Leipus et al. *Journal of Inequalities and Applications* (2023) 2023:32
https://doi.org/10.1186/s13660-023-02941-1

Journal of Inequalities and Applications
a SpringerOpen Journal

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On the distribution-tail behaviour of the product of normal random variables

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Abstract

In this paper we consider the product $\Pi_n = \prod_{k=1}^{n} \xi_k$ of n independent normally distributed zero mean random variables $\xi_1, \dots, \xi_n$. We derive an asymptotic formula for the survival probability $\mathbb{P}(\Pi_n > x)$, as $x \to \infty$, with the first remaining term.

MSC: 60E05; 62G32

Keywords: Distribution tail; Product of random variables; Normal distribution; Saddle point method; Asymptotic formulas

1 Introduction

The normal distribution is the most popular continuous distribution in probability theory due to its wide range of applications in various fields of science and practice. We say that a random variable (r.v.) ξ has normal distribution if its density function has the form

$$f_\xi(x) = \varphi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad x \in \mathbb{R},$$

where the parameter $\mu \in \mathbb{R}$ is the expectation and $\sigma > 0$ is the standard deviation of ξ. The distribution function (d.f.) of the normal r.v. ξ is

$$F_\xi(x) = \Phi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\} \mathrm{d}y.$$

In the case where $\mu = 0$ and $\sigma = 1$, the distribution above is called the *standard normal distribution*. In such a case,

$$\varphi(x) := \varphi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad \Phi(x) := \Phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \, \mathrm{d}y.$$

In this paper we deal with the products of normal random variables. In general, the theory for the products of independent random variables is far less developed than the mature theory for the sums of independent random variables (we mention the books [5, 12], and [24, Chap. 4]).

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In the case of two standard normal r.v.s., the exact expression for the distribution of their product was obtained almost a century ago. In [7, 28, 30], it was observed that for independent standard normal r.v.s ξ and η, the density function of the product $\xi\eta$ is

$$f_{\xi\eta}(x) = \frac{1}{\pi} K_0\bigl(|x|\bigr), \quad x \in \mathbb{R}, \tag{1}$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero, i.e.,

$$K_0(x) = \frac{1}{2} \int_0^\infty y^{-1} \exp\left\{-y - \frac{x^2}{4y}\right\} \mathrm{d}y, \quad x \ge 0.$$

Craig [7] also derived a closed-form expression for the exact density function $f_{\xi\eta}$ in the case of dependent normally distributed r.v.s ξ and η, showing that essentially $f_{\xi\eta}$ is of that form, reached when correlation $\mathrm{Corr}(\xi, \eta) = 0$, multiplied by an exponential function (see also [8, 13, 22]). Further developments of Craig's result can be found in numerous subsequent papers. Haldane [15] derived the cumulant generating function, the first four moments, and the first four cumulants of $\xi\eta$ in the general case. Aroian [2] showed that the distribution of $\xi\eta$ can be approximated by a normal distribution when the ratios of the means to standard deviations of the normally distributed multipliers are large. Aroian et al. [3] and Simon [23] provided expressions for the distribution of $\xi\eta$ with normally distributed multipliers in the special cases of parameters and dependence structures. Recently, Gaunt [14] (see also the references therein) successfully used the Stein method to obtain the exact expression of the density function not only for the product of zero-mean normal multipliers but also for sum of such independent summands.

Concerning applications of the distribution of the product of two (correlated or not) normal variables, we mention [4], where the authors showed that a Lagrangian power distribution in two-dimensional turbulence is well described by Craig's $\xi\eta$ distribution, and [19, 20] for testing of the indirect effect based on the distribution of the product of two normal variables. Along with the exact formula, the asymptotic behaviour of the tail distribution of the product of n normal r.v.s. is often of interest. An obvious application comes from the financial portfolio analysis, when calculating the compound return of the portfolio with normally distributed one-period returns and estimating tails of its distribution, as this could be beneficial for the portfolio decision making.

In Sect. 2 we formulate our main result, which is proved in Sect. 3.

2 Asymptotic behaviour of the product distribution tail

The asymptotic behaviour of the tail of the product of normal r.v.s. can be derived from exact formulas or using the saddle point method described in details by Butler [6], Fedoryuk [10], Jensen [16], for instance. Using the latter method described in [10], Arendarczyk and Dębicki [1] obtained the following result (see their Lemma 2.1) on the Weibull tail distributions.

