



The centers of discrete groups as stabilizers of dark matter

Darius Jurčiukonis^{1,*} and Luís Lavoura²

¹*University of Vilnius, Institute of Theoretical Physics and Astronomy, Saulėtekio av. 3, LT-10222 Vilnius, Lithuania*

²*Universidade de Lisboa, Instituto Superior Técnico, CFTP, Av. Rovisco Pais 1, P-1049-001 Lisboa, Portugal*

*E-mail: darius.jurciukonis@tfai.vu.lt

Received October 31, 2022; Revised December 14, 2022; Accepted December 27, 2022; Published January 11, 2023

.....
 The most usual option to stabilize dark matter (DM) is a \mathbb{Z}_2 symmetry. In general, though, DM may be stabilized by any \mathbb{Z}_N with $N \geq 2$. We consider the way that \mathbb{Z}_N is a subgroup of the internal-symmetry group G of a model; we entertain the possibility that \mathbb{Z}_N is the center of G , yet G is not of the form $\mathbb{Z}_N \times G'$, where G' is a group smaller (i.e., of lower order) than G . We examine all the discrete groups of order smaller than 2001 and we find that many of them cannot be written as the direct product of a cyclic group and some other group, yet they have a non-trivial center that might be used in model building to stabilize DM.

Subject Index B31, B40, B46

1. Introduction

The lightest dark matter (DM) particle ought to be stable (i.e., unable to decay), or at least it should have a lifetime of the order of the age of the Universe. If it is stable, then there is an unbroken cyclic \mathbb{Z}_N symmetry that is non-trivial (i.e., it has $N \geq 2$), such that standard matter is invariant under \mathbb{Z}_N while DM is not; the \mathbb{Z}_N charge different from 1 of the lightest DM particle prevents it from decaying to standard matter, which has \mathbb{Z}_N charge 1.

The most usual option in model building is $N = 2$. However, some authors have considered possibilities $N > 2$. For DM stabilized by a \mathbb{Z}_3 symmetry, see Refs. [1–9]. Larger cyclic groups have been used to stabilize DM, like \mathbb{Z}_4 and \mathbb{Z}_6 [10–12], \mathbb{Z}_5 [13], or a general \mathbb{Z}_N [14–21].

The \mathbb{Z}_N that stabilizes DM may be the center of a larger internal-symmetry group G .¹ The simplest possibility consists in G being a discrete group of order² O that is isomorphic to the direct product $\mathbb{Z}_N \times G'$, where G' is a group of order O/N . In that case, all the irreducible representations (“irreps”) of G consist of the product of an irrep of \mathbb{Z}_N (which is 1D, because \mathbb{Z}_N is Abelian and Abelian groups have 1D irreps) and an irrep of G' ; standard matter must be

¹The center of a group G is its Abelian subgroup formed by the elements of G that commute with all the elements of G .

²The order of a discrete group is the number of its elements.

placed in the trivial representation³ of \mathbb{Z}_N while DM is placed in non-trivial representations of \mathbb{Z}_N .

However, discrete groups G that cannot be written as the direct product of a cyclic group and a smaller group may also have a non-trivial \mathbb{Z}_N center. If that happens, then once again an irrep of G may represent \mathbb{Z}_N either trivially or in non-trivial fashion (namely, when some elements of \mathbb{Z}_N are represented by a phase f with $f \neq 1$ but $f^N = 1$). If \mathbb{Z}_N remains unbroken when G is (either softly or spontaneously) broken, and if there are particles with \mathbb{Z}_N charge different from 1, then those particles play the role of DM, while the particles with \mathbb{Z}_N value 1 are standard matter.

This possibility was recently called to our attention by Ref. [22], where a group G of order 81, named $\Sigma(81)$,^{4,5} was used as the internal symmetry of a model. The authors of Ref. [22] rightly pointed out that “[some] irreducible representations [of $\Sigma(81)$] form a closed set under tensor products, implying that if every Standard Model field transforms as [one of those representations], then any field transforming as [a representation that is not in that closed set] will belong to the dark sector. The lightest among them will then be a dark matter candidate.”

As a matter of fact, this mechanism had already been suggested before, namely, in Ref. [25]. There, it was noted that some discrete subgroups of $SU(2)$ have subsets of irreps that are closed under tensor products, and this fact might be used to stabilize DM.⁶

In this paper we make a survey of all the discrete groups of order $O \leq 2000$, except groups with either $O = 512$, $O = 1024$, or $O = 1536$. We select the groups that cannot be written as the direct product of a non-trivial cyclic symmetry and a smaller group, and that moreover have at least one *faithful* irreducible representation (“firrep”). We identify the center \mathbb{Z}_N of each of those groups, and also the dimensions D of their firreps. We construct various tables with the integers O , N , and D . We find that very many discrete groups, especially those that are not subgroups of any continuous group $SU(D)$, have centers \mathbb{Z}_N with $N \geq 2$, and N is sometimes quite large.

