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On Discrete Approximation of Analytic Functions by Shifts of the Lerch Zeta Function

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Abstract: The Lerch zeta function is defined by a Dirichlet series depending on two fixed parameters. In the paper, we consider the approximation of analytic functions by discrete shifts of the Lerch zeta function, and we prove that, for arbitrary parameters and a step of arithmetic progression, there is a closed non-empty subset of the space of analytic functions defined in the critical strip such that its functions can be approximated by discrete shifts of the Lerch zeta function. The set of those shifts is infinite, and it has a positive density. For the proof, the weak convergence of probability measures in the space of analytic functions is applied.

Keywords: approximation of analytic functions; Hurwitz zeta function; Lerch zeta function; weak convergence

MSC: 11M35

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1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all positive integers, non-negative integers, integers, real and complex numbers, respectively, and let $s = \sigma + it$ be a complex variable. The Lerch zeta function $L(\lambda, \alpha, s)$ with fixed parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$ is defined, for $\sigma > 1$ if $\lambda \in \mathbb{Z}$ and for $\sigma > 0$ if $\lambda \notin \mathbb{Z}$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

For $\lambda \in \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1.$$

Moreover, the equalities

$$L\left(\lambda, \frac{1}{2}, s\right) = \zeta(s)(2^s - 1), \quad \lambda \in \mathbb{Z},$$

and

$$L\left(\frac{1}{2}, 1, s\right) = \zeta(s)\left(1 - 2^{1-s}\right),$$

where $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function, hold. Moreover, the Lerch zeta function has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1 in the case $\lambda \in \mathbb{Z}$, and is an entire function in the case $\lambda \notin \mathbb{Z}$.

The function $L(\lambda, \alpha, s)$ was introduced independently by M. Lerch [1] and R. Lipschitsz [2]. M. Lerch also proved the functional equation for $L(\lambda, \alpha, s)$

$$L(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left(\exp\left\{ \frac{\pi i s}{2} - 2\pi i \alpha \lambda \right\} L(-\alpha, \lambda, s) + \exp\left\{ -\frac{\pi i s}{2} + 2\pi i \alpha (1-\lambda) \right\} L(\alpha, 1-\lambda, s) \right)$$

which is valid for all $0 < \lambda < 1$ and $s \in \mathbb{C}$; here, $\Gamma(s)$ denotes the Euler gamma function. Thus, the Lerch zeta function is an interesting analytic object that depends on two parameters and generalizes the classical zeta functions $\zeta(s)$ and $\zeta(s, \alpha)$. The analytic theory of the function $L(\lambda, \alpha, s)$ is given in [3]; its analytic properties depend on the arithmetic of the parameters λ and α.

In this paper, we are interested in the approximation of analytic functions by shifts of Lerch zeta functions $L(\lambda, \alpha, s + i\tau)$, $\tau \in \mathbb{R}$. Recall that the latter property of zeta functions, called universality, was discovered by S.M. Voronin [4], who proved that if the function $f(s)$ is continuous nonvanishing in the disc $|s| \leqslant r$, $0 < r < 1/4$, and analytic in the interior of that disc, then, for every $\varepsilon > 0$, $\tau = \tau(\varepsilon) \in \mathbb{R}$ exists such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau \right) - f(s) \right| < \varepsilon.$$

The universality of the Hurwitz zeta function with rational parameter α was considered by Voronin [5], B. Bagchi [6], and S.M. Gonek [7]. In this case, the investigation of universality for $\zeta(s, \alpha)$ reduces to that of joint universality for Dirichlet L-functions. The simplest case is of transcendental α because then the set

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}$$

is linearly independent over the field of rational numbers $\mathbb{Q}$. In this case, the universality of $\zeta(s, \alpha)$ was obtained by Gonek [7] and Bagchi [6]. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H(K)$ with $K \in \mathcal{K}$ be the class of continuous functions on K that are analytic in the interior of K. Let meas$A$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, we can join the above results of [5–7] to the following final result.

Theorem 1. *Suppose that the parameter α is transcendental or rational $\neq 1, 1/2$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

The transcendence of the parameter α in Theorem 1 can be replaced by the linear independence over $\mathbb{Q}$ for the set $L(\alpha)$.

