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Random convolution of inhomogeneous distributions with $\mathcal{O}$-exponential tail

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Abstract Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (not necessarily identically distributed), and η be a counting random variable independent of this sequence. We obtain sufficient conditions on $\{\xi_1, \xi_2, \ldots\}$ and η under which the distribution function of the random sum $S_\eta = \xi_1 + \xi_2 + \cdots + \xi_\eta$ belongs to the class of $\mathcal{O}$-exponential distributions.

Keywords Heavy tail, exponential tail, $\mathcal{O}$-exponential tail, random sum, random convolution, inhomogeneous distributions, closure property
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1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s) $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v. η and r.v.s $\{\xi_1, \xi_2, \ldots\}$ are independent. Let $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $n \in \mathbb{N}$, be the partial sums, and let

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$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the random sum of $\{\xi_1, \xi_2, \ldots\}$.

We are interested in conditions under which the d.f. of S_η

$$F_{S_\eta}(x) = \mathbb{P}(S_\eta \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(\eta = n)\mathbb{P}(S_n \leqslant x)$$

belongs to the class of $\mathcal{O}$-*exponential* distributions.

According to Albin and Sunden [1] or Shimura and Watanabe [15], *a d.f. F belongs to the class of $\mathcal{O}$-exponential distributions $\mathcal{OL}$ if*

$$0 < \liminf_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} < \infty$$

for all $a \in \mathbb{R}$, where $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$, is the tail of a d.f. F.

Note that if $F \in \mathcal{OL}$, then $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$.

It is obvious that a d.f. F belongs to the class $\mathcal{OL}$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty \tag{1}$$

or, equivalently, if and only if

$$\sup_{x\geqslant 0} \frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty.$$

The last condition shows that class $\mathcal{OL}$ is quite wide. We further describe some more popular subclasses of $\mathcal{OL}$ for which we will present some results on the random convolution of distributions from these subclasses.

A d.f. F is said to belong to the class $\mathcal{L}$ of long-tailed d.f.s if for every fixed $a > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = 1.$$

A d.f. F is said to belong to the class $\mathcal{L}(\gamma)$ of exponential distributions with some $\gamma > 0$ if for any fixed $a > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x+a)}{\overline{F}(x)} = \mathrm{e}^{-a\gamma}.$$

A d.f. F belongs to the class $\mathcal{D}$ (or *has a dominatingly varying tail*) *if for every fixed $a \in (0, 1)$, we have*

$$\limsup_{x\to\infty} \frac{\overline{F}(xa)}{\overline{F}(x)} < \infty.$$

A d.f. F supported on the interval $[0, \infty)$ *belongs to the class $\mathcal{S}$* (or *is subexponential*) *if*

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2,$$

where, as usual, $*$ denotes the convolution of d.f.s.

A d.f. F supported on the interval $[0,\infty)$ belongs to the class $\mathcal{S}^*$ (or is strongly subexponential) if

$$\mu := \int\limits_{[0,\infty)} x\mathrm{d}F(x) < \infty \quad \text{and} \quad \int\limits_0^x \overline{F}(x-y)\overline{F}(y)\mathrm{d}y \underset{x\to\infty}{\sim} 2\mu\overline{F}(x).$$

If a d.f. F is supported on $\mathbb{R}$, then F belongs to some of the classes $\mathcal{S}$ or $\mathcal{S}^*$ if $F^+(x) = F(x)\mathbb{1}_{\{[0,\infty)\}}(x)$ belongs to the corresponding class.

The presented definitions, together with Lemma 2 of Chistyakov [2], Lemma 9 of Denisov et al. [5], Lemma 1.3.5(a) of Embrechts et al. [9], and Lemma 1 of Kaas and Tang [11], imply that

$$\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{OL},\ \mathcal{D} \subset \mathcal{OL},\quad \bigcup_{\gamma>0} \mathcal{L}(\gamma) \subset \mathcal{OL}.$$

Now we present a few known results on when the d.f. F_{S_η} belongs to some class. The first result about subexponential distributions was proved by Embrechts and Goldie (Theorem 4.2 in [8]) and Cline (Theorem 2.13 in [3]).

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. ξ with subexponential d.f. F_ξ. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{E}(1+\delta)^\eta < \infty$ for some $\delta > 0$, then $F_{S_\eta} \in \mathcal{S}$.*

In the case of strongly subexponential d.f.s, the following result, which involves weaker restrictions on the r.v. η, can be derived from Theorem 1 of Denisov et al. [6] and Corollary 2.36 of Foss et al. [10].

Theorem 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent copies of a nonnegative r.v. ξ with strongly subexponential d.f. F_ξ and finite mean $\mathbb{E}\xi$. Let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. If $\mathbb{P}(\eta > x/c) \underset{x\to\infty}{=} o(\overline{F}_\xi(x))$ for some $c > \mathbb{E}\xi$, then $F_{S_\eta} \in \mathcal{S}^*$.*

Similar results for classes $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{OL}$ can be found in the papers of Leipus and Šiaulys [12] and Danilenko and Šiaulys [4]. We further present Theorem 6 from [12].