Theorem 2.1 *Let ξ_1 and ξ_2 be two independent, nonnegative r.v.s such that*

$$\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\sim} A_1 x^{\gamma_1} \exp\bigl\{-\beta_1 x^{\alpha_1}\bigr\}, \qquad \overline{F}_{\xi_2}(x) \underset{x\to\infty}{\sim} A_2 x^{\gamma_2} \exp\bigl\{-\beta_2 x^{\alpha_2}\bigr\},$$

where $A_i > 0$, $\gamma_i \in \mathbb{R}$, $\beta_i > 0$, $\alpha_i > 0$, $i = 1, 2$. *Then*

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} Ax^\gamma \exp\{-\beta x^\alpha\},$$

where

$$\begin{aligned}
\alpha &= \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},\\
\beta &= \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left(\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}+\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right),\\
\gamma &= \frac{\alpha_1\alpha_2+2\alpha_1\gamma_2+2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)},\\
A &= \sqrt{2\pi}\frac{A_1A_2}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}}(\alpha_2\beta_2)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}.
\end{aligned}$$

In particular, this theorem implies that if ξ_1, ξ_2, ξ_3 are i.i.d. standard normal r.v.s, then

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}, \qquad \overline{F}_{\xi_1\xi_2\xi_3}(x) \underset{x\to\infty}{\sim} \frac{2}{\sqrt{6\pi}}x^{-1/3}\mathrm{e}^{-(3/2)x^{2/3}}. \tag{2}$$

Let us describe the derivation of the above formulas. At first, suppose $n = 2$. Random variables ξ_1, ξ_2 are distributed according to a symmetric law Φ. Therefore,

$$\overline{F}_{\xi_1\xi_2}(x) = \mathbb{P}(\xi_1\xi_2 > x) = 2\mathbb{P}(\xi_1^+\xi_2^+ > x) = 2\overline{F}_{\xi_1^+\xi_2^+}(x), \quad x > 0, \tag{3}$$

where here and everywhere below η^+ denotes the positive part of an r.v. η.

For $x > 0$, by partial integration, we obtain (see also [11, Sect. 7.1])

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)\mathrm{e}^{-x^2/2} \le \overline{F}_{\xi_1}(x) \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}\mathrm{e}^{-x^2/2}. \tag{4}$$

To derive an asymptotic formula for $\overline{F}_{\xi_1^+\xi_2^+}(x)$, we apply Theorem 2.1 with

$$\alpha_1 = \alpha_2 = 2, \qquad \beta_1 = \beta_2 = \frac{1}{2}, \qquad \gamma_1 = \gamma_2 = -1, \qquad A_1 = A_2 = \frac{1}{\sqrt{2\pi}}.$$

After some calculations we get

$$\overline{F}_{\xi_1^+\xi_2^+}(x) \underset{x\to\infty}{\sim} \frac{1}{2\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}. \tag{5}$$

The asymptotics in (5), together with equality (3), implies the first formula in (2).

In the case of $n = 3$, similarly to the formula (3), it holds that

$$\overline{F}_{\xi_1\xi_2\xi_3}(x) = 4\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x), \quad x > 0. \tag{6}$$

To derive the asymptotics for $\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x)$, we can apply Theorem 2.1 again by using the asymptotic formula (5) and the estimate (4). According to these relations, the parameters in Theorem 2.1 are the following:

$$\alpha_1 = 1, \quad \beta_1 = 1, \quad \gamma_1 = -\frac{1}{2}, \quad A_1 = \frac{1}{2\sqrt{2\pi}},$$
$$\alpha_2 = 2, \quad \beta_2 = \frac{1}{2}, \quad \gamma_2 = -1, \quad A_2 = \frac{1}{\sqrt{2\pi}}.$$

Using the formulas of Theorem 2.1, we derive

$$\overline{F}_{\xi_1^+\xi_2^+\xi_3^+}(x) \underset{x\to\infty}{\sim} Ax^{\gamma}\mathrm{e}^{-\beta x^{\alpha}},$$

with $\alpha = 2/3$, $\beta = 3/2$, $\gamma = -1/3$, and $A = 1/(2\sqrt{6\pi})$. To obtain the second formula in (2), it is enough to use formula (6).

Note that the product of normally distributed r.v.s (or Weibull r.v.s in general), which are light-tailed, can produce heavy-tailed distributions, as seen from (2). We refer to [9, 17, 18, 21, 27] and the references therein for various extensions related to this phenomenon.

In our paper, we derive a more precise formula for the tail of the product of normal r.v.s. The following two statements are the main results of the paper.

Theorem 2.2 *Let $\xi_1, \xi_2, \dots$ be i.i.d. r.v.s such that, for each k, ξ_k is distributed according to Φ and let $\Pi_n := \prod_{k=1}^n \xi_k$. Then*

$$\overline{F}_{\Pi_n}(x) = \frac{2^{(n/2)-1}}{\sqrt{\pi n}} x^{-1/n} \exp\left\{-\frac{n}{2}x^{2/n}\right\}\left(1 + O_n\left(x^{-2/n}\right)\right), \tag{7}$$

where the constant in O_n depends on n.

