Nonlinear Analysis: Modelling and Control, Vol. 28, No. 1, 97–115
https://doi.org/10.15388/namc.2023.28.30208

On the non-closure under convolution for strong subexponential distributions*

Dimitrios Konstantinides[a], **Remigijus Leipus**[b], **Jonas Šiaulys**[c]

[a]Department of Statistics and Actuarial-Financial Mathematics,
University of the Aegean, Karlovassi,
GR-83 200 Samos, Greece
konstant@aegean.gr

[b]Institute of Applied Mathematics, Vilnius University,
Naugarduko 24, Vilnius LT-03225, Lithuania
remigijus.leipus@mif.vu.lt

[c]Institute of Mathematics, Vilnius University,
Naugarduko 24, Vilnius LT-03225, Lithuania
jonas.siaulys@mif.vu.lt

Received: May 13, 2022 / **Revised:** December 12, 2022 / **Published online:** December 23, 2023

Abstract. In this paper, we consider the convolution closure problem for the class of strong subexponential distributions, denoted as $\mathscr{S}^*$. First, we show that, if $F, G \in \mathscr{L}$, then inclusions of $F * G$, FG, and $pF + (1-p)G$ for all (some) $p \in (0,1)$ into the class $\mathscr{S}^*$ are equivalent. Then, using examples constructed by Klüppelberg and Villasenor [The full solution of the convolution closure problem for convolution-equivalent distributions, *J. Math. Anal. Appl.*, 41:79–92, 1991], we show that $\mathscr{S}^*$ is not closed under convolution.

Keywords: class of strong subexponential distributions, class of subexponential distributions, convolution closure.

1 Introduction and the main result

Throughout the paper, we will say that a distribution F is on $\mathbb{R} := (-\infty, \infty)$ if $\overline{F}(x) := 1 - F(x) > 0$ for all x; we will say that a distribution F is on $\mathbb{R}_+$ if its support is contained in $\mathbb{R}_+ := [0, \infty)$ and $\overline{F}(x) > 0$ for all x. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 1$, we write $a(x) \asymp b(x)$ if $0 < \liminf_{x\to\infty} a(x)/b(x) \leqslant \limsup_{x\to\infty} a(x)/b(x) < \infty$. For any two distribution F and G, by $F * G$ we denote their convolution:

$$F * G(x) = \int_{-\infty}^{\infty} F(x-y)\,\mathrm{d}G(y), \quad x \in \mathbb{R}.$$

*The last two authors are supported by the Research Council of Lithuania, grant No. S-MIP-20-16.

We say that distribution F on $\mathbb{R}$ belongs to the class of *long-tailed* distributions, denoted $\mathscr{L}$, if its right tail $\overline{F} = 1 - F$ satisfies

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$$

for any $y > 0$. We say that distribution F on $\mathbb{R}$ belongs to the *subexponential* class of distributions, denoted $\mathscr{S}$, if $F \in \mathscr{L}$ and

$$\lim_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 2. \tag{1}$$

The class of distributions, characterized by (1), was introduced by Chistyakov [2] and later, in more general setup, by Athreya and Ney [1] and Chover et al. [3, 4].

A distribution F on $\mathbb{R}$ is said to belong to the *strong subexponential* class of distributions, introduced by Klüppelberg [9] and denoted $\mathscr{S}^*$, if $\mu_F = \int_0^\infty \overline{F}(y)\,\mathrm{d}y \in (0,\infty)$ and

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y = 2\mu_F. \tag{2}$$

The properties of class $\mathscr{S}^*$ and related classes were studied in [6, Sect. 3.4], [8–14], [18–20], and other papers. In particular, it is well known that, under $\mu_F < \infty$, it holds that $\mathscr{S}^* \subset \mathscr{S} \subset \mathscr{L}$.

In the following theorem, we present equivalent conditions for the convolution $F * G$ to be in the class $\mathscr{S}^*$ under the initial assumption $F, G \in \mathscr{L}$.

Theorem 1. *Suppose that F and G are two distributions on $\mathbb{R}$. Let $F, G \in \mathscr{L}$. Then the following statements are equivalent*:

(i) $F * G \in \mathscr{S}^*$,
(ii) $FG \in \mathscr{S}^*$,
(iii) $pF + (1-p)G \in \mathscr{S}^*$ *for some* $0 < p < 1$,
(iv) $pF + (1-p)G \in \mathscr{S}^*$ *for all* $0 < p < 1$.

Moreover, any of these equivalent statements implies the relations

$$\overline{F*G}(x) \sim \overline{F}(x) + \overline{G}(x), \tag{3}$$

$$\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y \sim \mu_G\overline{F}(x) + \mu_F\overline{G}(x), \tag{4}$$

where $\mu_F := \int_0^\infty \overline{F}(y)\,\mathrm{d}y$, $\mu_G := \int_0^\infty \overline{G}(y)\,\mathrm{d}y$.

In the corollary below the assumption $F, G \in \mathscr{L}$ of Theorem 1 is replaced by a stricter assumption $F, G \in \mathscr{S}^*$. In this case, the asymptotic relation (4) is equivalent to any of statements (i)–(iv) of Theorem 1 (see also [7, Thm. 3], which refers to [10]).

Corollary 1. *Suppose that F and G are two distributions on $\mathbb{R}$. Let $F, G \in \mathscr{S}^*$. Then any of statements* (i)–(iv) *of Theorem* 1 *is equivalent to* (4).

Proof. We need only to prove that (4) implies (iii) of Theorem 1. Obviously,

$$\begin{aligned} I &:= \int_0^x \frac{1}{2}(\overline{F} + \overline{G})(x-y)\frac{1}{2}(\overline{F} + \overline{G})(y)\,\mathrm{d}y \\ &= \frac{1}{4}\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y + \frac{1}{2}\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y + \frac{1}{4}\int_0^x \overline{G}(x-y)\overline{G}(y)\,\mathrm{d}y \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Since $F, G \in \mathscr{S}^*$, by relation (4), we get

$$\begin{aligned} &\limsup_{x\to\infty} \frac{I}{(\mu_F + \mu_G)(\overline{F}(x) + \overline{G}(x))} \\ &\quad \leqslant \limsup_{x\to\infty} \max\left\{ \frac{I_1}{\mu_F \overline{F}(x)}, \frac{I_2}{\mu_G \overline{F}(x) + \mu_F \overline{G}(x)}, \frac{I_3}{\mu_G \overline{G}(x)} \right\} \leqslant \frac{1}{2}. \end{aligned}$$

Similarly, we obtain that

$$\liminf_{x\to\infty} \frac{I}{(\mu_F + \mu_G)(\overline{F}(x) + \overline{G}(x))} \geqslant \frac{1}{2}.$$

The derived estimates imply that

$$\begin{aligned} &\int_0^x \frac{1}{2}(\overline{F} + \overline{G})(x-y)\frac{1}{2}(\overline{F} + \overline{G})(y)\,\mathrm{d}y \\ &\quad \sim (\mu_F + \mu_G)\frac{1}{2}(\overline{F} + \overline{G})(x) = 2\mu_{(F+G)/2}\overline{(F+G)/2}(x), \end{aligned}$$

and, consequently, $(F+G)/2 \in \mathscr{S}^*$ by definition. □

We use Theorem 1 for proving main Theorem 2 on the convolution non-closure of class $\mathscr{S}^*$. Indeed, by Theorem 1, for distributions $F, G \in \mathscr{S}^*$, their convolution $F * G$ is not in $\mathscr{S}^*$ if and only if $pF + (1-p)G$ is not in $\mathscr{S}^*$ for some $p \in (0,1)$. Hence, in order to prove Theorem 2, we construct two distributions F and G such that $F/2 + G/2 \notin \mathscr{S}^*$ or, equivalently, $F * G \notin \mathscr{S}^*$.