Theorem 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s with common d.f. $F_\xi \in \mathcal{L}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$ having d.f. F_η. If $\overline{F}_\eta(\delta x) \underset{x\to\infty}{=} o(\sqrt{x}\,\overline{F}_\xi(x))$ for each $\delta \in (0,1)$, then $F_{S_\eta} \in \mathcal{L}$.*

In all presented results, r.v.s $\{\xi_1, \xi_2, \ldots\}$ are identically distributed. In this work, we consider independent, but not necessarily identically distributed, r.v.s. As was noted, we restrict our consideration on the class $\mathcal{OL}$. In fact, in this paper, we generalize the results of [4]. If $\{\xi_1, \xi_2, \ldots\}$ may be not identically distributed, then various collections of conditions on r.v.s $\{\xi_1, \xi_2, \ldots\}$ and η imply that $F_{S_\eta} \in \mathcal{OL}$. The rest of the paper is organized as follows. In Section 2, we formulate our main results. In Section 3, we present all auxiliary assertions, and the detailed proofs of the main results are presented in Section 4. Finally, a few examples of $\mathcal{O}$-exponential random sums are described in Section 5.

2 Main results

In this section, we formulate our main results. The first result describes the situation where the tails of d.f.s F_{ξ_k} for large indices k are uniformly comparable with itself at the points x and $x-1$ for all $x \in [0,\infty)$.

Theorem 4. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent nonnegative random variables with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{OL}$ if the following three conditions are satisfied.*

- *For some $\kappa \in \operatorname{supp}(\eta) \setminus \{0\} = \{n \in \mathbb{N} : \mathbb{P}(\eta = n) > 0\}$, $F_{\xi_\kappa} \in \mathcal{OL}$.*
- *For each $k \in \operatorname{supp}(\eta)$, $k \leqslant \kappa$, either $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ or $F_{\xi_k} \in \mathcal{OL}$.*
- $\sup\limits_{x\geqslant 0}\sup\limits_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} < \infty$.

Since each d.f. from the class $\mathcal{OL}$ is comparable with itself, the next assertion follows immediately from Theorem 4.

Corollary 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent nonnegative random variables with common d.f. $F_\xi \in \mathcal{OL}$. Then the d.f. of random sum F_{S_η} is $\mathcal{O}$-exponential for an arbitrary counting r.v. η.*

Our second main assertion is dealt with counting r.v.s having finite support.

Theorem 5. *Let $\{\xi_1, \xi_2, \ldots, \xi_D\}$, $D \in \mathbb{N}$, be independent nonnegative random variables with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots F_{\xi_D}\}$, and let η be a counting r.v. independent of $\{\xi_1, \xi_2, \ldots, \xi_D\}$. Then $F_{S_\eta} \in \mathcal{OL}$ under the following three conditions.*

- $\mathbb{P}(\eta \leqslant D) = 1$.
- *For some $\kappa \in \operatorname{supp}(\eta) \setminus \{0\}$, $F_{\xi_\kappa} \in \mathcal{OL}$.*
- *For each $k \in \{1, 2, \ldots, D\}$, either $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ or $F_{\xi_k} \in \mathcal{OL}$.*

Our last main assertion describes the case where the tails of d.f.s F_{ξ_k} are comparable at x and $x-1$ asymptotically and uniformly with respect to large indices k. In this case, conditions are more restrictive for a counting r.v.

Theorem 6. *Let $\{\xi_1, \xi_2, \ldots\}$ be independent nonnegative random variables with d.f.s $\{F_{\xi_1}, F_{\xi_2}, \ldots\}$, and let η be a counting r.v. d.f. F_η independent of $\{\xi_1, \xi_2, \ldots\}$. Then $F_{S_\eta} \in \mathcal{OL}$ if the following five conditions are satisfied.*

- *For some $\kappa \in \operatorname{supp}(\eta) \setminus \{0\}$, $F_{\xi_\kappa} \in \mathcal{OL}$.*
- *For each $k \in \operatorname{supp}(\eta)$, $k \leqslant \kappa$, either $\lim\limits_{x\to\infty} \frac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0$ or $F_{\xi_k} \in \mathcal{OL}$.*
- $\limsup\limits_{x\to\infty}\sup\limits_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} < \infty$.
- $\limsup\limits_{k\to\infty} \frac{1}{k}\sum_{l=1}^{k} \sup\limits_{x\geqslant 0}(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x)) < 1$.
- *For each $\delta \in (0,1)$, $\overline{F}_\eta(\delta x) = O(\sqrt{x}\,\overline{F}_{\xi_\kappa}(x))$.*

3 Auxiliary lemmas

In this section, we present all assertions that we use in the proofs of our main results. We present some of auxiliary results with proofs. The first assertion can be found in [7] (see Eq. (2.12)).

