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Joint Approximation of Analytic Functions by Shifts of the Riemann Zeta-Function Twisted by the Gram Function II

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Abstract: Let t_τ be a solution to the equation $\theta(t) = (\tau - 1)\pi$, $\tau > 0$, where $\theta(t)$ is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting points $s = 1/2$ and $s = 1/2 + it$. t_τ is called the Gram function. In the paper, we consider the approximation of collections of analytic functions by shifts of the Riemann zeta-function $(\zeta(s + it_\tau^{\alpha_1}), \dots, \zeta(s + it_\tau^{\alpha_r}))$, where $\alpha_1, \dots, \alpha_r$ are different positive numbers, in the interval $[T, T + H]$ with $H = o(T)$, $T \to \infty$, and obtain the positivity of the density of the set of such shifts. Moreover, a similar result is obtained for shifts of a certain absolutely convergent Dirichlet series connected to $\zeta(s)$. Finally, an example of the approximation of analytic functions by a composition of the above shifts is given.

Keywords: Gram function; joint universality; Riemann zeta-function; weak convergence

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1. Introduction

The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, one of the most important analytic objects of mathematics, is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and has the analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Denote by $\mathbb{P}$ the set of all prime numbers. In virtue of the main arithmetic theorem, the function $\zeta(s)$ equivalently can be defined, for $\sigma > 1$, by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Among many interesting properties and problems of the function $\zeta(s)$, the universality occupies a particular place. The latter property was discovered by S.M. Voronin [1] and, roughly speaking, means that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact sets of the strip D with connected complements, and $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then it is convenient to state a modern version of the Voronin theorem in the following form, see, for example, [2]. Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0. \quad (1)$$

Here, meas$A$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The above inequality shows that there are infinitely many shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$. Moreover, the positivity of a lower density of the set of approximating shifts $\zeta(s+i\tau)$ can be replaced by that of a density for all but at most countably many $\varepsilon > 0$ [3]. B. Bagchi proved [4] that the famous Riemann hypothesis, which asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$, is equivalent to inequality (1) with $f(s) = \zeta(s)$.

A joint version of universality for $\zeta(s)$ was obtained in [5,6] using generalized shifts $\zeta(s+i\varphi(\tau))$ with certain functions $\varphi(\tau)$. Let $a_1 = 1, a_2, \dots, a_r$ be real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$, and, for $j = 1, \dots, r$, $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, in [5], it was proved that, for every $a \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ia\alpha_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

In [6], the shifts $\zeta(s+i\varphi_j(\tau))$ with $\varphi_j(\tau) = \tau^{\alpha_j}(\log\tau)^{\beta_j}$, $\alpha_j, \beta_j \in \mathbb{R}$, were used.

Recall one more type of possible shifts. As usual, denote by $\Gamma(s)$ the Euler gamma-function, and by $\theta(t)$, $t > 0$, the increment of the argument of the function $\pi^{s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. The function $\theta(t)$ is monotonically increasing and unbounded from above for $t \geqslant t^* = 6.2898\dots$; hence, the equation

$$\theta(t) = (\tau - 1)\pi, \quad \tau \geqslant 0, \tag{2}$$

has the unique solution t_τ. The function t_τ with $\tau = n \in \mathbb{N}$ was considered by J.-P Gram [7] in connection with imaginary parts γ_n of nontrivial zeros of the Riemann zeta-function. Therefore, t_n are called the Gram points, and t_τ with arbitrary $\tau \geqslant 0$ is the Gram function. In [8], the joint universality of $\zeta(s)$ using shifts $\zeta(s+it_\tau^{\alpha_j})$ was considered. More precisely, suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers, and, for $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, the main result of [8] states that, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon\right\} > 0. \tag{3}$$

Moreover, "lim inf" can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

All stated above examples of universality theorems for $\zeta(s)$ are not effective in the sense that any value of τ in approximating shifts is not known. In a wider sense, the effectivization of universality for zeta-functions is understood as an indication of intervals as short as possible containing values of τ with an approximating property. An example of universality theorems in short intervals was given in [9]. Let

$$a = \max_{1\leqslant j\leqslant r} |a_j|^{-1} \quad \text{and} \quad \widehat{a} = \max_{1\leqslant j\leqslant r} |a_j|.$$

Theorem 1 ([9]). *Suppose that $a_1, \dots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$, and $a(T\widehat{a})^{1/3}(\log T\widehat{a})^{26/15} \leqslant H \leqslant T$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{\tau \in [T,T+H] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

*Moreover, "*lim inf*" can be replaced by "*lim*" for all but at most countably many $\varepsilon > 0$.*

The aim of this paper is to obtain a version of Theorem 1 for shifts $\zeta(s+it_\tau^{\alpha_j})$. This aim is motivated by an easier possibility to detect approximating shifts in short intervals.

Let $\alpha_1, \dots, \alpha_r$ be the same numbers as in (3). Without a loss of generality, we may suppose that $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. We will use the notation

$$\psi_{\alpha_j}(\tau) = \left(t_\tau^{\alpha_j}\right)^{1/3} \left(\log t_\tau^{\alpha_j}\right)^{26/15},$$

and write $(t_T^{\alpha_j})'$ in place of $(t_\tau^{\alpha_j})'_{\tau=T}$, $j = 1, \dots, r$.

Theorem 2. *Suppose that* $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H_0(K_j)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{\tau \in [T, T+H] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + it_\tau^{a_j}) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

The next theorem is devoted to the approximation of analytic functions by shifts of certain absolutely convergent Dirichlet series. Let $\theta > 0$ be a fixed number, $u > 0$ and

$$v_u(m) = \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}, \quad m \in \mathbb{N}.$$

Because $v_u(m)$ decreases exponentially with respect to m, the series

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}$$

is absolutely convergent for $\sigma > \sigma_0$, with arbitrary finite σ_0.

Theorem 3. *Suppose that* $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$, *and* $u_T \to \infty$ *and* $u_T \ll \exp\{o(T/\log T)^{\alpha_1}\}$. *For* $j = 1, \dots, r$, *let* $K_j \in \mathcal{K}$ *and* $f_j(s) \in H_0(K_j)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{\tau \in [T, T+H] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta_{u_T}(s + it_\tau^{a_j}) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

Theorem 2 can be generalized for certain compositions. We give one example. Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$, for different $a_1, \dots, a_k \in \mathbb{C}$

$$H_{a_1,\dots,a_k}(D) = \left\{g \in H(D) : (g(s) - a_j)^{-1} \in H(D), j = 1, \dots, k\right\},$$

and $H(K)$, $K \in \mathcal{K}$, be the class of continuous functions on K that are analytic in the interior of K.

