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Parametric Identification of Linear Systems Followed by Non-Invertible Piecewise Nonlinearities

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Abstract. The aim of the given paper is the development of an approach for parametric identification of *Wiener* systems with static non-invertible function, i.e., when the linear part with unknown parameters is followed by piecewise linear nonlinearity with negative slopes. It is shown here that the problem of identification of a nonlinear *Wiener* system could be reduced to a linear parametric estimation problem by a simple input-output data reordering and by a following data partition into three data sets. A technique based on ordinary least squares (*LS*) is proposed here for the separate estimation of parameters of linear and nonlinear parts of the *Wiener* system, including the unknown threshold of piecewise nonlinearity, by processing respective particles of input-output observations. The simulation results are given.

Introduction

A lot of physical systems are naturally described as *Wiener* systems, i.e., when the linear system is followed by a static nonlinearity [1 – 3]. Frequently, nonlinearities of actuator devices occur on the output of the system to be controlled that limit the system performance considerably [2, 3]. Therefore, *Wiener* models, consisting of a linear dynamic block followed by a static nonlinear one, are considered to be suitable for a broad spectrum of nonlinearities [4]. A special class of such systems is piecewise affine *Wiener* systems, consisting of some subsystems, between which occasional switchings happen at different time moments [3]. Assuming the nonlinearity to be piecewise linear, one could let the linear part of the *Wiener* system be represented by different regression functions with some parameters that are unknown. In such a case, observations of an output of the *Wiener* system could be partitioned into distinct data sets according to different descriptions. However, the boundaries the set of observations depend on the value of the unknown threshold $d-$ observations are divided into regimes subject to whether the some observed threshold variable is smaller or larger than d. Thus, there arises a problem, first, to find a way to partition available data, second, to calculate the estimates of parameters of regression functions by processing particles of observations to be determined, and, third, to get the unknown threshold d. It is known, that various compensators have been tried to adjust the performance of control systems by reducing parasitic effects of a nonlinearity. On the other hand, we describe here the approach based on reconstruction of the unmeasurable internal intermediate signal, acting between both blocks of the *Wiener* system, without designing special and complex enough compensators [5]. Afterwards, instead of measurable output of the *Wiener* system, affected by the piecewise nonlinearity, the reconstructed signal, free of the parasitic effects, could be used for parametric identification of the linear time invariant (*LTI*) system.

Here the same problem of the *Wiener* system is considered as in the article [6], except for recursive identification. The identification method in [6] is a direct application of the well-known recursive *LS* (*RLS*) algorithm [4], extended with the estimation of internal variables, some of which appear both linearly and nonlinearly.

In this paper, the initial data partition allows us to separate the parametric identification problem into two parts that are related by the internal signal to be restored. Thus, the nonlinearity of variables and some known problems related with it are avoided. In the first part of the article the parameters of the *FIR* model are estimated. Then, the unknown intermediate signal between two blocks is reconstructed. Finally, the parameters of the linear block based on the samples of the reconstructed signal are evaluated. In the second part the values of negative linear segment slopes are calculated by processing respective segments of partitioned input-output data with missing observations. At last, it is shown here how to improve the initial estimate of the threshold. The recursive method proposed in [6] enables the on-line estimation of the parameters of linear block transfer function and the parameters characterizing the non-invertible piecewise-linear nonlinearity and their changes during the process. In our paper we do not use the recursive expressions. However, forgetting factors applied to reduce an old information in [6] can be used here, too. Moreover, it is clearly shown that it is possible to identify the *Wiener* system parametrically, even if the static piecewise nonlinearity is non-invertible. It is obvious, that such a method based on the data partition can be applied to find initial parameter estimates based on short-length input-output measurements. They can be used by the recursive parametric identification of *Wiener* systems.

In Section 2, a statement of the problem is presented. In Section 3, the general method is given for determining an auxiliary signal that corresponds to the extracted version of the internal one. Section 4 presents the simulation results of the *Wiener* system to be identified parametrically. Section 5 contains conclusions.

Statement of the problem. The *Wiener* system consists of a linear part followed by a static non-invertible nonlinearity $f(\cdot,\boldsymbol{\eta})$ with the vector of parameters $\boldsymbol{\eta}$. The linear part of *Wiener* system is dynamic, time invariant, causal, and stable. It can be represented by *LTI* (linear, time-invariant) dynamic system with the transfer function $G(q^{-1},\boldsymbol{\Theta})$ as a rational function of the form

$$G(q^{-1},\boldsymbol{\Theta})=\frac{b_1 q^{-1}+\ldots+b_m q^{-m}}{1+a_1 q^{-1}+\ldots+a_m q^{-m}}=\frac{B(q^{-1},\mathbf{b})}{1+A(q^{-1},\mathbf{a})}, \quad (1)$$

with a finite number of parameters

$$\boldsymbol{\Theta}^T=(\mathbf{b}^T,\mathbf{a}^T)=(b_1,\ldots,b_m,a_1,\ldots,a_m),\ \mathbf{b}^T=(b_1,\ldots,b_m),\ \mathbf{a}^T=(a_1,\ldots,a_m), \quad (2)$$

that are determined from the set Ω of permissible parameter values $\boldsymbol{\Theta}$. Here q^{-1} is a time-shift operator, the set Ω is restricted by conditions on the stability of the respective difference equation. The unmeasurable intermediate signal

$$x(k)=\frac{B(q^{-1},\mathbf{b})}{1+A(q^{-1},\mathbf{a})}u(k), \quad (3)$$

generated by the linear part of the *Wiener* system $\forall k=\overline{1,N}$ as a response to the input $u(k)\,\forall k=\overline{1,N}$ is acting on the static non-invertible nonlinear part $f(\cdot,\boldsymbol{\eta})$ as follows

$$y(k)=f(x(k),\boldsymbol{\eta})+e(k). \quad (4)$$

Here $e(k)$ is a measurement noise.