Lemma 1. *Let F and G be two d.f.s satisfying $\overline{F}(x) > 0$, $\overline{G}(x) > 0$, $x \in \mathbb{R}$. Then*

$$\frac{\overline{F * G}(x-t)}{\overline{F * G}(x)} \leqslant \max\left\{ \sup_{y \geqslant v} \frac{\overline{F}(y-t)}{\overline{F}(y)}, \sup_{y \geqslant x-v+t} \frac{\overline{G}(y-t)}{\overline{G}(y)} \right\}$$

for all $x \in \mathbb{R}$, $v \in \mathbb{R}$, and $t > 0$.

The following assertion is the well-known Kolmogorov–Rogozin inequality for concentration functions. Recall that the Lévy concentration function or simply concentration function of a r.v. X is the function

$$Q_X(\lambda) = \sup_{x \in \mathbb{R}} \mathbb{P}(x \leqslant X \leqslant x + \lambda), \quad \lambda \in [0, \infty).$$

The proof of the next lemma can be found in [14] (Theorem 2.15).

Lemma 2. *Let $X_1, X_2, \ldots, X_n$ be independent r.v.s, and let $Z_n = \sum_{k=1}^n X_k$. Then, for all $n \in \mathbb{N}$,*

$$Q_{Z_n}(\lambda) \leqslant A\lambda \left\{ \sum_{k=1}^{n} \lambda_k^2 \big(1 - Q_{X_k}(\lambda_k)\big) \right\}^{-1/2},$$

where A is an absolute constant, and $0 < \lambda_k \leqslant \lambda$ for each $k \in \{1, 2, \ldots, n\}$.

The following assertion describes sufficient conditions under which the d.f. of two independent r.v.s belongs to the class $\mathcal{OL}$.

Lemma 3. *Let X_1 and X_2 be independent r.v.s with d.f.s F_{X_1} and F_{X_2}, respectively. Then the d.f. $F_{X_1} * F_{X_2}$ of the sum $X_1 + X_2$ is $\mathcal{O}$-exponential if $F_{X_1} \in \mathcal{OL}$ and one of the following two conditions holds:*

- $$\lim_{x \to \infty} \frac{\overline{F}_{X_2}(x)}{\overline{F}_{X_1}(x)} = 0, \tag{2}$$
- $F_{X_2} \in \mathcal{OL}$.

Proof. We split the proof into three parts.

I. First, suppose that $\mathbb{P}(X_2 \leqslant D) = 1$ for some $D > 0$. In this case, condition (2) holds evidently.

For each real x, we have

$$\overline{F_{X_1} * F_{X_2}}(x) = \mathbb{P}(X_1 + X_2 > x) = \int\limits_{(-\infty, D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y).$$

Hence, for such x,

$$\begin{aligned}
\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} &= \frac{\int_{(-\infty,D]} \overline{F}_{X_1}(x-1-y) \frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)} \mathrm{d}F_{X_2}(y)}{\int_{(-\infty,D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)} \\
&\leqslant \frac{\int_{(-\infty,D]} \sup_{y \leqslant D} \frac{\overline{F}_{X_1}(x-1-y)}{\overline{F}_{X_1}(x-y)} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)}{\int_{(-\infty,D]} \overline{F}_{X_1}(x-y) \mathrm{d}F_{X_2}(y)} \\
&= \sup_{z \geqslant x-D} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}.
\end{aligned}$$

This estimate implies that

$$\begin{aligned}\limsup_{x\to\infty}\frac{\overline{F_{X_1}*F_{X_2}}(x-1)}{\overline{F_{X_1}*F_{X_2}}(x)} &\leqslant \limsup_{x\to\infty}\sup_{z\geqslant x-D}\frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}\\ &=\limsup_{y\to\infty}\frac{\overline{F}_{X_1}(y-1)}{\overline{F}_{X_1}(y)}\\ &<\infty\end{aligned}$$

because $F_{X_1}\in\mathcal{OL}$. So, $F_{X_1}*F_{X_2}\in\mathcal{OL}$ as well.

II. Now let us consider the case where condition (2) holds but $\overline{F}_{X_2}(x)>0$ for all $x\in\mathbb{R}$. For each real x, we have

$$\overline{F_{X_1}*F_{X_2}}(x)=\int_{-\infty}^{\infty}\overline{F_{X_1}}(x-y)\mathrm{d}F_{X_2}(y).$$

Therefore,

$$\begin{aligned}\overline{F_{X_1}*F_{X_2}}(x-1)&=\Bigg(\int\limits_{(-\infty,x-M]}+\int\limits_{(x-M,\infty)}\Bigg)\overline{F}_{X_1}(x-1-y)\mathrm{d}F_{X_2}(y)\\ &\leqslant\int\limits_{(-\infty,x-M]}\overline{F}_{X_1}(x-1-y)\frac{\overline{F}_{X_1}(x-y)}{\overline{F}_{X_1}(x-y)}\mathrm{d}F_{X_2}(y)+\overline{F}_{X_2}(x-M)\\ &\leqslant\sup_{z\geqslant M}\frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}\int\limits_{(-\infty,x-M]}\overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)+\overline{F}_{X_2}(x-M)\end{aligned}$$

for all M,x such that $0<M<x-1$. In addition, for such M and x, we obtain

$$\overline{F_{X_1}*F_{X_2}}(x)\geqslant\int\limits_{(-\infty,x-M]}\overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y),$$

$$\begin{aligned}\overline{F_{X_1}*F_{X_2}}(x)&\geqslant\int\limits_{(M,\infty)}\overline{F}_{X_1}(x-y)\mathrm{d}F_{X_2}(y)\\ &\geqslant\overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M).\end{aligned}$$