Corollary 2.1 *Let $\widehat{\xi}_1, \widehat{\xi}_2, \dots$ be independent r.v.s such that, for each k, $\widehat{\xi}_k$ is distributed according to Φ_{0,σ_k} and let $\widehat{\Pi}_n := \prod_{k=1}^n \widehat{\xi}_k$. Then*

$$\overline{F}_{\widehat{\Pi}_n}(x) = \frac{2^{(n/2)-1}(\sigma^{(n)})^{1/n}}{\sqrt{\pi n}} x^{-1/n} \exp\left\{-\frac{n(\sigma^{(n)})^{-2/n}}{2}x^{2/n}\right\}\left(1 + O_n\left(x^{-2/n}\right)\right),$$

where $\sigma^{(n)} := \prod_{k=1}^n \sigma_k$ and the constant in O_n depends on n and $\sigma^{(n)}$.

Remark 2.1 The assertion of Corollary 2.1 follows from Theorem 2.2 immediately, because the r.v. $\xi_k = \widehat{\xi}_k/\sigma_k$ is distributed according to Φ for each k and

$$\overline{F}_{\widehat{\Pi}_n}(x) = \mathbb{P}(\widehat{\Pi}_n > x) = \mathbb{P}\left(\Pi_n > \frac{x}{\sigma^{(n)}}\right) = \overline{F}_{\Pi_n}\left(\frac{x}{\sigma^{(n)}}\right).$$

Remark 2.2 Note that the density of product $\widehat{\Pi}_n$, as in Corollary 2.1, can be written in terms of the Meijer G-function (see [25, 26]),

$$f_{\Pi_n}(x) = \frac{1}{(2\pi)^{n/2}\sigma^{(n)}} G_{0,n}^{n,0}\left(2^{-n}\left(x/\sigma^{(n)}\right)^2 | 0, \dots, 0\right).$$

In the case $n = 2$, this formula gives the modified Bessel function of order zero,

$$f_{\xi_1\xi_2}(x) = \frac{1}{\pi\sigma_1\sigma_2} K_0\left(\frac{|x|}{\sigma_1\sigma_2}\right), \quad x \in \mathbb{R},$$

cf. (1). In our proof, we use a different approach based on the *saddle point* method which does not require knowledge of G-functions and their properties. Moreover, it allows obtaining higher-order terms in the result (7).

3 Proof of the main theorem

We start with the formulation of an auxiliary lemma, which we use repeatedly in our proof. The lemma below can be obtained by applying the saddle point method to a real integral of a special form. The proof of the lemma can be found in [29] (see Theorem 1, Chap. II).

Lemma 3.1 *Let h and g be two real functions defined on an interval $[a,b)$ (b can be finite or infinite) such that*:

$$\text{(i)}\quad h(z)=h(a)+\sum_{k=0}^{N}a_k(z-a)^{k+\mu}+o\big((z-a)^{N+\mu}\big),$$

$$g(z)=\sum_{k=0}^{N}b_k(z-a)^{k+\alpha-1}+o\big((z-a)^{N+\alpha-1}\big),$$

$$h'(z)=\sum_{k=1}^{N}(k+\mu)a_k(z-a)^{k+\mu-1}+o\big((z-a)^{N+\mu-1}\big)$$

as $z\searrow a$ for all $N\geq 1$, where $a_0\neq 0$, $b_0\neq 0$, $\mu>0$, and $\alpha>0$;

$$\text{(ii)}\quad h(z)>h(a)\quad \textit{for } z\in(a,b), \textit{ and}$$

$$\inf_{z\in[a+\delta,b)}\big(h(z)-h(a)\big)>0\quad \textit{for each } \delta>0;$$

$$\text{(iii)}\quad h' \textit{ and } g\quad \textit{are continuous in a neighbourhood of point } a.$$

If the integral $\int_a^b g(z)\mathrm{e}^{-xh(z)}\,\mathrm{d}z$ converges absolutely for all sufficiently large x, then

$$\int_a^b g(z)\mathrm{e}^{-xh(z)}\,\mathrm{d}z=\mathrm{e}^{-xh(a)}\left(\sum_{k=0}^{N}\Gamma\left(\frac{k+\alpha}{\mu}\right)d_k x^{-(k+\alpha)/\mu}+O\big(x^{-(N+\alpha+1)/\mu}\big)\right)$$

for all $N\in\mathbb{N}$, where $\Gamma(\cdot)$ denotes the Gamma function and coefficients d_k are expressible in terms of a_k and b_k.

