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Feller chains and random functions

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Abstract. We prove that each Feller transition probability is the one-dimensional distribution of some stochastically continuous random function. We also introduce the notion of a regular random function and show, on one hand, that every random function has a regular modification, and on the other hand, that the composition of independent regular stochastically continuous random functions is stochastically continuous as well.

Keywords: Markov chains, random functions, stochastic continuity.

1 Introduction and main results

In this paper we continue the investigation of general Markov chains, begun in [4]. This time we assume that the state space E of the chain is a separable metric space and the mapping $x \mapsto P(x,\cdot)$ (where P denotes the transition probability of the chain) is a continuous function from E to $\Pi(E)$, the space of all probabilities on E, endowed with the topology of weak convergence. In such a case we call P a *Feller transition probability* and the corresponding Markov chain a *Feller chain*.

Feller chains are closely related to random functions. If E and F are two separable metric spaces then by a *random function* from E to F we call a measurable function $f\colon \Omega\times E \to F$, where (Ω, P) is some fixed probability space. If f is a random function, we denote its value at (ω, x) by $f(\omega)x$. The argument ω is often omitted and then fx is a random element of F. If ω is fixed then $f(\omega)$ is an ordinary function from E to F. It is called a *realization* of f.

Each random function f induces a transition probability from E to F defined by

$$P(x,B) = \mathsf{P}\{fx \in B\}, \quad B \subset F \text{ measurable.} \tag{1}$$

We call P the (one-dimensional) *distribution* of f. It is well-known that if F is a Borel subset of some Polish space (for simplicity we call such F a *Borel space*) then each transition probability from E to F is the distribution of some random function.

A random function f is called *continuous* if all its realizations are continuous functions. It is easily seen that the distribution of any continuous random function is a Feller transition probability. [1] considered the inverse problem and gave the conditions for existence of a continuous random function with given distribution P. However, in many cases such random function does not exist.

Example 1. Let $E = F = [0;1]$ and $P(x,\cdot) = (1-x)\delta_0 + x\delta_1$, where δ_y denotes the probability concentrated at y. Clearly, P is a Feller kernel. Suppose that it is the distribution of a continuous random function f, defined on some probability space (Ω, P). The set $N = \{(\omega, x) \mid 0 < f(\omega)x < 1\}$ is measurable and all its sections $N(\cdot, x)$ are null sets; therefore $(\mathsf{P} \times \lambda)(N) = 0$, where λ denotes the Lebesgue measure on E. Then there exists a measurable W with $\mathsf{P}(W) = 1$ such that, for all $\omega \in W$, $f(\omega)$ is a continuous function and $\lambda(N(\omega, \cdot)) = 0$.

If $\omega \in W$ then the set $\{x \mid 0 < f(\omega)x < 1\}$ is open and of null Lebesgue measure, i.e. it is empty. Hence $f(\omega)$ is a continuous function taking only 0 and 1 values. Since E is connected, $f(\omega)$ is constant. Therefore there exists a measurable partition (W_0, W_1) of W such that $f(\omega)x = 0$ for all $\omega \in W_0$ and all x, and $f(\omega)x = 1$ for all $\omega \in W_1$ and all x. Then $x = \mathsf{P}\{fx = 1\} = \mathsf{P}(W_1)$ for all $x \in [0; 1]$, a contradiction.

Example 1 shows that the concept of a continuous random function is too strong for analysis of Feller transition probabilities. It appears that the right notion is that of a *stochastically continuous random function.* f is called stochastically continuous if

$$x_n \to x \Rightarrow fx_n \to_p fx, \tag{2}$$

where $\to_p$ denotes convergence in probability. Clearly, the distribution of a stochastically continuous random function is a Feller transition probability (because convergence in probability implies convergence in distribution). The following converse statement is not so obvious.

Theorem 1. *If E is a separable metric space and F a Borel space then each Feller transition probability from E to F is the distribution of some stochastically continuous random function.*

A usual way to define a Markov chain via random functions is the following. We start from some random function f from E to E, defined on some probability space (I, λ), where I is a separable metric space. Next, we take an iid sequence (ϵ_i) of random elements of I, defined on some probability space (Ω, P) and distributed according λ, and set $f_i(\omega)x = f(\epsilon_i(\omega))x$. Then any sequence $(f_i \cdots f_1 x)$, $x \in E$, is a Markov chain with transition probability P, which is the distribution of the random function f. Random functions $f_i \cdots f_1$ are called *forward iterations* of f.