Theorem 2. *Distribution class $\mathscr{S}^*$ is not closed under convolution, i.e. there exist distributions $F, G \in \mathscr{S}^*$ such that $F * G \notin \mathscr{S}^*$.*

Remark 1. The first counterexample for the closure of the subexponential class with respect to convolution was provided by Leslie [16]. Another counterexample for the closure of the convolution equivalent class of distributions with respect to convolution was given a few years later by Klüppelberg and Villasenor [10].

2 Proof of Theorem 1

2.1 Auxiliary lemmas

Before proving our main result, we state two auxiliary lemmas.

Lemma 1. *Suppose that F and G are two distributions on $\mathbb{R}$.*

(i) *If $F \in \mathscr{S}^*$, $G \in \mathscr{L}$, and $\overline{G}(x) \asymp \overline{F}(x)$, then $G \in \mathscr{S}^*$.*
(ii) *If $F \in \mathscr{L}$, $G \in \mathscr{L}$, then $F * G \in \mathscr{L}$.*
(iii) *If $F \in \mathscr{L}$, $G \in \mathscr{L}$, and $F * G \in \mathscr{S}$, then $\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x)$.*

Proof. For the proof of part (i), see [9, Thm. 2.1(b)] (see also [6, Thm. 3.25]). For part (ii), see [5, Thm. 3(b)] (see also [6, Cor. 2.42] or [17, Lemma 4.2]). Part (iii) can be found in Theorem 1.1 of Leipus and Šiaulys [15]. □

Lemma 2. *Suppose F is distribution on $\mathbb{R}$ with finite μ_F. The following statements are equivalent*:

(i) *$F \in \mathscr{S}^*$.*
(ii) *$F \in \mathscr{L}$ and*

$$\lim_{v\to\infty} \limsup_{x\to\infty} \int_v^{x-v} \frac{\overline{F}(x-y)\overline{F}(y)}{\overline{F}(x)} \, \mathrm{d}y = 0.$$

Proof. An equivalent assertion by choosing a special form function instead of the additional variable v is given in [6, Them. 3.24]. For the sake of completeness, we briefly present the main steps of the lemma proof. According to considerations in [6] (see the proof of Theorem 3.24), [7] (see the proof of Lemma 4), and [9] (see the proof of Theorem 3.2(b)), the assertion of the lemma follows from the estimate

$$\int_0^{x/2} \frac{\overline{F}(x-z)\overline{F}(z)}{\overline{F}(x)} \, \mathrm{d}z \geqslant \int_0^{y} \overline{F}(z) \, \mathrm{d}z + \frac{\overline{F}(x-y)}{\overline{F}(x)} \int_y^{x/2} \overline{F}(z) \, \mathrm{d}z, \quad x > 2y > 0,$$

implying that

$$1 \leqslant \frac{\overline{F}(x-y)}{\overline{F}(x)} \leqslant \frac{(\overline{F}(x))^{-1} \int_0^{x/2} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z - \int_0^y \overline{F}(z) \, \mathrm{d}z}{\int_0^{x/2} \overline{F}(z) \, \mathrm{d}z - \int_0^y \overline{F}(z) \, \mathrm{d}z},$$

and from equality

$$\int_0^{x} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z = 2\int_0^{v} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z + \int_v^{x-v} \overline{F}(x-z)\overline{F}(z) \, \mathrm{d}z,$$

where $x > 2v > 0$. □

2.2 Proof of the theorem

(ii) $\Rightarrow$ (i) Assume that $FG \in \mathscr{S}^*$. Lemma 1(ii) implies $F * G \in \mathscr{L}$, thus the proof will follow from
$$\overline{F * G}(x) \asymp \overline{FG}(x) \tag{5}$$
and Lemma 1(i). To prove (5), assume that X and Y are independent random variables with distributions F and G, correspondingly, and write
$$\begin{aligned}\overline{F * G}(x) &= \mathbf{P}(X + Y > x) \geqslant \mathbf{P}\bigl(\{X > x,\, Y > 0\} \cup \{X > 0,\, Y > x\}\bigr)\\ &= \overline{G}(0)\overline{F}(x) + \overline{F}(0)\overline{G}(x) - \overline{F}(x)\overline{G}(x),\\ \overline{FG}(x) &= \overline{F}(x) + \overline{G}(x) - \overline{F}(x)\overline{G}(x).\end{aligned}$$

Thus,
$$\begin{aligned}\liminf_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} &\geqslant \liminf_{x\to\infty} \frac{\overline{G}(0)\overline{F}(x) + \overline{F}(0)\overline{G}(x) - \overline{F}(x)\overline{G}(x)}{\overline{F}(x) + \overline{G}(x) - \overline{F}(x)\overline{G}(x)}\\ &\geqslant \min\bigl\{\overline{F}(0), \overline{G}(0)\bigr\} > 0. \end{aligned}\tag{6}$$

On the other hand,
$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} &= \limsup_{x\to\infty} \frac{\mathbf{P}(X_1 + Y_2 > x)}{\mathbf{P}(X \vee Y > x)}\\ &\leqslant \limsup_{x\to\infty} \frac{\mathbf{P}(X_1 \vee Y_1 + X_2 \vee Y_2 > x)}{\mathbf{P}(X \vee Y > x)},\end{aligned}\tag{7}$$
where (X_1, Y_1) and (X_2, Y_2) are independent copies of (X, Y).

Since, by (ii), $F_{X\vee Y} \in \mathscr{S}^* \subset \mathscr{S}$, we have
$$\mathbf{P}(X_1 \vee Y_1 + X_2 \vee Y_2 > x) \sim 2\mathbf{P}(X \vee Y > x). \tag{8}$$

Hence, by (6)–(8),
$$\min\bigl\{\overline{F}(0), \overline{G}(0)\bigr\} \leqslant \liminf_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{FG}(x)} \leqslant 2$$
and (5) follows.

(i) $\Rightarrow$ (ii) Let $F * G \in \mathscr{S}^*$. Since $\mathscr{S}^* \subset \mathscr{S}$, by Lemma 1(iii),
$$\overline{F * G}(x) \sim \overline{F}(x) + \overline{G}(x) \sim \overline{FG}(x),$$
which further implies $FG \in \mathscr{L}$ by the above second equivalence. Therefore, $FG \in \mathscr{S}^*$ follows from Lemma 1(i) immediately.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows because of Lemma 1(i).