Wong [29] also provided the explicit forms of the first three coefficients:

$$d_0=\frac{b_0}{\mu a_0^{\alpha/\mu}},\qquad d_1=\left(\frac{b_1}{\mu}-\frac{(\alpha+1)a_1b_0}{\mu^2a_0}\right)\frac{1}{a_0^{(\alpha+1)/\mu}},$$

$$d_2=\left(\frac{b_2}{\mu}-\frac{(\alpha+2)a_1b_1}{\mu^2a_0}+\big((\alpha+\mu+2)a_1^2-2\mu a_0a_2\big)\frac{(\alpha+2)b_0}{2\mu^3a_0^2}\right)\frac{1}{a_0^{(\alpha+2)/\mu}}.$$

Proof of Theorem 2.2 We use induction on n. The estimate (4) implies the assertion of Theorem 2.2 in the case $n=1$.

Suppose $n = 2$. By the symmetry of the distribution Φ, for $x > 0$, we have

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x) &= 2\int_0^\infty \overline{\Phi}\left(\frac{x}{y}\right)\varphi(y)\,\mathrm{d}y \\ &= \frac{2}{\sqrt{2\pi}}\left(\int_0^{2x^{1/2}} + \int_{2x^{1/2}}^\infty\right)\overline{\Phi}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\ &=: I_1 + I_2. \end{aligned} \tag{8}$$

Obviously,

$$\begin{aligned} I_2 &\le \frac{1}{\sqrt{2\pi}}\int_{2x^{1/2}}^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\int_{2x}^\infty (2u)^{-1/2}\mathrm{e}^{-u}\,\mathrm{d}u \\ &\le C_{11}x^{-1/2}\mathrm{e}^{-2x} \end{aligned} \tag{9}$$

for some positive constant C_{11}.

For integral I_1, due to the first inequality in (4), we have

$$\begin{aligned} I_1 &\le \frac{1}{\pi}\int_0^{2x^{1/2}} \frac{y}{x}\mathrm{e}^{-\frac{1}{2}(\frac{x^2}{y^2}+y^2)}\,\mathrm{d}y = \frac{1}{2\pi}\int_0^4 \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\left(\int_0^1 + \int_1^4\right)\mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\,\mathrm{d}u =: I_{11} + I_{12}, \end{aligned} \tag{10}$$

where the variable change $y = \sqrt{xu}$ was used in the second step.

Using Lemma 3.1 for integral I_{12} (with $\mu = 2$, $\alpha = 1$, $a_0 = b_0 = 1$, $a_1 = -1$, and $b_1 = 0$), we get

$$\begin{aligned} I_{12} &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\Gamma\left(\frac{1}{2}\right)\left(\frac{2}{x}\right)^{1/2}\frac{1}{2} + \Gamma(1)\frac{2}{x}\frac{1}{2} + O\left(x^{-3/2}\right)\right) \\ &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\sqrt{\frac{\pi}{2}}x^{-1/2} + x^{-1} + O\left(x^{-3/2}\right)\right), \end{aligned} \tag{11}$$

because $u + \frac{1}{u}|_{u=1} = 2$, and

$$u + \frac{1}{u} = 2 + \sum_{k=0}^{N}(-1)^{k+2}(u-1)^{k+2} + o\left((u-1)^{N+2}\right) \tag{12}$$

for all $N \ge 1$ due to the Taylor formula.

In a similar way, using Lemma 3.1 (with $\mu = 2$, $\alpha = 1$, $a_0 = b_0 = 1$, $a_1 = -1$, and $b_1 = -2$), we derive

$$\begin{aligned} I_{11} &= \frac{1}{2\pi}\int_0^1 \mathrm{e}^{-\frac{x}{2}(u+\frac{1}{u})}\,\mathrm{d}u = \frac{1}{2\pi}\int_1^\infty \mathrm{e}^{-\frac{x}{2}(v+\frac{1}{v})}\frac{1}{v^2}\,\mathrm{d}v \\ &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\Gamma\left(\frac{1}{2}\right)\left(\frac{2}{x}\right)^{1/2}\frac{1}{2} + \Gamma(1)\frac{2}{x}\left(-\frac{1}{2}\right) + O\left(x^{-3/2}\right)\right) \\ &= \frac{\mathrm{e}^{-x}}{2\pi}\left(\sqrt{\frac{\pi}{2}}x^{-1/2} - x^{-1} + O\left(x^{-3/2}\right)\right), \end{aligned} \tag{13}$$

because of (12) and the following decomposition:

$$\frac{1}{\nu^2}=\sum_{k=0}^{N}(-1)^k(k+1)(\nu-1)^k+o\big((\nu-1)^N\big),$$

according to the Taylor formula.