This paper is organized as follows. In Sect. 2 we explain, through the well-known cases of $SU(2)$ and $SU(3)$, that some groups have a center \mathbb{Z}_N with $N \geq 2$ and some irreps of those groups represent \mathbb{Z}_N trivially while other irreps do not.⁷ In Sect. 3 we make a systematic survey of the centers of all the discrete groups G of order up to 2000 that cannot be written as the direct product of a cyclic group and another group and that have some faithful irreducible representation.⁸ In Sect. 4 we briefly state our conclusions. As an appendix to this paper, comprehensive listings of the groups that we have studied are available online at <https://github.com/jurciukonis/GAP-group-search>.

³The trivial representation of a group is the one where all the group elements are mapped onto the unit matrix.

⁴The group $\Sigma(81)$ cannot be written as a direct product $\mathbb{Z}_3 \times G'$, G' being a group of order 27. Rather, $\Sigma(81)$ (which has SmallGroups identifier [81, 7]) is of the form $(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$; i.e., it is a semi-direct product.

⁵The group $\Sigma(81)$ was used in model building by E. Ma [23]. See also Ref. [24].

⁶In the course of the present investigation we have found that this indeed happens for *all* the discrete subgroups of $SU(2)$, except the trivial subgroup.

⁷Hurried readers may skip Sect. 2.

⁸We do not survey groups of order either 512, 1024, or 1536, because there are impractically very many groups of those orders.

2. $SU(2)$ and $SU(3)$

2.1 $SU(2)$

The defining representation of $SU(2)$ consists of the 2×2 unitary matrices with determinant 1. One such matrix is

$$A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

This is proportional to the unit matrix and therefore it commutes with all the 2×2 matrices; in particular, it commutes with all the matrices in the defining representation of $SU(2)$. Hence, in any irrep of $SU(2)$, A_2 must be represented by a matrix that commutes with all the matrices in that irrep. Schur’s first lemma (see, e.g., Ref. [26]) states that any matrix that commutes with all the matrices in an irrep of a group must be proportional to the unit matrix. Therefore, in the D -dimensional irrep of $SU(2)$,⁹ A_2 must be represented by a multiple of the $D \times D$ unit matrix $\mathbb{1}_D$. But, as is clear from Eq. (1), $(A_2)^2 = \mathbb{1}_2$ is the unit element of $SU(2)$ in the defining representation. This property must be reproduced in the D -dimensional irrep of $SU(2)$. One hence concludes that, in that irrep,

$$A_2 \mapsto (-1)^{q_D} \times \mathbb{1}_D, \tag{2}$$

where q_D is an integer that is either 0 or 1 modulo 2. The integer q_D depends on the irrep.

The irreps of $SU(2)$ wherein A_2 is mapped onto the unit matrix, i.e., the ones for which q_D is 0 modulo 2, are unfaithful.¹⁰ Those are the integer-spin irreps. They have odd D and are faithful irreps of the quotient group

$$SU(2)/\mathbb{Z}_2 \cong SO(3). \tag{3}$$

In Eq. (3),

$$\mathbb{Z}_2 = \{\mathbb{1}_2, A_2\} \tag{4}$$

is the center of $SU(2)$; i.e., it is the subset of $SU(2)$ elements (in the defining representation) that commute with *all* the elements of $SU(2)$; it is a \mathbb{Z}_2 subgroup of $SU(2)$.

The D -dimensional irreps of $SU(2)$ with even D are the half-integer-spin representations and represent $SU(2)$ faithfully; i.e., they map $A_2 \mapsto -\mathbb{1}_D$.

Let us consider the tensor product of the irreps of $SU(2)$ with dimensions D_1 and D_2 . Clearly, $A_2 \mapsto (-1)^{q_{D_1}} \times \mathbb{1}_{D_1}$ in the irrep with dimension D_1 and $A_2 \mapsto (-1)^{q_{D_2}} \times \mathbb{1}_{D_2}$ in the irrep with dimension D_2 . In the product representation, which is in general reducible,

$$A_2 \mapsto (-1)^{q_{D_1}+q_{D_2}} \times \mathbb{1}_{D_1+D_2}. \tag{5}$$

Therefore, the subset of the irreps of $SU(2)$ that have $q_D = 0$ modulo 2 is closed under tensor products. This property of the irreps of $SU(2)$ also holds for the irreps of discrete subgroups of $SU(2)$. If one such subgroup contains A_2 in its defining representation, then a D -dimensional irrep of that subgroup must represent A_2 either by $\mathbb{1}_D$ or by $-\mathbb{1}_D$, and the subset of irreps that represent A_2 by unit matrices is closed under tensor products. It was suggested in Ref. [25] that this property may be used to stabilize DM. In that suggestion, nature possesses an internal symmetry under a discrete subgroup of $SU(2)$ that contains in its defining irrep the matrix A_2 ; standard matter sits in a (in general, reducible) representation of that internal symmetry where A_2 is mapped onto the unit matrix, while DM is in a representation of the internal symmetry

⁹As is well known, $SU(2)$ has one and only one D -dimensional irrep for each integer D .

¹⁰An unfaithful representation of a group represents two or more distinct elements by the same matrix.

in which A_2 is mapped onto minus the unit matrix. Then, any collection of standard-matter particles will be invariant under the transformation represented in the defining representation by A_2 , which implies that the lightest DM particle, which changes sign under that transformation, is stable. Dark matter is stabilized by the \mathbb{Z}_2 symmetry in Eq. (4); that \mathbb{Z}_2 symmetry is the center of the internal-symmetry group of the model.