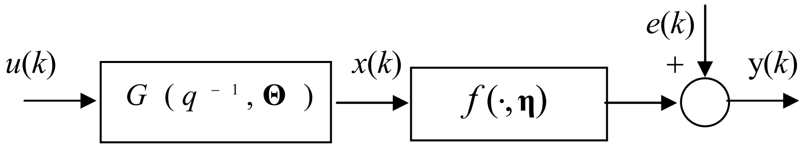

Fig. 1. A *Wiener* system, consisting of *LTI* system with $G(q^{-1},\boldsymbol{\Theta})$ (1) with parameters (2) and a nonlinearity $f(\cdot,\boldsymbol{\eta})$ (5), (Fig. 2) [6, 7]. The signal $\{x(k)\}$ is acting between the *LTI* system and $f(\cdot,\boldsymbol{\eta})$. Only samples of signals $\{u(k)\}$ and noisy $\{y(k)\}$ are available.

The nonlinear part of the *Wiener* system is a non-invertible piecewise linear nonlinearity with negative slopes as follows [7, 8]

$$f(x(k),\boldsymbol{\eta}) = \begin{cases} d_0 - d_1 x(k) & if\ x(k) > d \\ x(k) & if\ -d < x(k) \le d \\ c_0 - c_1 x(k) & if\ x(k) \le -d \end{cases} \tag{5}$$

that could be partitioned into three functions: $f(x(k),\boldsymbol{\alpha}) = d_0 - d_1 x(k)$, $f(x(k),d) = x(k)$, and $f(x(k),\boldsymbol{\beta}) = c_0 - c_1 x(k)$. Note that the function $f(x(k),\boldsymbol{\alpha})$ has only positive values, when $x(k) > d$, $f(x(k),d)$ has arbitrary positive, as well as negative values, when $-d < x(k) \le d$, and $f(x(k),\boldsymbol{\beta})$ has only negative values, when $x(k) < -d$.

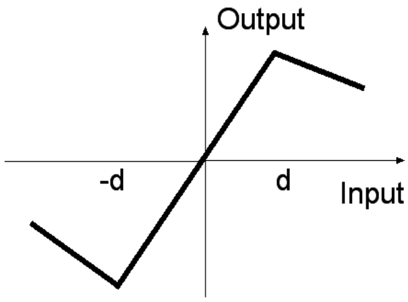

Fig. 2. Static non-invertible piecewise nonlinearity $f(\cdot,\boldsymbol{\eta})$.

Here $x(k) \equiv x(k,\boldsymbol{\Theta})$; $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$ are vectors of parameters:

$$\begin{aligned} \boldsymbol{\alpha}^T &= (d_0, d_1),\ d_0 = d(1+d_1), 0 < d_1 < d, \\ \boldsymbol{\beta}^T &= (c_0, c_1),\ c_0 = -d(1+c_1), 0 < c_1 < d, \\ \boldsymbol{\eta}^T &= (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T). \end{aligned} \tag{6}$$

The measurement noise $e(k) \equiv \zeta(k)\ \forall k = \overline{1,N}$ is added to the output $f(x(k),\boldsymbol{\eta})\ \forall k = \overline{1,N}$, respectively, $\{\zeta(k)\}$ is sequence of independent *Gaussian* variables with mean values $E\{\zeta(k)\} = 0$, and $E\{\zeta(k)\zeta(k+\tau)\} = \sigma_\zeta^2 \delta(\tau)$; $E\{\cdot\}$ is a mean value, σ_ζ^2 is variance of $\{\zeta(k)\}$, $\delta(\tau)$ is the *Kronecker* delta function.

The aim of the given paper is to estimate parameters of the linear and nonlinear parts (1), (5), respectively, avoiding the parasitic effects of nonlinear distortions, induced by the non-invertible nonlinearity (5) (Fig. 2), that appear in the noisy output $\{y(k)\}$ of the *Wiener* system (see Fig. 1).

The input-output data reordering. To calculate estimates it is needed to determine an auxiliary signal $\hat{x}(k)\,\forall k = \overline{1,N}$ (the estimate of unmeasurable $\{x(k)\}$) having no parasitic distortions. To solve such a problem one could approximate the linear part (1) of the *Wiener* system by the finite impulse response (*FIR*) model of the form [8, 9]

$$y(k) = \gamma_0 + \gamma_1 u(k) + \ldots + \gamma_\mu u(k-\mu+1) + e(k) \tag{7}$$

$\forall k = \overline{\mu, N}$ or using the expression in a matrix form

$$\mathbf{V} = \boldsymbol{\Pi}\boldsymbol{\gamma} + \mathbf{e}. \tag{8}$$

Here

$$\mathbf{V} = (y(\mu),\ y(\mu+1),\ldots,y(N-1),\ y(N))^T \tag{9}$$

is the $(N-\mu+1)\times 1$ vector, consisting only of observations of the non-noisy input $\{u(k)\}$;

$$\mathbf{\Pi}=\begin{bmatrix} 1 & u(\mu) & \dots & u(2) & u(1) \\ 1 & u(\mu+1) & \dots & u(3) & u(2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & u(N) & \dots & u(N-\mu+2) & u(N-\mu+1) \end{bmatrix} \tag{10}$$

is the full rank regression matrix $(N-\mu+1)\times(\mu+1)$, consisting only of observations of the non-noisy input $\{u(k)\}$, $\boldsymbol{\gamma}^T=(\gamma_0,\gamma_1,\dots,\gamma_\mu)$ is the vector of unknown parameters of *FIR* model (7);

$$\mathbf{n}=(e(\mu),e(\mu+1),\dots,e(N-1),e(N))^T \tag{11}$$

is the vector $(N-\mu+1)\times 1$, consisting of the values $\{e(k)\}$; μ is the order of *FIR* filter (7) that can be arbitrarily large, but fixed (μ can be detected experimentally by simulation). The reason for the use of the *FIR* model is as follows: the dependence of some regressors on the process output will be facilitated, and the assumption of ordinary least squares (*LS*) that the regressors depend only on the non-noisy input signal, will be satisfied.