The case of algebraic irrational α is the most difficult problem. In [8], a certain approximation to the universality of $\zeta(s, \alpha)$ with all parameters α was proposed. Let $H(D)$ be the space of the analytic on D functions equipped with the topology of uniform convergence on compacta. Then, it was proved in [8] that there exists a closed non-empty set $F_\alpha \subset H(D)$ whose functions are approximated by shifts $\zeta(s + i\tau, \alpha)$.

More general weighted universality theorems for zeta functions with some classes of weight functions were obtained, see, for example, [9–11].

All of the above-mentioned results on the approximation of analytic functions are of a continuous type. Additionally, discrete versions of the above statements are considered. Let card$A$ denote the number of elements of the set A. The following result is known, see [6,12,13].

Theorem 2. *For α rational $\neq 1$ or $1/2$, let $h > 0$ be arbitrary, and for transcendental α let h be such that the number $\exp\{(2\pi)/h\}$ is rational. Let K and $f(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \text{card}\left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Note that in [13] a more general case of periodic Hurwitz zeta functions is discussed.

The transcendence of α can be replaced [14] by the linear independence over $\mathbb{Q}$ for the set

$$\left\{ (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.$$

The discrete version of theorem from [8] was obtained in [15]. The joint generalizations of theorems from [8,15] are given in [16,17], respectively.

Recently, A. Sourmelidis and J. Steuding proved [18] a very deep universality result for $\zeta(s,\alpha)$ with algebraic irrational parameter α. They obtained that, for all but finitely many algebraic irrationals α, a shift $\zeta(s+i\tau,\alpha)$ approximating a given analytic function exists on discs of the strip D.

Universality theorems for the approximation of analytic functions by generalized shifts of the Hurwitz zeta function were given in [19,20]. Additionally, the universality of the function $\zeta(s,\alpha)$ follows from the joint Mishou type universality theorems for $\zeta(s)$ and $\zeta(s,\alpha)$; see, for example, [21–23].

The list of works on the approximation of analytic functions by shifts of the Lerch zeta function $L(\lambda,\alpha,s)$ with $\lambda \notin \mathbb{Z}$ is not extensive. The first theorem of such a kind was obtained in [24], see also [25].

Theorem 3. *Suppose that α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0,T] : \sup_{s\in K} |L(\lambda,\alpha,s+i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The latter theorem in [26] was extended for some compositions $F(L(\lambda,\alpha,s))$, where $F : H(D) \to H(D)$ are certain continuous operators.

Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta function $\zeta(s,\alpha;\mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and has meromorphic continuation to the whole complex plane with possible simple pole at point $s = 1$. If $0 < \lambda < 1$ is rational, then the sequence $\{e^{2\pi i\lambda m} : m \in \mathbb{N}_0\}$ is periodic. Therefore, the Lerch zeta function with rational parameter λ is a partial case of the periodic Hurwitz zeta function. Thus, the results of universality for $\zeta(s,\alpha;\mathfrak{a})$ also remain valid for $L(\lambda,\alpha,s)$ with rational α. From [13], the following theorem follows.

Theorem 4. *Suppose that the parameters λ and α are rational and transcendental, respectively, and $h > 0$ is such that $\exp\{(2\pi)/h\}$ is rational. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \text{card}\left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |L(\lambda,\alpha,s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

A similar corollary follows from the universality of the function $\zeta(s,\alpha;\mathfrak{a})$ with rational parameter α [27].

More attention is devoted to joint universality theorems for Lerch zeta functions. We mention the papers [28–32]. In the joint case, usually the algebraic independence of the parameters $\alpha_1, \dots, \alpha_r$ is required, i. e., that $\alpha_1, \dots, \alpha_r$ are not roots of any polynomial $p(s_1, \dots, s_r)$ with rational coefficients.

The problem of algebraic irrational parameter α also remains unsolved in the case $\lambda \notin \mathbb{Z}$. Therefore, in [33], some kind of approximation of analytic functions by shifts $L(\lambda, \alpha, s + i\tau)$ was proposed, namely, it was proved that a closed non-empty set $F_{\lambda,\alpha} \subset H(D)$ exists whose functions are approximated by $L(\lambda, \alpha, s + i\tau)$. All theorems on the approximation of analytic functions by shifts of zeta functions mentioned above are not effective in the sense that any concrete shift with approximating property is not known. In this situation, discrete shifts have a certain advantage over continuous ones because the number of discrete shifts is countable. Discrete shifts are also more convenient not only for the estimation of analytic functions but also in physics; see, for example, [34,35]. Therefore, the aim of this paper is a discrete version of the paper [33].

