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On stability in the maximum norm of difference scheme for nonlinear parabolic equation with nonlocal condition

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Abstract. We construct and analyze the backward Euler method for one nonlinear one-dimensional parabolic equation with nonlocal boundary condition. The main objective of this article is to investigate the stability and convergence of the difference scheme in the maximum norm. For this purpose, we use the M-matrices theory. We describe some new approach for the estimation of the error of solution and construct the majorant for it. Some conclusions and discussion of our approach are presented.

Keywords: parabolic equation, nonlocal boundary conditions, M-matrices, stability analysis.

1 Introduction

In this paper, we solve the following nonlinear parabolic equation with nonlocal boundary condition using finite difference method:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(u) + p(x,t), \quad x \in (0,1),\ t \in (0,T], \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

$$u(0,t) = 0, \quad \gamma \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}, \quad \gamma \in (0,1),\ t \in (0,T]. \tag{3}$$

The main objective is to investigate the stability and convergence of the difference scheme in the maximum norm, which is defined for any vector $V = \{V_i\}$, $i = \overline{1,N}$, by the formula

$$\|\mathbf{V}\| = \max_{1 \leqslant i \leqslant N} |V_i|.$$

Investigation of parabolic equations with nonlocal condition of type (3) has started in the 1970s (see e.g. [13, 14]) when the new models for nonlocal problems with nonlocal conditions of various types had been massively created. Boundary value problems for parabolic equations with nonlocal conditions of type (3) arise, for example, in exploring diffusion of particles in turbulent plasma, as well as in investigation of heat conduction in a thin heated rod when the flow change law is specified at the ends of the rod [13].

The initial research results for the linear parabolic equation with nonlocal condition (3) were associated with the stability and convergence of the difference scheme in the maximum norm in the case of $\gamma = 1$ [15, 16]. This research area of nonlocal problems, however, was not developed. Later, stability of finite difference schemes for parabolic equations with nonlocal condition of type (3) was investigated in other more complex norms. These specific norms comply with specific properties of difference operators with nonlocal conditions. In article [11] the following norm is defined:

$$\|u\|_{\mathbf{D}} = (\mathbf{D}u, u)^{1/2}, \tag{4}$$

where $\mathbf{D} = (\mathbf{MM}^*)^{-1}$ is the positive definite matrix, and $\mathbf{M}$ is the matrix formed by the eigenvectors (and, possibly, adjoint vectors) of difference problem.

The stability of parabolic equations with nonlocal condition of type (3) using energy norm (4) was investigated in [9, 10, 24]. The stability of finite difference schemes for parabolic and hyperbolic equations with integral boundary conditions in the norm $\|u\|_{\mathbf{D}}$ is widely investigated in [18, 20, 23, 25]. The convergence of finite difference method for Poisson equation with nonlocal condition of (3) type is proved in [6, 30].

M-matrices theory is started to use for theoretical investigation and solving of problems with nonlocal conditions for the last few years. This approach is used to prove the convergence of iterative methods for the systems of nonlinear difference equations [4, 22, 28], also to prove stability of difference schemes in the norm $\|u\|_{\mathbf{D}}$ [19, 24]. Stability and convergence in the maximum norm of finite difference schemes for differential equation with integral boundary condition using M-matrices theory are proved in [5, 26].

In this article, we use the M-matrices theory for the system of difference equations approximating differential problem (1)–(3).

The structure of the paper is following. The difference problem is formulated and the structure of error of discrete approximation is considered in Section 2. The connection between difference problem and M-matrices is briefly described in Section 3. Next, the main result of this paper about convergence and stability of difference scheme in the maximum norm is investigated in Sections 4 and 5. Comments and generalizations are presented in Section 6.

2 Difference problem and approximation error

We solve differential problem (1)–(3) using finite difference method. We assume that the solution $u(x,t)$ of (1)–(3) exists, is unique, and the derivatives $\partial^k u/\partial x^k$, $k = \overline{1,4}$, and $\partial^l u/\partial t^l$, $l = 1, 2$, are continuous and bounded. We define

$$\max\left|\frac{\partial^k u}{\partial x^k}\right| \leqslant M_k, \quad k = 3, 4; \qquad \max\left|\frac{\partial^2 u}{\partial t^2}\right| \leqslant C_2.$$

Also, we suppose the following assumption is valid for function $f(u)$,

Hypothesis 1. $\partial f/\partial u \geqslant 0$ for all values of $(x,t) \in [0,1] \times [0,T]$ and $u(x,t)$.