It could be emphasized that from the engineering point of view it is assumed here that no less than 50 persentage observations $\{y(k)\}$ are concentrated on the middle-set that corresponds to the condition $-d<x(k)\le d$ and approximately by 25 percentage or less on any side set with conditions $x(k)>d$ and $x(k)\le -d$, respectively. Let us rearrange now the true output data $y(k),\forall k=\overline{1,N}$ in an ascending order of their values, assuming that measurement noise $e(k)$ $\forall k=\overline{1,N}$ is absent, parameters, and the threshold d of non-invertible nonlinearity (5) are known. We could do that by interchanging equations in the initial system (8). Note that the interchange of equations does not influence the accuracy of *LS* parameter estimates $\hat{\boldsymbol{\gamma}}^T=(\hat{\gamma}_0,\hat{\gamma}_1,\dots,\hat{\gamma}_\mu)$ to be calculated. Thus, the true observations $y(k),\forall k=\overline{1,N}$ of the reordered output $\tilde{y}(\tilde{k})\,\forall\tilde{k}=\overline{1,N}$ of the *Wiener* system in an unnoisy or slightly noisy frame should be partitioned into three data sets: left-hand side data set (N_1 samples) with values less than or equal to negative *d*, middle data set (N_2 samples) with values higher than negative *d* but lower or equal to *d*, and right-hand side data set (N_3 samples) with values higher than *d*. Here $N=N_1+N_2+N_3$, $\tilde{k}$ is any integer k rearranged in an ascending order, dependent on the reordered values of observations $\{y(k)\}$, e.g. $\tilde{k}=5$ while true $k=10$. Then, assuming that a static nonlinearity is present in the given *Wiener* system (Fig. 1), the vector $\mathbf{V}$ and the matrix $\mathbf{\Pi}$ should be partitioned into three data sets: the left-hand data set $\mathbf{V}_1=\mathbf{\Pi}_1\boldsymbol{\gamma}$, the middle data set $\mathbf{V}_2=\mathbf{\Pi}_2\boldsymbol{\gamma}$, and the right-hand data set $\mathbf{V}_3=\mathbf{\Pi}_3\boldsymbol{\gamma}$, according to the three regimes of the static saturation-like nonlinearity with negative slopes. The left-hand side data set $\mathbf{V}_1$ (N_1 samples) consists of the reordered $\tilde{y}(\tilde{k}),\forall\tilde{k}=\overline{1,N_1}$, equal or less than negative d, the middle data set $\mathbf{V}_2$ (N_2 samples) consists of the reordered values $\tilde{y}(\tilde{k}),\forall\tilde{k}=\overline{1,N_2}$ higher than negative d, but lower or equal than d, and the right-hand side data set $\mathbf{V}_3$ (N_3 samples) consists of the reordered values $\tilde{y}(\tilde{k}),\forall\tilde{k}=\overline{1,N_3}$ more than d. Thus, initial system (7) is reordered into a system

$$\overline{\mathbf{V}}=\overline{\mathbf{\Pi}}\boldsymbol{\gamma}+\overline{\mathbf{e}} \tag{12}$$

with $\overline{\mathbf{V}}=[\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3]^T$, and $\overline{\mathbf{\Pi}}=[\mathbf{\Pi}_1,\mathbf{\Pi}_2,\mathbf{\Pi}_3]^T$ by simply interchanging equations in the initial system of linear equations (8). Here $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3$ are $N_1\times 1$, $N_2\times 1$ and $N_3\times 1$ vectors, respectively, and $\mathbf{\Pi}_1,\mathbf{\Pi}_2,\mathbf{\Pi}_3$ are $N_1\times(\mu+1)$, $N_2\times(\mu+1)$ and $N_3\times(\mu+1)$ matrices, correspondingly.

Hence, the observations with the highest and positive values will be concentrated on the right-hand side set, while the observations with the lowest and negative values on the left-hand side one. It could be noted that on boundaries the small portions of observations of the middle data set of

$\tilde{y}(\tilde{k})\,\forall\tilde{k}=\overline{1,N_2}$ are mixed together with some portions of observations of the left-hand side and right-hand side data sets, respectively, due to negative slopes of the nonlinearity (5). In general case (a noisy environment, unknown parameters $\mathbf{\Theta}$ and $\mathbf{\eta}$ and the threshold d) it is imperative for the efficient parametric identification of the Wiener system that such ambiguities are resolved. On the other hand, one can avoid this problem assuming here that slightly less than 50% unmixed observations are concentrated on the middle-set and approximately by 20% on any side set. The observations of the middle data set of $\{\tilde{y}(\tilde{k})\}$ are coincident with the respective reordered observations of the intermediate signal $\{x(k)\}$ in the absence of the measurement noise $\{e(k)\}$. Therefore, one could get unmixed observations of $\{\tilde{y}(\tilde{k})\}$ simply by choosing the upper interval bound lower than the 75 percentage and the lower interval bound higher than the 25 percentage of the sampled reordered observations of $\{\tilde{y}(\tilde{k})\}$. Next, let us reconstruct an unmeasurable intermediate signal $x(k)\,\forall k=\overline{1,N}$, using the middle data set $\tilde{y}(\tilde{k})$ that is, really, reordered in an ascending order of their values $\{y(k)\}$ with small portions of missing observations within it that belong to the left-hand and right-hand side sets of the reordered data. To estimate the parameters $\boldsymbol{\gamma}^T=(\gamma_0,\gamma_1,\ldots,\gamma_{\mu-1})$ of *FIR* model (7), we can use the expression of the form

$$\hat{\boldsymbol{\gamma}}=(\mathbf{\Pi}_2^T\mathbf{\Pi}_2^T)^{-1}\mathbf{\Pi}_2^T\mathbf{V}_2^T. \tag{13}$$