Theorem 5. *Suppose that the parameters λ, α and the number $h > 0$ are arbitrary. Let K be a compact set of the strip D. Then, a closed non-empty set $F_{\lambda,\alpha,h} \subset H(D)$ exists such that, for $f(s) \in F_{\lambda,\alpha,h}$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \operatorname{card}\left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |L(\lambda, \alpha, s + ikh) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

Since the case $\lambda \in \mathbb{Z}$ corresponds [15], we consider only the case $0 < \lambda < 1$. Note that the type of the second assertion was proposed independently in [36,37], see also [38].

A proof of Theorem 5 is based on a probabilistic limit theorem in the space of analytic functions $H(D)$.

2. Mean Square Estimate

We recall that notation $a \ll_\theta b$, $b > 0$, means that a constant $c = c(\theta) > 0$, not the same in all recurrences exists such that $|a| \leqslant cb$.

Lemma 1. *Suppose that $1/2 < \sigma < 1$ is fixed. Then, for arbitrary λ and α,*

$$\int_{-T}^{T} |L(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha,\sigma} T, \quad T > 0.$$

Proof. The lemma follows from Theorem 3.3.1 of [3], where the asymptotic formula for the mean square of $L(\lambda, \alpha, s)$ is given. □

Lemma 1, together with the Cauchy integral formula, implies the following estimate for the mean square of the derivative of $L(\lambda, \alpha, s)$.

Lemma 2. *Suppose that $1/2 < \sigma < 1$ is fixed. Then, for arbitrary λ and α,*

$$\int_{-T}^{T} |L'(\lambda, \alpha, \sigma + it)|^2 \, \mathrm{d}t \ll_{\lambda,\alpha,\sigma} T, \quad T > 0.$$

Since we consider the discrete case, we need an estimate for the discrete mean square of $L(\lambda, \alpha, s)$. For this, we will apply Lemmas 1 and 2 and the following Gallagher lemma; see, for example, Lemma 1.4 of [39], which connects continuous and discrete mean squares of some differentiable functions.

Lemma 3. *Let T_0, $T_0 \geqslant \delta > 0$ and $\mathcal{T}$ be a finite non-empty set lying in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$N_\delta(\tau) = \sum_{\substack{t \in \mathcal{T} \\ |t-\tau|<\delta}} 1.$$

Suppose that $S(t)$ is a continuous function on $[T_0, T_0 + T]$ having a continuous derivative on $(T_0, T_0 + T)$. Then, the inequality

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t + \left(\int_{T_0}^{T_0+T} |S(t)|^2 \, \mathrm{d}t \int_{T_0}^{T_0+T} |S'(t)|^2 \, \mathrm{d}t \right)^{1/2}$$

is valid.

Lemma 4. *Suppose that $1/2 < \sigma < 1$ is fixed. Then, for arbitrary $0 < \lambda < 1$, α, $h > 0$ and $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} |L(\lambda, \alpha, \sigma + ikh + it)|^2 \ll_{\lambda,\alpha,\sigma,h} N(1 + |t|).$$

Proof. In Lemma 3, we put $\delta = h$, $T_0 = h$, $T = Nh$ and $\mathcal{T} = \{2h, \dots, Nh\}$. Then, $N_h(\tau) = 1$. Therefore, an application of Lemma 3 gives

$$\sum_{k=2}^{N} |L(\lambda, \alpha, \sigma + ikh + it)|^2 \leqslant \frac{1}{h} \int_h^{Nh} |L(\lambda, \alpha, \sigma + i\tau + it)|^2 \, \mathrm{d}t$$
$$+ \left(\int_h^{Nh} |L(\lambda, \alpha, \sigma + i\tau + it)|^2 \, \mathrm{d}\tau \int_h^{Nh} |L'(\lambda, \alpha, \sigma + i\tau + it)|^2 \, \mathrm{d}\tau \right)^{1/2}. \quad (1)$$

