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Gama skirstinių sandaugos uodegos elgesys

Tail Behaviour of the Product of Gamma Distributions

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Santrauka

Sakykime $\xi_1, \xi_2, \ldots, \xi_n$ yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, o $\Pi_n := \prod_{k=1}^{n} \xi_k$ yra šių dydžių sandauga. Be to, sakykime, kad visi atsitiktiniai dydžiai pasiskirstę pagal gama skirstinį. Šiame darbe nagrinėjamas asimptotinis tikimybės $P(\Pi_n > x)$ elgesys. Įrodomos dvi teoremos skirstinių sandaugos asimptotinei formulei. Teoremų įrodymui taikomas matematinės indukcijos metodas.

Raktiniai žodžiai : Gama skirstinys, uodegos pasiskirstymo funkcija, balno taškas, asimptotinės funkcijos, atsitiktinių dydžių sandauga.

Tail Behaviour of the Product of Gamma Distributions

Abstract

Let $\xi_1, \xi_2, \ldots, \xi_n$ be a sequence of independent identically distributed random variables and let $\Pi_n := \prod_{k=1}^{n} \xi_k$ be the product of the variables. Also, we assume that all random variables are gamma-distributed. This paper considers the asymptotic behaviour for the probability $P(\Pi_n > x)$. We will prove 2 theorems for the asymptotic formula of product of distributions. All theorems are proved by the mathematical induction method.

Key words : Gamma distribution, tail distribution function, saddle point method, asymptotic functions, product of random variables.

Turinys

1. Įvadas

Pasigilinus į literatūrą, galima teigti, jog nepriklausomų atsitiktinių dydžių sandaugos teorija yra kur kas mažiau tyrinėta negu nepriklausomų atsitiktinių dydžių sumų teorija ([6], [11]). Didžioji dalis sandaugos teorijos literatūros yra skirta normaliesiems atsitiktiniams dydžiams (pvz., [5], [10]), tačiau galima rasti straipsnių ir apie gama atsitiktinius dydžius.

Nadarajah ir Kotz ([8], [9]) savo darbuose gavo tikslias pasiskirstymo funkcijas sudauginę atitinkamai du gama ir beta bei gama ir Veibulio skirstinius, o Malik [4] išreiškė dviejų apibendrintų gama skirstinių tankio funkciją. Springer ir Thompson [1], naudodami Melino transformaciją, gavo tikslią n gama skirstinų sandaugos formulę, kuri yra išreiškiama per Meijerio G-funkciją. Šio straipsnio 1 teoremoje yra gauta, kad:

$$f_{\Pi_n}(x) = \frac{1}{n\,\Gamma(\alpha)} G_{0N}^{N0}\left(x \mid \alpha-1, \alpha-1, \cdots, \alpha-1\right) = \frac{1}{2\pi n\Gamma(\alpha)} \int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma^n(s+\alpha-1)ds,$$

čia f_{Π_n} *yra mūsų nagrinėjamos gama skirstinių sandaugos tankio funkcija,* c *bet koks teigiamas skaičius, o* G *yra Meijerio* G *- funkcija, apibrėžiama lygybe*

$$G_{pq}^{mn}\left(z \mid \begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} \frac{\prod_{j=1}^{m}\Gamma\left(s+b_j\right)\cdot\prod_{j=1}^{n}\Gamma\left(1-a_j-s\right)}{\prod_{j=n+1}^{p}\Gamma\left(s+a_j\right)\cdot\prod_{j=m+1}^{q}\Gamma\left(1-b_j-s\right)}ds.$$

Šiame darbe bus tiriamas asimptotinis gama skirstinių sandaugos uodegos funkcijos elgesys, taikant balno taško metodą bei Arendarczyk ir Debicki [2] įrodytą teoremą. Balno taško metodui bus naudojama pagalbinė Wong [7] lema, kuri padės įvertinti specialios formos integralą. Suformuluosime dvi teoremas skirstinių sandaugos asimptotinei formulei. Pirmoje teoremoje gausime bendrojo nario išraišką, o antroje - bendrojo nario su liekana išraišką. Abi teoremos įrodomos matematinės indukcijos metodu.

