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Article

A Generalized Discrete Bohr–Jessen-Type Theorem for the Epstein Zeta-Function

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Abstract: Suppose that Q is a positive defined $n \times n$ matrix, and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$ with $\underline{x} \in \mathbb{Z}^n$. The Epstein zeta-function $\zeta(s;Q)$, $s = \sigma + it$, is defined, for $\sigma > \frac{n}{2}$, by the series $\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s}$, and it has a meromorphic continuation to the whole complex plane. Let $n \geqslant 4$ be even, while $\varphi(t)$ is an increasing differentiable function with a continuous monotonic bounded derivative $\varphi'(t)$ such that $\varphi(2t)(\varphi'(t))^{-1} \ll t$, and the sequence $\{a\varphi(k)\}$ is uniformly distributed modulo 1. In the paper, it is obtained that $\frac{1}{N}\#\{N \leqslant k \leqslant 2N : \zeta(\sigma + i\varphi(k);Q) \in A\}$, $A \in \mathcal{B}(\mathbb{C})$, for $\sigma > \frac{n-1}{2}$, converges weakly to an explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.

Keywords: Epstein zeta-function; limit theorem; weak convergence; Haar measure

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1. Introduction

Denote by $s = \sigma + it$ a complex variable, by Q a positive defined quadratic $n \times n$, $n \in \mathbb{N}$, matrix, and put $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$ for $\underline{x} \in \mathbb{Z}^n$. In [1], Epstein introduced the zeta-function

$$\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s}, \quad \sigma > \frac{n}{2}.$$

Moreover, he obtained the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s;Q) = \sqrt{\det Q}\,\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2} - s\right)\zeta\left(\frac{n}{2} - s; Q^{-1}\right), \quad s \in \mathbb{C},$$

where $\Gamma(s)$ is the Euler gamma-function, which meromorphically continued the function $\zeta(s;Q)$ to the whole complex plane with the unique simple pole at the point $s = \frac{n}{2}$, finding that

$$\operatorname{Res}_{s=\frac{n}{2}}\zeta(s;Q) = \pi^{\frac{n}{2}}\left(\Gamma\left(\frac{n}{2}\right)\sqrt{\det Q}\right)^{-1}.$$

The above equation recalls the functional equation for the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

discovered by Riemann in [2] and having the form

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The aim of Epstein was a zeta-function as general as possible having the functional equation of the Riemann type. Later, it turned out that the function $\zeta(s;Q)$ is an interesting analytic object having practical applications, for example, in chemistry [3] and mathematical physics [4,5].

The value distribution of the function $\zeta(s;Q)$, as other zeta-functions, is complicated; it is difficult to say something on its concrete values. Therefore, for characterization of the behavior of $\zeta(s;Q)$, the Bohr idea realized in [6,7] can be applied. This idea is very simple: given some set, consider how often the values of $\zeta(s;Q)$ belong to that set. Such a procedure leads to probabilistic limit theorems on weakly convergent probability measures. The first theorems of such a type for the function $\zeta(s;Q)$ were obtained in [8,9]. Throughout the paper, we suppose that $Q[\underline{x}] \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$. In this case, the function $\zeta(s;Q)$, for $\sigma > \frac{n}{2}$, has the representation [10]

$$\zeta(s;Q) = \zeta(s;E_Q) + \zeta(s;F_Q),$$

where $\zeta(s;E_Q)$ and $\zeta(s;F_Q)$ are the zeta-functions of a certain Eisenstein series and of a modular form of weight $\frac{n}{2}$, respectively. Moreover, we require that $n \geqslant 4$ and $n = 2\mathbb{N}$. Then, it is known by [11,12] that, for $\sigma > \frac{n-1}{2}$,

$$\zeta(s;Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L\Big(s - \frac{n}{2} + 1, \psi_l\Big) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \tag{1}$$

where

$$L(s,\chi_k) = \sum_{m=1}^{\infty} \frac{\chi_k(m)}{m^s} \quad \text{and} \quad L(s,\psi_l) = \sum_{m=1}^{\infty} \frac{\psi_l(m)}{m^s}$$

are Dirichlet L-functions with characters χ_k and ψ_l, respectively, a_{kl} are certain complex numbers, and the series

$$\sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}$$

is absolutely convergent for $\sigma > \frac{n-1}{2}$.