Theorem 4. *Suppose that* $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$, *and* $\Phi : H^r(D) \to H(D)$ *is a continuous operator such that* $\Phi(S^r) \supset H_{a_1,\dots,a_k}(D)$. *For* $k = 1$, *let* $K \in \mathcal{K}$, *and* $f(s) \in H(K)$ *and* $f(s) \neq a_1$ *on* K. *For* $k \geqslant 2$, *let* $K \subset D$ *be a compact set and* $f(s) \in H_{a_1,\dots,a_k}(D)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H} \text{meas}\left\{\tau \in [T, T+H] : \sup_{s\in K} |\Phi(\zeta(s + it_\tau^{a_1}), \dots, \zeta(s + it_\tau^{\alpha_r})) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many* $\varepsilon > 0$.

Let

$$\Phi(g_1, \dots, g_r) = \cos(g_1 + \cdots + g_r).$$

Consider the equation

$$\frac{e^{iw} + e^{-iw}}{2} = f$$

with $f(s) \in H_{1,-1}(D)$. It is easily seen that

$$w = \frac{1}{i} \log(f \pm \sqrt{f^2 - 1}).$$

Thus, we have

$$\Phi\left(\frac{1}{i} \log(f + \sqrt{f^2 - 1}), 0, \dots, 0\right) = f.$$

Because $(1/i \log(f + \sqrt{f^2 - 1}), 0, \dots, 0) \in S^r$, the inclusion $\Phi(S^r) \supset H_{-1,1}(D)$ is valid. Therefore, by Theorem 4, the functions of the set $H_{-1,1}(D)$ are approximated by shifts $\cos(\zeta(s + it_\tau^{\alpha_1}) + \cdots + \zeta(s + it_\tau^{\alpha_r}))$.

2. Mean Square Estimates

First, we recall the asymptotics for the function t_τ and its derivative as $\tau \to \infty$.

Lemma 1 ([10])**.** *Suppose that t_τ, $\tau \geqslant 0$, is the unique solution of Equation (2) satisfying $\theta(t_\tau) > 0$. Then, for $\tau \to \infty$,*

$$t_\tau = \frac{2\pi\tau}{\log \tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1 + o(1))\right)$$

and

$$t'_\tau = \frac{2\pi}{\log \tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1 + o(1))\right).$$

Note that, in view of Lemma 1,

$$\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} = o(T), \quad T \to \infty.$$

Let $\alpha > 0$ be one of the numbers $\alpha_1, \dots, \alpha_r$, and H satisfy the hypotheses of Theorem 2. We will obtain the upper bound for

$$I(T, H, \sigma, t) \overset{def}{=} \int_T^{T+H} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, d\tau$$

with fixed σ and $t \in \mathbb{R}$. Recall that the notation $g \ll_\theta f$, $f > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|g| \leqslant c(\theta) f$.

Lemma 2. *Suppose that $1/2 < \sigma \leqslant 13/22$ is fixed, $t \in \mathbb{R}$ and $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$. Then,*

$$I(T, H, \sigma, t) \ll_\sigma H(1 + |t|).$$

Proof. Let $T^{(\kappa+\lambda+1)/2(\kappa+1)}(\log T)^{(2+\kappa)/(\kappa+1)} \leqslant H \leqslant T$, where (κ, λ) is an exponential pair such that $2\sigma \leqslant 1 + \lambda - \kappa$. Then, by Theorem 7.1 of [11], uniformly in $\widehat{H}$

$$\int_{T-\widehat{H}}^{T+\widehat{H}} |\zeta(\sigma + it)|^2 \, dt \ll_\sigma \widehat{H}. \tag{4}$$

Taking $(\kappa, \lambda) = (4/11, 6/11)$ gives, for $1/2 < \sigma \leqslant 13/22$, uniformly in $\widehat{H}$, $T^{1/3}$ $(\log T)^{26/15} \leqslant \widehat{H} \leqslant T$, the bound (4).

In view of Lemma 1, for large τ, the function t_τ^α is increasing, while the function $(t_\tau^\alpha)'$ is increasing and decreasing for $\alpha > 1$ and $0 < \alpha \leqslant 1$, respectively. Therefore, for $\alpha > 1$,

$$\begin{aligned}
\int_T^{T+H} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \,\mathrm{d}\tau &= \int_T^{T+H} \frac{1}{(t_\tau^\alpha)'} |\zeta(\sigma + it_\tau^\sigma + it|^2 \,\mathrm{d}(t_\tau^\alpha) \\
&= \int_T^{T+H} \frac{1}{(t_\tau^\alpha)'} \,\mathrm{d}\left(\int_T^{t_\tau^\alpha + t} |\zeta(\sigma + iu|^2 \,\mathrm{d}u\right) \\
&= \frac{1}{(t_T^\alpha)'} \int_T^{\xi} \mathrm{d}\left(\int_T^{t_\tau^\alpha + t} |\zeta(\sigma + iu|^2 \,\mathrm{d}u\right) \\
&= \frac{1}{(t_T^\alpha)'} \int_{t_T^\alpha + t}^{t_\xi^\alpha + t} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \\
&\leqslant \frac{1}{(t_T^\alpha)'} \int_{t_T^\alpha - |t|}^{t_{T+H}^\alpha + |t|} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \\
&= \frac{1}{(t_T^\alpha)'} \int_{t_T^\alpha - |t|}^{t_T^\alpha + H(t_\xi^\alpha)' + |t|} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \\
&\leqslant \frac{1}{(t_T^\alpha)'} \int_{t_T^\alpha - H(t_{2T}^\alpha)' - |t|}^{t_T^\alpha + H(t_{2T}^\alpha)' + |t|} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u,
\end{aligned} \tag{5}$$

where $T \leqslant \xi \leqslant T + H$. Here, we write $(t_\xi^\alpha)' = (t_\tau^\alpha)'_{\tau=\xi}$. Now, we apply (4) with t_T^α in place of T and with $H(t_{2T}^\alpha)'$ in place of H. We have

$$H(t_{2T}^\alpha)' + |t| \geqslant \psi_{\alpha_1}(T) \frac{(t_{2T}^\alpha)'}{(t_T^{\alpha_1})'} \geqslant \psi_\alpha(T) \frac{(t_T^\alpha)'}{(t_T^\alpha)'} = \psi_\alpha(T).$$