Sakome, kad atsitiktinis dydis ξ *yra pasiskirstęs pagal gama skirstinį, jeigu jo pasiskirstymą nusako tankio funkcija*

$$f_\xi(x) = \frac{x^{\alpha-1}e^{-\beta x}\beta^\alpha}{\Gamma(\alpha)}, \quad x > 0, \text{ parametrai } \alpha > 0, \beta > 0.$$

Taigi a.d. ξ su gama skirstiniu pasiskirstymo funkcija $F_\xi(x) = \mathbb{P}(\xi \leq x)$ turi pavidalą:

$$F_\xi(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-\beta y}dy, \ x > 0.$$

Gerai žinoma, kad eksponentinis, Erlango ir χ^2 skirstiniai yra atskiri gama skirstinio atvejai. Eksponentinis skirstinys gaunamas parinkus parametrą $\alpha = 1$, Erlango skirstinys gaunamas parinkus $\alpha \in \mathbb{N}$, o $\chi^2 \sim Gamma\left(\frac{v}{2}, \frac{1}{2}\right)$, kur v yra skirstinio χ^2 laisvės laipsnių skaičius.

Atsitiktinio dydžio ξ skirstinio uodegos funkcija $\overline{F}(x)$ tenkina tokią asimptotinę lygybę (žr. įrodymą 4):

$$\overline{F}_\xi(x) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(x^{-1}\right)\right) \tag{1}$$

2. Skirstinių sandaugos asimptotinė formulė

Arendarczyk ir Debicki [7], pasinaudoję balno taško metodu, suformulavo teoremą 2.1, kuria naudojantis galima įvertinti Veibulio tipo uodegų skirstinių sandaugą. Mūsų teoremos 2.2 įrodymui naudosime tokį Arendarczyk ir Debicki [7] teiginį

Teorema 2.1 *Tarkime, kad ξ_1 ir ξ_2 yra du nepriklausomi, neneigiami atsitiktiniai dydžiai tokie, kad*

$$\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\sim} C_1 x^{\gamma_1} \exp\left\{-\beta_1 x^{\alpha_1}\right\}, \quad \overline{F}_{\xi_2}(x) \underset{x\to\infty}{\sim} C_2 x^{\gamma_2} \exp\left\{-\beta_2 x^{\alpha_2}\right\},$$

kur $C_i > 0, \gamma_i \in \mathbb{R}, \beta_i > 0, \alpha_i > 0, i = 1, 2$. Tada

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} C x^{\gamma} \exp\left\{-\beta x^{\alpha}\right\}, \ kur$$

$$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$

$$\beta = \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left(\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right),$$

$$\gamma = \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2\left(\alpha_1+\alpha_2\right)},$$

$$C = \sqrt{2\pi}\frac{C_1C_2}{\sqrt{\alpha_1+\alpha_2}} \left(\alpha_1\beta_1\right)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}} \left(\alpha_2\beta_2\right)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}.$$

Teorema 2.2 *Tegul $\xi_1, \xi_2, \ldots$ yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, tokie, kad $\forall k, \xi_k \sim Gamma\,(\alpha; \beta)$ ir tegul $\Pi_n := \prod_{k=1}^{n} \xi_k$. Tuomet*

$$\overline{F}_{\Pi_n}(x) \underset{x\to\infty}{\sim} \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \frac{\beta^{n(\alpha-1)+\frac{n-1}{2}}}{\Gamma^n(\alpha)} x^{\alpha-\frac{n+1}{2n}} e^{-n\beta x^{\frac{1}{n}}} \tag{2}$$

TEOREMOS ĮRODYMAS

Pagal 2.1 teoremą nesunkiai galime pastebėti, kad jei $n = 2$, $n = 3$ ir $n = 4$, tai

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} \sqrt{\pi} \frac{\beta^{2(\alpha-1)+\frac{1}{2}}}{\Gamma^2(\alpha)} x^{\alpha-\frac{3}{4}} \exp\left\{-2\beta x^{\frac{1}{2}}\right\},$$

$$\overline{F}_{\xi_1\xi_2\xi_3}(x) \underset{x\to\infty}{\sim} \frac{2\pi}{\sqrt{3}} \frac{\beta^{3(\alpha-1)+1}}{\Gamma^3(\alpha)} x^{\alpha-\frac{2}{3}} \exp\left\{-3\beta x^{\frac{1}{3}}\right\},$$

$$\overline{F}_{\xi_1\xi_2\xi_3\xi_4}(x) \underset{x\to\infty}{\sim} \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{4}} \frac{\beta^{4(\alpha-1)+\frac{3}{2}}}{\Gamma^4(\alpha)} x^{\alpha-\frac{5}{8}} \exp\left\{-4\beta x^{\frac{1}{4}}\right\}.$$

Iškeliame hipotezę mūsų nagrinėjamai funkcijai:

$$\overline{F}_{\Pi_n}(x) \underset{x\to\infty}{\sim} \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \frac{\beta^{n(\alpha-1)+\frac{n-1}{2}}}{\Gamma^n(\alpha)} x^{\alpha-\frac{n+1}{2n}} \exp\left\{-n\beta x^{\frac{1}{n}}\right\}$$