By Lemmas 1 and 2,

$$\int_h^{Nh} |L(\lambda, \alpha, \sigma + i\tau + it)|^2 \, \mathrm{d}\tau \ll_{\lambda,\alpha,\sigma,h} N(1 + |t|)$$

and

$$\int_h^{Nh} |L'(\lambda, \alpha, \sigma + i\tau + it)|^2 \, \mathrm{d}\tau \ll_{\lambda,\alpha,\sigma,h} N(1 + |t|).$$

Thus, in view of (1),

$$\sum_{k=2}^{N} |L(\lambda, \alpha, \sigma + i\tau + it)|^2 \ll_{\lambda,\alpha,\sigma,h} N(1 + |t|). \quad (2)$$

Suppose that $0 < \lambda < 1$ and $|t| \leqslant \pi\lambda x$, where $x > 0$ is some real number. Then, by Theorem 3.1.2 of [3],

$$L(\lambda, \alpha, s) = \sum_{0 \leqslant m \leqslant x} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m + \alpha)^s} + O_{\sigma,\lambda,\alpha}(x^{-\sigma}).$$

Therefore,

$$L(\lambda, \alpha, \sigma + it) \ll_{\lambda,\alpha,\sigma} (1 + |t|)^{1/2}.$$

This shows that

$$\sum_{k=0}^{1} |L(\lambda, \alpha, \sigma + ikh + it)|^2 \ll_{\lambda,\alpha,\sigma,h} N(1 + |t|)$$

and the lemma follows from (2). □

We will apply Lemma 4 for the approximation in the mean of the function $L(\lambda, \alpha, s)$ by an absolutely convergent Dirichlet series. Let $\theta > 0$ be a fixed number, and

$$v_n(m, \alpha) = \exp\left\{ -\left(\frac{m + \alpha}{n} \right)^{\theta} \right\}, \quad m \in \mathbb{N}_0, \ n \in \mathbb{N}.$$

Define the series

$$L_n(\lambda, \alpha, s) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m} v_n(m, \alpha)}{(m + \alpha)^s}.$$

Since $v_n(m, \alpha)$ decreases exponentially with respect to m, the latter series is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. The following integral representation is valid.

Lemma 5. *Let $\theta_1 > 1/2$ and*

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta} \right) n^s.$$

Then, for $\sigma > 1/2$,

$$L_n(\lambda, \alpha, s) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} L(\lambda, \alpha, s + z) l_n(z) \frac{\mathrm{d}z}{z}. \tag{3}$$

A proof of lemma is given in [3], p. 87.

Lemma 6. *Suppose that K is a compact set of the strip D. Then, for all λ, α, and $h > 0$,*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} |L(\lambda, \alpha, s + ikh) - L_n(\lambda, \alpha, s + ikh)| = 0.$$

Proof. $\varepsilon > 0$ exists such that, for all $s = \sigma + it \in K$, the inequalities $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ hold. Let $\theta_1 = 1/2 + \varepsilon$ and $\theta_2 = 1/2 + \varepsilon - \sigma < 0$. Since the integration function in (3) has a simple pole at the point $z = 0$, the residue theorem implies

$$L_n(\lambda, \alpha, s) - L(\lambda, \alpha, s) = \frac{1}{2\pi i} \int_{\theta_2 - i\infty}^{\theta_2 + i\infty} L(\lambda, \alpha, s + z) l_n(z) \frac{\mathrm{d}z}{z}.$$

Hence, for $s \in K$,

$$\begin{aligned}
&L_n(\lambda, \alpha, s + ikh) - L(\lambda, \alpha, s + ikh) \\
&\qquad = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L\left(\lambda, \alpha, \frac{1}{2} + \varepsilon + it + ikh + i\tau \right) \frac{l_n(1/2 + \varepsilon - \sigma + i\tau)}{1/2 + \varepsilon - \sigma + i\tau} \, \mathrm{d}\tau \\
&\qquad = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L\left(\lambda, \alpha, \frac{1}{2} + \varepsilon + ikh + i\tau \right) \frac{l_n(1/2 + \varepsilon - s + i\tau)}{1/2 + \varepsilon - s + i\tau} \, \mathrm{d}\tau \\
&\qquad \ll \int_{-\infty}^{\infty} \left| L\left(\lambda, \alpha, \frac{1}{2} + \varepsilon + ikh + i\tau \right) \right| \sup_{s \in K} \left| \frac{l_n(1/2 + \varepsilon - s + i\tau)}{1/2 + \varepsilon - s + i\tau} \right| \mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}|L(\lambda,\alpha,s+ikh)-L_n(\lambda,\alpha,s+ikh)| \\
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+ikh+i\tau\right)\right|\right)\sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\right|\mathrm{d}\tau \\
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{N+1}\sum_{k=0}^{N}\left|L\left(\lambda,\alpha,\frac{1}{2}+\varepsilon+ikh+i\tau\right)\right|^2\right)^{1/2}\sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\right|\mathrm{d}\tau \\
&\stackrel{\text{def}}{=} I.
\end{aligned} \tag{4}$$