Finally, relation (3) holds by Lemma 1(iii). It remains to prove relation (4). First, observe that any of the equivalent statements in (i)–(iv) from Theorem 1 implies the existence of finite μ_F and μ_G. Further, for $M > 0$ and $x > 2M$, we have

$$\begin{aligned}\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y &= \int_0^M \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y + \int_M^{x-M} \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y \\ &\quad + \int_0^M \overline{F}(y)\overline{G}(x-y)\,\mathrm{d}y \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} &\geqslant \liminf_{x\to\infty} \frac{J_1+J_3}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} \\ &\geqslant \liminf_{x\to\infty} \min\left\{\frac{J_1}{\mu_G\overline{F}(x)}, \frac{J_3}{\mu_F\overline{G}(x)}\right\} \\ &\geqslant \min\left\{\frac{\int_0^M \overline{G}(y)\,\mathrm{d}y}{\mu_G}, \frac{\int_0^M \overline{F}(y)\,\mathrm{d}y}{\mu_F}\right\}. \end{aligned} \tag{9}$$

Letting $M \to \infty$, we get from (9) that

$$\liminf_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} \geqslant 1. \tag{10}$$

For the corresponding upper bound, we obtain

$$\begin{aligned}\limsup_{x\to\infty} \frac{\int_0^x \overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)} &\leqslant \limsup_{x\to\infty} \max\left\{\frac{J_1}{\mu_G\overline{F}(x)}, \frac{J_3}{\mu_F\overline{G}(x)}\right\} \\ &\quad + \limsup_{x\to\infty} \frac{J_2}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}. \end{aligned} \tag{11}$$

By condition $F \in \mathscr{L}$, we get

$$\limsup_{x\to\infty} \frac{J_1}{\mu_G\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x-M)}{\overline{F}(x)} \frac{1}{\mu_G} \int_0^M \overline{G}(y)\,\mathrm{d}y = \frac{1}{\mu_G} \int_0^M \overline{G}(y)\,\mathrm{d}y.$$

Now, letting $M \to \infty$, we obtain

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{J_1}{\mu_G\overline{F}(x)} \leqslant 1. \tag{12}$$

Similarly, condition $G \in \mathscr{L}$ implies

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{J_3}{\mu_F\overline{G}(x)} \leqslant 1. \tag{13}$$

Further, according to Theorem 1(iii), we have that $(F+G)/2 \in \mathscr{S}^*$. Hence, due to Lemma 2,

$$\begin{aligned}
&\lim_{M\to\infty}\limsup_{x\to\infty}\frac{J_2}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}\\
&\quad\leqslant \frac{1}{\min\{\mu_F,\mu_G\}}\lim_{M\to\infty}\limsup_{x\to\infty}\frac{\int_M^{x-M}\overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\overline{F}(x)+\overline{G}(x)}\\
&\quad\leqslant \frac{2}{\min\{\mu_F,\mu_G\}}\lim_{M\to\infty}\limsup_{x\to\infty}\int_M^{x-M}\frac{\frac{1}{2}(\overline{F}+\overline{G})(x-y)\frac{1}{2}(\overline{F}+\overline{G})(y)}{\frac{1}{2}(\overline{F}+\overline{G})(x)}\,\mathrm{d}y\\
&\quad= 0.
\end{aligned}\tag{14}$$

Estimates (11)–(14) imply

$$\limsup_{x\to\infty}\frac{\int_0^x\overline{F}(x-y)\overline{G}(y)\,\mathrm{d}y}{\mu_G\overline{F}(x)+\mu_F\overline{G}(x)}\leqslant 1.\tag{15}$$

Hence, the desired relation (4) of the theorem follows immediately from (10) and (15). Theorem 1 is proved.

3 Proof of Theorem 2

3.1 Auxiliary lemmas

In this subsection, we present two additional lemmas, which play a crucial role in the proof of Theorem 2. The statement of the first lemma is similar to that in Lemma 2. Note that equivalent condition for $F \in \mathscr{S}^*$ does not require additional condition $F \in \mathscr{L}$, comparing to Lemma 2.

Lemma 3. *Suppose F is distribution of $\mathbb{R}$ such that $\mu_F<\infty$. Then $F\in\mathscr{S}^*$ if and only if*

$$\lim_{x\to\infty}\int_0^{x/2}\frac{\overline{F}(x-y)-\overline{F}(x)}{\overline{F}(x)}\,\overline{F}(y)\,\mathrm{d}y=0.$$

Proof. The proof is similar to the proof of Lemma 3 from [10]. Obviously, equality (2) is equivalent to

$$\frac{\int_0^{x/2}\overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y}{\overline{F}(x)}=\mu_F.$$

Thus,

$$\begin{aligned}
F\in\mathscr{S}^* \quad&\Longleftrightarrow\quad \lim_{x\to\infty}\int_0^{x/2}\frac{\overline{F}(x-y)}{\overline{F}(x)}\,\overline{F}(y)\,\mathrm{d}y=\lim_{x\to\infty}\int_0^{x/2}\overline{F}(y)\,\mathrm{d}y\\
&\Longleftrightarrow\quad \lim_{x\to\infty}\int_0^{x/2}\biggl(\frac{\overline{F}(x-y)}{\overline{F}(x)}-1\biggr)\overline{F}(y)\,\mathrm{d}y=0.
\end{aligned}$$

□

The second lemma is a technical result about behaviour of the special sequences.

Lemma 4. *Let* $\{a_n,\, n \geqslant 1\}$ *be an unboundedly increasing sequence of positive numbers, and let*

$$h_n := \max\bigl\{k\colon\ (k+1)! \leqslant a_n(\log a_n)^\beta\bigr\}$$

with some positive $\beta > 0$. *Then, for all sufficiently large* n,

$$\frac{\log a_n}{\log\log a_n} \leqslant h_n \leqslant \frac{2\log a_n}{\log\log a_n}. \tag{16}$$

Proof. The proof is constructed along to similar lines as in Lemma 5 from [10]. Namely, the Stirling's formula implies that

$$\log(k+1)! = (k+1)\log(k+1) - (k+1) - O(\log k)$$

for $k \to \infty$. Define

$$\widehat{h}_n = \frac{2\log a_n}{\log\log a_n}.$$

For some positive constant c_1 and for sufficiently large n, we have

$$\begin{aligned}\log(\widehat{h}_n+1)! &\geqslant (\widehat{h}_n+1)\log(\widehat{h}_n+1) - (\widehat{h}_n+1) - c_1\log\widehat{h}_n \geqslant \frac{9}{10}\,\widehat{h}_n\log\widehat{h}_n\\ &= \frac{9}{5}\,\log\bigl(a_n(\log a_n)^\beta\bigr)\frac{\log a_n}{\log\log a_n}\,\frac{\log 2 + \log\log a_n - \log\log\log a_n}{\log a_n + \beta\log\log a_n}\\ &\geqslant \log\bigl(a_n(\log a_n)^\beta\bigr),\end{aligned}$$

which implies the upper bound in (16).