Jeigu $n = 1$, $n = 2$, tai hipotezė teisinga. Sakykime, kad hipotezė teisinga, kai $n = m$. Įrodysime, kad f-lė teisinga, kai $n = m + 1$. Kadangi

$$\prod_{k=1}^{m+1} \xi_k = \xi_{m+1} \cdot \prod_{k=1}^{m} \xi_k,$$

tai atvejui $m+1$ galime taikyti 2.1 teoremą. Nagrinėjamu atveju turime:

$$\begin{aligned}
\alpha_1 &= \frac{1}{m}, & \alpha_2 &= 1,\\
\beta_1 &= N\beta, & \beta_2 &= 1,\\
\gamma_1 &= \alpha - \frac{m+1}{2m}, & \gamma_2 &= \alpha - 1,\\
C_1 &= \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{m(\alpha-1)+\frac{m-1}{2}}}{\Gamma^m(\alpha)}, & C_2 &= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}$$

Pagal 2.1 teoremą nesunkiai galime pastebėti, kad:

$$\alpha = \frac{\frac{1}{m}\cdot 1}{\frac{1}{m}+1} = \frac{1}{m+1},$$

$$\beta = (m\beta)^{\frac{1}{\frac{1}{m}+1}} \beta^{\frac{\frac{1}{m}}{\frac{1}{m}+1}} \left(\left(\frac{1}{m}^{\frac{1}{\frac{1}{m}+1}}\right) + \left(m^{\frac{\frac{1}{m}}{\frac{1}{m}+1}}\right)\right) = \beta\,(m+1)\,,$$

$$\gamma = \frac{\frac{1}{m} + 2\cdot\frac{1}{m}(\alpha-1) + 2\cdot 1\cdot(\alpha - \frac{m+1}{2m})}{2(\frac{1}{m}+1)} = \alpha - \frac{m+2}{2(m+1)}$$

$$\begin{aligned}
C &= \sqrt{2\pi}\,\frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{m(\alpha-1)+\frac{m-1}{2}}}{\Gamma^m(\alpha)} \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \frac{\sqrt{m}}{\sqrt{m+1}} \left(\frac{1}{m}m\beta\right)^{\frac{1}{2(\frac{1}{m}+1)}} \beta^{\frac{\frac{1}{m}}{2(\frac{1}{m}+1)}}\\
&= \frac{(2\pi)^{\frac{m}{2}}}{\sqrt{m+1}} \frac{\beta^{(m+1)(\alpha-1)+\frac{m}{2}}}{\Gamma^{(m+1)}(\alpha)}
\end{aligned}$$

Gautos išraiškos rodo, kad formulė teisinga ir $n = m+1$. Pagal indukcijos metodą gauname, jog formulė teisinga bet kuriam $n \in \mathbb{N}$.

3. Skirstinių sandaugos asimptotinė formulė su liekana

Šiame skyrelyje patikslinsime tą pačią 2.2 teoremą. Naudodamiesi balno taško metodu, gausime asimptotinę tikimybės $P(\Pi_n > x)$ išraišką su liekana.

Teorema 3.1 *Tegul $\xi_1, \xi_2, \ldots$ yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, tokie, kad $\forall k, \xi_k \sim Gamma\,(\alpha;\beta)$ ir tegul $\Pi_n := \prod_{k=1}^n \xi_k$. Tuomet*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \frac{\beta^{n(\alpha-1)+\frac{n-1}{2}}}{\Gamma^n(\alpha)} x^{\alpha-\frac{n+1}{2n}} e^{-n\beta x^{\frac{1}{n}}} \left(1 + O_n\left(x^{-\frac{1}{n}}\right)\right), \tag{3}$$

kur konstanta simbolyje O_n priklauso nuo n.

Prieš pradedant teoremos įrodymą, suformuluosime pagalbinę lemą, kurią naudosime daugelį kartų savo įrodyme. Ši lema gali būti gauta naudojant balno taško metodą specialios formos realiam integralui. 3.1 Lemos įrodymas yra aprašytas [3] (žr. Teorema 1 Sekcijoje 2).