Example: The quaternion group Q_8 is the order-eight subgroup of $SU(2)$ formed, in its defining 2D irrep, by the matrices¹¹

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}; \tag{6a}$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{6b}$$

$$AB = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \tag{6c}$$

$$A_2 = A^2 = B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (A_2)^2 = A^4 = B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6d}$$

The center of this group consists of the \mathbb{Z}_2 in Eq. (4), cf. line (6d); yet, Q_8 is not the direct product of that \mathbb{Z}_2 and any order-four group. The quaternion group has five irreps: the 2D one in Eqs. (6) and the four 1D ones

$$\mathbf{1}_{rs} : \quad A \mapsto r, \quad B \mapsto s, \tag{7}$$

where both r and s may be either 1 or -1 . Clearly, in all the 1D irreps $A_2 = A^2 = B^2$ is mapped onto 1, while in the 2D irrep A_2 is mapped onto $-\mathbb{1}_2$. In model building, standard matter might sit in singlet irreps of Q_8 while DM would be placed in doublets of Q_8 . We envisage, for instance, an extension of the SM with *global* symmetry Q_8 and four Higgs doublets $H_{1,2,3,4}$ that are singlets of Q_8 as

$$H_1 : \mathbf{1}_{++}, \quad H_2 : \mathbf{1}_{+-}, \quad H_3 : \mathbf{1}_{-+}, \quad H_4 : \mathbf{1}_{--}. \tag{8}$$

If there are in the scalar potential quadratic terms $H_1^\dagger H_2, H_1^\dagger H_3, H_1^\dagger H_4, H_2^\dagger H_3, H_2^\dagger H_4, H_3^\dagger H_4$, and their Hermitian conjugates, then the symmetry Q_8 is *softly* broken—but its center \mathbb{Z}_2 is preserved, because $H_{1,2,3,4}$ are all invariant under it. If either H_2, H_3 , or H_4 acquire a VEV, then the symmetry Q_8 is *spontaneously* broken—but its center is, once again, preserved. If additionally there is in the model some matter (either fermionic or bosonic) placed in doublets of Q_8 , then the lightest particle arising from that matter would be a DM candidate.

2.2 $SU(3)$

The defining representation of $SU(3)$ consists of the 3×3 unitary matrices with determinant 1 and includes the matrix

$$A_3 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} = \omega \times \mathbb{1}_3, \tag{9}$$

where $\omega = \exp(2i\pi/3)$. The Abelian group

$$\mathbb{Z}_3 = \left\{ \mathbb{1}_3, A_3, (A_3)^2 \right\} \tag{10}$$

¹¹In Eqs. (6) and below, we separate the classes of each group through semicolons.

forms the center of $SU(3)$ in the defining representation. The matrix A_3 commutes with all the matrices in the defining representation of $SU(3)$ and satisfies $(A_3)^3 = \mathbb{1}_3$. Therefore, in a D -dimensional irrep of $SU(3)$,

$$A_3 \mapsto \omega^{q_D} \times \mathbb{1}_D, \tag{11}$$

where q_D is an integer that depends on the irrep and may be either 0, 1, or 2 modulo 3.¹² Irreps with $q_D = 0$ (like the octet and the decuplet) have A_3 represented by $\mathbb{1}_D$ and are unfaithful representations of $SU(3)$. Irreps with either $q_D = 1$ (like the triplet) or $q_D = 2$ (like the sextet and the anti-triplet) are faithful. Clearly, if $A_3 \mapsto \omega^{q_{D_1}} \times \mathbb{1}_{D_1}$ in an irrep with dimension D_1 and $A_3 \mapsto \omega^{q_{D_2}} \times \mathbb{1}_{D_2}$ in an irrep with dimension D_2 , then in the product representation

$$A_3 \mapsto \omega^{q_{D_1}+q_{D_2}} \times \mathbb{1}_{D_1+D_2}. \tag{12}$$

Therefore, the irreps with $q_D = 0$ form a closed set under tensor products. There is a selection rule in tensor products of irreps of $SU(3)$, similar to the selection rule in tensor products of irreps of $SU(2)$, but with the group \mathbb{Z}_3 of Eq. (10) in $SU(3)$ instead of the group \mathbb{Z}_2 of Eq. (4) in $SU(2)$.

This also holds for many—but not all—the discrete subgroups of $SU(3)$. The three matrices in Eq. (10) may all belong to the defining representation of a discrete subgroup of $SU(3)$; when that happens, a D -dimensional irrep of that subgroup possesses a q_D -value, defined by Eq. (11). The q_D -values help determine the tensor products of irreps of the subgroup. This may be used to explain the stability of DM: if nature had an internal symmetry that was a discrete subgroup of $SU(3)$ that contained the matrix A_3 in its defining representation and that stayed unbroken, then standard matter would sit in irreps of that subgroup with $q_D = 0$ while DM would be in irreps with either $q_D = 1$ or $q_D = 2$; the lightest DM particle would then automatically be stable.