For the definition of the limit measure in limit theorems, one topological group is used. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$, by $\mathbb{P}$ the set of all prime numbers, and define

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. Then, Ω with the product topology is a compact topological Abelian group. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(p)$ the pth component of an element $\omega \in \Omega$, $p \in \mathbb{P}$, and extend the function $\omega(p)$ to the set $\mathbb{N}$ by the equality

$$\omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, for $\sigma > \frac{n-1}{2}$, define the complex-valued random element

$$\begin{aligned} \zeta(s,\omega;Q) &= \sum_{j=1}^{J}\sum_{l=1}^{L} \frac{a_{jl}\omega(j)\omega(l)}{j^s l^s} L(s,\omega,\chi_j) L\Big(s - \frac{n}{2} + 1, \omega, \psi_l\Big) \\ &+ \sum_{m=1}^{\infty} \frac{f_Q(m)\omega(m)}{m^s}, \end{aligned} \tag{2}$$

where
$$L(s,\omega,\chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega(m)}{m^s} \quad \text{and} \quad L(s,\omega,\psi_l) = \sum_{m=1}^{\infty} \frac{\psi_l(m)\omega(m)}{m^s}.$$
For $A \in \mathcal{B}(\mathbb{C})$, define
$$P_{T,\sigma;Q}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma+it;Q) \in A\}.$$
Then, in [8], it was proved that for fixed $\sigma > \frac{n-1}{2}$, the measure $P_{T,\sigma;Q}$ converges weakly to the measure
$$P_{\sigma;Q}(A) \stackrel{\text{def}}{=} m_H\{\omega \in \Omega : \zeta(\sigma,\omega;Q) \in A\}$$
as $T \to \infty$. In [9], the discrete version of the latter result was given, i.e., that
$$P_{N,\sigma,h;Q}(A) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta(\sigma+ikh;Q) \in A\},$$
for every $h > 0$, also converges weakly to the measure $P_{\sigma;Q}$ as $N \to \infty$. Here, $\#A$ denotes the cardinality of the set A. In [13], the weak convergence of the measure
$$P_{T,\sigma;Q}(A) = \frac{1}{T}\text{meas}\{t \in [T,2T] : \zeta(\sigma+i\varphi(t);Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$
with a certain differentiable function $\varphi(t)$ was considered.

The aim of this paper is to extend a theorem of [9] by using a more general discrete set than an arithmetic progression $\{mh\}$. For this, we introduce the following class $U(T_0)$ of functions. We say that $\varphi \in U(T_0)$ if:

1° $\varphi(t)$ is a real valued positive increasing to $+\infty$ on $[T_0,\infty]$ function, where T_0 is a fixed sufficient large number;

2° $\varphi(t)$ has a continuous monotonic bounded derivative $\varphi'(t)$ satisfying the estimate
$$\varphi(2t)\frac{1}{\varphi'(t)} \ll t, \quad t \geqslant T_0;$$

3° a sequence $\{a\varphi(k) : k \geqslant N_0\}$, $N_0 = [T_0]+1$, with every real $a \neq 0$ is uniformly distributed modulo 1.

We recall that a sequence $\{a_k : k \geqslant k_0\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if, for every interval $[a,b) \subset [0,1)$,
$$\lim_{n\to\infty} \frac{1}{n-k_0+1} \sum_{k=k_0}^{n} I_{[a,b)}(\{a_k\}) = b-a,$$
where $\{a_k\}$ denotes the fractional part of a_k, and $I_{[a,b)}$ is the indicator function of $[a,b)$. Moreover, $a \ll_\xi b$, $b > 0$ means that there exists a constant $c = c(\xi) > 0$ such that $|a| \leq cb$.

It is known [14] (Example 2.7) that the sequence $\{a\sqrt{k}\log k\}$, $a \in \mathbb{R}\setminus\{0\}$, $k \geqslant 2$, is uniformly distributed modulo 1. Thus, the function $\varphi(t) = \sqrt{t}\log t$ lies in the class $U(T_0)$. Denote by $\{t_k\}$ the sequence of the Gram numbers; for a definition, see [15–17]. Then, $\varphi(\tau) = t_\tau$ also is an element of the class $U(T_0)$.

For $A \in \mathcal{B}(\mathbb{C})$, define
$$P_{N,\sigma;Q}(A) = \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \zeta(\sigma+i\varphi(k);Q) \in A\}.$$

The main result of the paper is the following theorem.

Theorem 1. *Suppose that $\sigma > \frac{n-1}{2}$ is fixed, and $\varphi \in U(T_0)$. Then, $P_{N,\sigma;Q}$ converges weakly to the measure $P_{\sigma;Q}$ as $N \to \infty$.*

We notice that the linear set $\{kh : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ is not uniformly distributed modulo 1. Therefore, Theorem 1 is different from the main theorem of [9]. Moreover, the requirement 3° of the class $U(T_0)$ plays a crucial role for the identification of the limit measure in Theorem 1. More precisely, this requirement ensures that the limit measure in Lemma 2 coincides with the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$. Furthermore, the latter fact is used in other limit relations.