Here $\hat{\boldsymbol{\gamma}}^T=(\hat{\gamma}_0,\hat{\gamma}_1,\ldots,\hat{\gamma}_{\mu-1})$ is a vector $\mu\times 1$ of the estimates of parameters $\boldsymbol{\gamma}^T=(\gamma_0,\gamma_1,\ldots,\gamma_{\mu-1})$, respectively. Then, the estimate $\{\hat{x}(k)\}$, of the intermediate unmeasurable signal $\{x(k)\}$, could be extracted by

$$\hat{x}(k)=\hat{\gamma}_0u(k)+\hat{\gamma}_1u(k-1)+\ldots+\hat{\gamma}_{\mu-1}u(k-\mu+1), \tag{14}$$

avoiding parasitic influence of the nonlinearity (5). Here the true values $\gamma_0,\gamma_1,\ldots,\gamma_{\mu-1}$ are replaced by their estimates $\hat{\boldsymbol{\gamma}}^T=(\hat{\gamma}_0,\hat{\gamma}_1,\ldots,\hat{\gamma}_{\mu-1})$, respectively, calculated by Eq. (13). Note that the result of this step is the auxiliary signal (14) that is a reconstructed version of the intermediate unmeasurable signal $\{x(k)\}$, that acts between linear and nonlinear parts of the *Wiener* system. Now, let us calculate the estimates of the parameters of the transfer function $G(q^{-1},\mathbf{\Theta})$ according to

$$\hat{\mathbf{\Theta}}=(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{U}, \tag{15}$$

Here

$$\hat{\mathbf{\Theta}}^T=(\hat{\mathbf{b}}^T,\hat{\mathbf{a}}^T)=(\hat{b}_1,\ldots,\hat{b}_m,\hat{a}_1,\ldots,\hat{a}_m),\ \hat{\mathbf{b}}^T=(\hat{b}_1,\ldots,\hat{b}_m),\quad \hat{\mathbf{a}}^T=(\hat{a}_1,\ldots,\hat{a}_m), \tag{16}$$

are $2m\times 1,\ m\times 1, m\times 1$ vectors of the estimates of parameters (2), respectively;

$$\mathbf{X}=\begin{bmatrix} u(m+\eta) & \ldots\ u(\eta+1) & -\hat{x}(m+\eta) & \ldots\ -\hat{x}(\eta+1) \\ u(m+\eta+1) & \ldots\ u(\eta+2) & -\hat{x}(m+\eta+1) & \ldots\ -\hat{x}(\eta+2) \\ \vdots & \vdots & \vdots & \vdots \\ u(\tilde{N}_2-1) & \ldots\ u(\tilde{N}_2-m) & -\hat{x}(\tilde{N}_2-1) & \ldots\ -\hat{x}(\tilde{N}_2-m) \end{bmatrix} \tag{17}$$

is the $(\tilde{N}_2-m-\eta)\times 2\,m$ matrix, consisting of samples of the input $u(k)\,\forall k=\overline{m+\eta+1,\tilde{N}_2-1}$ and the auxiliary signal $\hat{x}(k)\,\forall k=\overline{m+\eta+1,\tilde{N}_2-1}$ of the form (14), $\mathbf{U}=(u(m+\eta+1),\ldots,u(\tilde{N}_2))^T$ is a $(\tilde{N}_2-m-\eta)\times 1$ vector of output observations, $\eta,\tilde{N}_2$ are integers, corresponding to the starting- and final-point in the middle data set, respectively, besides, $0 < \eta << N_2, \eta < \widetilde{N}_2 < N_2$.

Then, the second and, the main estimate $\widetilde{x}(k), \forall k = \overline{\mu, N}$ of the internal signal is calculated by

$$\widetilde{x}(k) = G(q^{-1}, \hat{\mathbf{\Theta}})u(k) = \frac{\hat{b}_1 q^{-1} + \ldots + \hat{b}_m q^{-m}}{1 + \hat{a}_1 q^{-1} + \ldots + \hat{a}_m q^{-m}} u(k) = \frac{\hat{B}(q^{-1}, \hat{\mathbf{b}})}{1 + \hat{A}(q^{-1}, \hat{\mathbf{a}})} u(k) \tag{18}$$

that can be compared with its previous version (14). Estimates of the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta},$ and $\boldsymbol{\eta}$ like vector of estimates $\hat{\mathbf{\Theta}}$ are calculated by the ordinary *LS*, too. In such a case, we can use two systems of linear equations as follows: first system

$$\begin{bmatrix} \sum_{j=1}^{N_1} \widetilde{y}(j) \\ \sum_{j=1}^{N_1} \widetilde{x}(j)\widetilde{y}(j) \end{bmatrix} = \begin{bmatrix} N_1 & \sum_{j=1}^{N_1} \widetilde{x}(j) \\ \sum_{j=1}^{N_1} \widetilde{x}(j) & \sum_{j=1}^{N_1} \widetilde{x}^2(j) \end{bmatrix} \begin{bmatrix} \hat{c}_0 \\ \hat{c}_1 \end{bmatrix} \tag{19}$$

that minimizes the sum

$$J(c_0, c_1) = \sum_{i=1}^{N_1} \left\{ \widetilde{y}(i) + c_0 - c_1 \widetilde{x}(i) \right\}^2 = \min, \tag{20}$$

and the second one

$$\begin{bmatrix} \sum_{j=1}^{N_3} \widetilde{y}(j) \\ \sum_{j=1}^{N_3} \widetilde{x}(j)\widetilde{y}(j) \end{bmatrix} = \begin{bmatrix} N_3 & \sum_{j=1}^{N_3} \widetilde{x}(j) \\ \sum_{j=1}^{N_3} \widetilde{x}(j) & \sum_{j=1}^{N_3} \widetilde{x}^2(j) \end{bmatrix} \begin{bmatrix} \hat{d}_0 \\ \hat{d}_1 \end{bmatrix} \tag{21}$$

that minimizes

$$J(d_0, d_1) = \sum_{i=1}^{N_3} \left\{ \widetilde{y}(i) + d_0 - d_1 \widetilde{x}(i) \right\}^2 = \min. \tag{22}$$