Remark 1. Hypothesis 1 is inherent to the variety of convection-reaction-diffusion mathematical models. For example, the source function $f(S)$ for some processes in bioreactors has the following form [17]:

$$f(S) = \frac{V_{\max} S}{K_a + S},$$

where $S(x,t)$ is the product, $V_{\max}$ and K_a are nonnegative constants (for more details, see also [27]).

Let U be the finite difference approximation of u. We denote

$$U_i^n = U\big(x_i, t^n\big),$$

where $x_i = ih$, $i = \overline{0,N}$, $h = 1/N$; $t^n = n\tau$, $n = \overline{0,M}$, $\tau = T/M$; $N, M \in \mathbb{Z}$.

We also denote

$$\partial_x U_i^n = \frac{U_{i+1}^n - U_i^n}{h}, \qquad \partial_x^2 U_i^n = \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2},$$
$$\partial_{\bar{x}} U_i^n = \frac{U_i^n - U_{i-1}^n}{h}, \qquad \partial_{\bar{t}} U_i^n = \frac{U_i^n - U_i^{n-1}}{\tau}.$$

We approximate equation (1) by difference one using the backward Euler method

$$\partial_{\bar{t}} U_i^n = \partial_x^2 U_i^n - f_i^n\big(U_i^n\big) + p_i^n, \quad i = \overline{1, N-1}. \tag{5}$$

The approximation error is

$$r_i(h,\tau) = R_{1,i}(h) + R_{2,i}(\tau), \quad i = \overline{1, N-1}, \tag{6}$$

where

$$\big|R_{1,i}^{(h)}\big| \leqslant \frac{h^2 M_4}{12}, \qquad \big|R_{2,i}^{(\tau)}\big| \leqslant \frac{\tau C_2}{2}. \tag{7}$$

To approximate nonlocal condition (3) with accuracy $O(h^2)$, we rewrite it in the following form:

$$\gamma\left(\partial_x u_0^n - \frac{h}{2}\frac{\partial^2 u_0^n}{\partial x^2} - \frac{h^2}{6}\frac{\partial^3 \tilde{u}_0^n}{\partial x^3}\right) = \partial_{\bar{x}} U_N^n + \frac{h}{2}\frac{\partial^2 u_N^n}{\partial x^2} - \frac{h^2}{6}\frac{\partial^3 \tilde{u}_N^n}{\partial x^3}.$$

Assume that differential equation (1) is defined not only in the interval $x \in (0,1)$, but also on the boundaries $x = 0$ and $x = 1$. Now, substitute into latter equality expressions of $\partial^2 u_0^n/\partial x^2$ and $\partial^2 u_N^n/\partial x^2$ from (1). Using $\partial U_0^n/\partial t$ and $\partial U_N^n/\partial t$ approximations of order $O(\tau^2)$, after elementary rearrangements, we have

$$\gamma\left(\partial_x u_0^n - \frac{h}{2} f_0^n\big(u_0^n\big) + \frac{h}{2} p_0^n\right) = \partial_{\bar{x}} u_N^n + \frac{h}{2}\big(\partial_{\bar{t}} u_N^n + f_N^n\big(u_N^n\big) - p_N^n + r_N(h,\tau)\big), \tag{8}$$

where

$$r_N = R_{1,N}(h) + R_{2,N}(\tau), \tag{9}$$

$$\left|R_{1,N}(h)\right| \leqslant \frac{h(\gamma+1)M_3}{3}, \qquad \left|R_{2,N}(\tau)\right| \leqslant \frac{\tau(\gamma+1)C_2}{2}. \tag{10}$$

Excluding approximation error terms from equality (8), we get the equation, which approximates nonlocal condition (3) with accuracy $|r_n(h,\tau)| = O(h^2 + h\tau)$. We rewrite this equation in the following form:

$$\partial_{\bar t} U_N^n = \frac{2}{h}\left(\gamma \partial_x U_0^n - \partial_{\bar x} U_N^n\right) - \gamma f_0^n\left(U_0^n\right) - f_N^n\left(U_N^n\right) + \gamma p_0^n + p_N^n. \tag{11}$$

Remark 2. Difference equation (11) approximating nonlocal condition (3) in the case $f(u) = 0$ and $p(t,x) = 0$ is usually provided without derivations and comments [9, 11]. We notice that in process of reworking equation (8) into form (11), we divided terms by h. Therefore, the difference equation (11) differs from differential one by $O(h+\tau)$.