It is well known that, for large $|t|$,

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\},\quad c>0,$$

uniformly in every interval $\sigma_1\leqslant\sigma\leqslant\sigma_2$, $\sigma_1<\sigma_2$. Therefore, by the definition of $l_n(s)$, for $s\in K$,

$$\frac{l_n(1/2+\varepsilon-s+i\tau)}{1/2+\varepsilon-s+i\tau}\ll_\theta n^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|\tau-t|\right\}\ll_{\theta,K} n^{-\varepsilon}\exp\{-c_1|\tau|\},\quad c_1>0.$$

This and Lemma 4 imply

$$I\ll_{\theta,K,\lambda,\alpha,\varepsilon,h} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\{-c_1|\tau|\}\,\mathrm{d}\tau\ll_{\theta,K,\lambda,\alpha,\varepsilon,h} n^{-\varepsilon},$$

and estimate (4) proves the lemma. □

A sequence of compact sets $\{K_l : l\in\mathbb{N}\}\subset D$ exists such that

$$D=\bigcup_{l=1}^{\infty}K_l,$$

$K_l\subset K_{l+1}$ for all $l\in\mathbb{N}$, and every compact set $K\subset D$ lies in some K_l. For $g_1,g_2\in H(D)$, define

$$\rho(g_1,g_2)=\sum_{l=1}^{\infty}2^{-l}\frac{\sup_{s\in K_l}|g_1(s)-g_2(s)|}{1+\sup_{s\in K_l}|g_1(s)-g_2(s)|}.$$

Then, ρ is a metric in the space $H(D)$ inducing its topology of uniform convergence on compact set.

The definition of the metric ρ together with Lemma 6 lead to the following lemma.

Lemma 7. *The equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho(L(\lambda,\alpha,s+ikh),L_n(\lambda,\alpha,s+ikh))=0$$

holds for all λ, α, and $h>0$.

3. Probabilistic Results

In this section, we will prove a limit theorem on the weak convergence of probability measures in the space $H(D)$. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$, and for $A \in \mathcal{B}(H(D))$, define

$$P_{N,\lambda,\alpha,h}(A) = \frac{1}{N+1}\operatorname{card}\{0 \leqslant k \leqslant N : L(\lambda, \alpha, s+ikh) \in A\}.$$

Theorem 6. *Let λ, α and $h > 0$ be arbitrary. Then, on $(H(D), \mathcal{B}(H(D)))$, a probability measure $P_{\lambda,\alpha,h}$ exists such that $P_{N,\lambda,\alpha,h}$ converges weakly to $P_{\lambda,\alpha,h}$ as $N \to \infty$.*

Before the proof of Theorem 6, we will prove limit theorems in some auxiliary spaces. Let S^1 denote the circle $\{s \in \mathbb{C} : |s| = 1\}$. Define the set

$$\Omega = \prod_{p \in \mathbb{N}_0} S^1_m,$$

where $S^1_m = S^1$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. For $A \in \mathcal{B}(\Omega)$, define

$$Q_{N,\alpha,h}(A) = \frac{1}{N+1}\operatorname{card}\Big\{0 \leqslant k \leqslant N : \Big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\Big) \in A\Big\}.$$

Lemma 8. *On $(\Omega, \mathcal{B}(\Omega))$, a probability measure $Q_{\alpha,h}$ exists such that $Q_{N,\alpha,h}$ converges weakly to $Q_{\alpha,h}$ as $N \to \infty$.*