Lema 3.1 *Tarkime, kad h ir g yra dvi realios funkcijos apibrėžtos intervale $[a,b)$ (b gali būti baigtinis arba begalinis), tokios, kad:*

(i) $$h(z)=h(a)+\sum_{k=0}^{N}a_k(z-a)^{k+\mu}+o\left((z-a)^{N+\mu}\right)$$
$$g(z)=\sum_{k=0}^{N}b_k(z-a)^{k+\alpha-1}+o\left((z-a)^{N+\alpha-1}\right)$$
$$h'(z)=\sum_{k=1}^{N}(k+\mu)a_k(z-a)^{k+\mu-1}+o\left((z-a)^{N+\mu-1}\right)$$

kai $z\searrow a$ visiems $N\geq 1$, kur $a_0\neq 0, b_0\neq 0, \mu>0$ ir $\alpha>0$;

(ii) $h(z)>h(a)$ *for* $z\in(a,b)$, *ir*

$$\inf_{z\in[a+\delta,b)}(h(z)-h(a))>0 \text{ kiekvienam } \delta>0;$$

(iii) h' and g yra tolydžios taško a aplinkoje.

Jeigu integralas $\int_a^b g(z)\mathrm{e}^{-xh(z)}\mathrm{d}z$ konverguoja absoliučiai visiems dideliems x, tada

$$\int_a^b g(z)\mathrm{e}^{-xh(z)}\mathrm{d}z=\mathrm{e}^{-xh(a)}\left(\sum_{k=0}^{N}\Gamma\left(\frac{k+\alpha}{\mu}\right)d_kx^{-(k+\alpha)/\mu}+O\left(x^{-(N+\alpha+1)/\mu}\right)\right)$$

visiems $N\in\mathbb{N}$, kur $\Gamma(\cdot)$ žymi gamą funkciją ir koeficientai d_k yra išreiškiami per a_k ir b_k. Pateikiame pirmų trijų d_k koeficientų tikslias išraiškas:

$$d_0=\frac{b_0}{\mu a_0^{\alpha/\mu}},\quad d_1=\left(\frac{b_1}{\mu}-\frac{(\alpha+1)a_1b_0}{\mu^2a_0}\right)\frac{1}{a_0^{(\alpha+1)/\mu}},$$
$$d_2=\left(\frac{b_2}{\mu}-\frac{(\alpha+2)a_1b_1}{\mu^2a_0}+\left((\alpha+\mu+2)a_1^2-2\mu a_0a_2\right)\frac{(\alpha+2)b_0}{2\mu^3a_0^2}\right)\frac{1}{a_0^{(\alpha+2)/\mu}}.$$

TEOREMOS ĮRODYMAS

Pagal (1) matome, kad teiginys tenkinamas, kai $n=1$. Tarkime, kad $n=2$. Tuomet $\forall x>0$, turime, kad

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x)&=\int_0^\infty\overline{F}\left(\frac{x}{y}\right)f(y)dy=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty\overline{F}\left(\frac{x}{y}\right)y^{\alpha-1}e^{-\beta y}dy\\&=\frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)}x^{\alpha-1}\int_0^\infty\exp\left\{-\beta\left(\frac{x}{y}+y\right)\right\}\left(1+O\left(\frac{y}{x}\right)\right)dy\\&=\frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)}x^{\alpha-1}\left(\int_0^{2x^{\frac{1}{2}}}+\int_{2x^{\frac{1}{2}}}^\infty\right)\exp\left\{-\beta\left(\frac{x}{y}+y\right)\right\}\left(1+O\left(\frac{y}{x}\right)\right)dy\\&:=I_1+I_2.\end{aligned}\tag{4}$$

Galime lengvai pastebėti, kad

$$I_2 \leq \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{2x^{\frac{1}{2}}}^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_2^\infty x^{\frac{\alpha}{2}} u^{\alpha-1} \exp\left\{-\beta u \sqrt{x}\right\} \leq C_1 x^{\frac{\alpha}{2}} \exp\left\{-2\beta\sqrt{x}\right\} \tag{5}$$

kažkokiai teigiamai konstantai C_1.

Integralui I_1, pritaikius kintamojo keitinį $y = u\sqrt{x}$, gauname, kad

$$\begin{aligned} I_1 &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-1} \int_0^{2x^{\frac{1}{2}}} \exp\left\{-\beta\left(\frac{x}{y}+y\right)\right\} \left(1+O\left(\frac{y}{x}\right)\right) dy \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \int_0^2 \exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\} \left(1+O\left(ux^{-\frac{1}{2}}\right)\right) du \\ &\leq \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \int_0^2 \exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\} du \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \left(\int_0^1 + \int_1^2\right) \exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\} du =: I_{11} + I_{12}. \end{aligned} \tag{6}$$