The estimates $\hat{c}_0, \hat{c}_1$ and $\hat{d}_0, \hat{d}_1$ are determined by solving eq. (19) and (21) as follows

$$\hat{c}_0 = [\sum_{j=1}^{N_1} \widetilde{x}(j) \sum_{j=1}^{N_1} \widetilde{y}(j) - \sum_{j=1}^{N_1} \widetilde{x}(j)\widetilde{y}(j) \sum_{j=1}^{N_1} \widetilde{x}(j)] / D_1, \tag{23}$$

$$\hat{c}_1 = [\sum_{j=1}^{N_1} \widetilde{x}(j) \sum_{j=1}^{N_1} \widetilde{y}(j) - N_1 \sum_{j=1}^{N_1} \widetilde{x}(j)\widetilde{y}(j)] / D_2, \tag{24}$$

$$\hat{d}_0 = [\sum_{j=1}^{N_3} \widetilde{x}(j) \sum_{j=1}^{N_3} \widetilde{y}(j) - \sum_{j=1}^{N_3} \widetilde{x}(j)\widetilde{y}(j) \sum_{j=1}^{N_3} \widetilde{x}(j)] / D^*_1, \tag{25}$$

$$\hat{d}_1 = [\sum_{j=1}^{N_3} \widetilde{x}(j) \sum_{j=1}^{N_3} \widetilde{y}(j) - N_3 \sum_{j=1}^{N_3} \widetilde{x}(j)\widetilde{y}(j)] / D^*_2, \tag{26}$$

$$D_1 = N_1 \sum_{j=1}^{N_1} \tilde{x}^2(j) - \sum_{j=1}^{N_1} \tilde{x}(j) \sum_{j=1}^{N_1} \tilde{x}(j),\ D^*_1 = N_3 \sum_{j=1}^{N_3} \tilde{x}^2(j) - \sum_{j=1}^{N_3} \tilde{x}(j) \sum_{j=1}^{N_3} \tilde{x}(j),$$

$$D_2 = \sum_{j=1}^{N_1} \tilde{x}(j) \sum_{j=1}^{N_1} \tilde{x}(j) - N_1 \sum_{j=1}^{N_1} \tilde{x}^2(j),\ D^*_2 = \sum_{j=1}^{N_3} \tilde{x}(j) \sum_{j=1}^{N_3} \tilde{x}(j) - N_3 \sum_{j=1}^{N_3} \tilde{x}^2(j),$$

respectively, where side-sets particles of $\tilde{y}(k)\,\forall k = \overline{1, N_1}$ and $\forall k = \overline{1, N_3}$, and associated samples of auxiliary signal $\{\tilde{x}(k)\}$ are used. At last, the estimate of the threshold d on the right-hand side ($\hat{d}_r$) and left-hand side ($\hat{d}_l$) sets are found according to

$$\hat{d}_r = \hat{d}_0 /(1 - \hat{d}_1),\ \hat{d}_l = \hat{c}_0 /(1 - \hat{c}_1), \tag{27}$$

respectively. Note that in Eq. (19) – (26) the reordered versions of $\tilde{x}(j), \forall k = \overline{1, N_1}$ and $\tilde{x}(j), \forall k = \overline{1, N_3}$ are substituted. To estimate the parameters of *FIR* model (5) recursively, one can use the well-known ordinary recursive *LS* (*RLS*). Now, instead of the signal $y(k), \forall k = \overline{1, N}$, that has parasitic effects induced by the static nonlinearity, the current values of $\hat{x}(k)\,\forall k = \overline{1, N}$ calculated by (18), will be applied for the parametric identification of the basic *LTI* system. It is easy to understand that equations (13), (15) as well as (23) – (26) can be transformed into recursive form. In such a case, the on-line solution is possible, too [9, 10].

Simulation results. The sum of sinusoids

$$u(k) = \sum_{i=1}^{15} \sin(i\pi k/10 + \phi_i), \tag{28}$$

and white *Gaussian* noise with variance 1 were generated as inputs to the linear block of the *Wiener* system (see Fig. 1)

$$G(q^{-1}, \mathbf{\Theta}) = \frac{b_1 q^{-1}}{1 + a_1 q^{-1}}, \tag{29}$$

respectively [7]. Here the true values of parameters (1) are: $b_1 = 1, a_1 = 0.7$. In Eq.(28) the stochastic variables $\phi_i\ \forall i = \overline{1,20}$ with a uniform distribution on $[0, 2\pi]$ were chosen. The true intermediate signal $\{x(k)\}$ of the *Wiener* system was given by Eq. (3). Afterwards, the signal $\{x(k)\}$ passes the non-invertible nonlinearity of the form [6, 7]

$$f(\cdot, \mathbf{\eta}) = \begin{cases} 1.1 - 0.1x(k) & if\ x(k) > 1 \\ x(k) & if\ -1 < x(k) \le 1 \\ -1.1 - 0.1x(k) & if\ x(k) \le -1 \end{cases} \tag{30}$$

that produces the output (4). First of all, N=100 data samples have been generated without additive measurement noise. The initial estimate of the static nonlinearity, calculated by processing $\{y(k)\}$ and $\{\hat{x}^{(1)}(k)\}$ is obtained. It is shown in Fig. 3. Here $\{\hat{x}^{(1)}(k)\}$ is reconstructed according to Eq. (14). In such a case, the initial estimate of the threshold d can be determined, too. It follows from Fig. 3, where the output $\{y(k)\}$ in dependence of $\{\hat{x}^{(1)}(k)\}$ is shown, that the initial value of the estimate $\hat{d} = 1$. The abovementioned signals to be used are shown in Fig.4. In Table 1 estimates

$\hat{b}_1$, $\hat{a}_1$ are shown for different inputs. In the first experiment (second and third columns in Table 1) they were calculated by means of ordinary *LS* by processing 100 samples of $\{u(k), x(k)\}$ (see Fig.1). In the second experiment (fourth and fifth columns in Table 1) observations of $\{x(k)\}$ were replaced by respective observations of $\{y(k)\}$ (see Fig.1).

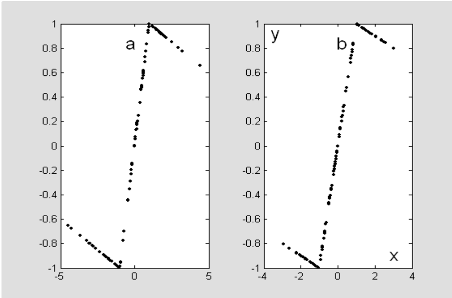

Fig.3. The output (y) of the Wiener system in dependence of the signal of the form (14) (x).
Input: (28) (a), white *Gaussian* noise (b).