The discrete generalized shifts were used in universality theorems for zeta- and L-functions. For example, in a series of works, the sequence $\{\gamma_k : k \in \mathbb{N}\}$ of imaginary parts of nontrivial zeros of the Riemann zeta-function in generalized shifts was used; see [18,19] and references therein. For the proof of universality, a limit theorem with generalized shifts $\zeta(s + ik\gamma_k; \mathfrak{a})$ for periodic zeta-functions in the space of analytic functions was obtained in [18]. For its proof, a conjecture that, for $c > 0$, the estimate

$$\sum_{\substack{\gamma_k \leqslant T \\ |\gamma_k - \gamma_l| < \frac{c}{\log T}}} \sum_{\gamma_l \leqslant T} 1 \ll T \log T, \quad T \to \infty,$$

is valid, was applied. This estimate is closely connected to the well-known Montgomery pair correlation conjecture for the sequence $\{\gamma_k\}$. This example shows that the use of generalized shifts $\zeta(s + i\gamma_k)$ also stimulates investigations of the very important sequence $\{\gamma_k\}$. On the other hand, universality theorems for zeta-functions are not effective in the sense that concrete shifts approximating a given analytic function are not known. Therefore, using generalized shifts extends a possibility for the detection of approximating shifts. In particular, this is important for universality theorems in short intervals, see, for example, [20]. We observe that the sequence $\{\varphi(k)\}$ in shifts $\zeta(\sigma + i\varphi(k); Q)$ does not have any relation to $\{\gamma_k\}$. The definition of the class $U(T_0)$ is based on the differentiability notion, while $\{\gamma_k\}$ is a discrete complicatedly distributed sequence. Theorem 1, as the results of [8,9,13], is theoretical: it describes not only the value distribution of $\zeta(s; Q)$ on the complex plane but also extends the space of applications of probabilistic methods. It is difficult to say something on its practical applications; however, since theorems of such a kind give the frequency of values of $\zeta(s; Q)$, the theorem at least potentially, may find some place, say, in crystallography [3]. We may expect more applications from universality theorems for $\zeta(s; Q)$, which are based on limit theorems in the space of analytic functions. These theorems will form a new chapter of the theory of the function $\zeta(s; Q)$, and this, this is our most immediate task.

Note that discrete value-distribution results of zeta and L-functions are complicated but more important than continuous because they contain more individual information. Historically, first of all, discrete results are connected to zero-distribution, mean values of various arithmetical functions and their limit theorems, moments and various limit characterizations, see [21–23]. For example, Selberg considered moments and related limit theorems of probabilistic type of objects involving Gram numbers and their modifications [16,24,25]. Voronin began [26] to study discrete shifts $\zeta(\sigma + ihk)$, $\frac{1}{2} < \sigma < 1$, $k \in \mathbb{N}$; for two classes of numbers $h > 0$, Reich [27] proposed discrete universality theorems on the approximation of analytic functions by shifts $\zeta(s + ikh)$. Significant progress was made by Bagchi [28]: he proved probabilistic limit theorems for $\zeta(s)$, Hurwitz zeta-functions, Dirichlet L-functions and other Dirichlet series in the space of analytic functions and applied them for proving the universality. Since that moment,discrete probabilistic limit theorems have been widely cultivated. For example, Kačinskaitė obtained [29] discrete limit theorems for the Matsumoto zeta-functions, Atstopienė presented [30] modified discrete universality theorems for $\zeta(s)$ and Hurwitz zeta-functions, Ignatavičiūtė studied [31] discrete value-distribution of Lerch zeta-functions, and the second author proved [32] discrete limit theorems for general Dirichlet series. In all the above works, discrete linear shifts were used. More information can be found in the survey papers [33,34], see also [35].

To prove Theorem 1, first, we will prove a discrete limit lemma on the torus Ω; then, we will consider the weak convergence of a measure defined by a certain function involving absolutely convergent Dirichlet series. Using some estimates between $\zeta(s;Q)$ and the above function, we will deduce Theorem 1.

2. The Case of Ω

For $A \in \mathcal{B}(\Omega)$, define

$$R_{N,\varphi}(A) = \frac{1}{N+1}\#\Big\{N \leqslant k \leqslant 2N : \Big(p^{-i\varphi(k)} : p \in \mathbb{P}\Big) \in A\Big\}.$$

This section is devoted to the weak convergence of $R_{N,\varphi}$ as $N \to \infty$.

For convenience, we recall the Weyl criterion on a uniformly distribution modulo 1; see, for example, [14].