Naudojantis 3.1 lema, integralui I_{11} (su $\mu = 2$, $\alpha = 1$, $a_0 = b_0 = 1$, $a_1 = -1$, $b_1 = -2$ ir atitinkamai išskaičiuotais parametrais $d_0 = \frac{1}{2}$ ir $d_1 = -\frac{1}{2}$) gauname, kad

$$\begin{aligned} I_{11} &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \int_0^1 \exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\} du \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \int_1^\infty \frac{1}{v^2} \exp\left\{-\beta\sqrt{x}\left(\frac{1}{v}+v\right)\right\} du \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \exp\left\{-2\beta\sqrt{x}\right\} \\ &\times \left(\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\left(\beta\sqrt{x}\right)^{-\frac{1}{2}} + \Gamma\left(1\right)\left(-\frac{1}{2}\right)\left(\beta\sqrt{x}\right)^{-1} + O\left(x^{-\frac{3}{4}}\right)\right) \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \exp\left\{-2\beta\sqrt{x}\right\} \left(\frac{\sqrt{\pi}}{2}\left(\beta\sqrt{x}\right)^{-\frac{1}{2}} - \frac{\left(\beta\sqrt{x}\right)^{-1}}{2} + O\left(x^{-\frac{3}{4}}\right)\right), \end{aligned} \tag{7}$$

kadangi $v + \frac{1}{v}\big|_{v=1} = 2$ ir

$$v + \frac{1}{v} = 2 + \sum_{k=0}^N (-1)^{k+2}(v-1)^{k+2} + o\left((v-1)^{N+2}\right), \tag{8}$$

$$\frac{1}{v^2} = \sum_{k=0}^N (-1)^k (k+1)(v-1)^k + o\left((v-1)^N\right), \tag{9}$$

kiekvienam $N \geq 1$ dėl Teiloro formulės.

Identišku principu, naudojantis 3.1 lema ir (8), integralui I_{12} (su $\mu = 2$, $\alpha = 1$, $a_0 = b_0 = 1$, $a_1 = -1$, $b_1 = 0$ ir atitinkamai išskaičiuotais parametrais $d_0 = d_1 = \frac{1}{2}$) gauname, kad

$$\begin{aligned} I_{12} &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \int_1^2 \exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\} du \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \exp\left\{-2\beta\sqrt{x}\right\} \left(\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\left(\beta\sqrt{x}\right)^{-\frac{1}{2}} + \Gamma\left(1\right)\frac{1}{2}\left(\beta\sqrt{x}\right)^{-1} + O\left(x^{-\frac{3}{4}}\right)\right) \\ &= \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O\left(x^{-\frac{1}{2}}\right)\right) \exp\left\{-2\beta\sqrt{x}\right\} \left(\frac{\sqrt{\pi}}{2}\left(\beta\sqrt{x}\right)^{-\frac{1}{2}} + \frac{\left(\beta\sqrt{x}\right)^{-1}}{2} + O\left(x^{-\frac{3}{4}}\right)\right). \end{aligned}$$

$$\tag{10}$$

Įsistačius gautus (6), (7) bei (10) įverčius gauname, kad

$$I_1 = \sqrt{\pi}\, \frac{\beta^{2(\alpha-1)+\frac{1}{2}}}{\Gamma^2(\alpha)}\, x^{\alpha-\frac{3}{4}}\, \exp\left\{-2\beta x^{\frac{1}{2}}\right\}\left(1+O\left(x^{-\frac{1}{2}}\right)\right) \tag{11}$$

Galiausiai, įsistatę (5) ir (11) įverčius į (4), gauname, kad

$$\overline{F}_{\xi_1\xi_2}(x) \le \sqrt{\pi}\, \frac{\beta^{2(\alpha-1)+\frac{1}{2}}}{\Gamma^2(\alpha)}\, x^{\alpha-\frac{3}{4}}\, \exp\left\{-2\beta x^{\frac{1}{2}}\right\}\left(1+C_2 x^{-\frac{1}{2}}\right) \tag{12}$$

su teigiama konstanta C_2.

Nagrinėkime apatinį rėžį funkcijai $\overline{F}_{\xi_1\xi_2}(x)$. Naudojantis (4), (7) bei (10) nesunkiai galime pastebėti, kad