It is obvious (see Table 1), that the nonlinearity was influenced on the accuracy of estimates, significantly, for example, $\hat{a}_1$ changed even sign. Afterwards, the *LS* problem (13) was solved using 42 and 55 rearranged samples of the output $\{y(k)\}$ of the *Wiener* system for both inputs (see Figures 5 – 8), excluding zeros. In such a case, the whole number of *FIR* filter (7) parameters $\mu = 14$ has been chosen experimentally by multifold simulation. Values of their estimates are given in Table 2 in the absence of additive noise.

Table 1. Estimates of parameters $b_1 = 1, a_1 = 0.7$.

Input	$\hat{b}_1$	$\hat{a}_1$	$\hat{b}_1$	$\hat{a}_1$
(28)	1.	0.7	0.2185	-0.1112
white noise	1.	0.7	0.5258	0.6281

The first estimate $\{\hat{x}^{(1)}(k)\}$ of internal unmeasurable signal $\{x(k)\}$ was reconstructed according to (14), replacing unknown true values of parameters by their estimates $\hat{\gamma}_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_{\mu-1}$ (see second column of Table 2). Then, the estimates of the parameters of the transfer function $G(q^{-1}, \mathbf{\Theta})$ were calculated by ordinary LS according to (15). We obtained: $\hat{b}_1 = 1.0012, \hat{a}_1 = 0.6998$ for the set of harmonics (28), and $\hat{b}_1 = 0.9985, \hat{a}_1 = 0.7$ for the white Gaussian noise on the input of the Wiener system. The second estimate $\hat{x}^{(2)}(k) \forall k = \overline{1, N}$ of internal signal $x(k) \forall k = \overline{1, N}$ was generated by Eq.(18), using previously calculated values $\hat{b}_1 = 0.9985, \hat{a}_1 = 0.7$.

The accuracy of the estimates of the intermediate signal, calculated by formulas (14) and (18), is more or less similar except for the first 15 samples, when the FIR model (7) was used. If $\{\hat{x}(k)\}$ has been obtained, then it is simple to separate different particles of samples that belong to distinct side-sets. Then, the estimates $\hat{c}_0, \hat{c}_1, \hat{d}_0, \hat{d}_1$ of the parameters c_0, c_1, d_0, d_1 were calculated by (19), (21), respectively. In such a case, the rearranged observations of $\{\hat{x}(k)\}$ and $\{y(k)\}$ were substituted in formula (19) and estimates of c_0, c_1 were determined by (23), (24). Afterwards, estimates of d_0, d_1 were calculated by (25), (26). It should be noted that $N_1 = 29, N_3 = 28$ for the periodical signal (26) (Fig. 2a), and $N_1 = 20, N_3 = 24$ for the Gaussian white noise. (see Fig. 3a) were used to calculate the estimates $\hat{c}_0, \hat{c}_1, \hat{d}_0, \hat{d}_1$, respectively (see Table 3).

In order to determine how realizations of different process- and measurement noises affect the accuracy of estimation of unknown parameters, we have used the Monte Carlo simulation with 10 data samples, each containing 100 pairs of input-noisy-output observations. 10 experiments with the different realizations of the measurement noise $e(k)\,\forall k=\overline{1,100}$ have been carried out. The intensity of noise was assured by choosing respective signal-to-noise ratios (*SNR*) (the square root of the ratio of signal and noise variances). For the noise *SNR* is defined as $10\log_{10}\frac{\sigma_w^2}{\sigma_e^2}$, where σ_w^2 is the variance of the signal difference $y(k)-e(k)$. Here σ_e^2 is a variance of the additive Gaussian noise (see Fig. 1). As inputs for given nonlinearity white Gaussian noise with variance 1 was chosen. Then, in each experiment $N=100$ data samples have been generated with additive measurement noise according to $y(k)=w(k)+\lambda e(k),\ \forall k=\overline{1,N},$ respectively. The values of λ was chosen so that *SNR* for the measurement noise was equal to 50. In each experiment the estimates of parameters were calculated. During the Monte Carlo simulation averaged values of estimates of the parameters and of the threshold and their confidence intervals $\Delta=\pm t_{\alpha/2;L-1}\frac{\hat{\sigma}}{\sqrt{L}}$, in which $\hat{\sigma}$ is the estimate of the standard deviation and α is the significance level, were determined, too. The value $t_{\alpha/2;L-1}$ is the point of Student's distribution with *L*-1 degrees of freedom which cuts $\alpha/2$ part of the distribution. In case α =0.05 and *L*= 100 we find from Student's distribution table [11] that $t_{0.025;99}=1.990.$ In Table 4, for each input the averaged estimates of parameters $b_1=1.0;\ a_1=-0.70$ of the simulated Wiener system (Fig.1) with the linear part and the piecewise non-invertible nonlinearity (30) with parameters $c_0=-1.1; c_1=-0.1; d_0=1.1;\ d_1=-0.1.$ and, with their confidence intervals are presented. It ought to be noted that in each experiment here as the input was fixed white *Gaussian* noise sequence chosen. The estimate of the threshold *d* on the right-hand side and left-hand side sets are found by Eq. (27): $\hat{d}_l=-1.18\pm0.04;\ \hat{d}_r=0.99\pm0.02.$