Example: The group A_4 is the order-12 subgroup of $SU(3)$ formed, in its defining representation, by the matrices

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A^2BA = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad ABA^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{13a}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad BAB = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}; \tag{13b}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^2B = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$BA^2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad ABA = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad A^3 = B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13c}$$

Neither the matrix A_3 nor $(A_3)^2$ belong to the defining representation of A_4 ; the center of A_4 is trivial; i.e., it is formed just by the unit element. The four irreps $\mathbf{3}$, $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{1}''$ of A_4 do not

¹²The value of q_D is the “trality” of the irrep (see, e.g., Ref. [27]).

have any selection rule in their tensor products. Thus, the group A_4 is of no use to explain the stability of DM.

3. Group search

3.1 Motivation

The defining representation of $SU(D)$ consists of the $D \times D$ unitary matrices with determinant 1. It is obvious that, in this representation, the center of $SU(D)$ is formed by the D diagonal matrices

$$\Delta \times \mathbb{1}_D, \Delta^2 \times \mathbb{1}_D, \Delta^3 \times \mathbb{1}_D, \dots, \Delta^D \times \mathbb{1}_D = \mathbb{1}_D, \tag{14}$$

where $\Delta = \exp(2i\pi/D)$. Thus, the center of $SU(D)$ is a \mathbb{Z}_D group. Any discrete group that has a firrep formed by matrices that belong to $SU(D)$ may contain in that representation either

- all the matrices in Eq. (14),
- only the last one of them,
- or—if D is not a prime number and may be divided by an integer m different from both 1 and D —the $m^{\text{th}}, 2m^{\text{th}}, \dots, D^{\text{th}}$ matrices in Eq. (14).

In general, if m is an integer that divides D and $\mu = \exp(2i\pi/m)$, then there is a cyclic symmetry \mathbb{Z}_m given, in the defining representation of $SU(D)$, by

$$\mathbb{Z}_m = \{ \mu \times \mathbb{1}_D, \mu^2 \times \mathbb{1}_D, \mu^3 \times \mathbb{1}_D, \dots, \mu^m \times \mathbb{1}_D = \mathbb{1}_D \}. \tag{15}$$

Some discrete subgroups of $SU(D)$ may then have \mathbb{Z}_m as their center.

Thus, discrete subgroups of $SU(D)$ that are not subgroups of any $U(D')$ with $D' < D$ may have very few centers. For instance, a discrete subgroup of $SU(10)$ that is not a subgroup of any $U(D')$ with $D' < 10$ may only have center $\mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_{10}$, or the trivial group; and a discrete subgroup of $SU(11)$ that is not a subgroup of any $U(D')$ with $D' < 11$ may only have center \mathbb{Z}_{11} or the trivial group.

Example: Consider the discrete group generated by two transformations b and c that obey

$$c^8 = e, \quad b^4 = c^4, \quad c^2bc^2 = b, \quad c^3b = b^3c, \tag{16}$$

where e is the identity transformation. There is a 4D irreducible representation of Eqs. (16) as

$$b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

Both matrices in Eqs. (17) are orthogonal and have determinant +1; therefore this group (see Ref. [28]) is a subgroup of both $SO(4)$ and $SU(4)$. One easily sees that in the representation (17)

$$b^4 \mapsto \text{diag}(-1, -1, -1, -1), \tag{18}$$

while b^2 is not mapped onto a diagonal matrix. Hence, this subgroup of $SU(4)$ has center \mathbb{Z}_2 generated by b^4 . One finds that the defining conditions (16) allow two inequivalent doublet representations:

$$\mathbf{2}_1 : \quad b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

and

$$\mathbf{2}_2 : \quad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}$$

Additionally, the conditions (16) have eight inequivalent singlet representations:

$$\mathbf{1}_p : b \mapsto i^p, c \mapsto i^p \quad \text{and} \quad \mathbf{1}_{4+p} : b \mapsto i^p, c \mapsto -i^p, \quad \text{where } p \in \{0, 1, 2, 3\}. \tag{21}$$

In irreps (19)–(21) the transformation b^4 is represented by the unit matrix instead of minus the unit matrix as in Eq. (18); therefore, those irreps are unfaithful. This group thus has only one firrep—the quadruplet (17)—and 10 unfaithful inequivalent irreps—the two doublets (19) and (20) and the eight singlets (21). If one had a theory with this group as internal symmetry and in that theory the only scalars that acquired VEVs were placed in unfaithful irreps of the group, then the internal symmetry would get spontaneously broken to the \mathbb{Z}_2 generated by b^4 . Alternatively, the theory might have soft-breaking terms of types either $\mathbf{1}_p^\dagger \mathbf{1}_q$ (with $p \neq q$) or $\mathbf{2}_1^\dagger \mathbf{2}_2$, and then the discrete group would be broken softly but its \mathbb{Z}_2 subgroup would remain unbroken. Any fields in such a theory placed in quadruplets of the symmetry group might then take the role of DM.

Discrete subgroups of $U(D)$ do not bear the constraint that the determinants of the matrices in their defining representations should be 1. As a consequence, if

$$\mathbb{Z}_t = \{ \theta \times \mathbb{1}_D, \theta^2 \times \mathbb{1}_D, \theta^3 \times \mathbb{1}_D, \dots, \theta^t \times \mathbb{1}_D = \mathbb{1}_D \}, \tag{22}$$

where $\theta = \exp(2i\pi/t)$, is the center of a discrete subgroup of $U(D)$, then there appears to be *a priori* no restriction on t .