Lemma 1. *A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is a uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

Lemma 2. *Suppose that $\varphi \in U(T_0)$. Then, $R_{N,\varphi}$ converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We have to prove that the Fourier transform $g_{N,\varphi}(\underline{k})$, $\underline{k} = (k_p \in \mathbb{Z},\, p \in \mathbb{P})$, of $R_{N,\varphi}$ converges weakly to the Fourier transform of the measure m_H as $N \to \infty$. The transform $g_{N,\varphi}(\underline{k})$ is given by

$$g_{N,\varphi}(\underline{k}) = \int\limits_{\Omega} \prod_{p\in\mathbb{P}}{}' \omega^{k_p}(p)\mathrm{d}R_{N,\varphi},$$

where "$'$" means that only a finite number of integers k_p are distinct from zero. Therefore, by the definition of $R_{N,\varphi}$,

$$\begin{aligned} g_{N,\varphi}(\underline{k}) &= \frac{1}{N+1}\sum_{k=N}^{2N} \prod_{p\in\mathbb{P}}{}' p^{-ik_p\varphi(k)} \\ &= \frac{1}{N+1}\sum_{k=N}^{2N} \exp\Big\{-i\varphi(k)\sum_{p\in\mathbb{P}}{}' k_p \log p\Big\}. \end{aligned} \tag{3}$$

Let $\underline{0}$ denotes a collection consisting of zeros. Obviously,

$$g_{N,\varphi}(\underline{0}) = 1. \tag{4}$$

Now, suppose that $\underline{k} \neq \underline{0}$. Since the logarithms of prime numbers are linearly independent over the field of rational numbers $\mathbb{Q}$, we have

$$a \overset{\text{def}}{=} \sum_{p\in\mathbb{P}}{}' k_p \log p \neq 0.$$

Therefore, in view of 3° of the class $U(T_0)$, the sequence

$$\Big\{\frac{a}{2\pi}\varphi(k) : k \geqslant \mathbb{N}_0\Big\}$$

is uniformly distributed modulo 1. Hence, Lemma 1 implies

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{k=N}^{2N} \exp\{-ia\varphi(k)\} = 0.$$

This, (3) and (4) show that

$$\lim_{N\to\infty} g_{N,\varphi}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and the lemma is proved. □

3. Case of Absolute Convergence

In this section, we will obtain a limit lemma for a function connected to $\zeta(s;Q)$, however, with representation of type (2) involving only absolutely convergent Dirichlet series.

Define

$$v_M(m) = \exp\left\{-\left(\frac{m}{M}\right)^{\theta}\right\}, \quad m, M \in \mathbb{N},$$

with a fixed $\theta > 0$. Since $v_M(m)$ exponentially decreases with respect to m, the series

$$L_M(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)v_M(m)}{m^s}$$

for every Dirichlet character is absolutely convergent in the half-plane $\sigma > \sigma_0$ with arbitrary finite σ_0. Let

$$\zeta_M(s;Q) = \sum_{j=1}^{J}\sum_{l=1}^{L} \frac{a_{jl}}{j^s l^s} L(s,\chi_j) L_M\left(s - \frac{n}{2} + 1, \psi_l\right) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s},$$

From the above remarks, we have the absolute convergence for $\sigma > \frac{n-1}{2}$ of the series for $\zeta_M(s;Q)$. Moreover, for $\omega \in \Omega$, let

$$\zeta_M(s,\omega;Q) = \sum_{j=1}^{J}\sum_{l=1}^{L} \frac{a_{jl}}{j^s l^s} L(s,\omega,\chi_j) L_M\left(s - \frac{n}{2} + 1, \omega, \psi_l\right) + \sum_{m=1}^{\infty} \frac{f_Q(m)\omega(m)}{m^s},$$

where

$$L_M\left(s - \frac{n}{2} + 1, \omega, \varphi_l\right) = \sum_{m=1}^{\infty} \frac{\psi_l(m)\omega(m)v_M(m)}{m^s}.$$

Then, clearly, the series for $\zeta_M(s,\omega;Q)$, as that for $\zeta_M(s;Q)$, is also absolutely convergent for $\sigma > \frac{n-1}{2}$.

For $A \in \mathcal{B}(\mathbb{C})$, define

$$P_{N,M,\sigma,\varphi;Q}(A) = \frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \zeta_M(\sigma + i\varphi(k);Q) \in A\}.$$

To prove the weak convergence for $P_{N,M,\sigma,\varphi;Q}$, we use the function $u_{M,\sigma;Q}(\omega) : \Omega \to \mathbb{C}$ defined by

$$u_{M,\sigma;Q}(\omega) = \zeta_M(\sigma,\omega;Q).$$

Let the probability measure $V_{M,\sigma;Q}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ be given by

$$V_{M,\sigma;Q} = m_H u_{M,\sigma;Q}^{-1},$$

where

$$m_H u_{M,\sigma;Q}^{-1}(A) = m_H(u_{M,\sigma;Q}^{-1}A), \quad A \in \mathcal{B}(\mathbb{C}).$$

We observe that the measure $V_{M,\sigma;Q}$ is defined correctly because, in view of absolute convergence of the series for $\zeta_M(s,\omega;Q)$, the function $u_{M,\sigma;Q}$ is continuous, hence, $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{C}))$-measurable.