We complement the system of difference equations (5), (11) by following equations:

$$U_0^n = 0, \quad n \geqslant 0; \qquad U_i^0 = \varphi_i, \quad i = \overline{0,N}.$$

As a result, we get the system of difference equations approximating problem (1)–(3)

$$\begin{aligned} \partial_{\bar t} U_i^n &= \partial_x^2 U_i^n - f_i^n\left(U_i^n\right) + p_i^n, \quad i = \overline{1,N-1}, \\ \partial_{\bar t} U_N^n &= \frac{2}{h}\left(\gamma \partial_x U_0^n - \partial_{\bar x} U_N^n\right) - \gamma f_0^n\left(U_0^n\right) - f_N^n\left(U_N^n\right) + \gamma p_0^n + p_N^n, \\ U_0^n &= 0, \qquad U_i^0 = \varphi_i, \quad i = \overline{0,N}. \end{aligned} \tag{12}$$

3 M-matrices and systems of difference equations

We investigate the system of difference equations (12) using theory of M-matrices. We provide the definition and some properties of M-matrices for this purpose [1, 29].

We use the notation $\mathbf{A} > 0$ ($\mathbf{A} \geqslant 0$) if $a_{kl} > 0$ ($a_{kl} \geqslant 0$) for all k, l. Also, $\mathbf{A} < \mathbf{B}$ ($\mathbf{A} \leqslant \mathbf{B}$) if $a_{kl} < b_{kl}$ ($a_{kl} \leqslant b_{kl}$). Similar notation is used for vectors. There exist a couple of equivalent definitions of M-matrices. We provide one that, in our opinion, corresponds to the specifics of systems of difference equations.

Definition 1. A square matrix with real elements $\mathbf{A} = \{a_{kl}\}$, $k, l = 1, 2, \ldots, m$, is called an M-matrix if $a_{kl} \leqslant 0$ when $k \neq l$ and the inverse $\mathbf{A}^{-1}$, whose all elements are nonnegative ($\mathbf{A}^{-1} \geqslant 0$), exists.

It follows from the definition that $a_{kk} > 0$.

We point out several typical properties of the M-matrices that are used to investigate system (12).

Proposition 1. *If $\mathbf{A}_1$ is an M-matrix and $\mathbf{A}_2 \geqslant \mathbf{A}_1$ and, additionally, all nondiagonal elements of the matrix $\mathbf{A}_2$ are nonpositive, then $\mathbf{A}_2$ is also an M-matrix and $\mathbf{A}_2^{-1} \leqslant \mathbf{A}_1^{-1}$.*

Proposition 2. *If $a_{kl} \leqslant 0$ when $k \neq l$, then two next statements are equivalent:*

(i) *The matrix $\mathbf{A}^{-1}$ exists, and $\mathbf{A}^{-1} \geqslant 0$;*

(ii) *The real parts of each eigenvalue of $\mathbf{A}$ are positive*: $\operatorname{Re}\lambda(\mathbf{A}) > 0$.

Now we start to investigate the system of difference equations (12). We define

$$z_i^n = u_i^n - U_i^n,$$

where U_i^n is the solution of difference problem (12), $u_i^n = u(x_i, t^n)$ is exact solution of differential problem (1)–(3). Using system (12) and expressions of approximation errors (6), (9), we get

$$\begin{aligned}
\partial_{\bar t} z_i^n &= \partial_x^2 z_i^n - d_i^n z_i^n + r_i(h,\tau), \quad i = \overline{1, N-1},\\
\partial_{\bar t} z_N^n &= \frac{2}{h}\bigl(\gamma \partial_x z_0^n - \partial_{\bar x} z_N^n\bigr) - d_N^n z_N^n + r_N(h,\tau),\\
z_0^n &= 0, \qquad z_i^0 = 0,
\end{aligned} \tag{13}$$