Suppose that $H(t_{2T}^\alpha)' + |t| \leqslant t_T^\alpha$. Then, by (4) and (5),

$$I(T, H, \sigma, t) \ll_\sigma \frac{H(t_{2T}^\alpha)' + |t|}{(t_T^\alpha)'} \leqslant_{\sigma,\alpha} H + \frac{|t|}{(t_T^\alpha)'} \ll_{\sigma,\alpha} H\left(1 + \frac{|t|}{\psi_{\alpha_1}(T)}\right) \ll_{\sigma,\alpha} H(1 + |t|). \tag{6}$$

Now, let $H(t_{2T}^\alpha)' + |t| > t_T^\alpha$. Then, we apply the well-known estimate

$$\int_{-T}^{T} |\zeta(\sigma + it)|^2 \,\mathrm{d}t \ll_\sigma T. \tag{7}$$

In this case, we have $t_T^\alpha + H(t_{2T}^\alpha)' + |t| < 2(H(t_{2T}^\alpha)' + |t|)$ and $t_T^\alpha - H(t_{2T}^\alpha)' - |t| > -2(H(t_{2T}^\alpha)' + |t|)$. Therefore, estimates (5) and (7) imply the bound

$$I(T, H, \sigma, \alpha) \ll \int_{-2(H(t_{2T}^\alpha)' + |t|)}^{2(H(t_{2T}^\alpha)' + |t|)} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \ll_\sigma \frac{H((t_{2T}^\alpha)' + |t|)}{(t_T^\alpha)'} \ll_\sigma H(1 + |t|).$$

This, together with (6), proves the lemma in the case $\alpha > 1$.

If $\alpha \leqslant 1$, then, in virtue of Lemma 1, the function $(t_\tau^\alpha)'$ is decreasing for large τ. Therefore,

$$\begin{aligned}
\int_T^{T+H} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \,\mathrm{d}\tau &= \frac{1}{(t_{T+H}^\alpha)'} \int_{t_\xi^\alpha + t}^{t_{T+H}^\alpha + t} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \\
&\leqslant \frac{1}{(t_{2T}^\alpha)'} \int_{t_T^\alpha - |t|}^{t_{T+H}^\alpha + |t|} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u \\
&\leqslant \frac{1}{(t_{2T}^\alpha)'} \int_{t_T^\alpha - H(t_T^\alpha)' - |t|}^{t_T^\alpha + H(t_T^\alpha)' + |t|} |\zeta(\sigma + iu)|^2 \,\mathrm{d}u.
\end{aligned} \tag{8}$$

We observe that

$$H(t_T^\alpha)' + |t| \geqslant \psi_{\alpha_1}(T) \frac{(t_T^\alpha)'}{(t_T^{\alpha_1})'} \geqslant \psi_\alpha(T)$$

because $\alpha \geqslant \alpha_1$. Thus, if $H(t_T^\alpha)' + |t| \leqslant t_T^\alpha$, then, by (4) and (8),

$$I(T,H,\sigma,t) \ll_\sigma \frac{H(t_T^\alpha)' + |t|}{(t_{2T}^\alpha)'} \ll_{\sigma,\alpha} H + \frac{|t|}{(t_{2T}^\alpha)'} \ll_{\sigma,\alpha} H\left(1 + \frac{|t|(t_T^\alpha)'}{\psi_{\alpha_1}(T)(t_{2T}^\alpha)'}\right) \ll_{\sigma,\alpha} H(1+|t|). \tag{9}$$

If $H(t_T^\alpha)' + |t| > t_T^\alpha$, then, similarly as above, we have by (7)

$$I(T,H,\sigma,t) \ll \int_{-2(H(t_T^\alpha)'+|t|)}^{2(H(t_T^\alpha)'+|t|)} |\zeta(\sigma+iu)|^2 \,\mathrm{d}u \ll_\sigma \frac{H((t_T^\alpha)'+|t|)}{(t_{2T}^\alpha)'} \ll_\sigma H(1+|t|).$$

This and (9) prove the lemma in the case $\alpha \leqslant 1$. □

We will apply Lemma 2 for estimation of the distance between the shifts $\zeta(s+it_\tau^\alpha)$ and $\zeta_u(s+it_\tau^\alpha)$.

Lemma 3. *Suppose that $K \subset D$ is a compact set, and $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$. Then, there exists $\varepsilon > 0$ and $C > 0$ such that*

$$\frac{1}{H}\int_T^{T+H} \sup_{s\in K} |\zeta(s+it_\tau^\alpha) - \zeta_u(s+it_\tau^\alpha)| \,\mathrm{d}\tau \ll_{\varepsilon,\theta,K} u^{-\varepsilon} + u^{1/2-2\varepsilon}\exp\left\{-C\left(\frac{T}{\log T}\right)^\alpha\right\}.$$

Proof. Denote

$$l_u(s) = \frac{s}{\theta}\Gamma\left(\frac{s}{\theta}\right)u^s.$$

The Mellin formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s}\,\mathrm{d}s = \mathrm{e}^{-b}, \quad a,b>0,$$

leads to the representation, see, for example, [2],

$$\zeta_u(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z)\frac{l_u(z)}{z}\,\mathrm{d}z, \tag{10}$$

where $\theta_1 > 1/2$. Because $K \subset D$, there exists $1/11 \geqslant \varepsilon > 0$ such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1-\varepsilon$ for $s = \sigma + it \in K$. Thus, for $s \in K$, we have $0 > 1/2 + \varepsilon - \sigma \stackrel{def}{=} \theta_2$, and take $\theta_1 = 1/2 + \varepsilon$. Then, the representation (10) and the residue theorem show that, for $s \in K$,

$$\zeta_u(s) - \zeta(s) = \frac{1}{2\pi i}\int_{\theta_2-i\infty}^{\theta_2+i\infty} \zeta(s+z)\frac{l_u(z)}{z}\,\mathrm{d}z + \frac{l_u(1-s)}{1-s}$$