Example: The discrete group $\mathbb{Z}_8 \times \mathbb{Z}_2$ has order 16 and SmallGroups identifier [16, 6]. In its defining representation it is formed by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}; \tag{23a}$$

$$\begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ -1 & 0 \end{pmatrix}; \tag{23b}$$

$$\begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & -1 \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}; \tag{23c}$$

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}; \quad \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}; \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{23d}$$

This is the firrep $\mathbf{2}$ of $\mathbb{Z}_8 \times \mathbb{Z}_2$. The other inequivalent irreps of that group are the $\mathbf{2}^*$ (wherein each matrix of the $\mathbf{2}$ is mapped onto its complex-conjugate matrix) and eight inequivalent unfaithful singlet irreps. Most of the 2×2 unitary matrices (23) do not have determinant 1; therefore, $\mathbb{Z}_8 \times \mathbb{Z}_2$ is a subgroup of $U(2)$ but not of $SU(2)$. One sees in line (23d) that the center of $\mathbb{Z}_8 \times \mathbb{Z}_2$ is

$$\mathbb{Z}_4 = \left\{ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{24}$$

Thus, while discrete subgroups of $SU(2)$ may have center either \mathbb{Z}_1 or \mathbb{Z}_2 , discrete subgroups of $U(2)$ enjoy further possibilities, for instance \mathbb{Z}_4 .

Motivated by this observation that discrete subgroups of $U(D)$ may in general have diverse centers, in our work we have surveyed many discrete groups in order to find out their centers and also which groups $U(D)$ they are subgroups of.

3.2 GAP and SmallGroups

GAP [29] is a computer algebra that provides a programming language and includes many functions that implement various algebraic algorithms. It is supplemented by libraries containing a large amount of data on algebraic objects. With GAP it is possible to study groups and their representations, to display the character tables, to find the subgroups of larger groups, to identify groups given through their generating matrices, and so on.

GAP allows access to the SmallGroups library [30]. This library contains all the finite groups of order less than 2001, but for order 1024—because there are many thousands of millions of groups of order 1024. SmallGroups also contains some groups for some specific orders larger than 2000. In SmallGroups the groups are ordered by their orders; for each order, the complete list of non-isomorphic groups is given. Each discrete group of order smaller than 2001 is labeled $[O, n]$ by SmallGroups, where $O \leq 2000$ is the order of the group and $n \in \mathbb{N}$ is an integer that distinguishes between the non-isomorphic groups of the same order.

3.3 Procedure

We have surveyed all the discrete groups of order $O \leq 2000$ in the SmallGroups library, except the groups of order either 512, 1024, or 1536.¹³ We have discarded all the groups that are isomorphic to the direct product of a smaller (i.e., of lower order) group and a cyclic group.¹⁴ We have used GAP to find out all the irreps of each remaining group, and then to ascertain whether those irreps are faithful or not. We have discarded all the groups that do not have any firrep.¹⁵ We have thus obtained 87 349 non-isomorphic groups, which are all listed in our tables available at the site <https://github.com/jurciukonis/GAP-group-search>. We have looked only at the firreps of each group; non-faithful irreps, and all reducible representations, were neglected. We have computed the determinants of the matrices of each firrep in order to find out whether all those determinants are 1 or not. We have also looked for all the matrices in the firreps that

¹³Rather exceptionally, we have included in our search four groups of order 1536 that are known to have 3D firreps, according to our previous paper [31].

¹⁴SmallGroups itself informs us about the structure of each group, namely, whether it is isomorphic to the direct product of smaller groups. We have found that there are, however, at least two exceptions. One of them is the group with SmallGroups identifier [180, 19]; SmallGroups informs us that this is the group $GL(2, 4)$ but omits the well-known fact that $GL(2, 4)$ is isomorphic to $\mathbb{Z}_3 \times A_5$, where A_5 is the group of the even permutations of five objects. (Thus, [180, 19] is discarded in our search, because it is the direct product of A_5 and the cyclic group \mathbb{Z}_3 .) The other exception is the dihedral groups D_O of order $O = 12 + 8p$, where p is an integer, namely, the groups D_{12} , D_{20} , D_{28} , and so on. (SmallGroups instead uses the notation $D_{O/2}$ for these groups; namely, it uses D_6 , D_{10} , D_{14} , and so on.) It is easy to check analytically that these specific D_O groups are isomorphic to $\mathbb{Z}_2 \times D_{O/2}$, but SmallGroups omits this fact. We have used a method, suggested to us by Gábor Horváth, to check whether *any* group $[O, n]$ is a direct product of smaller groups, namely the succession of GAP commands $G := \text{SmallGroup}(O, n)$ and $\text{ListX}(\text{DirectFactorsOfGroup}(G), \text{StructureDescription})$.