Proof. Denote by $\omega(m)$ the mth component of an element $\omega \in \Omega$, $m \in \mathbb{N}_0$. Let $g_{N,\alpha,h}(\underline{k})$, $\underline{k} = (k_m : k_m \in \mathbb{Z},\ m \in \mathbb{N}_0)$ be the Fourier transform of the measure $Q_{N,\alpha,h}$, i.e.,

$$g_{N,\alpha,h}(\underline{k}) = \int\limits_{\Omega} \left({\prod_{m \in \mathbb{N}_0}}^{*} \omega^{k_m}(m) \right) \mathrm{d}Q_{N,\alpha,h},$$

where the star "$*$" shows that only a finite number of integers k_m are distinct from zero. By the definition of $Q_{N,\alpha,h}$, we have

$$\begin{aligned} g_{N,\alpha,h}(\underline{k}) &= \frac{1}{N+1}\sum_{k=0}^{N} {\prod_{m \in \mathbb{N}_0}}^{*} (m+\alpha)^{-ikhk_m} \\ &= \frac{1}{N+1}\sum_{k=0}^{N} \exp\left\{-ikh {\sum_{m \in \mathbb{N}_0}}^{*} k_m \log(m+\alpha)\right\}. \end{aligned} \tag{5}$$

Let

$$A_{\alpha,h} = \left\{\underline{k} : {\sum_{m \in \mathbb{N}_0}}^{*} k_m \log(m+\alpha) = 2\pi r,\ r \in \mathbb{Z}\right\}$$

and

$$B_{\alpha,h} = \left\{\underline{k} : {\sum_{m \in \mathbb{N}_0}}^{*} k_m \log(m+\alpha) \neq 2\pi r,\ r \in \mathbb{Z}\right\}.$$

Thus, by (5), we have

$$g_{N,\alpha,h}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in A_{\alpha,h}, \\ \dfrac{1-\exp\left\{-(N+1)h {\sum^{*}}_{m \in \mathbb{N}_0} k_m \log(m+\alpha)\right\}}{(N+1)\left(1-\exp\left\{-ih {\sum^{*}}_{m \in \mathbb{N}_0} k_m \log(m+\alpha)\right\}\right)} & \text{if } \underline{k} \in B_{\alpha,h}. \end{cases}$$

Hence,

$$\lim_{N\to\infty} g_{N,\alpha,h}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in A_{\alpha,h}, \\ 0 & \text{if } \underline{k} \in B_{\alpha,h}. \end{cases}$$

Therefore, $Q_{N,\alpha,h}$, as $N \to \infty$ converges weakly to the measure $Q_{\alpha,h}$ on $(\Omega, \mathcal{B}(\Omega))$ defined by the Fourier transform

$$g_{\alpha,h}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} \in A_{\alpha,h}, \\ 0 & \text{if } \underline{k} \in B_{\alpha,h}. \end{cases}$$

□

Now, for $A \in \mathcal{B}(H(D))$, define

$$P_{N,n,\lambda,\alpha,h}(A) = \frac{1}{N+1}\text{card}\{0 \leqslant k \leqslant N : L_n(\lambda, \alpha, s + ikh) \in A\}.$$

Lemma 9. *On $(H(D), \mathcal{B}(H(D)))$, a probability measure $V_{n,\lambda,\alpha,h}$ exists such that $P_{N,n,\lambda,\alpha,h}$ converges weakly to $V_{n,\lambda,\alpha,h}$ as $N \to \infty$.*

Proof. Define the function $u_{n,\lambda,\alpha} : \Omega \to H(D)$ by

$$u_{n,\lambda,\alpha}(\omega) = L_n(\lambda, \alpha, s, \omega),$$

where

$$L_n(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)v_n(m, \alpha)}{(m+\alpha)^s}.$$

Since $|\omega(m)| = 1$, the latter series, as for $L_n(\lambda, \alpha, s)$, is absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0. Hence, the function $u_{n,\lambda,\alpha}$ is continuous. Therefore, each probability measure P on $(\Omega, \mathcal{B}(\Omega))$ defines the unique probability measure $Pu^{-1}_{n,\lambda,\alpha}$, where

$$Pu^{-1}_{n,\lambda,\alpha}(A) = P(u^{-1}_{n,\lambda,\alpha}A), \quad A \in \mathcal{B}(H(D)).$$

Moreover, by the definition of $u_{n,\lambda,\alpha}$, we have

$$P_{N,n,\lambda,\alpha,h}(A) = \frac{1}{N+1}\text{card}\Big\{0 \leqslant k \leqslant N : \Big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\Big) \in u^{-1}_{n,\lambda,\alpha}A\Big\},\ A \in \mathcal{B}(H(D)).$$