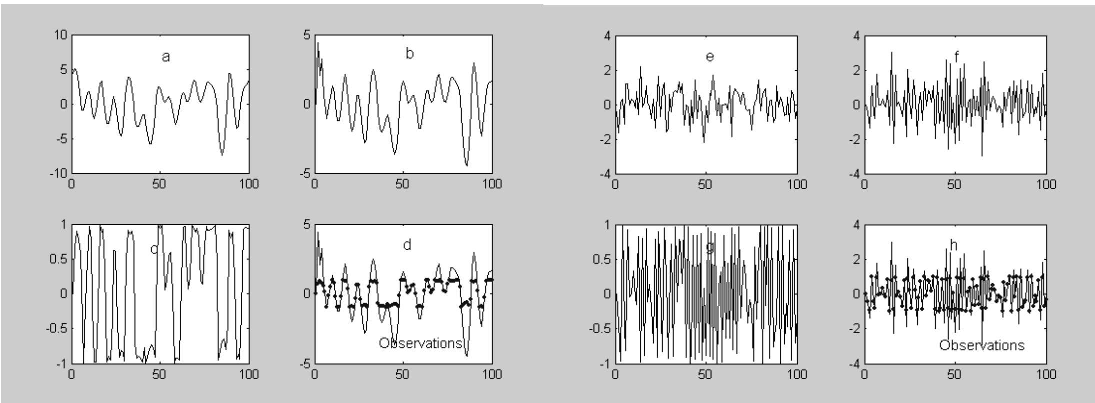

Fig. 4. Signals of the simulated system. Inputs: sum of sinusoids (25) (a), white noise (e). Parts (b), (c), (d) correspond to (a), and (f), (g), (h) – to (e). Internal signal: (b), (f). Outputs: (c), (g). Internal signals (b), (f) and outputs (c), (g) (dotted lines) together: (b), (c) – part (d); (f), (g) – part (h).

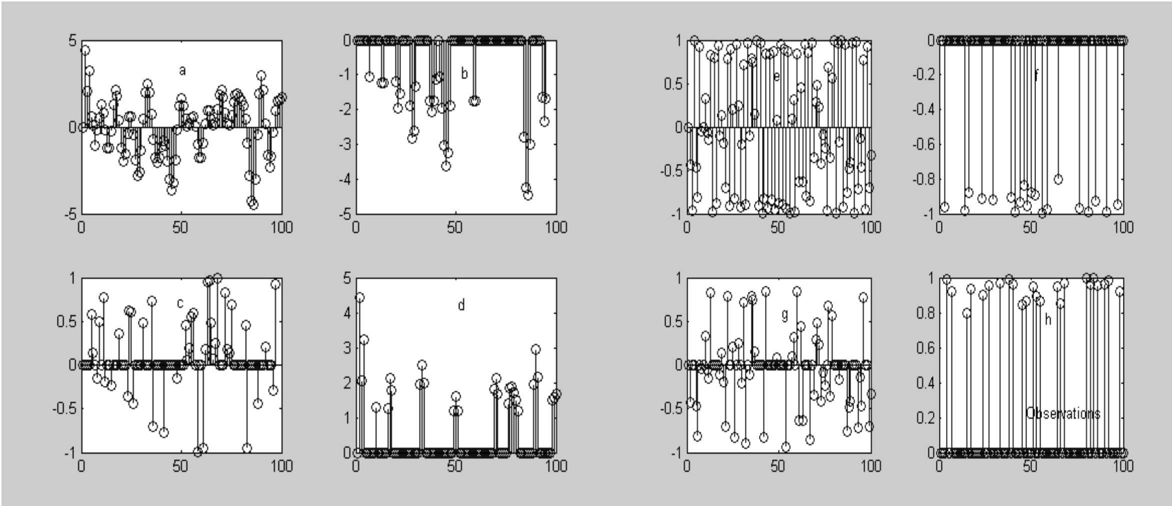

Fig.5. Samples of true $x(k)$ (a) and $y(k)$ (e), and their data sets: left (b,f); middle (c,g); right (d, h), respectively. Samples that belong to other data set, are equal to zeros. Input $u(k)$ of the form (28) (see Fig. 4a).

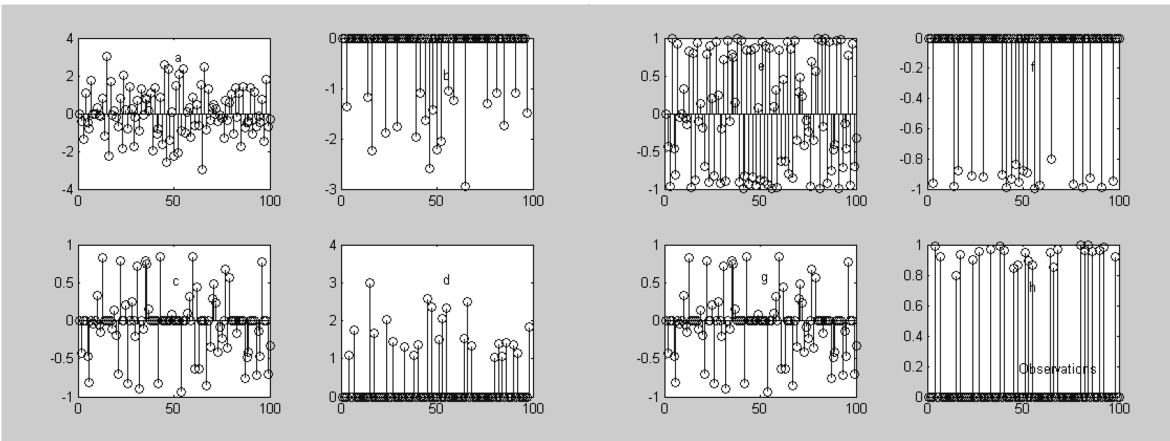

Fig.6. Samples of true $x(k)$ (a) and $y(k)$ (e), and their data sets: left (b,f); middle (c,g); right (d, h), respectively. Samples that belong to other data set, are equal to zeros. Input $u(k)$ white Gaussian noise (see Fig. 4e).