where $d_i^n = \partial f(\tilde u_i)/\partial u$, $i = \overline{1, N}$. System (13) is two-layered difference scheme. We write down this scheme in the matrix form

$$(\mathbf{A} + \tau \mathbf{D}_n) z^n = \mathbf{B} z^{n-1} + g^n, \tag{14}$$

where $\mathbf{A} = \mathbf{I} + \tau \boldsymbol{\Lambda}$, $\mathbf{B} = \mathbf{I}$, $\mathbf{I}$ is the identity matrix, $g^n = \tau t = (\tau r_1, \tau r_2, \dots, \tau r_N)^{\mathsf T}$, $\mathbf{D}_n$ is the diagonal matrix with elements d_i^n,

$$\boldsymbol{\Lambda} = \frac{1}{h^2}\begin{pmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -1 & 2 & -1\\
-2\gamma & 0 & \dots & 0 & -2 & 2
\end{pmatrix}.$$

Eigenvalue problem for the matrix $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda} V = \lambda V$$

is equivalent to difference eigenvalue problem

$$\begin{aligned}
&\partial_x^2 V_i + \lambda V_i = 0, \quad i = \overline{1, N-1},\\
&\frac{2}{h}\bigl(\partial_x V_0 - \partial_{\bar x} V_N\bigr) + \lambda V_N = 0, \qquad V_0 = 0.
\end{aligned} \tag{15}$$

All eigenvalues of problem (15) are positive [12]:

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_{2k-1} < \lambda_{2k} < \cdots < \lambda_{N-1} < \frac{4}{h^2},$$

$$\lambda_0 = \frac{4}{h^2}\sin^2\biggl(\frac{1}{2}\psi h\biggr), \qquad \lambda_{2k-1} = \frac{4}{h^2}\sin^2\biggl(\biggl(\pi k - \frac{1}{2}\psi\biggr)h\biggr),$$

$$\lambda_{2k} = \frac{4}{h^2}\sin^2\biggl(\biggl(\pi k + \frac{1}{2}\psi\biggr)h\biggr), \quad k = \overline{1, m},$$

where $\psi = \arccos\gamma$, $0 < \psi < \pi$, $m = (N-1)/2$ if N is odd, and $m = N/2$ if N is even.

Lemma 1. *If $\gamma \in (0,1)$, then the matrix $\mathbf{A} = \mathbf{I} + \tau\boldsymbol{\Lambda}$ is the M-matrix.*

Proof. It follows from the definition of the matrix $\mathbf{A}$ that all diagonal elements are positive and nondiagonal are nonnegative. Since all eigenvalues of the matrix $\mathbf{A}$ are positive, then all listed properties are suffiecient for the matrix $\mathbf{A}$ to be the M-matrix [29]. □

We use the result, proved in [5], to investigate stability and convergence of difference system (12).

Lemma 2. (*See* [5].) *Let V^n and W^n be solutions of difference equations*

$$(\mathbf{A} + \mathbf{D}_n)V^n = \mathbf{B}V^{n-1} + f^n, \quad n \geqslant 1,$$
$$\mathbf{A}W^n = \mathbf{B}W^{n-1} + g^n, \quad n \geqslant 1,$$

respectively, where $\mathbf{D}_n = \{d_{kk}^n\}$ is a diagonal matrix, $\mathbf{D}_n \geqslant 0$. If $\mathbf{A}$ is an M-matrix, $\mathbf{B} \geqslant 0$, $W^0 \geqslant 0$, $g^n \geqslant 0$ as $n \geqslant 1$, and $|V^0| \leqslant W^0$, $|f^n| \leqslant g^n$, then

$$|V^n| \leqslant W^n, \quad n \geqslant 1.$$

Certain analogue of Lemma 2 for elliptic equation is known for a long time [8] and is usually called comparison theorem [21]. Function $W(x)$ is called majorant or comparison function. One important feature of the comparison theorem is it is usually formulated as one of the most important corollaries of maximum principle. It means that comparison theorem is related to those systems of difference equations, which matrix is diagonally dominant. In this article, diagonal dominance of a matrix is not mandatory property for the selected investigation methodology of stability and convergence of systems of difference equations.