Thus, $P_{N,n,\lambda,\alpha,h} = Q_{N,\alpha,h}u^{-1}_{n,\lambda,\alpha}$. Since, under continuous mappings, the weak convergence of probability measures is preserved, see, for example, Theorem 4.1 of [40], the continuity of $u_{n,\lambda,\alpha}$ and Lemma 8 show that the measure $P_{N,n,\lambda,\alpha,h}$ converges weakly to $Q_{\alpha,h}u^{-1}_{n,\lambda,\alpha}$ as $N \to \infty$. Consequently, $V_{n,\lambda,\alpha,h} = Q_{\alpha,h}u^{-1}_{n,\lambda,\alpha}$. □

To prove the weak convergence for the measure $P_{N,\lambda,\alpha,h}$, we apply one lemma to the convergence in the distribution of random elements ($\xrightarrow{\mathcal{D}}$), see, for example, Theorem 3.2 of [40].

Lemma 10. *Let $(\mathbb{X}, d)$ be a separable metric space, and $\mathbb{X}$-valued random elements X_{kn} and Y_n, $k \in \mathbb{N}$, and $n \in \mathbb{N}$ be defined by the same probability space with measure P. Suppose that*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k, \qquad X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

moreover, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\{d(X_{kn}, Y_n) \geqslant \varepsilon\} = 0.$$

Then, we have $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Proof of Theorem 6. On a certain probability space with measure P, define the random variable $\theta_{N,h}$ by

$$P\{\theta_{N,h} = kh\} = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Define the $H(D)$-valued random elements

$$X_{N,n,\lambda,\alpha,h} = X_{N,n,\lambda,\alpha,h}(s) = L_n(\lambda, \alpha, s + i\theta_{N,h})$$

and

$$X_{N,\lambda,\alpha,h} = X_{N,\lambda,\alpha,h}(s) = L(\lambda, \alpha, s + i\theta_{N,h}).$$

Moreover, let $X_{n,\lambda,\alpha,h}$ be the $H(D)$-valued random element having the distribution $V_{n,\lambda,\alpha,h}$. Then, the statement of Lemma 9 can be written in the form

$$X_{N,n,\lambda,\alpha,h} \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,\lambda,\alpha,h}. \tag{6}$$

Now we recall some notions. The family of probability measures $\{P_n : n \in \mathbb{N}\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\varepsilon > 0$, a compact set $K = K_\varepsilon \subset \mathbb{X}$ exists such that, for all $n \in \mathbb{N}$,

$$P_n(K) > 1 - \varepsilon.$$

The family $\{P_n\}$ is called relatively compact if every subsequence of $\{P_n\}$ contains a weakly convergent subsequence. It is well known (Prokhorov's theorem, see, for example, [40]) that every tight family $\{P_n\}$ is relatively compact.

We will show that the sequence $\{V_{n,\lambda,\alpha,h} : n \in \mathbb{N}\}$ is tight. Using the Cauchy integral formula, we find

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_l} |L_n(\lambda, \alpha, s + ikh)| \leqslant R_{l,\lambda,\alpha,h} < \infty,$$

where K_l are compact sets from the definition of the metric ρ. Let $\varepsilon > 0$, and $M_l = M_{l,\lambda,\alpha,h} = 2^l \varepsilon^{-1} R_{l,\lambda,\alpha,h}$. Then, using relation (6), we obtain

$$\begin{aligned} P\left\{\sup_{s\in K_l} |X_{n,\lambda,\alpha,h}(s)| \geqslant M_l\right\} &\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} P\left\{\sup_{s\in K_l} |X_{n,\lambda,\alpha,h}(s)| > M_l\right\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M_l(N+1)} \sum_{k=0}^{N} \sup_{s\in K_l} |L_n(\lambda, \alpha, s + ikh)| \leqslant \frac{\varepsilon}{2^l} \end{aligned}$$

for all $l, n \in \mathbb{N}$. Therefore, putting

$$K = K_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K_l} |g(s)| \leqslant M_l,\ l \in \mathbb{N} \right\},$$

we have a compact set K in $H(D)$, and

$$P\{X_{n,\lambda,\alpha,h} \in K\} = 1 - P\{X_{n,\lambda,\alpha,h} \notin K\} \geqslant 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, by the definition of $X_{n,\lambda,\alpha,h}$, the family $\{V_{n,\lambda,\alpha,h}\}$ is tight; hence, it is relatively compact.