The obtained estimates imply that

$$\frac{\overline{F_{X_1}*F_{X_2}}(x-1)}{\overline{F_{X_1}*F_{X_2}}(x)}\leqslant\sup_{z\geqslant M}\frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}+\frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)\overline{F}_{X_2}(M)}$$

for all x and M such that $0<M<x-1$. Consequently,

$$\limsup_{x\to\infty}\frac{\overline{F_{X_1}*F_{X_2}}(x-1)}{\overline{F_{X_1}*F_{X_2}}(x)}\leqslant\sup_{z\geqslant M}\frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}+\frac{1}{\overline{F}_{X_2}(M)}\limsup_{x\to\infty}\frac{\overline{F}_{X_2}(x-M)}{\overline{F}_{X_1}(x-M)}$$

$$= \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}$$

for all positive M. Therefore,

$$\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)} < \infty$$

because F_{X_1} is $\mathcal{O}$-exponential. Consequently, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ by (1).

III. It remains to prove the assertion when both d.f.s F_{X_1} and F_{X_2} are $\mathcal{O}$-exponential. By Lemma 1 we have

$$\frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)} \right\}$$

for all x and M such that $0 < M < x - 1$. Therefore, for every positive M,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\quad\leqslant \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{x\to\infty} \sup_{z\geqslant x-M+1} \frac{\overline{F}_{X_2}(z-1)}{\overline{F}_{X_2}(z)} \right\} \\
&\quad= \max\left\{ \sup_{z\geqslant M} \frac{\overline{F}_{X_1}(z-1)}{\overline{F}_{X_1}(z)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)} \right\}.
\end{aligned}$$

Letting M tend to infinity, we get that

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\overline{F_{X_1} * F_{X_2}}(x-1)}{\overline{F_{X_1} * F_{X_2}}(x)} \\
&\quad\leqslant \max\left\{ \limsup_{M\to\infty} \frac{\overline{F}_{X_1}(M-1)}{\overline{F}_{X_1}(M)}, \limsup_{y\to\infty} \frac{\overline{F}_{X_2}(y-1)}{\overline{F}_{X_2}(y)} \right\} < \infty
\end{aligned}$$

because F_{X_1} and F_{X_2} belong to class $\mathcal{OL}$. Consequently, $F_{X_1} * F_{X_2} \in \mathcal{OL}$ due to requirement (1). Lemma 3 is proved. □

Lemma 4. *Let $\{X_1, X_2, \dots, X_n\}$ be independent nonnegative r.v.s with d.f.s $\{F_{X_1}, F_{X_2}, \dots, F_{X_n}\}$. Let $F_{X_1} \in \mathcal{OL}$ and suppose that, for each $k \in \{2, 3, \dots, n\}$, either $\lim_{x\to\infty} \frac{\overline{F}_{X_k}(x)}{\overline{F}_{X_1}(x)} = 0$ or $F_{X_k} \in \mathcal{OL}$. Then the d.f. $F_{X_1} * F_{X_2} * \cdots * F_{X_n}$ belongs to the class $\mathcal{OL}$.*

Proof. We use induction on n. If $n = 2$, then the statement follows from Lemma 3. Suppose that the statement holds if $n = m$, that is, $F_{X_1} * F_{X_2} * \cdots * F_{X_m} \in \mathcal{OL}$, and we will show that the statement is correct for $n = m + 1$.

Conditions of the lemma imply that $F_{X_{m+1}} \in \mathcal{OL}$ or

$$\lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F_{X_1} * F_{X_2} * \cdots * F_{X_m}}(x)} = \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 + \cdots + X_m > x)}$$

$$\leqslant \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\mathbb{P}(X_1 > x)} = \lim_{x\to\infty} \frac{\overline{F}_{X_{m+1}}(x)}{\overline{F}_{X_1}(x)} = 0.$$

So, using Lemma 3 again, we get

$$F_{X_1} * F_{X_2} * \cdots * F_{X_{m+1}} = (F_{X_1} * F_{X_2} * \cdots * F_{X_m}) * F_{X_{m+1}} \in \mathcal{OL}.$$

We see that the statement of the lemma holds for $n = m + 1$ and, consequently, by induction, for all $n \in \mathbb{N}$. The lemma is proved. □

4 Proofs of the main results

In this section, we present proofs of our main results.

Proof of Theorem 4. Conditions of Theorem and Lemma 4 imply that the d.f. $F_{S_\kappa}(x) = \mathbb{P}(S_\kappa \leqslant x)$ belongs to the class $\mathcal{OL}$. So, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} < \infty \tag{3}$$

or, equivalently,

$$\sup_{x\geqslant 0} \frac{\overline{F}_{S_\kappa}(x-1)}{\overline{F}_{S_\kappa}(x)} \leqslant c_1 \tag{4}$$

for some positive constant c_1.