Estimates (10), (11), and (13) imply that for some $C_{12}>0$,

$$I_1\le\frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}\big(1+C_{12}x^{-1}\big).\tag{14}$$

Substituting the derived estimates (9) and (14) into (8), we obtain

$$\overline{F}_{\xi_1\xi_2}(x)\le\frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}\big(1+C_2x^{-1}\big)\tag{15}$$

with a positive constant C_2.

Now let us consider the lower bound for $\overline{F}_{\xi_1\xi_2}(x)$. Equality (8) and the lower bound in (4) imply

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x)&\ge\frac{2}{\sqrt{2\pi}}\int_0^{2x^{1/2}}\overline{\Phi}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\,\mathrm{d}y\\&\ge\frac{1}{\pi}\int_0^{2x^{1/2}}\frac{y}{x}\left(1-\frac{y^2}{x^2}\right)\mathrm{e}^{-\frac12(y^2+\frac{x^2}{y^2})}\,\mathrm{d}y\\&\ge\frac{1}{2\pi}\int_0^4\left(1-\frac{u}{x}\right)\mathrm{e}^{-\frac{x}{2}(u+\frac1u)}\,\mathrm{d}u\\&\ge\frac{1}{2\pi}\left(1-\frac{4}{x}\right)\int_0^4\mathrm{e}^{-\frac{x}{2}(u+\frac1u)}\,\mathrm{d}u\\&=\left(1-\frac{4}{x}\right)(I_{11}+I_{12}).\end{aligned}$$

Thus, due to the estimates (11) and (13),

$$\overline{F}_{\xi_1\xi_2}(x)\ge\frac{1}{\sqrt{2\pi}}x^{-1/2}\mathrm{e}^{-x}\big(1-C_3x^{-1}\big)\tag{16}$$

for a positive constant C_3. Inequalities (15) and (16) imply the assertion of the theorem in the case $n=2$.

Let us suppose now that formula (7) holds for $n=m$, $m\ge2$, i.e., let

$$\overline{F}_{\Pi_m}(x)=\frac{2^{(m/2)-1}}{\sqrt{\pi m}}x^{-1/m}\exp\left\{-\frac{m}{2}x^{2/m}\right\}\big(1+O_m\big(x^{-2/m}\big)\big).\tag{17}$$

Obviously,

$$\begin{aligned}\overline{F}_{\Pi_{m+1}}(x)&=\frac{2}{\sqrt{2\pi}}\int_0^\infty\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\,\mathrm{d}y\\&=\frac{2}{\sqrt{2\pi}}\left(\int_0^{(m+1)x^{1/(m+1)}}+\int_{(m+1)x^{1/(m+1)}}^\infty\right)\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)\mathrm{e}^{-y^2/2}\,\mathrm{d}y\\&=:J_1+J_2.\end{aligned}\tag{18}$$

According to (4),

$$J_2 \le \frac{2}{\sqrt{2\pi}} \int_{(m+1)x^{1/(m+1)}}^{\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y \le C_{4m} x^{-\frac{1}{m+1}} \exp\left\{-\frac{(m+1)^2}{2} x^{2/(m+1)}\right\} \tag{19}$$

with a positive quantity C_{4m} depending only on m.

Using the induction hypothesis (17) and the variable change $y = x^{1/(m+1)} u^{1/2}$, we get

$$\begin{aligned} J_1 &\le \frac{2^{(m-1)/2}}{\pi\sqrt{m}} \int_0^{(m+1)x^{1/(m+1)}} \left(\frac{y}{x}\right)^{1/m} \\ &\quad \times \exp\left\{-\frac{1}{2}\left(m\left(\frac{x}{y}\right)^{2/m} + y^2\right)\right\}\left(1 + C_{5m}\left(\frac{y}{x}\right)^{2/m}\right)\mathrm{d}y \\ &= \frac{2^{(m-3)/2}}{\pi\sqrt{m}} \int_0^{(m+1)^2} u^{-(m-1)/(2m)} \\ &\quad \times \exp\left\{-\frac{1}{2}x^{2/(m+1)}\left(mu^{-1/m} + u\right)\right\}\left(1 + C_{5m}\left(x^{-2/(m+1)}u^{1/m}\right)\right)\mathrm{d}u \\ &\le \frac{2^{(m-3)/2}}{\pi\sqrt{m}}\left(1 + C_{6m}\left(x^{-2/(m+1)}\right)\right) \\ &\quad \times \int_0^{(m+1)^2} u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}x^{2/(m+1)}\left(mu^{-1/m} + u\right)\right\}\mathrm{d}u \\ &= \frac{2^{(m-3)/2}}{\pi\sqrt{m}}\left(1 + C_{6m}\left(x^{-2/(m+1)}\right)\right)(J_{11} + J_{12}), \end{aligned} \tag{20}$$