Notice that while using Lemma 2 for investigation of stability and convergence of specific difference scheme, it is not necessary to convert this scheme into (14) form, which is used in the proof of Lemma 2. In certain cases, it is easier to reformulate Lemma 2 than modify difference scheme.

So, now we rewrite difference scheme (13) in the matrix form slightly different from (14)

$$\frac{z^n - z^{n-1}}{\tau} = -\boldsymbol{\Lambda} z^n - \mathbf{D}_n z^n + r^n, \tag{16}$$

where z^n, r^n are Nth order vectors, $\boldsymbol{\Lambda}$ is the same matrix as in system (14). We reformulate Lemma 2 for this exact difference scheme.

Lemma 2'. *Let z^n be solution of* (16) *and W^n solution of equations*

$$\frac{W^n - W^{n-1}}{\tau} = -\boldsymbol{\Lambda} W^n + g^n. \tag{17}$$

Let $W^0 \geqslant 0$, $g^n \geqslant 0$ as $n \geqslant 1$. $\mathbf{D}_n = \{d_{kk}^n\}$ is the diagonal matrix, $\mathbf{D}_n \geqslant 0$. If $|z^0| \leqslant W^0$, $|r^n| \leqslant g^n$, then $|z^n| \leqslant W^n$, $n \geqslant 1$.

4 Stability of difference scheme

Now, we investigate stability of difference scheme using statement of Lemma 2.

We use general stability concept since the system of difference equations is a nonlinear problem. Indeed, assume two problems. One problem is system (12) with functions $p(x)$ and $\varphi(x)$. We denote the solution of this problem U_i^n. And other problem is the same system (12) with perturbed functions $\widetilde{p}(x)$ and $\widetilde{\varphi}(x)$. The solution of this problem is denoted as $\widetilde{U}_i^n$.

Definition 2. Difference scheme (12) is stable if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, not dependent on h and τ, such that

$$\left|U_i^n - \widetilde{U}_i^n\right| \leqslant \varepsilon$$

if $|p_i^n - \widetilde{p}_i^n| \leqslant \delta$, $|\varphi_i - \widetilde{\varphi}_i| \leqslant \delta$.

Note that such defined stability definition is usually called stability according to initial conditions and right-hand side stability.

We denote

$$\widetilde{z}_i^n = U_i^n - \widetilde{U}_i^n, \qquad \delta_i^0 = \varphi_i - \widetilde{\varphi}_i, \qquad \delta_i^n = p_i^n - \widetilde{p}_i^n.$$

Similarly to derivation of system (13), we have

$$\begin{aligned} \partial_{\bar{t}}\widetilde{z}_i^n &= \partial_x^2\widetilde{z}_i^n - d_i^n\widetilde{z}_i^n + \delta_i^n, \quad i = \overline{1, N-1}, \\ \partial_{\bar{t}}\widetilde{z}_N^n &= \frac{2}{h}\left(\gamma\partial_x\widetilde{z}_0^n - \partial_{\bar{x}}\widetilde{z}_N^n\right) - d_N^n\widetilde{z}_N^n + \delta_N^n, \\ \widetilde{z}_0^n &= 0, \qquad \widetilde{z}_i^0 = \delta_i^0. \end{aligned} \tag{18}$$

This system could also be rewritten in a form similar to (16)

$$\begin{aligned} \frac{\widetilde{z}^n - \widetilde{z}^{n-1}}{\tau} &= -\boldsymbol{\Lambda}\widetilde{z}^n - \mathbf{D}_n\widetilde{z}^n + \delta^n, \\ \widetilde{z}_0^n &= 0, \qquad \widetilde{z}^0 = \delta^0. \end{aligned} \tag{19}$$

Theorem 1. *The difference scheme* (12) *is stable in the maximum norm if* $\gamma \in (0, 1)$.

Proof. We define function

$$W(x,t) = \frac{\delta}{2}\left(2 - x^2 + \frac{2x}{1-\gamma}\right) + \frac{\delta t}{2}, \tag{20}$$

where δ is, so far, not defined number $\delta > 0$.