Similarly, using Stirling's formula again, for

$$\widetilde{h}_n = \frac{\log a_n}{\log\log a_n},$$

we obtain

$$\begin{aligned}\log(\widetilde{h}_n+1)! &\leqslant \widetilde{h}_n\log\widetilde{h}_n + c_2\log\widetilde{h}_n\\ &= \log a_n\biggl(1 - \frac{\log\log\log a_n}{\log\log a_n}\biggr)\biggl(1 + \frac{c_2}{\widetilde{h}_n}\biggr)\end{aligned}$$

with some positive c_2 and sufficiently large n. Therefore, for large n,

$$\log(\widetilde{h}_n+1)! \leqslant \log\bigl(a_n(\log a_n)^\beta\bigr),$$

which implies the lower bound in (16). Lemma is proved. □

3.2 Proof of the theorem

Define two distributions $\mathcal{F}$ and $\mathcal{G}$ with tails:

$$\overline{\mathcal{F}}(x) := \mathbf{1}_{(-\infty,\,6!)}(x) + (6!)^2\Biggl\{\sum_{n=6}^{\infty}\frac{1}{(n!)^2}\mathbf{1}_{[n!,\,(n+1)!-b_nd_n)}(x) + \sum_{n=6}^{\infty}\frac{1}{((n+1)!)^2}\biggl(1+\frac{(n+1)!-x}{d_n}\biggr)\mathbf{1}_{[(n+1)!-b_nd_n,\,(n+1)!)}(x)\Biggr\},$$

$$\overline{\mathcal{G}}(x) := \mathbf{1}_{(-\infty,8!)}(x) + (8!)^2\Biggl\{\sum_{n=3}^{\infty}\frac{1}{((2^n)!)^2}\mathbf{1}_{[(2^n)!,\,(2^n+1)!-\widehat{b}_n\widehat{d}_n)}(x) + \sum_{n=3}^{\infty}\frac{1}{((2^n+1)!)^2}\biggl(1+\frac{(2^n+1)!-x}{\widehat{d}_n}\biggr)\mathbf{1}_{[(2^n+1)!-\widehat{b}_n\widehat{d}_n,\,(2^n+1)!)}(x) + \sum_{n=3}^{\infty}\frac{1}{x^2}\mathbf{1}_{[(2^n+1)!,\,(2^{n+1})!)}(x)\Biggr\},$$

where $b_n := n^2+2n$, $d_n := (\log b_n)^3$, $\widehat{b}_n = b_{2^n} = 2^n(2^n+2)$, and $\widehat{d}_n = (\log\widehat{b}_n)^2$. The functions above are constructed according to the scheme presented in [10] and [16].

Because of Theorem 1, it suffices to prove that $\mathcal{F},\mathcal{G}\in\mathscr{S}^*$ and $(\mathcal{F}+\mathcal{G})/2\notin\mathscr{S}^*$. According to Lemma 3, we have to prove the following relations:

$$\mu_{\mathcal{F}}<\infty,\qquad \mu_{\mathcal{G}}<\infty, \tag{17}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{\overline{\mathcal{F}}(x-y)-\overline{\mathcal{F}}(x)}{\overline{\mathcal{F}}(x)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y = 0, \tag{18}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{\overline{\mathcal{G}}(x-y)-\overline{\mathcal{G}}(x)}{\overline{\mathcal{G}}(x)}\,\overline{\mathcal{G}}(y)\,\mathrm{d}y = 0, \tag{19}$$

$$\limsup_{x\to\infty}\int_0^{x/2}\frac{(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x-y)-(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x)}{(\overline{\mathcal{F}}+\overline{\mathcal{G}})(x)}\,(\overline{\mathcal{F}}+\overline{\mathcal{G}})(y)\,\mathrm{d}y \;>\; 0. \tag{20}$$

Denote

$$\varDelta_{\mathcal{F}}(x,y) := \frac{\overline{\mathcal{F}}(x-y)-\overline{\mathcal{F}}(x)}{\overline{\mathcal{F}}(x)},\qquad \varDelta_{\mathcal{G}}(x,y) := \frac{\overline{\mathcal{G}}(x-y)-\overline{\mathcal{G}}(x)}{\overline{\mathcal{G}}(x)}.$$

Proof of (17). According to definitions of $\overline{\mathcal{F}}(x)$ and $\overline{\mathcal{G}}(x)$,

$$\mu_{\mathcal{F}} = \int_0^{\infty}\overline{\mathcal{F}}(y)\,\mathrm{d}y \leqslant 6! + 6!\sum_{n=3}^{\infty}\frac{1}{(n!)^2}\bigl((n+1)!-n!\bigr) < 1238,$$

$$\mu_{\mathcal{G}} = \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y \leqslant 8! + 8!\sum_{n=3}^\infty \frac{1}{((2^n)!)^2}\big((2^n+1)! - (2^n)!\big) + 8!\sum_{n=3}^\infty \left(\frac{1}{(2^n+1)!} - \frac{1}{(2^{n+1})!}\right) < 98243,$$

implying (17). □

Proof of (18). Suppose that n is sufficiently large and let

$$(n+1)! - b_n d_n \leqslant x < (n+1)!. \tag{21}$$

For such x, we have

$$\Delta_{\mathcal{F}}(x,y) = \begin{cases} \frac{y}{d_n} & \text{if } 0 \leqslant y \leqslant x - ((n+1)! - b_n d_n), \\ \frac{x-((n+1)!-b_n d_n)}{d_n+(n+1)!-x} & \text{if } x - ((n+1)! - b_n d_n) < y \leqslant \frac{x}{2}. \end{cases}$$

Therefore, for x in (21), we have

$$J_{\mathcal{F}}(x) := \int_0^{x/2} \Delta_{\mathcal{F}}(x,y)\overline{\mathcal{F}}(y)\,\mathrm{d}y \leqslant \frac{1}{d_n}\int_0^{b_n d_n} y\overline{\mathcal{F}}(y)\,\mathrm{d}y + b_n \int_{b_n d_n}^\infty \overline{\mathcal{F}}(y)\,\mathrm{d}y =: K_1 + K_2. \tag{22}$$