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x) &\ge \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^{2x^{\frac{1}{2}}} \overline{F}\left(\frac{x}{y}\right) y^{\alpha-1}e^{-\beta y}dy = \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)}x^{\alpha-1}\int_0^{\infty}\exp\left\{-\beta\left(\frac{x}{y}+y\right)\right\}\left(1+O\left(\frac{y}{x}\right)\right)dy\\ &\ge \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)}x^{\alpha-1}\int_0^{2x^{\frac{1}{2}}}\exp\left\{-\beta\left(\frac{x}{y}+y\right)\right\}\left(1-\frac{C_3 y}{x}\right)dy\\ &\ge \frac{\beta^{2\alpha-1}}{\Gamma^2(\alpha)}\, x^{\alpha-\frac{1}{2}}\left(1-C_3x^{-\frac{1}{2}}\right)\left(\int_0^1+\int_1^2\right)\exp\left\{-\beta\sqrt{x}\left(\frac{1}{u}+u\right)\right\}du\\ &= \sqrt{\pi}\,\frac{\beta^{2(\alpha-1)+\frac{1}{2}}}{\Gamma^2(\alpha)}\, x^{\alpha-\frac{3}{4}}\, \exp\left\{-2\beta x^{\frac{1}{2}}\right\}\left(1-C_4x^{-\frac{1}{2}}\right)\end{aligned} \tag{13}$$

su teigiamomis konstantomis C_3, C_4. Iš nelygybių gauname, kad kai $n=2$, teorema teisinga.

Tarkime, kad formulė teisinga, kai $n=m$, $m\ge 2$ t.y. tegul

$$\overline{F}_{\Pi_m}(x) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}\,\frac{\beta^{m(\alpha-1)+\frac{m-1}{2}}}{\Gamma^m(\alpha)}\, x^{\alpha-\frac{m+1}{2m}}\, \exp\left\{-m\beta x^{\frac{1}{m}}\right\}\left(1+O_m\left(x^{-\frac{1}{m}}\right)\right). \tag{14}$$

Akivaizdu, kad

$$\begin{aligned}\overline{F}_{\Pi_{m+1}}(x) &= \int_0^\infty \overline{F}_{\Pi_m}\left(\frac{x}{y}\right)f(y)dy = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \overline{F}_{\Pi_m}\left(\frac{x}{y}\right)y^{\alpha-1}e^{-\beta y}dy\\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\int_0^{(m+1)x^{\frac{1}{m+1}}}+\int_{(m+1)x^{\frac{1}{m+1}}}^\infty\right)\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)y^{\alpha-1}e^{-\beta y}dy := J_1+J_2\end{aligned} \tag{15}$$

Nariui J_2 pagal (5) gauname, kad

$$\begin{aligned}J_2 &\le \frac{\beta^\alpha}{\Gamma(\alpha)}\int_{(m+1)x^{\frac{1}{m+1}}}^\infty y^{\alpha-1}e^{-\beta y}dy = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_{m+1}^\infty x^{\frac{\alpha}{m+1}}u^{\alpha-1}\exp\left\{-\beta u x^{\frac{1}{m+1}}\right\}du\\ &\le C_{4m}x^{\frac{\alpha}{m+1}}\exp\left\{-(m+1)\beta x^{\frac{1}{m+1}}\right\}\end{aligned} \tag{16}$$

su teigiama konstanta C_{4m} priklausiančia tik nuo m.

Naudojantis indukcijos hipoteze (14) ir kintamojo keitiniu $y = ux^{\frac{1}{m+1}}$, gauname, kad

$$
\begin{aligned}
J_1 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{(m+1)x^{\frac{1}{m+1}}} \overline{F}_{\Pi_m}\left(\frac{x}{y}\right) y^{\alpha-1} e^{-\beta y} dy \\
&= \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{m+1}{2m}} \int_0^{(m+1)x^{\frac{1}{m+1}}} y^{\frac{1-m}{2m}} \exp\left\{-\beta\left(y+m\left(\frac{x}{y}\right)^{\frac{1}{m}}\right)\right\}\left(1+O_m\left(\frac{y}{x}\right)^{\frac{1}{m}}\right) dy \\
&= \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{1}{2}} \int_0^{m+1} u^{\frac{1-m}{2m}} \exp\left\{-\beta x^{\frac{1}{m+1}}\left(u+mu^{-\frac{1}{m}}\right)\right\}\left(1+O_m\left(u^{\frac{1}{m}} x^{-\frac{1}{m+1}}\right)\right) du \\
&\le \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O_m\left(x^{-\frac{1}{m+1}}\right)\right) \\
&\quad \times \left(\int_0^1 + \int_1^{m+1}\right) u^{\frac{1-m}{2m}} \exp\left\{-\beta x^{\frac{1}{m+1}}\left(u+mu^{-\frac{1}{m}}\right)\right\} du =: J_{11} + J_{12}.
\end{aligned}
\tag{17}
$$