where C_{5m} and C_{6m} are positive quantities depending only on m, and

$$\begin{aligned} J_{11} &:= \int_0^1 u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}x^{2/(m+1)}\left(mu^{-1/m} + u\right)\right\}\mathrm{d}u, \\ J_{12} &:= \int_1^{(m+1)^2} u^{-(m-1)/(2m)} \exp\left\{-\frac{1}{2}x^{2/(m+1)}\left(mu^{-1/m} + u\right)\right\}\mathrm{d}u. \end{aligned}$$

Using Lemma 3.1 (with $\mu = 2$, $\alpha = 1$, $a_0 = (m+1)/(2m)$, $a_1 = -(m+1)(2m+1)/(6m^2)$, $b_0 = 1$, and $b_1 = -(m-1)/(2m)$), we obtain

$$\begin{aligned} J_{12} &= \exp\left\{-\frac{m+1}{2}x^{2/(m+1)}\right\}\left(\Gamma\left(\frac{1}{2}\right)\sqrt{\frac{m}{2(m+1)}}\left(\frac{x^{2/(m+1)}}{2}\right)^{-1/2}\right. \\ &\quad \left. + \Gamma(1)\frac{m+5}{2(m+1)}\left(\frac{x^{2/(m+1)}}{2}\right)^{-1} + O_m\left(x^{-3/(m+1)}\right)\right) \\ &= \exp\left\{-\frac{m+1}{2}x^{2/(m+1)}\right\}\left(\sqrt{\frac{\pi m}{m+1}}x^{-1/(m+1)}\right. \\ &\quad \left. + \frac{m+5}{2(m+1)}x^{-2/(m+1)} + O_m\left(x^{-3/(m+1)}\right)\right), \end{aligned} \tag{21}$$

because $mu^{-1/m} + u|_{u=1} = m + 1$, and

$$mu^{-1/m} + u = m + 1 + \sum_{k=0}^{N} (-1)^k \frac{\prod_{l=1}^{k+1}(1 + lm)}{(k + 2)!m^{k+1}} (u - 1)^{k+2} + o\bigl((u - 1)^{N+2}\bigr),$$

$$u^{-(m-1)/(2m)} = 1 + \sum_{k=1}^{N} \frac{(-1)^k \prod_{l=1}^{k}((2l - 1)m - 1)}{k!(2m)^k} (u - 1)^k + o\bigl((u - 1)^N\bigr)$$

for each $N \geq 1$ due to the Taylor formula.

Concerning the term J_{11}, after the variable change $u = 1/v$, we get

$$J_{11} = \int_1^\infty v^{-(3m+1)/(2m)} \exp\biggl\{-\frac{x^{2/(m+1)}}{2}\bigl(mv^{1/m} + v^{-1}\bigr)\biggr\}\, \mathrm{d}v.$$

Therefore, using Lemma 3.1 again (with $\mu = 2$, $\alpha = 1$, $a_0 = (m+1)/(2m)$, $a_1 = -(4m-1)(m+1)/(6m^2)$, $b_0 = 1$, and $b_1 = -(3m + 1)/(2m)$), similarly as in (21), we get that

$$\begin{aligned} J_{11} &= \exp\biggl\{-\frac{m+1}{2}x^{\frac{2}{m+1}}\biggr\}\biggl(\Gamma\biggl(\frac{1}{2}\biggr)\sqrt{\frac{m}{2(m + 1)}}\biggl(\frac{x^{2/(m+1)}}{2}\biggr)^{-1/2} \\ &\quad + \Gamma(1)\frac{m + 5}{6(m + 1)}\biggl(\frac{x^{2/(m+1)}}{2}\biggr)^{-1} + O_m\bigl(x^{-3/(m+1)}\bigr)\biggr) \\ &= \exp\biggl\{-\frac{m+1}{2}x^{2/(m+1)}\biggr\}\biggl(\sqrt{\frac{\pi m}{m + 1}}x^{-1/(m+1)} \\ &\quad - \frac{m + 5}{2(m + 1)}x^{-2/(m+1)} + O_m\bigl(x^{-3/(m+1)}\bigr)\biggr), \end{aligned} \tag{22}$$