Lemma 3. *Suppose that $\sigma > \frac{n+1}{2}$ is fixed and $\varphi \in U(T_0)$. Then, $P_{N,M,\sigma;Q}$ converges weakly to the measure $V_{M,\sigma;Q}$ as $N \to \infty$.*

Proof. By the definition of $u_{M,\sigma;Q}$,

$$u_{M,\sigma;Q}\Big(p^{-i\varphi(k)} : p \in \mathbb{P}\Big) = \zeta_M(\sigma + i\varphi(k); Q).$$

Therefore, the definitions of the measures $P_{N,M,\sigma,\varphi;Q}$ and $R_{N,\varphi}$ show that, for $A \in \mathcal{B}(\mathbb{C})$,

$$P_{N,M,\sigma,\varphi;Q}(A) = \frac{1}{N+1}\#\Big\{N \leqslant k \leqslant 2N : \Big(p^{-i\varphi(k)} : p \in \mathbb{P}\Big) \in u^{-1}_{M,\sigma;Q}A\Big\}.$$

Hence, $P_{N,M,\sigma,\varphi;Q} = R_{N,\varphi}u^{-1}_{M,\sigma;Q}$. Since the function $u_{M,\sigma;Q}$ is continuous, using a preservation of weak convergence under continuous mappings, see, for example, Theorem 5.1 of [36], and Lemma 2, we obtain that $P_{N,M,\sigma,\varphi;Q}$ converges weakly to $m_H u^{-1}_{M,\sigma;Q}$ as $N \to \infty$. □

4. Some Discrete Estimates

It is well known that mean square estimates play an important role for proofs of limit theorems for zeta-functions. In our case, we need estimates for

$$\sum_{k=N}^{2N} |\zeta(\sigma + i\varphi(k) + i\tau); Q)|^2, \quad \tau \in \mathbb{R}. \tag{5}$$

We do not have a direct proof for an estimate of the latter square; therefore, we will derive it from that of a similar continuous mean square

$$\int_T^{2T} |\zeta(\sigma + i\varphi(t) + i\tau); Q)|^2 \mathrm{d}t$$

using the following lemma.

Lemma 4. *Let $T, T_0 \geqslant \delta > 0$, $\mathcal{T} \subset [T_0 + \delta/2, T_0 + T - \delta/2]$ be a finite nonempty set, and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Suppose that a complex-valued function $S(t)$ is continuous on $[T_0, T_0 + T]$ and has a continuous derivative in the interior of that interval. Then

$$\sum_{\tau \in \mathcal{T}} N_\delta^{-1}(\tau)|S(\tau)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(t)|^2 \mathrm{d}t + \left(\int_{T_0}^{T_0+T} |S(t)|^2 \mathrm{d}t + \int_{T_0}^{T_0+T} |S'(t)|^2 \mathrm{d}t\right)^{1/2}.$$

Lemma 4 is called the Gallagher lemma; its proof can be found in [37] (Lemma 1.4).

In virtue of Lemma 4, for estimation of the discrete mean square (6), we need the estimates for continuous mean squares of the function $\zeta(s; Q)$ and its derivative.

We observe that the class $U(T_0)$ is a subclass of the class $W(T_0)$ introduced in [13]. Let $L(s, \chi)$ be an arbitrary Dirichlet L-function. Then, the Lemma 1 of [13] gives the following estimate.

Lemma 5. *Suppose that $\varphi \in U(T_0)$, $\sigma > \frac{1}{2}$ is fixed and $\tau \in \mathbb{R}$. Then*

$$\int_T^{2T} |L(\sigma + i\varphi(t) + i\tau, \chi)|^2 \mathrm{d}t \ll_{\sigma,\chi,\varphi} T(1 + |\tau|).$$

Lemma 6. *Suppose that $\varphi \in U(T_0)$, $\sigma > \frac{1}{2}$ is fixed, and $\tau \in \mathbb{R}$. Then*

$$\sum_{k=N}^{2N} |L(\sigma + i\varphi(k) + i\tau, \chi)|^2 \ll_{\sigma,\chi,\varphi} N(1 + |\tau|).$$

Proof. Let γ be a circle with center σ lying in the half-plane $\{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$. Then, by the Cauchy integral formula,

$$L'(\sigma + i\varphi(t) + i\tau, \chi) = \frac{1}{2\pi i} \int_\gamma \frac{L(z + i\varphi(t) + i\tau, \chi)}{(s - z)^2} \mathrm{d}z.$$