It is easily proved that if f is stochastically continuous then all f_i, as well as all iterations $f_i \cdots f_1$, are stochastically continuous. Indeed, the first assertion follows from the fact that $(fx, fx') \overset{d}{=} (f(\epsilon_i)x, f(\epsilon_i)x')$ while the second from Theorem 3 below. However, it is not true, in general, that the composition of stochastically continuous functions is stochastically continuous. Let us look at this phenomenon more closely.

Let E, F and G be separable metric spaces, f a random function from E to F and g a random function from F to G, both defined on the same probability space (Ω, P). The function $(\omega, x) \mapsto g(\omega)f(\omega)x$ is called *the composition of f and g* and is denoted by gf. It is measurable as a superposition of measurable functions $(\omega, x) \mapsto (\omega, f(\omega)x)$ and $(\omega, y) \mapsto g(\omega)y$. Therefore gf is a random function from E to G.

We are interested in whether gf is stochastically continuous if f and g are stochastically continuous. Let $x_n \to x$, then $fx_n \to_p fx$ and so gf is stochastically continuous if $Y_n \to_p Y$ implies $gY_n \to_p gY$. This condition is in fact necessary, if we want gf to be stochastically continuous for *any* stochastically continuous random function f. Indeed, we can take $E = \mathbb{N} \cup \{\infty\}$ and define $f(\omega)n = Y_n(\omega)$ for $n \in \mathbb{N}$ and

$f(\omega)\infty = Y(\omega)$. Hence we can forget about g and investigate, for what stochastically continuous functions f it is true that

$$X_n \to_p X \Rightarrow fX_n \to_p fX. \tag{3}$$

It is easily shown that (3) holds for any continuous f, but this result is not what we are seeking for. However, it appears that (3) fails to be true if f is not continuous. Here is a typical example.

Example 2. Let $E = F = [0;1]$, $\Omega = (0;1)$, P be the Lebesgue measure on Ω and

$$f(\omega)x = \begin{cases} 1 & \text{for } x = \omega; \\ 0 & \text{otherwise.} \end{cases}$$

For $\omega \in \Omega$ set $X_n(\omega) = (\omega - 1/n)^+$ and $X(\omega) = \omega$. Then $X_n \to X$, but $fX_n = 0$ while $fX = 1$.

Hoffmann-Jørgensen [3] considered the same problem and gave conditions on X for (3) to hold. It is required, roughly speaking, that almost surely X would be a continuity point of f. However, this result does not explain why iterations of a stochastically continuous random function are stochastically continuous. The first idea that comes into mind is to consider an independence assumption.

For any random function f, we denote by $\sigma(f)$ the σ-algebra generated by the random elements fx, $x \in E$. We say that a random element U of an arbitrary separable metric space is *independent of* f if the σ-algebras $\sigma(U)$ and $\sigma(f)$ are independent. If g is another random function, we call it independent of f if $\sigma(f)$ and $\sigma(g)$ are independent.

It is easily seen that iterations of a random function f are compositions of independent copies of f. Therefore we can ask if the composition of arbitrary independent stochastically continuous random functions is stochastically continuous. Similarly to above, the problem is reduced to the following question: does (3) hold if f is stochastically continuous and (X_n) independent of f (then, of course, X is also independent of f)? Unfortunately, the answer still is 'no': in Example 2 (X_n) is independent of f, because each fx is a degenerate random variable (almost surely $fx = 0$).

An appropriate solution to the problem (see Theorems 2–3 below) is given by the notion of a *regular* random function. We call a random function f regular if there exists a separable metric space I, a Borel function $\tilde{f}\colon I \times E \to F$ and a random element ϵ of I such that $\sigma(\epsilon) = \sigma(f)$ and, for all (ω, x),

$$f(\omega)x = \tilde{f}\big(\epsilon(\omega)\big)x. \tag{4}$$

Of course, $\tilde{f}$ is a random function from E to F defined on the probability space (I, λ), where λ is the distribution of ϵ. Moreover, the distributions of f and $\tilde{f}$ coincide. It is easily seen that all iterations of an arbitrary random function on (I, λ) are regular, provided I is a separable metric space.

Regularity implies that $\sigma(f)$ is countably generated. This allows us to construct non-regular random functions, see Example 3 below. We do not have any example of a non-regular random function f with $\sigma(f)$ countably generated.