because of simple poles of the integration function at the points $z=0$ and $z = 1-s$. Hence, for $s \in K$,

$$\begin{aligned}
\zeta_u(s+it_\tau^\alpha) - \zeta(s+it_\tau^\alpha) =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \zeta\left(\frac{1}{2}+\varepsilon+it+it_\tau^\alpha+iv\right)\frac{l_u(1/2+\varepsilon-\sigma+iv)}{1/2+\varepsilon-\sigma+iv}\,\mathrm{d}v \\
&+ \frac{l_u(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha} \\
\ll& \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iv\right)\right| \sup_{s\in K}\left|\frac{l_u(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|\mathrm{d}v \\
&+ \sup_{s\in K}\left|\frac{l_u(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha}\right|
\end{aligned}$$

after a shift $t+v \to v$. Therefore,

$$\frac{1}{H}\int_T^{T+H}\sup_{s\in K}|\zeta(s+it_\tau^\alpha)-\zeta_u(s+it_\tau^\alpha)|\,d\tau$$
$$\ll \int_{-\infty}^{\infty}\left(\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iv\right)\right|d\tau\right)\sup_{s\in K}\left|\frac{l_u(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv}\right|dv$$
$$+\frac{1}{H}\int_T^{T+H}\sup_{s\in K}\left|\frac{l_u(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha}\right|d\tau \stackrel{def}{=} I_1+I_2. \quad (11)$$

An application of Lemma 2 gives

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iv\right)\right|d\tau \leqslant \left(\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+it_\tau^\alpha+iv\right)\right|^2 d\tau\right)^{1/2}$$
$$\ll_\varepsilon (1+|v|)^{1/2} \ll_\varepsilon 1+|v|. \quad (12)$$

Using the well-known estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c>0, \quad (13)$$

which is uniform in any interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $\sigma_1<\sigma_2$, we find, for all $s\in K$,

$$\frac{l_u(1/2+\varepsilon-s+iv)}{1/2+\varepsilon-s+iv} \ll_\theta u^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|v-t|\right\} \ll_{\theta,K} u^{-\varepsilon}\exp\{-c_1|v|\}, \quad c_1>0.$$

This together with (12) yields

$$I_1 \ll_{\varepsilon,\theta,K} u^{-\varepsilon}\int_{-\infty}^{\infty}(1+|v|)\exp\{-c_1|v|\}\,dv \ll_{\varepsilon,\theta,K} u^{-\varepsilon}. \quad (14)$$

Similarly, as above, (13) implies, for all $s\in K$,

$$\frac{l_u(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha} \ll_\theta u^{1-\sigma}\exp\left\{-\frac{c}{\theta}|t_\tau^\alpha+t|\right\} \ll_{\theta,K} u^{1/2-2\varepsilon}\exp\{-c_2 t_\tau^\alpha\}, \quad c_2>0.$$

Therefore, by Lemma 1,

$$I_2 \ll_{\theta,K} u^{1/2-2\varepsilon}\frac{1}{H}\int_T^{T+H}\exp\{-c_2 t_\tau^\alpha\}\,d\tau \ll_{\theta,K} u^{1/2-2\varepsilon}\frac{1}{H}\int_T^{T+H}\exp\left\{-c_3\left(\frac{\tau}{\log\tau}\right)^\alpha\right\}d\tau$$
$$\ll_{\theta,K} u^{1/2-2\varepsilon}\exp\left\{-c_4\left(\frac{T}{\log T}\right)^\alpha\right\}, \quad c_3,c_4>0.$$

This, together with estimates (14) and (11), proves the lemma. □

3. Limit Theorems

In this section, we will prove a probabilistic joint limit theorem for the Riemann zeta-function twisted by the Gram function in short intervals. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$, $\underline{t}_\tau^{\underline{\alpha}}=(t_\tau^{\alpha_1},\dots,t_\tau^{\alpha_r})$,

$$\underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}})=(\zeta(s+it_\tau^{\alpha_1}),\dots,\zeta(s+it_\tau^{\alpha_r})).$$

Moreover, let $H(D)$ stand for the space of analytic functions on D equipped with the topology of uniform convergence on compacta, and

$$H^r(D)=\underbrace{H(D)\times\cdots\times H(D)}_{r}.$$

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,H}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \underline{\zeta}(s + i\underline{t}_\tau^{\alpha}) \in A\Big\},$$

and consider the weak convergence for $P_{T,H}$ as $T \to \infty$.

For the definition of a limit measure, we need one topological structure. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Define one more set

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Then, again, Ω^r is a compact topological Abelian group. Hence, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. By $\omega_j(p)$ denote the pth component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \ldots, r$, and by $\omega = (\omega_1, \ldots, \omega_r)$ denote the elements of Ω^r. On the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \omega) = (\zeta(s, \omega_1), \ldots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \ldots, r.$$

Note that the latter products, for almost all ω_j with respect to the Haar measure m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, are uniformly convergent on compact subsets of D, see, for example, [2], and define the $H(D)$-valued random elements. Because the Haar m_H is the product of the measures m_{jH}, $\underline{\zeta}(s, \omega)$ is the $H^r(D)$-valued random element. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega)$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H\Big\{\omega \in \Omega^r : \underline{\zeta}(s, \omega) \in A\Big\}, \quad A \in \mathcal{B}(H^r(D)).$$

Recall that the support $S_{P_{\underline{\zeta}}}$ of the measure $P_{\underline{\zeta}}$ is a minimal closed subset of $H^r(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The set $S_{P_{\underline{\zeta}}}$ consists of all elements $\underline{g} \in H^r(D)$ such that, for every open neighborhood $\underline{G}$ of $\underline{g}$, the inequality $P_{\underline{\zeta}}(\underline{G}) > 0$ is satisfied. Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Now, we state a limit theorem for $P_{T,H}$.

Theorem 5. *Suppose that $\psi_{\alpha_1}(T)((t_T^{\alpha_1})')^{-1} \leqslant H \leqslant T$. Then, $P_{T,H}$ converges weakly to $P_{\underline{\zeta}}$, as $T \to \infty$. Moreover, the support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

We divide the proof of Theorem 5 into several lemmas.