Define $k_n := \max\{k\colon\ (k+1)! \leqslant b_n d_n\}$ and write

$$\begin{aligned} K_1 &\leqslant \frac{(6!)^2}{2d_n}\Bigg\{1 + \sum_{k=6}^{k_n+1} \frac{1}{(k!)^2}\big(\big((k+1)! - b_k d_k\big)^2 - (k!)^2\big) \\ &\quad + \sum_{k=6}^{k_n+1} \frac{1}{((k+1)!)^2}\left(1 + \frac{(k+1)!}{d_k}\right)\big(\big((k+1)!\big)^2 - \big((k+1)! - b_k d_k\big)^2\big)\Bigg\} \\ &\leqslant \frac{(6!)^2}{2d_n}\left(1 + 3\sum_{k=6}^{k_n+1} b_k\right) = \frac{(6!)^2}{2d_n}\left(1 + 3\sum_{k=6}^{k_n+1}(k^2+2k)\right) \end{aligned} \tag{23}$$

because

$$\frac{1}{(k!)^2}\big(((k+1)! - b_k d_k)^2 - (k!)^2\big) \leqslant (k+1)^2 - 1 = b_k$$

and

$$\left(1 + \frac{(k+1)!}{d_k}\right)\big(\big((k+1)!\big)^2 - \big((k+1)! - b_k d_k\big)^2\big) \leqslant 2\big((k+1)!\big)^2 b_k.$$

Thus, by (23) and Lemma 4,

$$K_1 \leqslant (6!)^2 \frac{k_n^3}{d_n} \leqslant (6!)^2 \left(\frac{2\log b_n}{\log\log b_n}\right)^3 \frac{1}{(\log b_n)^3} = \frac{8(6!)^2}{(\log\log b_n)^3}. \tag{24}$$

For the second integral in (22), we have

$$\begin{aligned} K_2 &\leqslant b_n(6! - b_n d_n)^+ + b_n \sum_{k=k_n+1}^{\infty} \frac{6!}{(k!)^2}\bigl((k+1)! - k!\bigr) \\ &= b_n(6! - b_n d_n)^+ + b_n 6! \sum_{k=k_n}^{\infty} \frac{1}{k!} \\ &\leqslant b_n(6! - b_n d_n)^+ + \frac{24(6!)}{(\log\log b_n)^2 \log b_n} \end{aligned} \tag{25}$$

because of the following estimate:

$$\begin{aligned} b_n \sum_{k=k_n}^{\infty} \frac{1}{k!} &\leqslant \frac{\mathrm{e}}{k_n!}\frac{b_n d_n}{d_n} \leqslant \frac{\mathrm{e}}{k_n!}\frac{(k_n+2)!}{d_n} \\ &\leqslant \frac{2\mathrm{e}k_n^2}{d_n} \leqslant 6\left(\frac{2\log b_n}{\log\log b_n}\right)^2 \frac{1}{(\log b_n)^3} \\ &= \frac{24}{(\log\log b_n)^2 \log b_n}. \end{aligned}$$

Here we have used that, by definition of k_n, $b_n d_n \leqslant (k_n+2)! \leqslant 2k_n!$ and then applied Lemma 4. Substituting estimates (24)–(25) into (22), we get that for x from (21), it holds

$$J_{\mathcal{F}}(x) \leqslant \frac{c_1}{(\log\log b_n)^3} \tag{26}$$

for some positive constant c_1.

Now, consider x satisfying

$$(n+1)! \leqslant x < (n+2)! - b_{n+1}d_{n+1}. \tag{27}$$

We split this interval into three subintervals

$$(n+1)! \leqslant x < 2\bigl((n+1)! - b_n d_n\bigr), \tag{28}$$

$$2\bigl((n+1)! - b_n d_n\bigr) \leqslant x < 2(n+1)!, \tag{29}$$

$$2(n+1)! \leqslant x < (n+2)! - b_{n+1}d_{n+1} \tag{30}$$

and estimate $J_{\mathcal{F}}(x)$ in each case separately.

In case (28), we have

$$\varDelta_{\mathcal{F}}(x,y) = \begin{cases} \frac{y}{d_n} & \text{if}\ , 0 \leqslant y \leqslant x - (n+1)!, \\ \frac{(n+1)! - x + y}{d_n} & \text{if}\ x - (n+1)! < y \leqslant x - ((n+1)! - b_n d_n), \\ b_n & \text{if}\ x - ((n+1)! - b_n d_n) < y \leqslant \frac{x}{2}. \end{cases}$$

Since

$$\frac{(n+1)! - x + y}{d_n} \leqslant \min\left\{\frac{y}{d_n}, b_n\right\},$$

for the x from (28) and for $x - (n+1)! < y \leqslant x - ((n+1)! - b_n d_n)$, we get that $J_{\mathcal{F}}(x) \leqslant K_1 + K_2$ and estimate (26) holds again.

Consider now case (29). We have

$$\Delta_{\mathcal{F}}(x,y) = \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant x - (n+1)!, \\ \frac{(n+1)! - x + y}{d_n} & \text{if } x - (n+1)! < y \leqslant \frac{x}{2}. \end{cases}$$

Thus,

$$\begin{aligned} J_{\mathcal{F}}(x) &= \int\limits_{x-(n+1)!}^{x/2} \frac{(n+1)! - x + y}{d_n} \overline{\mathcal{F}}(y)\,\mathrm{d}y \\ &\leqslant b_n \int\limits_{b_n d_n}^{\infty} \overline{\mathcal{F}}(y)\,\mathrm{d}y = K_2 \leqslant \frac{c_2}{\log b_n (\log\log b_n)^2}, \end{aligned}$$

according to estimate (25), where c_2 is some positive constant.

Finally, in case (30), $\Delta_{\mathcal{F}}(x,y) = 0$ for all $0 \leqslant y \leqslant x/2$, implying $J_{\mathcal{F}}(x) = 0$.

Summarizing, estimate (26) holds for all x in (27) and for all sufficiently large n. This implies relation (18). □

Proof of (19). Suppose that n is sufficiently large and split the interval

$$(2^n)! \leqslant x < (2^{n+1})!$$

into following subintervals:

$$(2^n)! \leqslant x < 2(2^n)!, \tag{31}$$

$$2(2^n)! \leqslant x < (2^n + 1)! - \widehat{b}_n \widehat{d}_n, \tag{32}$$

$$(2^n + 1)! - \widehat{b}_n \widehat{d}_n \leqslant x < (2^n + 1)!, \tag{33}$$

$$(2^n + 1)! \leqslant x < 2((2^n + 1)! - \widehat{b}_n \widehat{d}_n), \tag{34}$$

$$2((2^n + 1)! - \widehat{b}_n \widehat{d}_n) \leqslant x < 2(2^n + 1)!, \tag{35}$$

$$2(2^n + 1)! \leqslant x < (2^{n+1})!. \tag{36}$$

As in the case of $\mathcal{F}$, for each subset above, we will obtain the exact expressions for $\Delta_{\mathcal{G}}(x,y)$ and then, the upper bounds for $J_{\mathcal{G}}(x)$.