Example 3. Let f be the random function from Example 2. Then $fx = 1_{\{x\}}$ and $\sigma(f)$ is generated by the one-points sets $\{x\}$, $x \in E$. It is easily shown that $\sigma(f)$ is the set of all $W \subset \Omega$ such that either W or W^c is countable. Suppose that $\sigma(f)$ is generated by some sequence (W_i). We can assume that all W_i are countable. Set $W^* = \bigcup_i W_i$. Then the σ-algebra

$$\mathcal{F} = \bigl\{W \mid W \subset W^* \text{ or } W^c \supset W^{*c}\bigr\}$$

contains all W_i and therefore $\sigma(f) \subset \mathcal{F}$. However, $\{x\} \notin \mathcal{F}$ for any $x \notin W^*$, a contradiction.

A random function f' is called a *modification* of a random function f if, for all x, almost surely $f'x = fx$.

Theorem 2. *If F is a Borel space then each random function from E to F has a regular modification.*

Theorem 3. 1. *If f is a regular stochastically continuous function from E to F and (X_n) a sequence of random elements of E, independent of f, then* (3) *holds.*

2. *If f and g are independent stochastically continuous random functions and g is regular then gf is stochastically continuous.*

2 Proofs

Proof of Theorem 1. Let $\bar{F}$ be a completion of the metric space F, ι denote the morphism $x \mapsto x$ from F to $\bar{F}$ and $\bar{P}(x,\cdot) = \iota^* P(x,\cdot)$. Since ι is continuous, the dual morphism ι^* from $\Pi(F)$ to $\Pi(\bar{F})$ is also continuous. Hence $\bar{P}$ is a Feller transition probability from E to $\bar{F}$.

Fernique [2] has shown that there exists a stochastically continuous random function $\mu \mapsto Y_\mu$ from $\Pi(\bar{F})$ to $\bar{F}$ such that the distribution of Y_μ is μ. Denote

$$\bar{f}(\omega)x = Y_{\bar{P}(x,\cdot)}(\omega).$$

If $x_n \to x$ then $\bar{P}(x_n,\cdot) \to \bar{P}(x,\cdot)$ and therefore

$$\bar{f}x_n = Y_{\bar{P}(x_n,\cdot)} \to_p Y_{\bar{P}(x,\cdot)} = \bar{f}x.$$

Hence $\bar{f}$ is stochastically continuous. Moreover, the distribution of $\bar{f}x$ is $\bar{P}(x,\cdot)$.

Since F is a Borel subset of $\bar{F}$, equality

$$f(\omega)x = \begin{cases} \bar{f}(\omega)x & \text{if } \bar{f}(\omega)x \in F; \\ y^* & \text{otherwise;} \end{cases}$$

defines a random function from E to F; here y^* is a fixed point in F. Since $\mathsf{P}\{\bar{f}x \in F\} = \bar{P}(x,F) = 1$, each fx almost surely equals $\bar{f}x$; hence f is stochastically continuous. Moreover, for all Borel $B \subset F$,

$$\mathsf{P}\{fx \in B\} = \mathsf{P}\{\bar{f}x \in B\} = \bar{P}(x,B) = P(x,B);$$

therefore the distribution of fx is $P(x,\cdot)$.

Proof of Theorem 2. Let $L^0(F)$ denote the set of all equivalence classes $\pmod{\mathsf{P}}$ of random elements of F endowed with the topology of convergence in probability. It is well-known that it can be metrized by the metric $\rho(Y,Y') = \mathsf{E}d(Y,Y')$, where d is a bounded metric of F. Let $\bar{F}$ denote the completion of F; then F is a Borel subset of $\bar{F}$.

By Theorem 4.2.1 of [5], the set $\{fx \mid x \in E\}$ is separable, therefore there exists a sequence $(A_{ni} \mid i \geq 1)$, $n \geq 1$, of measurable partitions of E such that $\operatorname{diam}\{fx \mid x \in A_{ni}\} \leq 2^{-n}$ for all $n, i \geq 1$. Without loss of generality we can assume that the nth partition is finer than the preceding, i.e. for each i there exists a j with $A_{ni} \subset A_{n-1,j}$. In each A_{ni} fix some x_{ni} so that $x_{n-1,j} \in A_{ni} \subset A_{n-1,j}$ implies $x_{ni} = x_{n-1,j}$. Let p_n denote the function from E to E, which maps each $x \in A_{ni}$ to x_{ni}. Define $f_n(\omega)x = f(\omega)p_n(x)$. For all Borel $B \subset F$, the sets

$$\big\{(\omega,x) \mid f_n(\omega)x \in B\big\} = \bigcup_i \big(\{fx_{ni} \in B\} \times A_{ni}\big)$$

is measurable, therefore f_n are random functions from E to F. By construction, for each fixed x_{mi}, $p_n(x_{mi}) = x_{mi}$ for $n \geq m$. This yields $f'(\omega)x_{mi} = f(\omega)x_{mi}$ for all ω, m and i.