From the relative compactness, we have that a subsequence $\{V_{n_r,\lambda,\alpha,h}\}$ and a probability measure $P_{\lambda,\alpha,h}$ on $(H(D), \mathcal{B}(H(D)))$ exist such that $V_{n_r,\lambda,\alpha,h}$ converges weakly to $P_{\lambda,\alpha,h}$ as $r \to \infty$. Thus, the relation

$$X_{n_r,\lambda,\alpha,h} \xrightarrow[r\to\infty]{\mathcal{D}} P_{\lambda,\alpha,h} \tag{7}$$

is true. Moreover, in view of Lemma 7, for $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \limsup_{N\to\infty} P\{\rho(X_{N,\lambda,\alpha h}, X_{n_r,\lambda,\alpha,h}) \geqslant \varepsilon\}$$
$$\leqslant \lim_{r\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^{N} \rho(L(\lambda,\alpha,s+ikh), L_{n_r}(\lambda,\alpha,s+ikh)) = 0.$$

This, (6), (7) and Lemma 10, show that

$$X_{N,\lambda,\alpha,h} \xrightarrow[N\to\infty]{\mathcal{D}} P_{\lambda,\alpha,h},$$

and, in view of the definition of $X_{N,\lambda,\alpha,h}$, the theorem is proved. □

4. Proof of Theorem 5

Theorem 5 follows easily from Theorem 6. Before its proof, recall a notion of the support of a probability measure. The support of a probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, where $\mathbb{X}$ is a separable space, is a minimal closed set $S_P \subset \mathbb{X}$ such that $P(S_P) = 1$. The set S_P consists of elements $x \in \mathbb{X}$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied.

Proof of Theorem 5. By Theorem 6, $P_{N,\lambda,\alpha,h}$ converges weakly to the measure $P_{\lambda,\alpha,h}$ on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$. Let $F_{\lambda,\alpha,h}$ be the support of $P_{\lambda,\alpha,h}$. Then, $F_{\lambda,\alpha,h}$ is a closed non-empty subset of the space $H(D)$.

For $f(s) \in F_{\lambda,\alpha,h}$ define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then, G_ε is an open neighbourhood of $f(s) \in F_{\lambda,\alpha,h}$. Therefore, by a property of the support

$$P_{\lambda,\alpha,h}(G_\varepsilon) > 0. \tag{8}$$

Thus, Theorem 6 and the equivalent weak convergence of probability measures in terms of open sets, see, for example, Theorem 2.1 of [40], imply

$$\liminf_{N\to\infty} P_{N,\lambda,\alpha,h}(G_\varepsilon) \geqslant P_{\lambda,\alpha,h}(G_\varepsilon) > 0.$$

This and the definitions of $P_{N,\lambda,\alpha,h}$ and G_ε prove the first assertion of the theorem.

To prove the second assertion of the theorem, we apply the equivalent of weak convergence of probability measures in term of continuity sets. Recall that A is a continuity set of the measure P if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

The boundary of the set G_ε lies in the set

$$\left\{ g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, the boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Hence, the set G_ε is a continuity set for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 6 and the equivalent of weak convergence of probability measures, see Theorem 2.1 of [40], give in view of (8)

$$\lim_{N\to\infty} P_{N,\lambda,\alpha,h}(G_\varepsilon) = P_{\lambda,\alpha,h}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This and the definitions of $P_{N,\lambda,\alpha,h}$ and G_ε prove the second assertion of the theorem. □

5. Discussion

In this paper, we obtain that the Lerch zeta function $L(\lambda, \alpha, s)$ has a discrete approximation property with arbitrary parameters λ and α. More precisely, we prove that a closed non-empty subset $F_{\lambda,\alpha,h}$ of the space of analytic functions on the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ exists such that, for every $\varepsilon > 0$, the set

$$\left\{ k : \sup_{s\in K} |L(\lambda, \alpha, s + ikh) - f(s)| < \varepsilon \right\}$$

has a positive lower density for all $h > 0$ and $f(s) \in F_{\lambda,\alpha,h}$. This shows that the latter set is infinite. It remains an open problem to identity the set $F_{\lambda,\alpha,h}$.

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