In case (31),

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant x - (2^n)!, \\ \frac{((2^n)!)^2 - (x-y)^2}{(x-y)^2} & \text{if } x - (2^n)! < y \leqslant \frac{x}{2}, \end{cases}$$

and, consequently,

$$J_{\mathcal{G}}(x) \leqslant \int_0^{x/2} \left(\left(\frac{x}{x-y} \right)^2 - 1 \right) \overline{\mathcal{G}}(y)\, \mathrm{d}y.$$

Since $(x/(x-y))^2 \leqslant 4$, by the dominated convergence theorem, we have that

$$\sup_{(2^n)! \leqslant x < 2(2^n)!} J_{\mathcal{G}}(x) = \epsilon_1(n) \to 0, \quad n \to \infty. \tag{37}$$

In case (32), $\Delta_{\mathcal{G}}(x,y) = 0$ for $0 \leqslant y \leqslant x/2$, implying

$$\sup_{2(2^n)! \leqslant x < (2^n+1)! - \widehat{b}_n \widehat{d}_n} J_{\mathcal{G}}(x) = 0. \tag{38}$$

In case (33), we have

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y}{(2^n+1)! + \widehat{d}_n - x} & \text{if } 0 \leqslant y < x - \bigl((2^n+1)! - \widehat{b}_n \widehat{d}_n\bigr), \\ \frac{x - ((2^n+1)! - \widehat{b}_n \widehat{d}_n)}{(2^n+1)! + \widehat{d}_n - x} & \text{if } x - ((2^n+1)! - \widehat{b}_n \widehat{d}_n) \leqslant y < \frac{x}{2}, \end{cases}$$

implying that

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \int_0^{x/2} \min\left\{ \frac{y}{\widehat{d}_n}, \widehat{b}_n \right\} \overline{\mathcal{G}}(y)\, \mathrm{d}y \leqslant \frac{1}{\widehat{d}_n} \int_0^{\widehat{d}_n \widehat{b}_n} y \overline{\mathcal{G}(y)}\, \mathrm{d}y + \widehat{b}_n \int_{\widehat{d}_n \widehat{b}_n}^{\infty} \overline{\mathcal{G}}(y)\, \mathrm{d}y \\ &=: L_1 + L_2. \end{aligned}$$

Analogously to k_n, define $\widehat{k}_n := \max\{k\colon\ (2^k+1)! \leqslant \widehat{b}_n \widehat{d}_n\}$. We get

$$\begin{aligned} L_1 &\leqslant \frac{1}{\widehat{d}_n} \int_0^{\widehat{b}_n \widehat{d}_n} y \Biggl\{ \mathbf{1}_{(-\infty, 8!)}(y) + (8!)^2 \sum_{k=3}^{\infty} \frac{1}{((2^k)!)^2} \mathbf{1}_{[(2^k)!, (2^k+1)! - \widehat{b}_k \widehat{d}_k)}(y) \\ &\quad + (8!)^2 \sum_{k=3}^{\infty} \frac{1 + \widehat{b}_k}{((2^k+1)!)^2} \mathbf{1}_{[(2^k+1)! - \widehat{b}_k \widehat{d}_k, (2^k+1)!)}(y) \\ &\quad + (8!)^2 \sum_{k=3}^{\infty} \frac{1}{y^2} \mathbf{1}_{[(2^k+1)!, (2^{k+1})!)}(y) \Biggr\}\, \mathrm{d}y \\ &\leqslant \frac{(8!)^2}{2\widehat{d}_n} + \frac{(8!)^2}{\widehat{d}_n} \sum_{k=3}^{\widehat{k}_n+1} \frac{1}{((2^k)!)^2} \int_{(2^k)!}^{(2^k+1)!} y\, \mathrm{d}y + \frac{(8!)^2}{\widehat{d}_n} \int_{9!}^{\widehat{b}_n \widehat{d}_n} \frac{1}{y}\, \mathrm{d}y \\ &\leqslant \frac{(8!)^2}{2\widehat{d}_n} + \frac{(8!)^2}{\widehat{d}_n} \log(\widehat{b}_n \widehat{d}_n) + \frac{(8!)^2}{2\widehat{d}_n} \sum_{k=3}^{\widehat{k}_n+1} \widehat{b}_k = \epsilon_2(n) \to 0, \quad n \to \infty, \end{aligned} \tag{39}$$

because for sufficiently large n,

$$\sum_{k=3}^{\widehat{k}_n+1} \widehat{b}_k \leqslant \frac{16}{3}\, 2^{2\widehat{k}_n} + 8 \cdot 2^{\widehat{k}_n} \leqslant 6 \cdot 2^{2\widehat{k}_n} \leqslant \frac{24(\log \widehat{b}_n)^2}{(\log\log \widehat{b}_n)^2}$$

due to Lemma 4.

For the integral L_2, we obtain

$$\begin{aligned} L_2 &\leqslant \widehat{b}_n\Biggl((8! - \widehat{b}_n\widehat{d}_n)^+ + (8!)^2 \sum_{k=\widehat{k}_n+1}^{\infty} \frac{(2^k+1)! - (2^k)!}{((2^k)!)^2} + (8!)^2 \int\limits_{\widehat{b}_n\widehat{d}_n}^{\infty} \frac{\mathrm{d}y}{y^2} \Biggr) \\ &\leqslant \widehat{b}_n\Biggl((8! - \widehat{b}_n\widehat{d}_n)^+ + (8!)^2 \sum_{k=\widehat{k}_n+1}^{\infty} \frac{1}{(2^k-1)!} + \frac{(8!)^2}{\widehat{b}_n\widehat{d}_n} \Biggr) \\ &\leqslant \widehat{b}_n(8! - \widehat{b}_n\widehat{d}_n)^+ + \frac{(8!)^2 \mathrm{e}\, \widehat{b}_n}{(2^{\widehat{k}_n+1}-1)!} + \frac{(8!)^2}{\widehat{d}_n} = \epsilon_3(n) \to 0, \quad n\to\infty, \end{aligned} \tag{40}$$

because $(2^{\widehat{k}_n+1}-1)! > \widehat{b}_n\widehat{d}_n$ for large n, according to definition of the sequence $\widehat{k}_n$. Relations (39)–(40) imply that

$$\sup_{2(2^n)! \leqslant x < (2^n+1)! - b_n d_n} J_{\mathcal{G}}(x) \leqslant \epsilon_2(n) + \epsilon_3(n) \to 0, \quad n \to \infty. \tag{41}$$

In case (34), we obtain

$$\varDelta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leqslant y \leqslant x-(2^n+1)!, \\ \frac{x^2(\widehat{d}_n+(2^n+1)!-x+y)}{\widehat{d}_n((2^n+1)!)^2} & \text{if } x-(2^n+1)! < y \leqslant x-((2^n+1)!-\widehat{b}_n\widehat{d}_n), \\ (\frac{x}{(2^n)!})^2 - 1 & \text{if } x-((2^n+1)!-\widehat{b}_n\widehat{d}_n) < y \leqslant \frac{x}{2}. \end{cases} \tag{42}$$

Hence,

$$\begin{aligned} J_{\mathcal{G}}(x) = &\int\limits_0^{x-(2^n+1)!} \frac{y(2x-y)}{(x-y)^2}\, \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ &+ \int\limits_{x-(2^n+1)!}^{x-((2^n+1)!-\widehat{b}_n\widehat{d}_n)} \biggl(\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 \biggr) \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ &+ \int\limits_{x-((2^n+1)!-\widehat{b}_n\widehat{d}_n)}^{x/2} \biggl(\biggl(\frac{x}{(2^n)!}\biggr)^2 - 1 \biggr) \overline{\mathcal{G}}(y)\,\mathrm{d}y. \end{aligned} \tag{43}$$