Nariui J_{12}, naudojantis 3.1 lema (su $\mu = 2$, $\alpha = 1$, $a_0 = \dfrac{m+1}{2m}$, $b_0 = 1$, $a_1 = \dfrac{-(m+1)(2m+1)}{6m^2}$, $b_1 = \dfrac{1-m}{2m}$ ir atitinkamai išskaičiuotais parametrais $d_0 = \sqrt{\dfrac{m}{2(m+1)}}$ bei $d_1 = \dfrac{m+5}{6(m+1)}$), turime, kad

$$
\begin{aligned}
J_{12} &= \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O_m\left(x^{-\frac{1}{m+1}}\right)\right) \exp\left\{-\beta(m+1)x^{\frac{1}{m+1}}\right\} \\
&\quad \times \left(\Gamma\left(\frac{1}{2}\right)\sqrt{\frac{m}{2(m+1)}}\left(\beta x^{\frac{1}{m+1}}\right)^{-\frac{1}{2}} + \Gamma(1)\frac{m+5}{6(m+1)}\left(\beta x^{\frac{1}{m+1}}\right)^{-1} + O\left(x^{-\frac{3}{2(m+1)}}\right)\right),
\end{aligned}
\tag{18}
$$

kadangi $u + mu^{-\frac{1}{m}}\Big|_{u=1} = m+1$ ir

$$
u + mu^{-\frac{1}{m}} = m+1+\sum_{k=0}^{N}(-1)^k \frac{\prod_{l=1}^{k+1}(1+lm)}{(k+2)!m^{k+1}}(u-1)^{k+2} + o\left((u-1)^{N+2}\right), \tag{19}
$$

$$
u^{\frac{1-m}{2m}} = 1+\sum_{k=1}^{N} \frac{(-1)^k \prod_{l=1}^{k}((2l-1)m-1)}{k!(2m)^k}(u-1)^k + o\left((u-1)^N\right), \tag{20}
$$

kiekvienam $N \ge 1$ dėl Teiloro formulės.
Nariui J_{11} taikome kintamojo keitinį $u = \dfrac{1}{v}$. Iš to pastebime, kad

$$
J_{11} = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O_m\left(x^{-\frac{1}{m+1}}\right)\right) \int_1^{\infty} v^{-\frac{3m+1}{2m}} \exp\left\{-\beta x^{\frac{1}{m+1}}\left(\frac{1}{v}+mv^{\frac{1}{m}}\right)\right\} dv.
$$

Dar kartą pasinaudoję 3.1 lema (su $\mu = 2$, $\alpha = 1$, $a_0 = \dfrac{m+1}{2m}$, $b_0 = 1$, $a_1 = \dfrac{-(4m-1)(m+1)}{6m^2}$, $b_1 = -\dfrac{3m+1}{2m}$ ir atitinkamai išskaičiuotais parametrais $d_0 = \sqrt{\dfrac{m}{2(m+1)}}$ bei $d_1 = -\dfrac{m+5}{6(m+1)}$), gauname, jog

$$
\begin{aligned}
J_{11} &= \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)} x^{\alpha-\frac{1}{2}} \left(1+O_m\left(x^{-\frac{1}{m+1}}\right)\right) \exp\left\{-\beta(m+1)x^{\frac{1}{m+1}}\right\} \\
&\quad \times \left(\Gamma\left(\frac{1}{2}\right)\sqrt{\frac{m}{2(m+1)}}\left(\beta x^{\frac{1}{m+1}}\right)^{-\frac{1}{2}} - \Gamma(1)\frac{m+5}{6(m+1)}\left(\beta x^{\frac{1}{m+1}}\right)^{-1} + O\left(x^{-\frac{3}{2(m+1)}}\right)\right),
\end{aligned}
\tag{21}
$$

kadangi $\dfrac{1}{v}+mv^{\frac{1}{m}}=m+1$ ir

$$\frac{1}{v}+mv^{\frac{1}{m}}=m+1+\sum_{k=0}^{N}\left(\frac{\prod_{l=1}^{k+1}(1-lm)}{(k+2)!m^{k+1}}+(-1)^k\right)(v-1)^{k+2}+o\left((v-1)^{N+2}\right), \tag{22}$$

$$v^{-\frac{3m+1}{2m}}=1+\sum_{k=1}^{N}\frac{(-1)^k\prod_{l=1}^{k}((2l+1)m+1)}{k!(2m)^k}(v-1)^k+o\left((v-1)^N\right), \tag{23}$$

kiekvienam $N\geq 1$ dėl Teiloro formulės.