Define

$$f'(\omega)x = \begin{cases} \lim_{n\to\infty} f_n(\omega)x & \text{if the limit exists;} \\ y^* & \text{otherwise;} \end{cases}$$

where y^* is a fixed point in F. It is easily seen that f_n are also random functions from E to $\bar{F}$. Since $\bar{F}$ is complete, the set

$$C = \big\{(\omega,x) \mid f_n(\omega)x \text{ converge in } \bar{F}\big\}$$

is measurable and $(\omega,x) \mapsto \lim f_n(\omega)x$ is a measurable function from C to $\bar{F}$. Then the set

$$\big\{(\omega,x) \mid f_n(\omega)x \text{ converge in } F\big\} = \big\{(\omega,x) \in C \mid \lim f_n(\omega)x \in F\big\}$$

is also measurable and therefore f' is a random function from E to $\bar{F}$. Since all its values lie in F, it is a random function from E to F.

For all $x \in E$, $\sum_n \rho(f_n x, fx) \leq \sum_n 2^{-n} < \infty$, therefore almost surely $f_n x \to fx$. Then f' is a modification of f.

Let α denote some bijection from $\mathbb{N}^2$ to $\mathbb{N}$. Let $s_j = x_{mi}$ for $j = \alpha(m,i)$,

$$\psi_n(x, y_1, y_2, \dots) = y_{\alpha(n,i)} \quad \text{for } x \in A_{ni}$$

and

$$\psi(x, y_1, y_2, \dots) = \begin{cases} \lim_{n\to\infty} \psi_n(x, y_1, y_2, \dots) & \text{if the limit exists;} \\ y^* & \text{otherwise.} \end{cases}$$

ψ_n are measurable functions from $E \times F^\infty$ to F and

$$f_n(\omega)x = \psi_n\big(x, f(\omega)s_1, f(\omega)s_2, \dots\big)$$

for all (ω,x). Then ψ is measurable as well and, for all (ω,x),

$$f'(\omega)x = \psi\big(x, f(\omega)s_1, f(\omega)s_2, \dots\big).$$

Hence f' is regular: we can write it in the form (4) with $I = F^\infty$, $\epsilon = (fs_i)$ and $\tilde{f}(y_1, y_2, \dots)x = \psi(x, y_1, y_2, \dots)$; moreover, $\sigma(f') = \sigma(\epsilon)$, because $fs_i = f's_i$ for all i.

Proof of Theorem 3 1. For some sequence $n_k \to \infty$, $X_{n_k} \to X$ almost surely, hence X is measurable with respect to the completion of the σ-algebra $\sigma(X_n \mid n \geq 1)$ and therefore (X_n, X) is independent of f. Let d denote a bounded metric in F. By regularity, $\mathsf{E}d(fX_n, fX) = \mathsf{E}\varphi(X_n, X)$, where $\varphi(x_1, x_2) = \mathsf{E}d(fx_1, fx_2)$. Since f is stochastically continuous, $x_n \to x$ implies $\varphi(x_n, x) \to 0$.

Let $X_n \to_p X$; we need to show that $\mathsf{E}d(fX_n, fX) \to 0$. Suppose the contrary, then there exists an $\varepsilon > 0$ and a sequence $n_k \to \infty$ such that almost surely $X_{n_k} \to X$ while $\mathsf{E}d(fX_{n_k}, fX) \geq \varepsilon$ for all k. We got a contradiction, because $X_{n_k} \to X$ implies $\varphi(X_{n_k}, X) \to 0$ and then $\mathsf{E}\varphi(X_{n_k}, X) \to 0$ by dominated convergence.

2. If $x_n \to x$ then $fx_n \to_p fx$ and, by the first statement of the theorem, $gfx_n \to gfx$.

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REZIUMĖ

Felerio grandinės ir atsitiktinės funkcijos

V. Kazakevičius

Įrodoma, kad kiekviena Felerio perėjimo tikimybė yra tam tikros stochastiškai tolydžios atsitiktinės funkcijos vienmatis skirstinys. Apibrėžiamos reguliarios atsitiktinės funkcijos ir įrodoma, kad bet kokia stochastiškai tolydi funkcija turi reguliarią modifikaciją ir kad nepriklausomų reguliarių stochastiškai tolydžių funkcijų kompozicija taip pat stochastiškai tolydi.

Raktiniai žodžiai: Markovo grandinės, atsitiktinės funkcijos, stochastinis tolydumas.