We start with a limit lemma on the space Ω^r. For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_{T,H}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \Big((p^{-t_\tau^{\alpha_1}} : p \in \mathbb{P}), \ldots, (p^{-t_\tau^{\alpha_r}} : p \in \mathbb{P})\Big) \in A\Big\}.$$

Lemma 4. *Under hypotheses of Theorem 5, $Q_{T,H}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We will apply the Fourier transform method. Denote by $g_{T,H}(\underline{k}_1, \ldots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \ldots, r$, the Fourier transform of $Q_{T,H}$, i.e.,

$$g_{T,H}(\underline{k}_1, \ldots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p) \right) \mathrm{d}Q_{T,H},$$

where the sign "$*$" shows that only a finite number of integers k_{jp} are distinct from zero. The definition of $Q_{T,H}$ implies

$$\begin{aligned} g_{T,H}(\underline{k}_1, \ldots, \underline{k}_r) &= \frac{1}{H} \int_T^{T+H} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-ik_{jp} t_\tau^{\alpha_j}} \right) \mathrm{d}\tau \\ &= \frac{1}{H} \int_T^{T+H} \exp\left\{ -i \sum_{j=1}^{r} t_\tau^{\alpha_j} \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p \right\} \mathrm{d}\tau. \end{aligned} \tag{15}$$

Let $\underline{0} = (0, \ldots, 0, \ldots)$. Obviously, by (15),

$$g_{T,H}(\underline{0}, \ldots, \underline{0}) = 1. \tag{16}$$

Denote

$$a_j = \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p,$$

and suppose that $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$. Then, there exists $j \in \{1, \ldots, r\}$ such that $\underline{k}_j \neq \underline{0}$. It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, $a_j \neq 0$. Let $j_0 = \max(j : a_j \neq 0)$, and

$$A(\tau) \stackrel{def}{=} \sum_{j=1}^{r} t_\tau^{\alpha_j} \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p = \sum_{j=1}^{r} a_j t_\tau^{\alpha_j}.$$

Because $\alpha_1 < \cdots < \alpha_r$, we have, by Lemma 1,

$$A'(\tau) = \sum_{j=1}^{r} \alpha_j a_j t_\tau^{\alpha_j - 1} t_\tau{}' = a_{j_0} (t_\tau^{\alpha_{j_0}})'(1 + o(1))$$

as $\tau \to \infty$. Hence,

$$(A'(\tau))^{-1} = \frac{1}{a_{j_0} (t_\tau^{\alpha_{j_0}})'} (1 + o(1))$$

as $\tau \to \infty$. Then,

$$\begin{aligned} \frac{1}{H} \int_T^{T+H} \cos A(\tau)\, \mathrm{d}\tau &= \frac{1}{H} \int_T^{T+H} \frac{1}{A'(\tau)} \, \mathrm{d} \sin A(\tau) \\ &= \frac{1}{a_{j_0} H} \int_T^{T+H} \frac{1}{(t_\tau^{\alpha_{j_0}})'} \, \mathrm{d} \sin A(\tau) + \frac{1}{H} \int_T^{T+H} \frac{o(1)}{(t_\tau^{\alpha_{j_0}})'} \, \mathrm{d} \sin A(\tau) \\ &\ll_{j_0} (H (t_T^{\alpha_{j_0}})')^{-1} + o(1) \ll_{j_0} (H (t_T^{\alpha_1})')^{-1} + o(1) = o(1) \end{aligned}$$

as $T \to \infty$. Similarly, we obtain that

$$\frac{1}{H} \int_T^{T+H} \sin A(\tau)\, \mathrm{d}\tau = o(1)$$

as $T \to \infty$. Therefore, the last two estimates and (15) show that

$$g_{T,H}(\underline{k}_1, \ldots, \underline{k}_r) = o(1), \quad T \to \infty.$$

Thus, by (16),

$$\lim_{T\to\infty} g_{T,H}(\underline{k}_1,\dots,\underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) = (\underline{0},\dots,\underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1,\dots,\underline{k}_r) \neq (\underline{0},\dots,\underline{0}). \end{cases}$$

Because the right-hand side of this equality is the Fourier transform of the measure m_H, the lemma is proved. □

Lemma 4 is a key lemma to obtain limit lemmas for Dirichlet series in the space $H^r(D)$. Extend the function $\omega_j(p)$ to the set $\mathbb{N}$ by

$$\omega_j(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad m \in \mathbb{N},\ j = 1,\dots r,$$

and, for $n \in \mathbb{N}$, define

$$\underline{\zeta}_n(s,\omega) = (\zeta_n(s,\omega_1),\dots,\zeta(s,\omega_r)),$$

where

$$\zeta_n(s,\omega_j) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega_j(m)}{m^s}, \quad j = 1,\dots,r.$$

The latter series is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. Therefore, the mapping $w_n : \Omega^r \to H^r(D)$ given by $w_n(\omega) = \underline{\zeta}_n(s,\omega)$ is continuous. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,H,n}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \underline{\zeta}_n(s + i\underline{t}_\tau^{\alpha}) \in A\Big\}.$$

The weak convergence of $P_{T,H,n}$ as $T \to \infty$ is based on one simple property of weak convergence of probability measures. Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two spaces, and $w : \mathbb{X}_1 \to \mathbb{X}_2$ a $(\mathcal{B}(\mathbb{X}_1), \mathcal{B}(\mathbb{X}_2))$-measurable mapping, i.e.,

$$w^{-1}\mathcal{B}(\mathbb{X}_2) \subset \mathcal{B}(\mathbb{X}_1).$$

Then, every probability measure P on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$ induces the unique probability measure Pw^{-1} on $(\mathbb{X}_2, \mathcal{B}(\mathbb{X}_2))$ defined by

$$Pw^{-1}(A) = P(w^{-1}A), \quad A \in \mathcal{B}(\mathbb{X}_2).$$

Moreover, it turns out that in such a situation, the weak convergence is preserved, i.e., the following lemma is valid; see, for example, [12], Theorem 5.1.

Lemma 5. *Suppose that P and P_n, $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}_1, \mathcal{B}(\mathbb{X}_1))$,$w : \mathbb{X}_1 \to \mathbb{X}_2$ a continuous mapping, and P_n converges weakly to P as $n \to \infty$. Then, $P_n w^{-1}$ converges weakly to Pw^{-1} as $n \to \infty$.*

Now, we state a limit lemma for $P_{T,H,n}$.