We observe that, for all $x \geqslant 0$,

$$\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} = \mathcal{J}_1(x) + \mathcal{J}_2(x), \tag{5}$$

where

$$\mathcal{J}_1(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta \leqslant \kappa)}{\mathbb{P}(S_\eta > x)},$$
$$\mathcal{J}_2(x) = \frac{\mathbb{P}(S_\eta > x-1, \eta > \kappa)}{\mathbb{P}(S_\eta > x)}.$$

Since $\kappa \in \operatorname{supp}(\eta)$, we obtain

$$\begin{aligned}\mathcal{J}_1(x) &= \frac{\sum_{n=0}^{\kappa} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum_{n=0}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)} \\ &\leqslant \frac{1}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)} \sum_{n=0}^{\kappa} \mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta = n) \\ &= \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \frac{\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(\eta = \kappa)}.\end{aligned}$$

Hence, it follows from (3) that

$$\limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \tag{6}$$

By Lemma 1 we have

$$\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)} \right\} \tag{7}$$

for all real x and M.

The third condition of the theorem implies that

$$\sup_{x\geqslant 0} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_2 \tag{8}$$

for all $k \in \mathbb{N}$ and some positive c_2.

If we choose $M = x/2$ in estimate (7), then, using (4), we get

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_1, c_2\} := c_3. \tag{9}$$

Applying Lemma 1 again, we obtain

$$\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{ \sup_{z\geqslant M} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z\geqslant x-M+1} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)} \right\}.$$

By choosing $M = x/2$ we get from inequalities (8) and (9) that

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_3.$$

Continuing the process, we find

$$\sup_{x\geqslant 0} \frac{\mathbb{P}(S_{\kappa+k} > x-1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_3$$

for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} \mathcal{J}_2(x) &= \frac{1}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa+k) \\ &\leqslant \frac{c_3}{\mathbb{P}(S_\eta > x)} \sum_{k=1}^{\infty} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa+k) \\ &\leqslant \frac{c_3 \mathbb{P}(S_\eta > x)}{\mathbb{P}(S_\eta > x)} = c_3 \end{aligned} \tag{10}$$

for all $x \geqslant 0$.

The obtained relations (5), (6), and (10) imply that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty.$$

Therefore, the d.f. F_{S_η} belongs to the class $\mathcal{OL}$ due to requirement (1). Theorem 4 is proved. □

Proof of Theorem 5. The statement of the theorem can be derived from Theorem 4 or proved directly. We present the direct proof of Theorem 5.

It is evident that $S_k = \xi_\kappa + \sum_{n=1,\ n\neq\kappa}^{k} \xi_n$ for each $k \geqslant \kappa$. Hence, by Lemma 4, $F_{S_k} \in \mathcal{OL}$ for all $\kappa \leqslant k \leqslant D$.

If $x \geqslant 1$, then we have

$$\begin{aligned}
\frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} &= \frac{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\sum_{\substack{n=1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)} \\
&\leqslant \frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa) + \sum_{\substack{n=\kappa+1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x-1)\mathbb{P}(\eta = n)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa) + \sum_{\substack{n=\kappa+1\\ n\in\mathrm{supp}(\eta)}}^{D} \mathbb{P}(S_n > x)\mathbb{P}(\eta = n)} \\
&\leqslant \max\left\{\frac{\mathbb{P}(S_\kappa > x-1)\mathbb{P}(\eta \leqslant \kappa)}{\mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa)},\ \max_{\substack{\kappa+1\leqslant n\leqslant D\\ n\in\mathrm{supp}(\eta)}} \frac{\mathbb{P}(S_n > x-1)}{\mathbb{P}(S_n > x)}\right\}, \qquad (11)
\end{aligned}$$

where in the last step we use the inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leqslant \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right\},$$

provided that $n \geqslant 1$ and $a_i, b_i > 0$ for $i \in \{1, 2, \ldots, n\}$.

Since $F_{S_n} \in \mathcal{OL}$ for all $n \geqslant \kappa$, we get from (11) that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_\eta > x-1)}{\mathbb{P}(S_\eta > x)} < \infty, \qquad (12)$$

and the statement of Theorem 5 follows. □

Proof of Theorem 6. As usual, it suffices to prove relation (12). If $x \geqslant 0$, then we have

$$\begin{aligned}
\mathbb{P}(S_\eta > x) &= \sum_{n=1}^{\infty} \mathbf{P}(S_n > x)\mathbb{P}(\eta = n) \\
&\geqslant \mathbb{P}(S_\kappa > x)\mathbb{P}(\eta = \kappa) \\
&\geqslant \overline{F}_{\xi_\kappa}(x)\mathbb{P}(\eta = \kappa). \qquad (13)
\end{aligned}$$