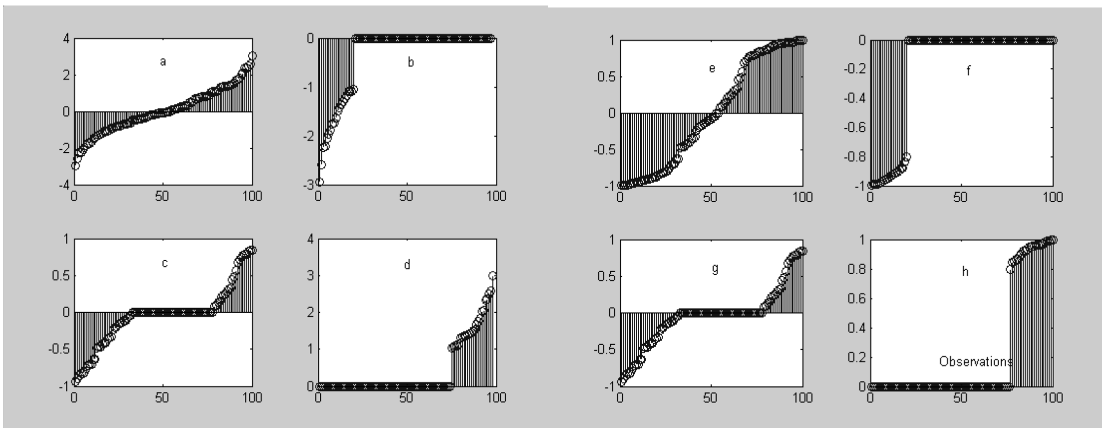

Fig. 7. The reordered of their values signals $x(k)$ and $y(k)$ (a)(see also Fig.6a, e, respectively) and their rearranged data sets: left (b, f); middle (c, g); right (d,h). Input (see Fig. 4e).

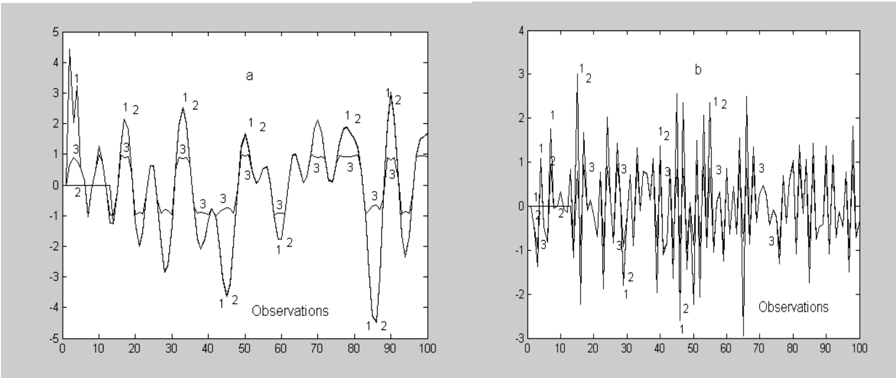

Fig. 8. The signals: true intermediate $\{x(k)\}$ (1), $\{\hat{x}^{(1)}(k)\}$(2), and $y(k)$ (3). Inputs: of the form (25), white Gaussian noise (b).

Table 2. Estimates of 10-th parameters of FIR system for distinct input signals.

Estimates	Input is of the form (28)	Input is white Gaussian noise
$\hat{\gamma}_0$	-1.0111	0.0020
$\hat{\gamma}_1$	11.4598	0.9949
$\hat{\gamma}_2$	-51.2717	-0.6979
$\hat{\gamma}_3$	165.7966	0.4862
$\hat{\gamma}_4$	-376.7118	-0.3409
$\hat{\gamma}_5$	653.1675	0.2383
$\hat{\gamma}_6$	-884.5980	-0.1689
$\hat{\gamma}_7$	951.8472	0.1172
$\hat{\gamma}_8$	-814.4884	-0.0821
$\hat{\gamma}_9$	549.6397	0.0626
$\hat{\gamma}_{10}$	-285.3381	-0.0384
$\hat{\gamma}_{11}$	108.7535	0.0282
$\hat{\gamma}_{12}$	-27.5263	-0.0211
$\hat{\gamma}_{13}$	3.6005	0.0146

Table 3. Estimates of parameters $c_0 = -1.1; c_1 = -0.1; d_0 = 1.1; d_1 = -0.1$.

Input	$\hat{c}_0$	$\hat{c}_1$	$\hat{d}_0$	$\hat{d}_1$
(28)	-1.0609	-0.0849	0.8911	0.0115
White noise	-1.0781	-0.0904	1.1025	-0.1020

The *Monte Carlo* simulation implies that the accuracy of parametric identification of the *Wiener* system with static non-invertible nonlinearity depends on the intensity of measurement noise.

Table 4. Estimates of parameters $c_0 = -1.1; c_1 = -0.1; d_0 = 1.1; d_1 = -0.1$. $SNR =$ 10.

$\hat{b}_1$	$\hat{a}_1$	$\hat{c}_0$	$\hat{c}_1$	$\hat{d}_0$	$\hat{d}_1$
0.99 ± 0.02	-0.70 ± 0.01	-1.08 ± 0.02	-0.09 ± 0.01	1.10 ± 0.03	-0.10 ± 0.01

Conclusions. The approach is presented here, based on the extraction of an unknown internal intermediate signal, acting between linear and nonlinear blocks of the *Wiener* system with a static non-invertible piecewise nonlinearity, avoiding complex enough compensators [9 – 10]. It is shown here that a problem of identification of *Wiener* systems (Fig. 1) could be essentially reduced by using *FIR* model, and a simple data rearrangement in an ascending order according to their values. Thus, the available data are partitioned into three data sets that correspond to distinct threshold regression models. Afterwards, the estimates of unknown parameters of linear regression models can be calculated by processing respective sets of the rearranged output and associated input observations. A technique, based on ordinary *LS*, is proposed here for estimating the parameters of linear and nonlinear parts of the *Wiener* system, including the unknown threshold of the piecewise nonlinearity, too. It is shown here (see Fig. 3) how at the beginning the initial estimate of the unknown threshold can be determined. During successive steps the unknown intermediate signal is reconstructed and the missing values of observations of output data particles are replaced by their estimates. Note that missing values can be retrieved by the approach given in [12]. Various results of numerical simulation (Fig. 2 – 8), including that of *Monte Carlo* (Table 4) prove the efficiency of the proposed approach for the parametric identification of *LTI* systems followed by static non-invertible piecewise nonlinearity in a noisy frame.

Finally, we can state that the successful parametric identification of the non-invertible piecewise-linear nonlinearity is a new element in the known approach (see [8-10]). It is obvious, that the proposed here data partition method can be used for different types of complex nonlinearities with piecewise-linear segments.

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