¹⁵There are groups for which all the faithful representations are reducible. An example is the group formed by the 32 matrices

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix},$$

where a, b, c , and d may be either 1 or -1 . This group—with SmallGroups identifier [32, 27] and structure $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ —has eight inequivalent singlet irreps and six inequivalent doublet irreps, but all of them are unfaithful. The defining representation of this group is, by definition, faithful, but it is reducible.

are proportional to the unit matrix and we have checked that those matrices form a group \mathbb{Z}_N for some integer N (which for some groups is just 1).

We have tried to answer the questions on how the following three integers are related:

- (1) The integer O that is the order of the discrete group G .
- (2) The integer N corresponding to the group \mathbb{Z}_N that is the center of G .
- (3) The integer D such that G has one or more firreps of dimension D .

We have also examined the question whether each D -dimensional firrep is equivalent to a representation through matrices of $SU(D)$.¹⁶

There are relatively few groups that have firreps with different dimensions. (For instance, A_5 has firreps of dimensions three, four, and five. On the other hand, the group $\Sigma(36 \times 3)$, that has SmallGroups identifier [108, 15], has irreps of dimensions 1, 3, and 4, but the 1D and 4D irreps are unfaithful—all the firreps have dimension 3.) We have found just 2787 such discrete groups, out of the total 87 349 groups that we have surveyed; they are collected in the table entitled Intersections at <https://github.com/jurciukonis/GAP-group-search>.

Computing time: The scan over the SmallGroups library to find the firreps of all possible dimensions constituted a computationally very expensive task. Our computations with GAP took about three months. It is difficult to estimate the total number of CPU hours (CPUH) spent in the computations, because various computers with different CPUs were used. Most of the time was consumed in the computation of the irreps of the groups. For example, the computation for group [1320, 15], took about 320 CPUH running on an Intel Xeon CPU @ 1.60 GHz or about 46 CPUH in the newer Intel i9-10850K CPU @ 3.60 GHz. Also, some groups of orders 1728 and 1920 require quite a few CPUH to find the irreps. Orders 768, 1280, and 1792 have more than one million non-isomorphic groups of each order and therefore require many CPUH to scan over all of them.

Example: The discrete group $GL(2, 3)$ has order 48 and SmallGroups identifier [48, 29]. By definition, it is the group generated by three transformations a , c , and d that satisfy (see Ref. [32])

$$a^4 = c^3 = d^2 = (cd)^2 = e, \quad (25a)$$

$$b^2 = a^2, \quad (25b)$$

$$b^3 = dad, \quad (25c)$$

$$bab^{-1} = dbd = a^{-1}, \quad (25d)$$

$$b = a^{-1}cac^{-1}, \quad (25e)$$

¹⁶All the irreps of discrete groups are equivalent to representations through unitary matrices, and therefore we know that the generators that GAP provides to us are equivalent to unitary generators, even though GAP often gives them in a non-unitary version. In order to know whether the generators belong to $SU(D)$ we just compute their determinants.

where e is the identity transformation and $b \equiv c^{-1}ac$. There is a faithful representation of these transformations through 2×2 unitary matrices:

$$a \mapsto \frac{1}{3} \begin{pmatrix} i\sqrt{3} & \sqrt{6}\omega \\ -\sqrt{6}\omega^2 & -i\sqrt{3} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (26)$$

The first two matrices (26) have determinant 1 while the third one has determinant -1 ; hence, we classify $GL(2, 3)$ as a subgroup of $U(2)$, but it is not a subgroup of $SU(2)$. On the other hand, there is another faithful irrep of $GL(2, 3)$, through 4×4 unitary matrices, all of them with determinant 1:

$$a \mapsto \frac{1}{9} \begin{pmatrix} -3\sqrt{3}i & 0 & 6i & -3\sqrt{2} \\ 0 & 3\sqrt{3}i & 3\sqrt{2} & -6i \\ 6i & -3\sqrt{2} & i\sqrt{3} & -2\sqrt{6} \\ 3\sqrt{2} & -6i & 2\sqrt{6} & -i\sqrt{3} \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 0 & -\omega^2 & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (27)$$

and $c \mapsto \text{diag}(\omega, \omega^2, 1, 1)$. Therefore, we classify $GL(2, 3)$ as a subgroup of both $U(2)$ and $SU(4)$, but $GL(2, 3)$ earns these two classifications *through different irreps*.

3.4 The discrete subgroups of $U(3)$ and $SU(3)$

In Ref. [31] a classification of the discrete subgroups of $SU(3)$ and of the discrete subgroups of $U(3)$ that are not subgroups of $SU(3)$ has been provided. All those subgroups were classified according to their generators and structures. In this subsection we give relations between the integer N characterizing the center \mathbb{Z}_N of each group and the identifiers of the group series defined in Ref. [31].

As explained in Sects. 2.2 and 3.1, the finite subgroups of $SU(3)$ can only have either trivial center or center \mathbb{Z}_3 ; thus, they have either $N = 1$ or $N = 3$. Explicitly, they have the following values of N :

- The groups $\Delta(3n^2)$, $\Delta(6n^2)$, and $C_{m,n}^{(k)}$ have $N = 1$ when n cannot be divided by three, and $N = 3$ when n is a multiple of three.
- The groups $D_{3,l}^{(1)}$ have $N = 3$.
- The exceptional groups $\Sigma(60)$ and $\Sigma(168)$ have $N = 1$.
- The exceptional groups $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$, and $\Sigma(360 \times 3)$ have $N = 3$.