Hence,

$$|L'(\sigma + i\varphi(t) + i\tau, \chi)|^2 \ll \int_\gamma \frac{|\mathrm{d}z|}{|s - z|^2} \int_\gamma |L(z + i\varphi(t) + i\tau, \chi)|^2 \, |\mathrm{d}z|.$$

Therefore, in view of Lemma 5,

$$\begin{aligned} \int_T^{2T} |L'(\sigma + i\varphi(t) + i\tau, \chi)|^2 \, dt &\ll_{\sigma,\chi,\varphi,\gamma} \int_\gamma |\mathrm{d}z| \int_T^{2T} |L(\mathrm{Re}z + i\varphi(t) + i\tau + i\mathrm{Im}z, \chi)|^2 |\mathrm{d}z| \\ &\ll_{\sigma,\chi,\varphi,\gamma} T(1 + |\tau|). \end{aligned} \tag{6}$$

Now, we are ready to apply Lemma 4. We take $\delta = 1$, $T_0 = N - 1$, $T = N + 2$, $\mathcal{T} = \{N, N + 1, \ldots, 2N\}$ and $S(t) = L(\sigma + i\varphi(t) + i\tau, \chi)$ in Lemma 4. Clearly, we have $N_\delta(x) = 1$. Thus, Lemmas 4 and 5, together with (6) imply

$$\begin{aligned} \sum_{k=N}^{2N} |L(\sigma + i\varphi(k) + i\tau, \chi)|^2 &\ll \int_{N-1}^{2N+1} |L(\sigma + i\varphi(t) + i\tau, \chi)|^2 \, \mathrm{d}t \\ &+ \left(\int_{N-1}^{2N+1} |L(\sigma + i\varphi(t) + i\tau, \chi)|^2 \, \mathrm{d}t \int_{N-1}^{2N+1} (\varphi'(t))^2 |L'(\sigma + i\varphi(t) + i\tau, \chi)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \\ &\ll_{\sigma,\chi,\varphi,\gamma} N(1 + |\tau|). \end{aligned}$$

□

Lemma 6 allows estimating the distance in the mean between $\zeta(s; Q)$ and $\zeta_M(s; Q)$.

Lemma 7. *Suppose that $\varphi \in U(T_0)$, $\sigma > \frac{1}{2}$ is fixed, and $\tau \in \mathbb{R}$. Then*

$$\lim_{M \to \infty} \limsup_{N \to \infty} \frac{1}{N + 1} \sum_{k=N}^{2N} |\zeta(\sigma + i\varphi(k); Q) - \zeta_M(\sigma + i\varphi(k); Q)| = 0.$$

Proof. First, we will deal with Dirichlet L-functions, and will prove that, for arbitrary Dirichlet character χ,

$$\lim_{M \to \infty} \limsup_{N \to \infty} \frac{1}{N + 1} \sum_{k=N}^{2N} |L(\sigma + i\varphi(k), \chi) - L_M(\sigma + i\varphi(k), \chi)| = 0. \tag{7}$$

For this, we will use the integral representation

$$L_M(s,\chi) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} L(s+z,\chi) l_M(z)\, dz, \tag{8}$$

where

$$l_M(z) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) M^z, \tag{9}$$

$\theta_1 > \frac{1}{2}$ and θ comes from the definition of $v_M(m)$, see, for example, [9] and the classical estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\} \text{ with } c > 0 \tag{10}$$

which is uniform in every interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $\sigma_1 < \sigma_2$. The representation (8) shows that for $\theta_2 > 0$,

$$\begin{aligned} &L_M(\sigma + i\varphi(k), \chi) - L(\sigma + i\varphi(k), \chi) \\ &\ll \int_{-\theta_2 - i\infty}^{-\theta_2 + i\infty} |L(\sigma + i\varphi(k) + z, \chi)||l_M(z)| \frac{|dz|}{|z|} + |R_M(\sigma + i\varphi(k))| \\ &\ll \int_{-\infty}^{\infty} |L(\sigma + i\varphi(k) - \theta_2 + i\tau, \chi)| \left| \frac{l_M(-\theta_2 + i\tau)}{-\theta_2 + i\tau} \right| d\tau + |R_M(\sigma + i\varphi(k))|, \end{aligned} \tag{11}$$

where

$$R_M(\sigma + i\varphi(k)) = \begin{cases} 0 & \text{if } \chi \text{ is a non-principal character mod } q, \\ \frac{l_M(1-\sigma-i\varphi(k))}{1-\sigma-i\varphi(k)} \prod_{p|q} \left(1 - \frac{1}{p}\right) & \text{otherwise.} \end{cases}$$