Lemma 6. *Under hypotheses of Theorem 5, $P_{T,H,n}$ converges weakly to the measure $V_n \overset{def}{=} m_H w_n^{-1}$ as $T \to \infty$.*

Proof. By the definition of w_n,

$$w_n\Big((p^{-it_\tau^{\alpha_1}} : p \in \mathbb{P}),\dots,(p^{-it_\tau^{\alpha_r}} : p \in \mathbb{P})\Big) = \underline{\zeta}_n(s + i\underline{t}_\tau^{\alpha}).$$

Therefore, the definition of $P_{T,H,n}$ implies, for $A \in \mathcal{B}(H^r(D))$,

$$P_{T,H,n}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \Big((p^{-it_\tau^{\alpha_1}} : p \in \mathbb{P}), \ldots, (p^{-it_\tau^{\alpha_r}} : p \in \mathbb{P})\Big) \in w_n^{-1}A\Big\}$$
$$= Q_{T,H}(w_n^{-1}A).$$

Thus, $P_{T,H,n} = Q_{T,H}w_n^{-1}$. Because the mapping w_n is continuous, this equality and Lemmas 4 and 5 prove the lemma. □

The measure V_n plays an important role for the proof of Theorem 5. Because V_n is independent on any hypotheses, we have the following statement: see proofs of Lemma 10 and Theorem 3 in [9].

Lemma 7. *The measure V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{\zeta}}$ is the set S^r.*

Before the proof of Theorem 5, we recall one lemma on convergence in distribution ($\xrightarrow{\mathcal{D}}$) of random elements; see, for example, [12], Theorem 4.2.

Lemma 8. *Suppose that the space $(\mathbb{X}, d)$ is separable, and the $\mathbb{X}$-valued random elements Y_n and X_{kn} are defined on the same probability space with measure μ. Let, for every k,*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k,$$

and

$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0,$$

then

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

Proof of Theorem 5. Let $\theta_{T,H}$ be a random variable defined on a certain probability space with measure μ, and uniformly distributed on $[T, T+H]$. Define the $H^r(D)$-valued random elements

$$X_{T,H,n} = X_{T,H,n}(s) = \underline{\zeta}_n(s + it^{\underline{\alpha}}_{\theta_{T,H}})$$

and

$$X_{T,H} = X_{T,H}(s) = \underline{\zeta}(s + it^{\underline{\alpha}}_{\theta_{T,H}}).$$

Moreover, denote by X_n the $H^r(D)$-valued random element having the distribution V_n. Then, by Lemma 7,

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}} \tag{17}$$

and, by Lemma 6,

$$X_{T,H,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n. \tag{18}$$

Now, recall the metric in $H^r(D)$. Let $\{K_l : l \in \mathbb{N}\} \subset D$ be a sequence of embedded compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and every compact set K of D lies in a certain K_l. For example, we can take K_l closed rectangles of D. Then,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric in $H(D)$ inducing the topology of uniform convergence on compacta. Taking

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1\leqslant j\leqslant r} \rho(g_{1j}, g_{2j}), \quad \underline{g}_k = (g_{k1}, \dots, g_{kr}) \in H^r(D),\ k = 1, 2,$$

we obtain a metric in $H^r(D)$ inducing the product topology.

Now, return to Lemma 3. Taking $u = n$, we find by Lemma 3 that, for every compact set $K \subset D$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta(s + it_\tau^\alpha) - \zeta_n(s + it_\tau^\alpha)| \,\mathrm{d}\tau = 0.$$

This, and the definitions of the metrics ρ and $\underline{\rho}$, imply

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \underline{\rho}(\underline{\zeta}(s + i\underline{t}_\tau^\alpha), \underline{\zeta}_n(s + i\underline{t}_\tau^\alpha)) \,\mathrm{d}\tau = 0.$$

Hence, by the definitions of $X_{T,H,n}$ and $X_{T,H}$, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\Big\{\underline{\rho}(X_{T,H}, X_{T,H,n}) \geqslant \varepsilon\Big\} \\
&\qquad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon H} \int_T^{T+H} \underline{\rho}(\underline{\zeta}(s + i\underline{t}_\tau^\alpha), \underline{\zeta}_n(s + i\underline{t}_\tau^\alpha)) \,\mathrm{d}\tau = 0.
\end{aligned}$$

The latter equality, together with relations (17) and (18), shows that all hypotheses of Lemma 8 are satisfied by the random elements X_n, $X_{T,H,n}$ and $X_{T,H}$. Therefore, we obtain that

$$X_{T,H} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

and this relation is equivalent to the assertion of the theorem. □

The weak convergence of probability measures has several equivalents; see, for example, [12], Theorem 2.1.

Lemma 9. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then, the following statements are equivalent:*

1° P_n converges weakly to P as $n \to \infty$;

2° For every open set $G \subset \mathbb{X}$,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

3° For every closed set $F \subset \mathbb{X}$,

$$\limsup_{n\to\infty} P_n(F) \leqslant P(F);$$

4° For every continuity set A of P (A is a continuity set of P if $P(\partial A) = 0$, where ∂A is the boundary of A),

$$\lim_{n\to\infty} P_n(A) = P(A).$$

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,H,u_T}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \underline{\zeta}_{u_T}(s + i\underline{t}_\tau^\alpha) \in A\Big\}.$$

Theorem 6. *Under hypotheses of Theorem 3, P_{T,H,u_T} converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. We preserve the notation of the proof of Theorem 5 for $\theta_{T,H}$ and $X_{T,H}$ and define one more $H^r(D)$-valued random element

$$X_{T,H,u_T} = X_{T,H,u_T}(s) = \underline{\zeta}_{u_T}(s + i\underline{t}^\alpha_{\theta_{T,H}}).$$

Let $\varepsilon > 0$ and a closed set $F \subset H^r(D)$ be fixed, and $F_\varepsilon = \{\underline{g} \in H^r(D) : \underline{\rho}(\underline{g}, F) \leqslant \varepsilon\}$, where $\underline{\rho}(\underline{g}, F) = \inf_{\underline{g}_1 \in F} \underline{\rho}(\underline{g}, \underline{g}_1)$. Then, the set F_ε is closed. Therefore, by Theorem 5 and 3° of Lemma 9,

$$\limsup_{T\to\infty} \mu\{X_{T,H} \in F_\varepsilon\} \leqslant P_{\underline{\zeta}}(F_\varepsilon). \tag{19}$$