Similarly, for $K \geqslant 2$ and $x \geqslant 2K$,

$$\mathbb{P}(S_\eta > x-1) = \sum_{n=1}^{\kappa} \mathbf{P}(S_n > x-1)\mathbb{P}(\eta = n)$$

$$
\begin{aligned}
&+ \sum_{1 \leqslant k \leqslant (x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x-1)\mathbb{P}(\eta = \kappa + k) \\
&+ \sum_{k > (x-1)/(K-1)} \mathbf{P}(x-1 < S_{\kappa+k} \leqslant x)\mathbb{P}(\eta = \kappa + k) \\
&+ \sum_{k > (x-1)/(K-1)} \mathbf{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\
&:= \mathcal{K}_1(x) + \mathcal{K}_2(x) + \mathcal{K}_3(x) + \mathcal{K}_4(x). \qquad (14)
\end{aligned}
$$

The distribution function F_{S_κ} belongs to the class $\mathcal{OL}$ due to Lemma 4. So, by estimate (6) we have

$$
\limsup_{x\to\infty} \frac{\mathcal{K}_1(x)}{\mathbb{P}(S_\eta > x)} = \limsup_{x\to\infty} \mathcal{J}_1(x) < \infty. \qquad (15)
$$

Now we consider the sum $\mathcal{K}_2(x)$. Since F_{S_κ} is $\mathcal{O}$-exponential, we have

$$
\sup_{x \geqslant 0} \frac{\mathbb{P}(S_\kappa > x-1)}{\mathbb{P}(S_\kappa > x)} \leqslant c_4
$$

with some positive constant c_4. On the other hand, the third condition of Theorem 6 implies that

$$
\sup_{x \geqslant c_5} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \leqslant c_6
$$

for some constants $c_5 > 2$, $c_6 > 0$ and all $k \in \mathbb{N}$.

By Lemma 1 (with $v = c_5$) we have

$$
\frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\left\{ \sup_{z \geqslant x - c_5 + 1} \frac{\mathbb{P}(S_\kappa > z-1)}{\mathbb{P}(S_\kappa > z)}, \sup_{z \geqslant c_5} \frac{\overline{F}_{\xi_{\kappa+1}}(z-1)}{\overline{F}_{\xi_{\kappa+1}}(z)} \right\}.
$$

Consequently,

$$
\sup_{x \geqslant c_5} \frac{\mathbb{P}(S_{\kappa+1} > x-1)}{\mathbb{P}(S_{\kappa+1} > x)} \leqslant \max\{c_4, c_6\} := c_7.
$$

Applying Lemma 1 again for the sum $S_{\kappa+2} = S_{\kappa+1} + \xi_{\kappa+2}$ (with $v = x/2 + 1/2$), we get

$$
\frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant \max\left\{ \sup_{z \geqslant \frac{x}{2} + \frac{1}{2}} \frac{\mathbb{P}(S_{\kappa+1} > z-1)}{\mathbb{P}(S_{\kappa+1} > z)}, \sup_{z \geqslant \frac{x}{2} + \frac{1}{2}} \frac{\overline{F}_{\xi_{\kappa+2}}(z-1)}{\overline{F}_{\xi_{\kappa+2}}(z)} \right\}.
$$

If $x \geqslant 2(c_5 - 1) + 1$, then $x/2 + 1/2 \geqslant c_5$. Therefore, by the last inequality we obtain that

$$
\sup_{x \geqslant 2(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+2} > x-1)}{\mathbb{P}(S_{\kappa+2} > x)} \leqslant c_7.
$$

Applying Lemma 1 once again (with $v = x/3 + 2/3$), we get

$$
\frac{\mathbb{P}(S_{\kappa+3} > x-1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant \max\left\{ \sup_{z \geqslant \frac{2x}{3} + \frac{1}{3}} \frac{\mathbb{P}(S_{\kappa+2} > z-1)}{\mathbb{P}(S_{\kappa+2} > z)}, \sup_{z \geqslant \frac{x}{3} + \frac{2}{3}} \frac{\overline{F}_{\xi_{\kappa+3}}(z-1)}{\overline{F}_{\xi_{\kappa+3}}(z)} \right\}.
$$

If $x \geqslant 3(c_5 - 1) + 1$, then $2x/3 + 1/3 \geqslant 2(c_5 - 1) + 1$ and $x/3 + 2/3 \geqslant c_5$. So, the last estimate implies

$$\sup_{x \geqslant 3(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+3} > x - 1)}{\mathbb{P}(S_{\kappa+3} > x)} \leqslant c_7.$$

Continuing the process, we can get that

$$\sup_{x \geqslant k(c_5-1)+1} \frac{\mathbb{P}(S_{\kappa+k} > x - 1)}{\mathbb{P}(S_{\kappa+k} > x)} \leqslant c_7 \tag{16}$$

for all $k \in \mathbb{N}$.