because $mv^{1/m} + v^{-1}|_{v=1} = m + 1$ and

$$mv^{1/m} + v^{-1} = m + 1 + \sum_{k=0}^{N} \biggl(\frac{\prod_{l=1}^{k+1}(1 - lm)}{(k + 2)!m^{k+1}} + (-1)^k\biggr)(v - 1)^{k+2} + o\bigl((v - 1)^{N+2}\bigr),$$

$$v^{-(3m+1)/(2m)} = 1 + \sum_{k=1}^{N} \frac{(-1)^k \prod_{l=1}^{k}((2l + 1)m + 1)}{k!(2m)^k} (v - 1)^k + o\bigl((v - 1)^N\bigr),$$

for each $N \geq 1$. Equations (18)–(22) imply that

$$\overline{F}_{\Pi_{m+1}}(x) \leq \frac{2^{(m+1)/2-1}}{\sqrt{\pi(m + 1)}}x^{-1/(m+1)} \exp\biggl\{-\frac{m+1}{2}x^{2/(m+1)}\biggr\} \times \bigl(1 + C_{7m}x^{-2/m+1}\bigr) \tag{23}$$

with a positive quantity C_{7m} depending only on m.

On the other hand, by hypothesis (17), similarly as in the derivation of (20), we get

$$\begin{aligned}\overline{F}_{\Pi_{m+1}}(x) &\geq \frac{2^{(m+1)/2-2}}{\pi\sqrt{m}}\bigl(1-C_{8m}x^{-2/(m+1)}\bigr)\\&\quad\times\int_0^{(m+1)^2}u^{-(m-1)/(2m)}\exp\biggl\{-\frac{x^{2/(m+1)}}{2}\bigl(mu^{-1/m}+u\bigr)\biggr\}\,\mathrm{d}u\\&=\frac{2^{(m+1)/2-2}}{\pi\sqrt{m}}\bigl(1-C_{8m}x^{-2/(m+1)}\bigr)(J_{11}+J_{12})\end{aligned}$$

with a positive quantity C_{8m}. Using estimates (21) and (22), we obtain

$$\begin{aligned}\overline{F}_{\Pi_{m+1}}(x) &\geq \frac{2^{(m+1)/2}-1}{\sqrt{\pi(m+1)}}x^{-1/(m+1)}\exp\biggl\{-\frac{m+1}{2}x^{2/(m+1)}\biggr\}\\&\quad\times\bigl(1-C_{9m}x^{-2/(m+1)}\bigr)\end{aligned}$$

with a positive quantity C_{9m} depending only on m, which together with (23) implies formula (7) in the case $n = m + 1$. The induction principle implies now the validity of (7) for all $n \in \mathbb{N}$. Theorem 2.2 is proved. □

4 A small simulation study

In this section, we consider two examples of products of normally distributed random variables. In both cases, we will compare the obtained theoretical results with the tail probabilities estimated by the Monte Carlo method.

Example 4.1 Consider the product Π_4 of four independent random variables $\xi_1, \xi_2, \xi_3, \xi_4$ distributed according to the normal law Φ. According to Theorem 2.2, there exist positive constants D_1 and D_2 such that for $x \geq D_2$,

$$\frac{1}{\sqrt{\pi}}x^{-1/4}\mathrm{e}^{-2\sqrt{x}}\biggl(1-\frac{D_1}{\sqrt{x}}\biggr)\leq\mathbb{P}(\Pi_4>x)\leq\frac{1}{\sqrt{\pi}}x^{-1/4}\mathrm{e}^{-2\sqrt{x}}\biggl(1+\frac{D_1}{\sqrt{x}}\biggr).$$

