the following equality is true:

$$\lambda_{kl} = \eta_k + \mu_l, \quad k, l = \overline{1, N-1}.$$

The eigenvalues of problem (3.13) are

$$\mu_l = \frac{4}{h^2}\sin^2\frac{l\pi h}{2}, \quad l = \overline{1, N-1},$$

and all of them are unconditionally positive. From Proposition 4 it follows that all eigenvalues of problem (3.12) are positive, if $\gamma < \frac{2}{1-\xi^2}$. When the values of parameters γ_1 and ξ_1 are such that $\gamma > \frac{2}{1-\xi^2}$, then Equation (2.11) has the root β_0, i.e.

$$\gamma_1 = f(\beta_0, \xi_1).$$

In this case one negative eigenvalue

$$\eta_1 = -\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right)$$

of one-dimentional eigenvalue problem (3.12) exists.

Further, when

$$|\eta_1| = \min_l \mu_l, \tag{3.14}$$

then one eigenvalue of two-dimensional eigenvalue problem is equal zero, and the rest of eigenvalues are positive.

From Equation (3.14) we get

$$\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right)=\frac{4}{h^2}\sin^2\frac{\pi h}{2}.$$

From here we express

$$\beta_0=\frac{2}{h}ln\left(\sin\frac{\pi h}{2}+\sqrt{\sin^2\frac{\pi h}{2}+1}\right). \tag{3.15}$$

We notice that $\beta_0 \approx \pi$, when h is small enough. Moreover, $\beta_0 < \pi$ and $\beta_0 \to \pi$, as $h \to 0$.

Now we can use Corollary 2 from Section 2 and rephrase the proposition for two-dimensional problem.

Corollary 3. Suppose, h is small enough, i.e. $h\gamma < 1$. We take any fixed value $\xi_0 \in [0, 1)$ and β_0 under formula (3.15). We define

$$\gamma_1 = f(\beta_0,\xi_0)=\frac{2}{h}\tanh\left(\frac{\beta_0 h}{2}\right)\frac{\sinh(\beta_0)}{\cosh(\beta_0)-\cosh(\beta_0\xi_0)}. \tag{3.16}$$

All the eigenvalues of two-dimensional eigenvalue problem (3.8)–(3.10) with parameters ξ_0 and γ are positive if and only if

$$\gamma < \gamma_1. \tag{3.17}$$

The propositions under which this conclusion is obtained and the conclusion itself could be demonstrated in the Table 1.

Table 1. The dynamics of the least eigenvalue λ_{11} depending on γ.

γ	$\gamma<\gamma_0$	$\gamma=\gamma_0$	$\gamma_0<\gamma<\gamma_1$	$\gamma=\gamma_1$	$\gamma>\gamma_1$
η_1	$\nexists\ \eta_1<0$ all $\eta_k>0$	$\nexists\ \eta_1<0$ $\exists\ \eta_k=0$	$\exists\ \eta_1<0$ $\lvert\eta_1\rvert<\mu_1$	$\exists\ \eta_1<0$ $\lvert\eta_1\rvert=\mu_1$	$\exists\ \eta_1<0$ $\lvert\eta_1\rvert>\mu_1$
λ_{11}	$\lambda_{11}>0$	$\lambda_{11}>0$	$\lambda_{11}>0$	$\lambda_{11}=0$	$\lambda_{11}<0$

In Table 1 the dynamics of the least eigenvalue λ_{11} of the problem (3.8)–(3.10) is demonstrated in the case when ξ_0 is fixed, and γ varies; here η_1 is the negative eigenvalue of the problem (3.12), μ_1 is the least eigenvalue of the problem (3.13), β_0 and γ_1 are calculated by formulas (3.15) and (3.16); $\gamma_0=\frac{2}{(1-\xi^2)}>2$.

If γ is negative, condition (3.17) is always fulfilled independently from the values of γ and ξ. So, when $\gamma<0$, all the eigenvalues of the problem (3.8)–(3.10) are positive for all values $\xi\in[0,1)$.

Now we return from the eigenvalue problem (3.8)–(3.10) to the system (3.7) of nonlinear equations. Diagonal elements of the matrix A of this system are positive and nondiagonal element are nonpositive (Proposition 11). Eigenvalues of matrix A are positive when $\gamma < \gamma_1$ (Corollary 3). The statement follows from these properties.

Proposition 12. *Suppose, $h\gamma < 1$. If for $\xi_0 \in [0, 1)$, the condition $0 < \gamma < \gamma_1$ is fulfilled, where γ_1 is defined by formula (3.16), and β_0 is defined by formula (3.15), then matrix A is an M-matrix.*

In Figures 9–10 the dependence of γ_1 from ξ is presented. When $\xi = 0$, then $\gamma_1 \approx 3.4234$ [35]. If ξ is increasing in the interval $[0, 1)$, then γ_1 slowly increases in the beginning of the interval and it increases fast on the points close to the value 1, approaching the value $\gamma_1 = \frac{2}{h}$.

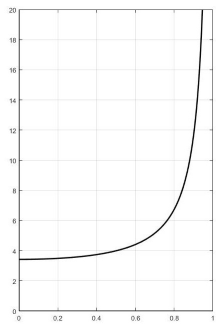

Figure 9. Grapfics of value γ_1 (formula (3.16)) depending on ξ. $\beta_0 = 3.1406$; $h = 0.02$.