Suppose that $\varepsilon > 0$ is such that $\sigma \geqslant \frac{1}{2} + \varepsilon$ and $\theta_2 = \sigma - \frac{1}{2} - \frac{\varepsilon}{2}$. Then, (10) leads to

$$\begin{aligned} &\frac{1}{N+1} \sum_{k=N}^{2N} |L(\sigma + i\varphi(k), \chi) - L_M(\sigma + i\varphi(k), \chi)| \\ &\ll \int_{-\infty}^{\infty} \frac{1}{N+1} \sum_{k=N}^{2N} \left| L\left(\frac{1}{2} + \frac{\varepsilon}{2} + i\varphi(k) + i\tau, \chi\right) \right| \left| \frac{l_M\left(-\sigma + \frac{1}{2} + \frac{\varepsilon}{2} + i\tau\right)}{-\sigma + \frac{1}{2} + \frac{\varepsilon}{2} + i\tau} \right| d\tau \\ &\quad + \frac{1}{N+1} \sum_{k=N}^{2N} |R_M(\sigma + i\varphi(k))| \stackrel{\text{def}}{=} I + Z. \end{aligned} \tag{12}$$

By Lemma 5,

$$\begin{aligned} \frac{1}{N+1} \sum_{k=N}^{2N} \left| L\left(\frac{1}{2} + \frac{\varepsilon}{2} + i\varphi(k) + i\tau, \chi\right) \right| &\leqslant \left(\frac{1}{N+1} \sum_{k=N}^{2N} \left| L\left(\frac{1}{2} + \frac{\varepsilon}{2} + i\varphi(k) + i\tau, \chi\right) \right|^2 \right)^{\frac{1}{2}} \\ &\ll_{\varepsilon,\chi,\varphi} (1 + |\tau|)^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Moreover, (9) and (10) give the estimate

$$\frac{l_M\left(-\sigma + \frac{1}{2} + \frac{\varepsilon}{2} + i\tau\right)}{-\sigma + \frac{1}{2} + \frac{\varepsilon}{2} + i\tau} \ll_\theta M^{-\sigma + \frac{1}{2} + \frac{\varepsilon}{2}} \exp\left\{-\frac{c}{\theta}|\tau|\right\} \ll_\theta M^{-\frac{\varepsilon}{2}} \exp\left\{-\frac{c}{\theta}|\tau|\right\}.$$

This and (13) show that

$$I \ll_{\varepsilon,\chi,\varphi,\theta} M^{-\frac{\varepsilon}{2}} \int_{-\infty}^{\infty} (1 + |\tau|)^{\frac{1}{2}} \exp\left\{-\frac{c}{\theta}|\tau|\right\} d\tau \ll_{\varepsilon,\chi,\varphi,\theta} M^{-\frac{\varepsilon}{2}}. \tag{14}$$

Similarly as above, the definition of $R_M(\sigma + i\varphi(k))$ and (10) give

$$R_M(\sigma + i\varphi(k)) \ll_\theta M^{1-\sigma} \exp\left\{-\frac{c}{\theta}\varphi(k)\right\} \ll_\theta M^{\frac{1}{2}-\varepsilon} \exp\left\{-\frac{c}{\theta}\varphi(k)\right\}.$$

Thus,

$$Z \ll_\theta \frac{M^{\frac{1}{2}-\varepsilon}}{N} \sum_{k=N}^{2N} \exp\left\{-\frac{c}{\theta}\varphi(k)\right\} \ll_\theta M^{\frac{1}{2}-\varepsilon} \exp\left\{-\frac{c}{\theta}\varphi(N)\right\}.$$

This together with (14) and (12) proves Equality (7).

By the representation (1), we have

$$\begin{aligned}
&\frac{1}{N+1} \sum_{k=N}^{2N} |\zeta(s + i\varphi(k); Q) - \zeta_M(\sigma + i\varphi(k); Q)| \\
&= \sum_{j=1}^{J} \sum_{l=1}^{L} \frac{1}{N+1} \sum_{k=N}^{2N} \left| \frac{a_{jl}}{j^{\sigma+i\varphi(k)} l^{\sigma+i\varphi(k)}} L(\sigma + i\varphi(k), \chi_j) \right| \\
&\quad \times \left| L\left(\sigma - \frac{n}{2} + 1 + i\varphi_k, \psi_l\right) - L_M\left(\sigma - \frac{n}{2} + 1 + i\varphi_k, \psi_l\right) \right| \\
&\ll \sum_{j=1}^{J} \sum_{l=1}^{L} \frac{|a_{jl}|}{j^\sigma l^\sigma} \frac{1}{N} \sum_{k=N}^{2N} \left| L\left(\sigma - \frac{n}{2} + 1 + i\varphi_k, \psi_l\right) - L_M\left(\sigma - \frac{n}{2} + 1 + i\varphi_k, \psi_l\right) \right|.
\end{aligned}$$

Therefore, this and (7) prove the lemma. □

5. Proof of Theorem 1

We will use for random elements the convergence in distribution ($\xrightarrow{\mathcal{D}}$). Recall that $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ if the distribution of X_n converges weakly to that of X as $n \to \infty$. In addition, if $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$ and X has the distribution P, then we write $X_n \xrightarrow[n\to\infty]{\mathcal{D}} P$. The latter notation comes from [6].