We can suppose that $K = c_5$ in representation (14). In such a case, it follows from inequality (16) that

$$\begin{aligned} \limsup_{x\to\infty} \frac{\mathcal{K}_2(x)}{\mathbb{P}(S_\eta > x)} &\leqslant \limsup_{x\to\infty} \frac{c_7}{\mathbb{P}(S_\eta > x)} \sum_{1 \leqslant k \leqslant \frac{x-1}{c_5-1}} \mathbb{P}(S_{\kappa+k} > x)\mathbb{P}(\eta = \kappa + k) \\ &\leqslant c_7. \end{aligned} \tag{17}$$

Since, obviously,

$$\limsup_{x\to\infty} \frac{\mathcal{K}_4(x)}{\mathbb{P}(S_\eta > x)} \leqslant 1, \tag{18}$$

it remains to estimate sum $\mathcal{K}_3(x)$. Using Lemma 2, we obtain

$$\mathcal{K}_3(x) \leqslant A \sum_{k > \frac{x-1}{c_5-1}} \mathbb{P}(\eta = \kappa + k)\Bigg(\sum_{l=1}^{k} \Big(1 - \sup_{x\in\mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x)\Big)\Bigg)^{-1/2}$$

with some absolute positive constant A. By the fourth condition of the theorem,

$$\frac{1}{k} \sum_{l=1}^{k} \sup_{x\in\mathbb{R}} \big(\overline{F}_{\xi_{\kappa+l}}(x - 1) - \overline{F}_{\xi_{\kappa+l}}(x)\big) \leqslant 1 - \Delta$$

for some $0 < \Delta < 1$ and all sufficiently large k. So, for such k,

$$\sum_{l=1}^{k} \Big(1 - \sup_{x\in\mathbb{R}} \mathbb{P}(x - 1 \leqslant \xi_{\kappa+l} \leqslant x)\Big) \geqslant k\Delta.$$

From the last estimate it follows that

$$\begin{aligned} \mathcal{K}_3(x) &\leqslant \frac{A}{\sqrt{\Delta}} \sum_{k > \frac{x-1}{c_5-1}} \frac{1}{\sqrt{k}} \mathbb{P}(\eta = \kappa + k) \\ &\leqslant \frac{A}{\sqrt{\Delta}} \sqrt{\frac{c_5 - 1}{x - 1}}\, \mathbb{P}\bigg(\eta > \kappa + \frac{x - 1}{c_5 - 1}\bigg) \end{aligned}$$

for sufficiently large x. Therefore,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\mathcal{K}_3(x)}{\mathbb{P}(S_\eta > x)} \\
&\leqslant \frac{A}{\sqrt{\Delta}} \frac{\sqrt{c_5 - 1}}{\mathbb{P}(\eta = \kappa)} \limsup_{x\to\infty} \frac{\overline{F}_\eta(\frac{x-1}{c_5-1})}{\sqrt{x-1}\,\overline{F}_{\xi_\kappa}(x-1)} \limsup_{x\to\infty} \frac{\overline{F}_{\xi_\kappa}(x-1)}{\overline{F}_{\xi_\kappa}(x)} \\
&< \infty
\end{aligned} \tag{19}$$

by estimate (13) and the last condition of the theorem. Representation (14) and estimates (15), (17), (18), and (19) imply the desired inequality (12). Theorem 6 is proved. □

5 Examples of $\mathcal{O}$-exponential random sums

In this section, we present three examples of random sums S_η for which the d.f.s F_{S_η} are $\mathcal{O}$-exponential.

Example 1. Let $\{\xi_1, \xi_2, \ldots\}$ be independent r.v.s. We suppose that the r.v. ξ_k for $k \in \{1, 2, \ldots, D\}$ is distributed according to the Pareto law with parameters k and α, that is,

$$\overline{F}_{\xi_k}(x) = \left(\frac{k}{k+x}\right)^\alpha, \quad x \geqslant 0,$$

where $k \in \{1, 2, \ldots, D\}$, $D \geqslant 1$, and $\alpha > 0$. In addition, we suppose that the r.v. ξ_{D+k} for each $k \in \mathbb{N}$ is distributed according to the exponential law with parameter λ/k, that is,

$$\overline{F}_{\xi_{D+k}}(x) = \mathrm{e}^{-\lambda x/k}, \quad x \geqslant 0.$$

It follows from Theorem 4 that the d.f. of the random sum S_η is $\mathcal{O}$-exponential for each counting r.v. η independent of $\{\xi_1, \xi_2, \ldots\}$ under the condition $\mathbb{P}(\eta = \kappa) > 0$ for some $\kappa \in \{1, 2, \ldots, D\}$ because:

- $F_{\xi_k} \in \mathcal{L} \subset \mathcal{OL}$ for each $k \leqslant \kappa$,
- $$\begin{aligned}
&\sup_{x\geqslant 0}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \\
&= \max\left\{\sup_{0\leqslant x\leqslant 1}\sup_{k\geqslant 1} \frac{1}{\overline{F}_{\xi_{\kappa+k}}(x)}, \sup_{x>1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}\right\} \\
&= \max\left\{\sup_{0\leqslant x\leqslant 1} \max\left\{\max_{1\leqslant k\leqslant D-\kappa}\left(\frac{\kappa+k+x}{\kappa+k}\right)^\alpha, \sup_{k\geqslant 1} \mathrm{e}^{\lambda x/k}\right\},\right. \\
&\quad \left.\sup_{x>1}\max\left\{\max_{1\leqslant k\leqslant D-\kappa}\left(\frac{\kappa+k+x}{\kappa+k+x-1}\right)^\alpha, \sup_{k\geqslant 1} \mathrm{e}^{\lambda/k}\right\}\right\} \\
&\leqslant \max\left\{2^\alpha, \mathrm{e}^\lambda\right\}.
\end{aligned}$$