In particular case of (34), when $(2^n+1)! \leqslant x < (2^n+1)! + \widehat{b}_n\widehat{d}_n$, by estimating the above integrals separately, we get

$$J_{\mathcal{G}}(x) \leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + \epsilon_4(n)$$
$$+ \frac{2}{\widehat{d}_n}\int_0^{\widehat{b}_n\widehat{d}_n} y\overline{\mathcal{G}}(y)\,\mathrm{d}y + 2\widehat{b}_n \int_{\widehat{b}_n\widehat{d}_n}^{2\widehat{b}_n\widehat{d}_n} \overline{\mathcal{G}}(y)\,\mathrm{d}y + \widehat{b}_n \int_{\widehat{b}_n\widehat{d}_n}^{\infty} \overline{\mathcal{G}}(y)\,\mathrm{d}y$$

for some vanishing function $\epsilon_4(n)$. Thus, for large n and for all $x \in [(2^n+1)!, (2^n+1)! + \widehat{b}_n\widehat{d}_n)$, we have that

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \epsilon_1(n) + \epsilon_4(n) + 3(L_1 + L_2) \\ &\leqslant \epsilon_1(n) + \epsilon_4(n) + 3\bigl(\epsilon_2(n) + \epsilon_3(n)\bigr) \to 0. \end{aligned} \tag{44}$$

For the remaining subinterval of (34), where $(2^n+1)! + \widehat{b}_n\widehat{d}_n \leqslant x < 2((2^n+1)! - \widehat{b}_n\widehat{d}_n)$, using expressions (42) and (43), we obtain

$$\begin{aligned} J_{\mathcal{G}}(x) &\leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + 3\widehat{b}_n \int_{\widehat{b}_n\widehat{d}_n}^{\infty} \overline{\mathcal{G}}(y)\,\mathrm{d}y \\ &\leqslant \epsilon_1(n) + 3\,\epsilon_3(n) \to 0. \end{aligned} \tag{45}$$

Relations (44) and (45) imply that

$$\sup_{2(2^n+1)! \leqslant x < 2((2^n+1)! - \widehat{b}_n\widehat{d}_n)} J_{\mathcal{G}}(x) \leqslant \epsilon_5(n) \tag{46}$$

with some vanishing function ϵ_5.

Consider now case (35). For such x,

$$\Delta_{\mathcal{G}}(x,y) = \begin{cases} \frac{y(2x-y)}{(x-y)^2} & \text{if } 0 \leqslant y \leqslant x - (2^n+1)!, \\ \frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 & \text{if } x - (2^n+1)! < y \leqslant \frac{x}{2}, \end{cases}$$

implying that

$$J_{\mathcal{G}}(x) = \int_0^{x-(2^n+1)!} \frac{y(2x-y)}{(x-y)^2}\,\overline{\mathcal{G}}(y)\,\mathrm{d}y$$
$$+ \int_{x-(2^n+1)!}^{x/2} \left(\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1\right)\overline{\mathcal{G}}(y)\,\mathrm{d}y.$$

In the case under consideration, we have that $x - (2^n + 1)! \geqslant \widehat{b}_n \widehat{d}_n$ and

$$\frac{x^2(\widehat{d}_n + (2^n+1)! - x + y)}{\widehat{d}_n((2^n+1)!)^2} - 1 \leqslant 4\,\frac{\widehat{d}_n + (2^n+1)! - x/2}{\widehat{d}_n} \leqslant \frac{\widehat{d}_n + \widehat{b}_n\widehat{d}_n}{\widehat{d}_n} \leqslant 5\widehat{b}_n.$$

The derived estimates yield

$$J_{\mathcal{G}}(x) \leqslant \int_0^\infty \left(\frac{x}{x-y}\right)^2 \mathbf{1}_{[0,x/2]}(y)\overline{\mathcal{G}}(y)\,\mathrm{d}y - \int_0^\infty \overline{\mathcal{G}}(y)\,\mathrm{d}y + 5L_2,$$

which implies that

$$\sup_{2((2^n+1)! - \widehat{b}_n\widehat{d}_n) \leqslant x < 2(2^n+1)!} J_{\mathcal{G}}(x) \leqslant \epsilon_1(n) + 5\epsilon_3(n). \tag{47}$$

Finally, consider case (36). For these x and for all $0 \leqslant y \leqslant x/2$,

$$\varDelta_{\mathcal{G}}(x,y) = \frac{y(2x-y)}{(x-y)^2}.$$

Thus,

$$\sup_{2(2^n+1)! \leqslant x < (2^{n+1})!} J_{\mathcal{G}}(x) \leqslant \epsilon_1(n). \tag{48}$$

The derived estimates (37), (38), (41), (46), (47), and (48) imply that

$$\lim_{n\to\infty} \sup_{(2^n)! \leqslant x < (2^{n+1})!} J_{\mathcal{G}}(x) \to 0,$$

showing the validity of (19).

It remains to prove inequality (20). Integral from this inequality is bounded from below by

$$J_{\mathcal{F},\mathcal{G}}(x) := \int_0^{x/2} \frac{\overline{\mathcal{G}}(x-y) - \overline{\mathcal{G}}(x)}{\overline{\mathcal{F}}(x) + \overline{\mathcal{G}}(x)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y.$$

Take $x_n := (2^n+1)!$. Then $\overline{\mathcal{F}}(x_n) = \overline{\mathcal{G}}(x_n) = 1/x_n^2$, implying that

$$J_{\mathcal{F},\mathcal{G}}(x_n) = \frac{1}{2}\int_0^{x_n/2} \frac{\overline{\mathcal{G}}(x_n-y) - \overline{\mathcal{G}}(x_n)}{\overline{\mathcal{G}}(x_n)}\,\overline{\mathcal{F}}(y)\,\mathrm{d}y \geqslant \frac{1}{2}\int_0^{\widehat{b}_n\widehat{d}_n} \varDelta_{\mathcal{G}}(x_n,y)\overline{\mathcal{F}}(y)\,\mathrm{d}y$$

for large n. According to (42),

$$\Delta_{\mathcal{G}}(x_n, y) = \frac{\widehat{d}_n + y}{\widehat{d}_n} \geqslant \frac{y}{\widehat{d}_n}.$$