It is easily seen that

$$\{X_{T,H,u_T} \in F\} \subset \{X_{T,H} \in F_\varepsilon\} \cup \{\underline{\rho}(X_{T,H}, X_{T,H,u_T}) \geqslant \varepsilon\},$$

thus

$$\mu\{X_{T,H,u_T} \in F\} \leqslant \mu\{X_{T,H} \in F_\varepsilon\} + \mu\{\underline{\rho}(X_{T,H}, X_{T,H,u_T}) \geqslant \varepsilon\}. \tag{20}$$

An application of Lemma 3 yields

$$\lim_{T\to\infty} \frac{1}{H} \int_T^{T+H} \underline{\rho}(\underline{\zeta}(s + i\underline{t}_\tau^\alpha), \underline{\zeta}_{u_T}(s + i\underline{t}_\tau^\alpha))\, d\tau = 0.$$

Hence,

$$\lim_{T\to\infty} \mu\{\underline{\rho}(X_{T,H}, X_{T,H,u_T}) \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon H} \int_T^{T+H} \underline{\rho}(\underline{\zeta}(s + i\underline{t}_\tau^\alpha), \underline{\zeta}_{u_T}(s + i\underline{t}_\tau^\alpha))\, d\tau = 0.$$

The latter equality and (19) and (20) imply

$$\limsup_{T\to\infty} \mu\{\{X_{T,H,u_T} \in F\} \leqslant P_{\underline{\zeta}}(F_\varepsilon),$$

and if $\varepsilon \to 0$,

$$\limsup_{T\to\infty} \mu\{\{X_{T,H,u_T} \in F\} \leqslant P_{\underline{\zeta}}(F),$$

which, together with 3° of Lemma 9, proves the theorem. □

For $A \in \mathcal{B}(H(D))$, define

$$P_{T,H,\Phi}(A) = \frac{1}{H}\text{meas}\Big\{\tau \in [T, T+H] : \Phi(\underline{\zeta}(s + i\underline{t}_\tau^\alpha)) \in A\Big\}.$$

Theorem 7. *Under hypotheses of Theorem 4, $P_{T,H,\Phi}$ converges weakly to $P_{\underline{\zeta}}\Phi^{-1}$. Moreover, the support of the measure $P_{\underline{\zeta}}\Phi^{-1}$ contains the closure of the set $H_{a_1,\dots,a_k}(D)$.*

Proof. Because $P_{T,H,\Phi} = P_{T,H}\Phi^{-1}$, and the operator Φ is continuous, the first assertion of the theorem follows from Theorem 5 and Lemma 5.

Let g be an arbitrary element of the set $\Phi(S^r)$, and G an open neighborhood of g. Because Φ is continuous, $\Phi^{-1}G$ is an open neighborhood of a certain element of the set S^r.

In view of Theorem 5, the set S^r is the support of the measure $P_{\underline{\zeta}}$; therefore, $P_{\underline{\zeta}}(\Phi^{-1}G) > 0$. Hence, $P_{\underline{\zeta}}\Phi^{-1}(G) > 0$. Moreover,

$$P_{\underline{\zeta}}\Phi^{-1}(\Phi(S^r)) = P_{\underline{\zeta}}(\Phi^{-1}\Phi(S^r)) = P_{\underline{\zeta}}(S^r) = 1.$$

However, the support of $P_{\underline{\zeta}}\Phi^{-1}$ is a closed set, and we have that the support of $P_{\underline{\zeta}}\Phi^{-1}$ contains the closure of the set $\Phi(S^r)$. Because, by a hypotheses of theorem, $\Phi(S^r) \supset H_{a_1,\dots,a_k}(D)$, we obtain that the support of $P_{\underline{\zeta}}\Phi^{-1}$ contains the closure of $H_{a_1,\dots,a_k}(D)$. The theorem is proved. □

4. Proofs of Approximation Theorems

The proofs of Theorems 2–4 are based on limit Theorems 5–7 and the Mergelyan theorem on approximation of analytic functions by polynomials [13]. For convenience, we state the latter theorem as the next lemma.

Lemma 10. *Let $K \subset \mathbb{C}$ be a compact set with connected complement and $g(s)$ a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Proof of Theorem 2. For polynomials $p_1(s), \dots, p_r(s)$, define the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

Because, by Theorem 5, $(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_r(s)})$ is an element of the support of the measure $P_{\underline{\zeta}}$, we have

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{21}$$

Now, using Lemma 10, we choose the polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2},$$

and define the set

$$\widehat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

Then, we have $G_\varepsilon \subset \widehat{G}_\varepsilon$. Therefore, by (21), $P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0$. Hence, by Theorem 5 and 2° of Lemma 9,

$$\liminf_{T\to\infty} P_{T,H}(\widehat{G}_\varepsilon) \geqslant P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0.$$

This, and the definitions of $P_{T,H}$ and $\widehat{G}_\varepsilon$, prove the first assertion of the theorem.

For the proof of the second assertion of the theorem, observe that the boundaries $\partial\widehat{G}_{\varepsilon_1}$ and $\partial\widehat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Therefore, $P_{\underline{\zeta}}(\partial\widehat{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. Hence, the set $\widehat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 5 and 4° of Lemma 9, we have

$$\lim_{T\to\infty} P_{T,H}(\widehat{G}_\varepsilon) = P_{\underline{\zeta}}(\widehat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$, and the definitions of $P_{T,H}$ and $\widehat{G}_\varepsilon$ yield the second assertion of the theorem. □

Proof of Theorem 3. We use Theorem 6 and repeat the proof of Theorem 2. □

Proof of Theorem 4. *The case* $k = 1$. By Lemma 10, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{22}$$

Because $f(s) \neq a_1$ on K, also $p(s) \neq a_1$ (if ε is small enough) on K. Therefore, by Lemma 10 again, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K} \left| p(s) - a_1 - \mathrm{e}^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{23}$$

The function $g_1(s) =^{def} \mathrm{e}^{q(s)} + a_1 \neq a_1$, thus $g_1(s) \in H_{a_1}(D)$. Therefore, in view of Theorem 7, the function $g_1(s)$ belongs to the support of the measure $P_{\underline{\zeta}}\Phi^{-1}$. Hence,

$$P_{\underline{\zeta}}\Phi^{-1}(G_{1\varepsilon}) > 0, \tag{24}$$

where

$$G_{1\varepsilon} = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - g_1(s)| < \frac{\varepsilon}{2} \right\}.$$

Let

$$\mathcal{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, in virtue of (22) and (23), we have $G_{1\varepsilon} \subset \mathcal{G}_\varepsilon$. Therefore, by (24),

$$P_{\underline{\zeta}}\Phi^{-1}(\mathcal{G}_\varepsilon) > 0. \tag{25}$$

This, Theorem 7 and 1° and 2° of Lemma 9 show that

$$\liminf_{T\to\infty} P_{T,H,\Phi}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\zeta}}\Phi^{-1}(\mathcal{G}_\varepsilon).$$

This and (25) prove the first assertion of the theorem.