The series of finite subgroups of $U(3)$ that are not subgroups of $SU(3)$ constructed in Ref. [31] have the following values of N :

- The groups $Y(m, j)$, $L(m)$, $J(m)$, and $\Theta(m)$ have $N = 3^m$.
- The groups $T_r^{(k)}(m)$, $\Delta(3n^2, m)$, $L_r^{(k)}(n, m)$, $P_r^{(k)}(m)$, $Q_r^{(k)}(m)$, $Q_r^{(k)'}(m)$, $X(m)$, $S_r^{(k)}(m)$, $S_r^{(k)'}(m)$, $Y_r^{(k)}(m)$, $V_r^{(k)}(m)$, $W(n, m)$, $Z(n, m)$, $Z'(n, m)$, $Z''(n, m)$, $\Upsilon(m)$, $\Upsilon'(m)$, and $\Omega(m)$ have $N = 3^{m-1}$.
- The groups $M_r^{(k)}$, $M_r^{(k)'}$, $J_r^{(k)}$, $Y(j)$, $\tilde{Y}(j)$, $V(j)$, and $D(j)$ have $N = 3$.
- The groups $U(n, m, j)$ have $N = 3^j$.
- The groups $S_4(j)$ have $N = 2^{j-1}$.
- The groups $\Delta(6n^2, j)$ have $N = 2^{j-1}$ when n cannot be divided by three, and $N = 3 \times 2^{j-1}$ when n is a multiple of three.
- The groups $\Delta'(6n^2, m, j)$, $H(n, m, j)$, $G(m, j)$, $\hat{\Xi}(m, j)$, and $\Pi(m, j)$ have $N = 3^m 2^{j-1}$.
- The groups $Z(n, m, j)$ and $Z'(n, m, j)$ have $N = 3^{m-1} 2^{j-1}$.
- The groups $\Xi(m, j)$ have $N = 3^m 2^{j-2}$.

In Ref. [31] a few more subgroups of $U(3)$ that are not subgroups of $SU(3)$ are mentioned, which could not be classified into any series. Their values of N are the following:

- The groups [729, 96], [729, 97], [729, 98], [1458, 663], [1458, 666], [1701, 130], and [1701, 131] have $N = 3$.
- The group [1296, 699] has $N = 6$.
- The groups [972, 170], [1701, 102], and [1701, 112] have $N = 9$.

4. Conclusions

In this paper we have pointed out that dark matter may be stabilized by a \mathbb{Z}_N cyclic group under which it has a non-trivial charge—contrary to standard matter, which is invariant under that \mathbb{Z}_N —and that that \mathbb{Z}_N may be the center of the larger internal-symmetry group G of nature, while G is not a direct product $\mathbb{Z}_N \times G'$. Thereafter we have performed an extensive and computationally very time-consuming search for the centers of discrete groups that cannot be written in the form $\mathbb{Z}_N \times G'$ and that have faithful irreducible representations. The following are our conclusions:

- We have found groups with centers \mathbb{Z}_N for $N \leq 162$.
- We have found groups with $N = 2^p \times 3^q$ for *all* the integers p and q such that $N \leq 162$.
- We have found groups with $N = 2^p \times 5$ for $0 \leq p \leq 3$.
- We have also found groups with $N = 7$, $N = 11$, $N = 14$, $N = 15$, and $N = 25$.
- The number N always divides the order O of the group. The integer O/N always has at least two prime factors; we have found groups with $O/N = 4, 6, 8, 9, 10, 12, 14$, and so on.

In the cases of some smallish groups, we have explicitly computed the way in which an element g of order¹⁷ N belonging to the \mathbb{Z}_N center of the group is represented in the various irreps. We have found that the sum of the squares of the dimensions of the irreps where g is represented by any N th root of unity times the unit matrix is always equal to O/N .

Acknowledgements

L.L. thanks Salvador Centelles Chuliá for a discussion on his paper [22]. We thank the GAP support team, specifically Bill Alombert, Stefan Kohl, and Gábor Horváth for helping us solve a problem with that software. D.J. thanks the Lithuanian Academy of Sciences for financial support through project DaFi2021. L.L. thanks the Portuguese Foundation for Science and Technology for support through projects UIDB/00777/2020 and UIDP/00777/2020, and also CERN/FIS-PAR/0004/2019, CERN/FIS-PAR/0008/2019, and CERN/FIS-PAR/0002/2021.

Funding

Open Access funding: SCOAP³.

References

- 1 E. Ma, Phys. Lett. B **662**, 49 (2008) [[arXiv:0708.3371 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 2 I. P. Ivanov and V. Keus, Phys. Rev. D **86**, 016004 (2012) [[arXiv:1203.3426 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 3 M. Aoki and T. Toma, J. Cosmol. Astropart. Phys. **1409**, 016 (2014) [[arXiv:1405.5870 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 4 G. Bélanger, K. Kannike, A. Pukhov, and M. Raidal, J. Cosmol. Astropart. Phys. **1301**, 022 (2013) [[arXiv:1211.1014 \[hep-ph\]](#)] [[Search inSPIRE](#)].