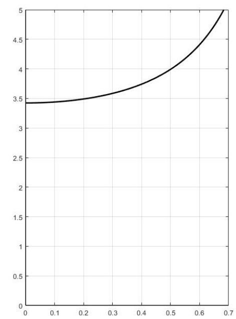

Figure 10. The part of Figure 9.

When in the differential problem (1.1)–(1.3) function f depends on the solution u, we formulate the presumptions which the function f should satisfy.

HYPOTHESIS 2. $0 \leq \frac{\partial f}{\partial u} \leq \beta < \infty$ for all u and $(x, y) \in \Omega$.

Proposition 13. *[28] If matrix A is an M-matrix and Hypothesis 2 is fulfilled then the unique solution of problem (3.7) exists.*

Convergence of iterative methods of the system (3.7) depends not only on the matrix A, but also on the matrix $A + D$, where D is a diagonal matrix with diagonal elements $\frac{\partial f_{ij}(\tilde{u}_{ij})}{\partial u_{ij}}$ in some intermediate points $\tilde{u}_{ij}$. We notice that under the Hypothesis 2, matrix $A + D$ also is an M-matrix. This follows from the following properties of M-matrix.

Proposition 14. *[36] If A is an M-matrix and D is a diagonal matrix with the nonnegative elements, then $A + D$ is an M-matrix.*

Taken separately, this means that for all the eigenvalues of matrix $A + D$ the following property is true:

$$Re\lambda(A + D) > 0.$$

So, when matrices A and $A + D$ are M-matrices there are many convergent explicit and implicit iterative methods for the system of equations (3.7), considered in [10, 28, 35].

But when $\gamma < 0$, then matrix A is not an M-matrix. It only possesses a property, that all the eigenvalues are positive. Thus, when all the eigenvalues are positive and $\gamma < 0$, convergence of iterative methods for system (3.7) remains as an open issue, at least, for the system of nonlinear equations. But in the case of linear equation (1.1) (i.e., $\frac{\partial f}{\partial u} = const > 0$), the positiveness of all eigenvalues of matrix A is the sufficient condition for convergence of many iterative methods [26, 35], including Peaceman-Rachford alternating direction method [29].

4 Conclusions and comments

When we solve the elliptic or parabolic equations with nonlocal conditions by the finite difference method, matrix of the system of difference equations very often is an M-matrix. It is sufficiently good property of nonsymmetric matrix. In the presence of this property it is possible in many cases to prove convergence of finite difference method [11, 31], to analyse stability of difference schemes and convergence of iterative methods for solution of the system of difference equations [28, 35].

In present paper basing on the investigation of spectrum structure of difference problem we obtained necessary and sufficient conditions for the matrix of the system would be an M-matrix. Thus, we noticed and emphasized few peculiarities of spectrum of difference eigenvalue problem characteristic for the problems with nonlocal conditions.

One of these peculiarities appears when the presumption is ignored, that solving the difference problem the step of the grid h should be sufficiently small in comparison with the value of parameter γ. At this point we present short comment. It is proved in [27] and in the present paper, that under some connection between parameters γ and h the negative eigenvalue in the spectrum of difference problem which is characteristic for differential problem could disappear. This property of difference eigenvalue problem would be estimated ambiguously. The matter is that when the negative eigenvalue is disappeared from the spectrum the properties of the spectrum even get better. From the other hand, as more properties of the differential problem are reflected in difference problem, as of better quality the approximation is. Such situation in difference problem occurred when the nonlocal boundary conditions were being investigated.

We present one more property of difference eigenvalue problem characteristic for the problems with nonlocal conditions.

When differential problem is formulated with classical (Dirichlet or Neumann type) boundary conditions, then usually there is no difference among difference eigenvalue problem and matrix eigenvalue problem. In principle both of these notions could be considered as synonymous.

But for the problems with nonlocal conditions this is not true. This depends not only on the form of nonlocal conditions but also on the values of parameters (or functions) in the nonlocal condition. For difference eigenvalue problem considered in the present paper the situation is similar when $h\gamma = 2$. When value of h is fixed and value of the parameter γ gradually increasing approaches to value of $\frac{2}{h}$ then negative eigenvalue approaches to $-\infty$. And analogously, when γ approaches to the value $\frac{2}{h}$ decreasing then one positive eigenvalue indefinitely grows, i.e. approaches $+\infty$, Thus, in other words, one eigenvalue of difference eigenvalue problem in the point $\gamma = \frac{2}{h}$ has a discontinuity from $-\infty$ to $+\infty$. Undoubtedly, there is no such analogous in the spectrum of matrix. Simply, when $h\gamma = 2$ the difference eigenvalue problem is not equivalent to the matrix eigenvalue problem.

By the investigation of difference eigenvalue problem and by the results of this investigation we would like to emphasize that this problem with nonlocal conditions is a separate object for investigation nonequivalent for the matrix eigenvalue problem. Many properties and peculiarities of the spectrum of difference eigenvalue problem with other type of nonlocal conditions are analyzed in [30]. Some examples are provided in that paper, where is demonstrated that the number of eigenvalues could depend on the value of parameter in nonlocal condition. Moreover, it could happen that spectrum of the difference eigenvalue problem could be continuous or empty set.

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