As in the proof of Theorem 2, the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}\Phi^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, by Theorem 7, 1° and 4° of Lemma 9 and (25), we have the limit

$$\lim_{T\to\infty} P_{T,H,\Phi}(\mathcal{G}_\varepsilon) = P_{\underline{\zeta}}\Phi^{-1}(G_\varepsilon)$$

exists and is positive for all but at most countably many $\varepsilon > 0$. This gives the second assertion of the theorem.

The case $k \geqslant 2$. Because $f(s) \in H_{a_1,\dots,a_k}(D)$, by Theorem 7, $f(s)$ is an element of the support of the measure $P_{\underline{\zeta}}\Phi^{-1}$. Thus, inequality (25) is valid, and it remains to repeat the above arguments of the case $k = 1$ for the set $\mathcal{G}_\varepsilon$. □

5. Conclusions

The Gram function t_τ, $\tau \geqslant 0$, is defined as a solution of the equation $\theta(t) = (\tau - 1)\pi$, $\tau \geqslant 0$, where $\theta(t)$ is the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. In the paper, the approximation theorems of a collection of analytic functions by shifts $(\zeta(s + it_\tau^{\alpha_1}), \dots, \zeta(s + it_\tau^{\alpha_r}))$ of the Riemann zeta-function, where $\alpha_1, \dots, \alpha_r$ are different positive numbers, are obtained. It is proved that the set of those shifts has a positive density in the intervals $[T, T + H]$ with $H = o(T)$ as $T \to \infty$. This shows that this set is infinite. A similar result is obtained for an absolutely convergent Dirichlet series $\zeta_{u_T}(s)$, where $u_T \to \infty$ as $T \to \infty$. Moreover, the approximation of the analytic functions by a composition $\Phi(\zeta(s + it_\tau^{\alpha_1}), \dots, \zeta(s + it_\tau^{\alpha_r}))$, where $\Phi : H^r(D) \to H(D)$ is a certain continuous operator, is obtained. The case of short intervals is one of the ways of effectivization of universality theorems for zeta-functions.

All the theorems of the paper are results of pure mathematics, more precisely, contributions to the theory of the Riemann zeta-function. On the other hand, they are a starting point for the development of some of the problems of the theory of $\zeta(s)$. One of the classical problems of $\zeta(s)$ is related to the value denseness of $\zeta(s)$. Let $1/2 < \sigma < 1$ be fixed. By the Bohr–Courant theorem [14], the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$

is dense in $\mathbb{C}$. Voronin proved [15] a more general result on the denseness in $\mathbb{C}^N$ of the set

$$\Big\{\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(N-1)}(\sigma + it) : t \in \mathbb{R}\Big\}.$$

The theorems of the paper allow to consider the denseness of more complicated sets, for example, of the set

$$\Big\{\Big(\zeta(\sigma + it_\tau^{\alpha_1}), \zeta'(\sigma + it_\tau^{\alpha_1}), \dots, \zeta^{(N_1-1)}(\sigma + it_\tau^{\alpha_1}), \dots, \\ \zeta(\sigma + it_\tau^{\alpha_r}), \zeta'(\sigma + it_\tau^{\alpha_r}), \dots, \zeta^{(N_r-1)}(\sigma + it_\tau^{\alpha_r})\Big) : \tau \in \mathbb{R}\Big\}$$

in $\mathbb{C}^{N_1+\dots+N_r}$, $N_j \in \mathbb{N}$, $j = 1, \dots, r$.

The second problem connected to the results of the paper is the independence of the function $\zeta(s)$. This problem was mentioned in the description of the 18th Hilbert problem presented during the ICM of 1900. A. Ostrowski proved [16] the algebraic-differential independence of $\zeta(s)$. Voronin obtained [17] the functional independence of $\zeta(s)$, i.e., he proved that if $F_0, F_1, \dots, F_m : \mathbb{C}^N \to C$ are continuous functions, and the equality

$$\sum_{l=0}^{m} s^l F_l\Big(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\Big) = 0$$

holds identically for $s \in \mathbb{C}$, then $F_l \equiv 0$ for $l = 0, 1, \dots, m$. We have a conjecture that the results of the paper can extend the latter Voronin theorem, including its joint version.

Finally, at it was mentioned in the introduction, the universality theorems for $\zeta(s)$ are closely connected to the Riemann hypothesis (RH) which is one of the seven most important Millenium problems of mathematics. Therefore, the development of various types of universality, maybe, leads to the proof or disproof of RH.

The theorems of the paper also have some practical application aspects connected to the estimation of complicated analytic functions. If α_1 is sufficiently large, then H can be small enough. Thus, the approximation value τ lies in a very short interval, and we can estimate $f_j(s)$ by using the inequality

$$\sup_{s \in K_j} \big|f_j(s)\big| \leqslant \sup_{s \in K_j} \Big|\zeta(s + it_\tau^{\alpha_j})\Big| + \varepsilon,$$

and the known estimates and continuity for $\zeta(s)$. For example, this can be applied for the estimation of multiple integrals over analytic curves in quantum mechanics, as it was done in a one-dimensional case in [18]. In general, universality theorems for $\zeta(s)$ can be applied in all fields of mathematics that use estimates of analytic functions.

Moreover, universality theorems can be used [19] in quantum mechanics for the description of the behaviour of particles.

We note that in the applications, discrete versions of universality theorems are more convenient. Therefore, our next paper will be devoted to a more complicated discrete generalization of the theorems of the paper.

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