The proof consists of two parts. In the first part, we explain how to choose $\delta = \delta(\varepsilon)$ such that

$$W(x,t) \leqslant \varepsilon$$

on all domain $x \in [0, 1]$, $t \in [0, T]$. In the second part, we prove that function $W(x,t)$ is majorant for solution's error $\widetilde{z}_i^n$ satisfying system (18).

(i) As $\gamma \in (0,1)$, then $2/(1-\gamma) > 2$. Therefore, $W(x,t)$ on the interval $x \in [0,1]$ is positive and monotonically increasing function of x. The same property for $W(x,t)$ is also valid on the interval $t \in [0,T]$. Hence,

$$W(x,t) \leqslant \frac{\delta}{2}\left(1 + \frac{2}{1-\gamma} + T\right).$$

Consequently, the inequality $W(x,t) \leqslant \varepsilon$ on the domain $x \in [0,1]$, $t \in [0,T]$ is valid if

$$\delta \leqslant \delta_0 = \frac{2\varepsilon}{1 + \frac{2}{1-\gamma} + T}. \tag{21}$$

(ii) Now, we construct the system of difference equations (17), whose solution is $W(x,t)$. $W(x,t)$ is defined by formula (20). To this end, we find g_i^n values using $W(x,t)$ expression

$$\begin{aligned} g_i^n &= \frac{3}{2}\delta > \delta, \quad i = \overline{2, N-1}, \\ g_1^n &= \frac{3}{2}\delta + \frac{W_0^n}{h^2} > \delta, \qquad g_N^n = \frac{3}{2}\delta + \gamma\delta > \delta, \end{aligned} \tag{22}$$

and also

$$\begin{aligned} g_0^n &= W_0^n > 0, \\ g_i^0 &= W_i^0 = \frac{\delta}{2}\left(2 - x_i^2 + \frac{2x_i}{1-\gamma}\right) \geqslant \delta. \end{aligned} \tag{23}$$

Next, we take two systems: system (19) for function $\tilde{z}_i^n$ and system (17) for function W_i^n. Values of g_i^n are defined by formulas (22), (23).

Suppose δ is fixed in system (19) and satisfies inequality (21). Also, suppose that for δ_i^n and δ_i^0 in system (17), the following equalities are valid:

$$\left|\delta_i^n\right| \leqslant \delta, \quad i = \overline{1, N}; \qquad \left|\delta_i^0\right| \leqslant \delta, \quad i = \overline{0, N}.$$

Then, all assumptions of Lemma 2' for the systems of difference equations (17), (19) are valid. Therefore,

$$\left|\tilde{z}_i^n\right| \leqslant W_i^n$$

for all i and n.

Now, combining both proof parts, we have that for every $\varepsilon > 0$, the following inequality is valid:

$$\left|\tilde{z}_i^n\right| \leqslant W_i^n \leqslant \varepsilon$$

if

$$\left|\delta_i^n\right| \leqslant \delta \leqslant \delta_0 = \frac{2\varepsilon}{1 + \frac{2}{1-\gamma} + T}, \quad i = \overline{0, N},\ n \geqslant 0.$$

Hence, according to Definition 2, it follows that difference scheme (12) is stable. □

5 Error estimation and convergence of difference scheme

Now, we investigate the convergence of difference scheme (12) using the same Lemma 2'.

We consider system (16) for the error estimate $z_i^n = u_i^n - U_i^n$, where u_i^n and U_i^n are solutions of differential and difference problems, accordingly. In accordance with error estimates (7) and (10), we have

$$\begin{aligned} |r_i| &\leqslant \frac{h^2 M_4}{12} + \frac{\tau C_2}{2}, \quad i = \overline{1, N-1}, \\ |r_N| &\leqslant \frac{h M_3(\gamma+1)}{3} + \frac{\tau C_2(\gamma+1)}{2}. \end{aligned} \tag{24}$$

According to received estimates (24), we define majorant $W(x,t)$

$$W(x,t) = \frac{h^2 M(\gamma+1)}{6}\left(2 - \frac{x^2}{2} + \frac{2x}{1-\gamma}\right) + \frac{\tau C_2(\gamma+1)t}{2}, \tag{25}$$

where $M = \max(M_3, M_4)$.