Consequently, denoting $\widetilde{k}_n = \max\{k\colon (k+1)! \leqslant \widehat{b}_n \widehat{d}_n\}$, we get

$$\begin{aligned}
J_{\mathcal{F},\mathcal{G}}(x_n) &\geqslant \frac{1}{2\widehat{d}_n} \int\limits_0^{\widehat{b}_n \widehat{d}_n} y\overline{\mathcal{F}}(y)\,\mathrm{d}y \\
&= \frac{1}{2\widehat{d}_n} \int\limits_0^{\widehat{b}_n \widehat{d}_n} y\Biggl\{\mathbf{1}_{(-\infty,6!)}(y) + (6!)^2 \sum_{k=6}^{\infty} \frac{1}{(k!)^2} \mathbf{1}_{[k!,(k+1)!-b_k d_k)}(y) \\
&\quad + (6!)^2 \sum_{k=6}^{\infty} \frac{1}{((k+1)!)^2} \biggl(1 + \frac{(k+1)! - y}{d_k}\biggr) \mathbf{1}_{[(k+1)!-b_k d_k,(k+1)!)}(y)\Biggr\}\,\mathrm{d}y \\
&\geqslant \frac{(6!)^2}{4\widehat{d}_n} \sum_{k=6}^{\widetilde{k}_n} \frac{((k+1)! - b_k d_k)^2 - (k!)^2}{(k!)^2} \\
&= \frac{(6!)^2}{4\widehat{d}_n} \sum_{k=6}^{\widetilde{k}_n} \biggl(\biggl(k + 1 - \frac{b_k d_k}{k!}\biggr)^2 - 1\biggr) \\
&\geqslant \frac{(6!)^2}{4\widehat{d}_n} \Biggl(\sum_{k=6}^{\widetilde{k}_n} k^2 - 2\sum_{k=6}^{\widetilde{k}_n} \frac{k+1}{k!}\, b_k d_k\Biggr).
\end{aligned}$$

Since the series

$$\sum_{k=6}^{\infty} \frac{k+1}{k!}\, b_k d_k$$

converges, we have that

$$J_{\mathcal{F},\mathcal{G}}(x_n) \geqslant c_2 \biggl(\frac{\widetilde{k}_n^3}{\widehat{d}_n} - c_3\biggr)$$

for large n with some positive constants c_2 and c_3. As $\widehat{b}_n \widehat{d}_n = \widehat{b}_n (\log \widehat{b}_n)^2$, applying Lemma 4 with $a_n = \widehat{b}_n$ and $\beta = 2$ to the sequence $\widetilde{k}_n$, we get

$$\frac{\widetilde{k}_n^3}{\widehat{d}_n} \geqslant \frac{\log \widehat{b}_n}{(\log\log \widehat{b}_n)^3} \to \infty, \quad n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} J_{\mathcal{F},\mathcal{G}}(x_n) = \infty,$$

and the desired inequality (20) follows. Theorem 2 is proved. □

Acknowledgment. The authors are deeply grateful to the anonymous referees for careful reading and valuable comments and suggestions, which helped to improve the earlier version of the paper.

References

1. K.B. Athreya, P.E. Ney, *Branching Processes*, Springer, New York, 1972, https://doi.org/10.1007/978-3-642-65371-1.
2. V.P. Chistyakov, A theorem on sums of independent, positive random variables and its applications to branching processes, *Theory Probab. Appl.*, **9**:640–648, 1964, https://doi.org/10.1137/1109088.
3. J. Chover, P. Ney, S. Wainger, Degeneracy properties of subcritical branching processes, *Ann. Probab.*, **1**:663–673, 1973, https://doi.org/10.1214/aop/1176996893.
4. J. Chover, P. Ney, S. Wainger, Functions of probability measures, *J. Anal. Math.*, **26**:255–302, 1973, https://doi.org/10.1007/BF02790433.
5. P. Embrechts, C.M. Goldie, On closure and factorization properties of subexponential and related distributions, *J. Aust. Math. Soc., Ser. A*, **29**:243–256, 1980, https://doi.org/10.1017/S1446788700021224.
6. S. Foss, D. Korshunov, S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions,* 2nd edn, Springer, New York, 2013, https://doi.org/10.1007/978-1-4419-9473-8.14.
7. J.L. Geluk, J.B.G. Frenk, Renewal theory for random variables with a heavy tailed distribution and finite variance, *Stat. Probab. Lett.*, **81**:77–82, 2011, https://doi.org/10.1016/j.spl.2010.09.021.
8. R. Kaas, Q. Tang, Note on the tail behavior of random walk maxima with heavy tails and negative drift, *N. Am. Actuar. J.*, **7**:57–61, 2003, https://doi.org/10.1080/10920277.2003.10596103.
9. C. Klüppelberg, Subexponential distributions and integrated tails, *J. Appl. Probab.*, **25**:132–141, 1988, https://doi.org/10.2307/3214240.
10. C. Klüppelberg, J.A. Villasenor, The full solution of the convolution closure problem for convolution-equivalent distributions, *J. Math. Anal. Appl.*, **41**:79–92, 1991, https://doi.org/10.1016/0022-247X(91)90291-7.
11. D. Konstantinides, R. Leipus, J. Šiaulys, A note on product-convolution for generalized subexponential distributions, *Nonlinear Anal. Model. Control*, **27**:1054–1067, 2022, https://doi.org/10.15388/namc.2022.27.29405.
12. D. Korshunov, Large-deviation probabilities for maxima of sums of independent random variables with negative mean and subexponential distribution, *Theory Probab. Appl.*, **46**:355–366, 2002, https://doi.org/10.1137/S0040585X97979019.
13. R. Leipus, J. Šiaulys, Asymptotic behaviour of the finite-time ruin probability under subexponential claim sizes, *Insur. Math. Econ.*, **40**:498–508, 2007, https://doi.org/10.1016/j.insmatheco.2006.07.006.
14. R. Leipus, J. Šiaulys, Asymptotic behaviour of the finite-time ruin probability in renewal risk models, *Appl. Stoch. Models Bus. Ind.*, **25**:309–321, 2009, https://doi.org/10.1002/asmb.747.

15. R. Leipus, J. Šiaulys, On a closure property of convolution equivalent class of distributions, *J. Math. Anal. Appl.*, **490**:124226, 2020, https://doi.org/10.1016/j.jmaa.2020.124226.
16. J.R. Leslie, On the non-closure under convolution of the subexponential family, *J. Appl. Probab.*, **26**:58–66, 1989, https://doi.org/10.2307/3214316.
17. K.W. Ng, Q.H. Tang, H. Yang, Maxima of sums of heavy-tailed random variables, *ASTIN Bull.*, **32**:43–55, 2002, https://doi.org/10.2143/AST.32.1.1013.
18. H. Qian, B. Geng, S. Wang, Tail asymptotics of randomly weighted sums of dependent strong subexponential random variables, *Lith. Math. J.*, **62**:113–122, 2022, https://doi.org/10.1007/s10986-021-09546-5.
19. H. Xu, Y. Wang, D. Cheng, C. Yu, On the closure under infinitely divisible distribution roots, *Lith. Math. J.*, **62**:259–287, 2022, https://doi.org/10.1007/s10986-022-09558-9.
20. Y. Yang, R. Leipus, J. Šiaulys, Y. Cang, Uniform estimates for the finite-time ruin probability in the dependent renewal risk model, *J. Math. Anal. Appl.*, **383**:215–225, 2011, https://doi.org/10.1016/j.jmaa.2011.05.013.