Proof of Theorem 1. Let the random variable θ_N be defined on the probability space with a measure P, and have the distribution

$$P\{\theta_N = k\} = \frac{1}{N+1}, \quad k = N, N+1, \dots, 2N.$$

For $\sigma > \frac{n-1}{2}$, define the complex-valued random elements

$$\begin{aligned}
X_{N,M} &= X_{N,M}(\sigma; Q) = \zeta_M(\sigma + i\varphi(\theta_N); Q), \\
X_N &= X_N(\sigma; Q) = \zeta(\sigma + i\varphi(\theta_N); Q),
\end{aligned}$$

and denote by $Y_M = Y_M(\sigma; Q)$ the complex-valued random element having the distribution $V_{M,\sigma;Q}$, where $V_{M,\sigma;Q}$ is from Lemma 3. The measure $V_{M,\sigma;Q}$ is independent of the function $\varphi(t)$. Therefore, it is known, see Formula (9) of [9], that

$$Y_M \xrightarrow[M\to\infty]{\mathcal{D}} P_{\sigma;Q}. \tag{15}$$

By Lemma 3, we have

$$X_{N,M} \xrightarrow[N\to\infty]{\mathcal{D}} Y_M. \tag{16}$$

Moreover, a simple application of Lemma 7 gives, for every $\varepsilon > 0$,

$$\lim_{M\to\infty} \limsup_{N\to\infty} P\{|X_N - X_{N,M}| \geqslant \varepsilon\}$$
$$\leqslant \lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{k=N}^{2N} |\zeta(\sigma + i\varphi(k); Q) - \zeta_M(\sigma + i\varphi(k); Q)| = 0.$$

The latter equality together with relations (15) and (16) shows that the random elements Y_M, $X_{N,M}$ and X_N satisfy the hypotheses of Theorem 4.2 of [36]. Thus, we obtain that

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P_{\sigma;Q},$$

and this is equivalent to the assertion of Theorem 1. □

6. Concluding Remarks

In the paper, we obtain a probabilistic characterization for the behavior of the Epstein zeta-function $\zeta(s;Q)$, $s = \sigma + it$, which is defined by the series

$$\zeta(s;Q) = \sum_{\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}} (\underline{x}^T Q \underline{x})^{-s}, \quad \sigma > \frac{n}{2},$$

where Q is a positive defined quadratic $n \times n$ matrix and has analytic continuation to the whole complex plane except for a simple pole at the point $s = \frac{n}{2}$.

More precisely, we prove a discrete probabilistic limit theorem of the Bohr–Jessen type in the sense of weak convergent probability measures on the complex plane for the $\zeta(s;Q)$ by using generalized shifts $\zeta(s + i\varphi(k); Q)$, i.e., that

$$\frac{1}{N+1}\#\{N \leqslant k \leqslant 2N : \zeta(s + i\varphi(k); Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the explicitly given probability measure as $N \to \infty$. The function $\varphi(t)$, $t \geqslant T_0 > 0$, is positive monotonically increasing to $+\infty$, and having a continuous monotonic bounded derivative $\varphi'(t)$ satisfying

$$\varphi(2t)\frac{1}{\varphi'(t)} \ll t, \quad t \geqslant T_0.$$

Moreover, we require that the sequence $\{a\varphi(k) : k \geqslant T_0\}$ would be distributed uniformly modulo 1 for every $a \in \mathbb{R} \setminus \{0\}$. Note that the condition $\varphi'(t) \leqslant C$, $t \geqslant T_0$ implies

$$\varphi(t)\varphi'(t) \ll t, \quad t \geqslant T_0,$$

and is needed for estimation of the second discrete moment

$$\sum_{k=N}^{2N} |\zeta(\sigma + i\varphi(k) + i\tau; Q)|^2, \quad t \in \mathbb{R}. \tag{17}$$

The used method is based on the Gallagher lemma (Lemma 4) connecting continuous and discrete mean squares of same differentiable functions. It is an interesting problem to obtain an estimate for Quantity (17) without using the Gallagher lemma.

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