Now, define coefficients g_i^n in system (17) so that the solution would be W_i^n defined by formula (25):

$$\begin{aligned} g_i^n &= \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{6}, \quad i = \overline{2, N-1}, \\ g_1^n &= \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)}{6} + \frac{W_0^n}{h^2}, \\ g_N^n &= \frac{\tau C_2(\gamma+1)}{2} + \frac{h^2 M(\gamma+1)^2}{6} + \frac{h M(\gamma+1)}{3}. \end{aligned} \tag{26}$$

Notice that $W_i^n > 0$ for all i and n values.

Theorem 2. *Let the following assumptions hold for the differential problem*:

(i) $\gamma \in (0,1)$;

(ii) $\partial f/\partial u \geqslant 0$ (*Hypothesis* 1);

(iii) *The solution of differential problem is smooth enough* (*the error estimates* (7) *and* (10) *are valid*).

Then for the error estimate $z_i^n = u_i^n - U_i^n$ (*here* u_i^n *is solution of problem* (1)–(3), U_i^n *is the solution of difference problem* (12)), *the following estimate is valid*:

$$\|z^n\|_C = \max_{1\leqslant i\leqslant N} |z_i| \leqslant C_1 h^2 + C_2 \tau, \tag{27}$$

where constants C_1 *and* C_2 *do not depend on* h *and* τ.

Proof. We consider two already defined problems: system of equations (16) with unknowns z_i^n and system (17) with unknowns W_i^n. For z_i^n, estimates (24) are valid. For system (17), values of g_i^n are defined by formulas (26). Assumptions of Lemma 2' are valid for both systems. Using these assumptions, we have

$$|z_i^n| \leqslant W_i^n$$

for all i and n.

Since function $\alpha(x) = 2 - x^2/2 + 2x/(1-\gamma)$ is positive and monotonically increasing on the interval $x \in [0,1]$, so

$$W_i^n \leqslant \frac{h^2 M(\gamma+1)}{6}\left(1 + \frac{2}{1-\gamma}\right) + \frac{\tau C_2(\gamma+1)T}{2}.$$

Consequently, the theorem is proved. □

6 Comments and generalizations

In this article, we investigated the stability and convergence of the difference scheme for one-dimensional nonlinear parabolic equation with nonlocal condition. The thorough study of this problem for linear elliptic equations with Dirichlet boundary conditions has started in 50–60's [2, 7, 21] (and first time noticed in [8]). Particularly, in [7] a lot of practical applications and comments on described methodology are given. It was noticed that there is no universal algorithm for defining majorant for specific problem.

It is interesting to note that in all above cited monographs (as well as in most other books), restrictions on the matrix of systems of difference equations $\mathbf{A}U = f$, $\mathbf{A} = \{a_{ij}\}$, $i, j = \overline{1, n}$, are in fact the same

$$a_{ii} > 0; \qquad a_{ij} \leqslant 0, \quad i \neq j; \qquad a_{ii} \geqslant \sum_{j=1}^{n} |a_{ij}|, \tag{28}$$

typical for the majority of simplest elliptical equations with Dirichlet boundary conditions. These restrictions are sufficient for the system of difference equations to satisfy the maximum principle. According to this principle, the solution U_{ij} of the system of difference equations can not reach the greatest positive and the smallest negative values in the inner domain.

Comparison theorem directly follows from maximum principle. We formulated similar proposition in Lemma 2. However, this lemma is proved in [5] without using maximum principle. The proof is based on the M-matrices theory. In other words, maximum principle is not a necessary condition for the estimation of error using comparison theorem. Therefore, third assumption of (28), namely, the diagonal dominance of the matrix is not necessary.

Further, using Lemma 2 proof techniques, it follows that comparison is valid if the matrix $\mathbf{A}$ of the system of difference equations has property $\mathbf{A}^{-1} \geqslant 0$. That is, if all elements of the inverse matrix are nonnegative [5]. Matrices with this property are called monotone (in the sense of Collatz [3, Chap. 3, Sect. 23]) [29].

So, this paper is a first step to a new reseach methodology — applications of monotone matrices for the differential equations with nonlocal conditions. The results of this paper give some hope that the assumption $a_{ij} \leqslant 0$, $i \neq j$, is not the necessary condition for the comparison theorem evaluation of the error.

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