Example 2. Let a r.v. η be uniformly distributed on $\{1, 2, \dots, D\}$, that is,

$$\mathbb{P}(\eta = k) = \frac{1}{D}, \quad k \in \{1, 2, \dots, D\},$$

for some $D \geqslant 2$. Let $\{\xi_1, \xi_2, \dots, \xi_D\}$ be independent r.v.s, where ξ_1 is exponentially distributed, and $\xi_2, \dots, \xi_D$ are uniformly distributed.

If the r.v. η is independent of the r.v.s $\{\xi_1, \xi_2, \dots, \xi_D\}$, then Theorem 5 implies that the d.f. of the random sum S_η is $\mathcal{O}$-exponential.

Example 3. Let $\{\xi_1, \xi_2, \dots\}$ be independent r.v.s, where $\{\xi_1, \xi_2, \dots, \xi_{\kappa-1}\}$ are finitely supported, $\kappa \geqslant 2$, and ξ_κ is distributed according to the Weibull law, that is,

$$\overline{F}_{\xi_\kappa}(x) = \mathrm{e}^{-\sqrt{x}}, \quad x \geqslant 0.$$

In addition, we suppose that the r.v. $\xi_{\kappa+k}$ for each $k = m^2$, $m \geqslant 2$, has the d.f. with tail

$$\overline{F}_{\xi_{\kappa+k}}(x) = \begin{cases} 1 & \text{if } x < 0, \\ \frac{1}{k} & \text{if } 0 \leqslant x < k, \\ \frac{1}{k}\mathrm{e}^{-(x-k)} & \text{if } x \geqslant k, \end{cases}$$

whereas for each remaining index $k \notin \{m^2, m \in \mathbb{N} \setminus \{1\}\}$, the r.v. $\xi_{\kappa+k}$ has the exponential distribution, that is,

$$\overline{F}_{\xi_{\kappa+k}}(x) = \mathrm{e}^{-x}, \quad x \geqslant 0.$$

If the counting r.v. η is independent of $\{\xi_1, \xi_2, \dots\}$ and is distributed according to the Poisson law with parameter λ, then it follows from Theorem 6 that the random sum S_η is $\mathcal{O}$-exponentially distributed because:

- $F_{\xi_\kappa} \in \mathcal{L} \subset \mathcal{OL}$;
- $\lim\limits_{x\to\infty} \dfrac{\overline{F}_{\xi_k}(x)}{\overline{F}_{\xi_\kappa}(x)} = 0 \quad \text{if } k = 1, 2, \dots, \kappa - 1;$
- $$\sup_{x\geqslant 1}\sup_{k\geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}$$
$$= \sup_{x\geqslant 1}\max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)}, \sup_{k\geqslant 1,\, k\neq m^2} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \right\}$$
$$= \sup_{x\geqslant 1}\max\left\{ \sup_{k\geqslant 1,\, k=m^2,\, m\geqslant 2} \left\{\mathbb{1}_{[1,k)}(x) + \mathrm{e}^{x-k}\mathbb{1}_{[k,k+1)}(x) + \mathrm{e}\mathbb{1}_{[k+1,\infty)}(x)\right\}, \right.$$
$$\left. \sup_{k\geqslant 1,\, k\neq m^2} \mathrm{e} \right\} = \mathrm{e};$$
- $$\limsup_{k\to\infty} \frac{1}{k}\sum_{l=1}^{k} \sup_{x\geqslant 0}\left(\overline{F}_{\xi_{\kappa+l}}(x-1) - \overline{F}_{\xi_{\kappa+l}}(x)\right)$$
$$= \limsup_{k\to\infty} \frac{1}{k}\left(\sum_{l=1,\, l=m^2}^{k} \left(1 - \frac{1}{l}\right) + \left(1 - \frac{1}{\mathrm{e}}\right) \sum_{l=1,\, l\neq m^2}^{k} 1 \right)$$

$$\leqslant \left(1 - \frac{1}{\mathrm{e}}\right);$$

- $$\overline{F}_\eta(x) < \left(\frac{\mathrm{e}\lambda}{x}\right)^x, \quad x > \lambda.$$

Here the last estimate is the well-known Chernof bound for the Poisson law (see, e.g., p. 97 in [13]).

As we can see, the r.v.s $\{\xi_1, \xi_2, \dots\}$ from the last example satisfy the conditions of Theorem 6, whereas the third condition of Theorem 4 does not hold because, in this case,

$$\sup_{x \geqslant 0} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0 \leqslant x < 1} \sup_{k \geqslant 1} \frac{\overline{F}_{\xi_{\kappa+k}}(x-1)}{\overline{F}_{\xi_{\kappa+k}}(x)} \geqslant \sup_{0 \leqslant x < 1} \sup_{k = m^2, m \geqslant 2} k = \infty.$$

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