¹⁷The order of an element g is the smallest integer o such that g^o is the unit element of the group.

- 5 R. Ding, Z.-L. Han, Y. Liao, and W.-P. Xie, *J. High Energy Phys.* **1605**, 030 (2016) [[arXiv:1601.06355 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 6 C.-W. Chiang and B.-Q. Lu, *J. High Energy Phys.* **2007**, 082 (2020) [[arXiv:1912.12634 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 7 K. Kannike, K. Loos, and M. Raidal, *Phys. Rev. D* **101**, 035001 (2020) [[arXiv:1907.13136 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 8 A. Hektor, A. Hryczuk, and K. Kannike, *J. High Energy Phys.* **1903**, 204 (2019) [[arXiv:1901.08074 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 9 P. Ko and Y. Tang, *Phys. Lett. B* **815**, 136181 (2021) [[arXiv:2006.15822 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 10 C. E. Yaguna and O. Zapata, *J. High Energy Phys.* **2110**, 185 (2021) [[arXiv:2106.11889 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 11 C. E. Yaguna and O. Zapata, *Phys. Rev. D* **105**, 095026 (2022) [[arXiv:2112.07020 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 12 S.-Y. Ho, P. Ko, and C.-T. Lu, *J. High Energy Phys.* **2203**, 005 (2022) [[arXiv:2201.06856 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 13 G. Bélanger, A. Pukhov, C. E. Yaguna, and O. Zapata, *J. High Energy Phys.* **2009**, 030 (2020) [[arXiv:2006.14922 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 14 H. K. Dreiner, C. Luhn, and M. Thormeier, *Phys. Rev. D* **73**, 075007 (2006) [[arXiv:hep-ph/0512163](#)] [[Search inSPIRE](#)].
- 15 B. Batell, *Phys. Rev. D* **83**, 035006 (2011) [[arXiv:1007.0045 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 16 G. Bélanger, K. Kannike, A. Pukhov, and M. Raidal, *J. Cosmol. Astropart. Phys.* **1204**, 010 (2012) [[arXiv:1202.2962 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 17 G. Bélanger, K. Kannike, A. Pukhov, and M. Raidal, *J. Cosmol. Astropart. Phys.* **1406**, 021 (2014) [[arXiv:1403.4960 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 18 D. Aristizabal Sierra, M. Dhen, C. S. Fong, and A. Vicente, *Phys. Rev. D* **91**, 096004 (2015) [[arXiv:1412.5600 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 19 C. E. Yaguna and O. Zapata, *J. High Energy Phys.* **2003**, 109 (2020) [[arXiv:1911.05515 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 20 C. Bonilla, S. Centelles-Chuliá, R. Cepedello, E. Peinado, and R. Srivastava, *Phys. Rev. D* **101**, 033011 (2020) [[arXiv:1812.01599 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 21 S. Centelles-Chuliá, R. Cepedello, E. Peinado, and R. Srivastava, *Chin. Phys. C* **44**, 083110 (2020) [[arXiv:1901.06402 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 22 S. Centelles Chuliá, R. Cepedello, and O. Medina, *J. High Energy Phys.* **2210**, 080 (2022) [[arXiv:2204.12517 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 23 E. Ma, *Phys. Lett. B* **649**, 287 (2007) [[arXiv:hep-ph/0612022](#)] [[Search inSPIRE](#)].
- 24 H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010) [[arXiv:1003.3552 \[hep-th\]](#)] [[Search inSPIRE](#)].
- 25 L. Lavoura, S. Morisi, and J. W. F. Valle, *J. High Energy Phys.* **2013**, 118 (2013) [[arXiv:1205.3442 \[hep-ph\]](#)] [[Search inSPIRE](#)].
- 26 J. P. Elliott and P. G. Dawber, *Symmetry in Physics, Volume 1: Principles and Simple Applications* (Macmillan Press, London, 1979), p. 55.
- 27 G. Khanna, S. Mukhopadhyay, R. Simon, and N. Mukunda, *Annals of Physics*, **253**, 1, 55 (1997) (available at: <https://core.ac.uk/download/pdf/291518359.pdf>).
- 28 T. Dokchitser, Group C4.D4, (available at: <https://people.maths.bris.ac.uk/~matyd/GroupNames/1/C4.D4.html>).
- 29 The GAP Group, GAP—groups, algorithms, programming—a system for computational discrete algebra. Version 4.11.1; 2022, (available at: <https://www.gap-system.org>).
- 30 E. B. Besche H. U. and E. A. O’Brien, The SmallGroups library. Version 1.5.1; 2022, (available at: <https://www.gap-system.org/Packages/smallgrp.html>).
- 31 D. Jurčiukonis and L. Lavoura, *Prog. Theor. Exp. Phys.* **2017**, 053A03 (2017) [[arXiv:1702.00005 \[math.RT\]](#)] [[Search inSPIRE](#)].
- 32 T. Dokchitser, Group GL(2, 3) (available at: [https://people.maths.bris.ac.uk/~matyd/GroupNames/1/GL\(2,3\).html](https://people.maths.bris.ac.uk/~matyd/GroupNames/1/GL(2,3).html)).