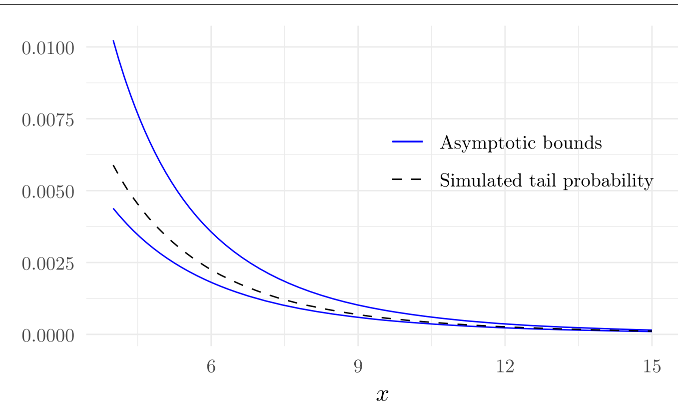

Figure 1 Simulated values and asymptotic bounds for the tail probability of Π_4

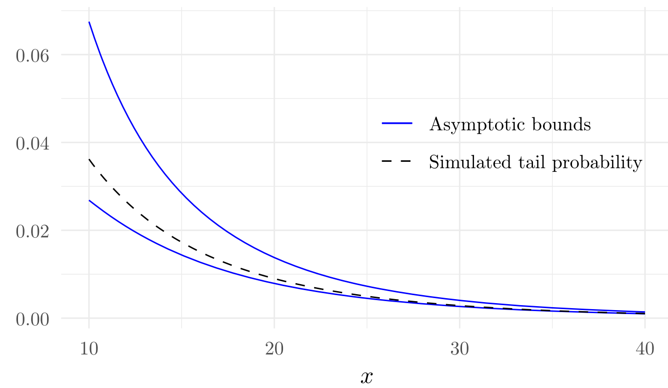

Figure 2 Simulated values and asymptotic bounds for the tail probability of $\widehat{\Pi}_3$

The simulated values of $\mathbb{P}(\Pi_4 > x)$, together with the upper and lower estimates (blue lines), are presented in Fig. 1. Several values of constants D_1 and D_2 can be selected from the obtained graphs. For instance, Fig. 1 shows that it is possible to take $D_1 = 0.8$ and $D_2 = 6$ in the case under consideration.

Example 4.2 Consider the product $\widehat{\Pi}_3$ of three independent random variables $\widehat{\xi}_1$, $\widehat{\xi}_2$, and $\widehat{\xi}_3$ distributed according to the normal laws $\Phi_{0,1}$, $\Phi_{0,2}$, and $\Phi_{0,3}$, respectively. Due to Corollary 2.1, there exist two positive constants $\widehat{D}_1$ and $\widehat{D}_2$ such that for $x \geq \widehat{D}_2$,

$$\mathbb{P}(\widehat{\Pi}_3 > x) \geq \left(1 - \frac{\widehat{D}_1}{\sqrt[3]{x^2}}\right)\sqrt{\frac{2}{3\pi}}\left(\frac{x}{6}\right)^{-1/3} \exp\left\{-\frac{3}{2}\left(\frac{x}{6}\right)^{2/3}\right\},$$

$$\mathbb{P}(\widehat{\Pi}_3 > x) \leq \left(1 + \frac{\widehat{D}_1}{\sqrt[3]{x^2}}\right)\sqrt{\frac{2}{3\pi}}\left(\frac{x}{6}\right)^{-1/3} \exp\left\{-\frac{3}{2}\left(\frac{x}{6}\right)^{2/3}\right\}.$$

The simulated values of $\mathbb{P}(\widehat{\Pi}_3 > x)$, together with the upper and lower bounds (blue lines), are presented in Fig. 2. In particular, Fig. 2 shows that it is possible to take $\widehat{D}_1 = 2$ and $\widehat{D}_2 = 15$ in this example.

5 Conclusions

In this paper, we obtained an asymptotic formula with the first leading-order term for the tail of the distribution of the product of independent normal laws. The main results of the work are formulated in Theorem 2.2 and Corollary 2.1. We considered normal random variables with zero means only, but in Corollary 2.1 we allowed the variances of these laws to be unconstrained. From the assertion of Theorem 2.2, it follows that for the product of n independent standard normal distributions Π_n:

$$\mathbb{P}(\Pi_n > x) \geq \left(1 - \frac{C_{2n}}{\sqrt[n]{x^2}}\right)\frac{1}{\sqrt{\pi n}}x^{-1/n} \exp\left\{-\frac{n}{2}x^{2/n}\right\},$$

$$\mathbb{P}(\Pi_n > x) \leq \left(1 + \frac{C_{3n}}{\sqrt[n]{x^2}}\right)\frac{1}{\sqrt{\pi n}}x^{-1/n} \exp\left\{-\frac{n}{2}x^{2/n}\right\}$$

if $x \geq C_{1n}$, where C_{1n}, C_{2n}, and C_{3n} are positive quantities depending only on n. It follows from the theorem that such three quantities exist. However, their structure remains unclear. Apparently, this requires a deeper analysis of the problem.

Acknowledgements
The authors would like to thank editor in chief and the anonymous referees for their valuable comments.

Funding
The research is supported by a grant No. S-MIP-20-16 from the Research Council of Lithuania.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
R.L. did a major contribution in conceptualization, supervision of the project and funding acquisition. J.Š. dealt with the methodology, investigation and theoretical part of the project. M.D. participated in the design of the manuscript and performed the simulation study. R.Z. performed the validation and writing the initial version of the manuscript. All authors read and approved the final manuscript.

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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 May 2022 Accepted: 13 February 2023 Published online: 24 February 2023

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