Iš nelygybių (16)-(18 ir (21) gauname

$$\overline{F}_{\Pi_{m+1}}(x)\leq\frac{(2\pi)^{\frac{m}{2}}}{\sqrt{m+1}}\frac{\beta^{(m+1)(\alpha-1)+\frac{m}{2}}}{\Gamma^{m+1}(\alpha)}\,x^{\alpha-\frac{m+2}{2(m+1)}}\,\exp\left\{-\left(m+1\right)\beta x^{\frac{1}{m+1}}\right\}\left(1+C_{5m}x^{-\frac{1}{m+1}}\right). \tag{24}$$

Nagrinėkime apatinį rėžį funkcijai $\overline{F}_{\Pi_{m+1}}(x)$. Pagal indukcijos hipotezę (14), panašiai kaip ir (17), gauname

$$\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) &\geq \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{(m+1)x^{\frac{1}{m+1}}}\overline{F}_{\Pi_m}\left(\frac{x}{y}\right)y^{\alpha-1}e^{-\beta y}dy\\
&=\frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}\frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)}x^{\alpha-\frac{m+1}{2m}}\\
&\quad\times\int_0^{(m+1)x^{\frac{1}{m+1}}}y^{\frac{1-m}{2m}}\exp\left\{-\beta\left(y+m\left(\frac{x}{y}\right)^{\frac{1}{m}}\right)\right\}\left(1+O_m\left(\frac{y}{x}\right)^{\frac{1}{m}}\right)dy\\
&\geq\frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}\frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)}x^{\alpha-\frac{m+1}{2m}}\\
&\quad\times\int_0^{(m+1)x^{\frac{1}{m+1}}}y^{\frac{1-m}{2m}}\exp\left\{-\beta\left(y+m\left(\frac{x}{y}\right)^{\frac{1}{m}}\right)\right\}\left(1-C_{6m}\left(\frac{y}{x}\right)^{\frac{1}{m}}\right)dy\\
&\geq\frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}\frac{\beta^{\alpha(m+1)-\frac{m+1}{2}}}{\Gamma^{m+1}(\alpha)}x^{\alpha-\frac{1}{2}}\left(1-C_{6m}\left(x^{-\frac{1}{m}}\right)\right)\\
&\quad\times\left(\int_0^1+\int_1^{m+1}\right)u^{\frac{1-m}{2m}}\exp\left\{-\beta x^{\frac{1}{m+1}}\left(u+mu^{-\frac{1}{m}}\right)\right\}du\\
&=\frac{(2\pi)^{\frac{m}{2}}}{\sqrt{m+1}}\frac{\beta^{(m+1)(\alpha-1)+\frac{m}{2}}}{\Gamma^{m+1}(\alpha)}\,x^{\alpha-\frac{m+2}{2(m+1)}}\,\exp\left\{-\left(m+1\right)\beta x^{\frac{1}{m+1}}\right\}\left(1-C_{7m}x^{-\frac{1}{m+1}}\right)
\end{aligned} \tag{25}$$

su teigiamomis konstantomis C_{6m}, C_{7m}, priklausančiomis nuo m. Gautos nelygybės rodo, jog formulė galioja, kai $n=m+1$. Pagal indukcijos metodą gauname, jog formulė teisinga bet kuriam $n\in\mathbb{N}$.

Pakeitus nagrinėjamo skirstinio parametrus galime gauti kitus tikimybinius skirstinius. Pavyzdžiui, jeigu $\eta \sim Gamma\,(1, \lambda)$, tai $\eta \sim Exp\,(\lambda)$, arba, jei $\eta \sim Gamma\left(\frac{v}{2}, \frac{1}{2}\right)$, tai $\eta \sim \chi_v^2$, ir t.t.. Įsistatę atitinkamus parametrus į (2), suformuluosime dvejas išvadas, gaudami skirstinių sandaugos formules eksponentiniam bei chi kvadrato skirstiniams.

Išvada 3.1 *Tegul $\eta_1, \eta_2, \ldots$ yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, tokie, kad $\forall k, \eta_k \sim Exp\,(\lambda)$ ir tegul $\Pi_n := \prod_{k=1}^n \eta_k$. Tuomet*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi\lambda)^{\frac{n-1}{2}}}{\sqrt{n}}\, x^{\frac{n-1}{2n}} \exp\left\{-n\lambda x^{\frac{1}{n}}\right\}\left(1 + O_n\left(x^{-\frac{1}{n}}\right)\right),$$

kur konstanta simbolyje O_n priklauso nuo n.

Išvada 3.2 *Tegul $\eta_1, \eta_2, \ldots$ yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, tokie, kad $\forall k, \eta_k \sim \chi_k^2$ ir tegul $\Pi_n := \prod_{k=1}^n \eta_k$. Tuomet*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \frac{x^{\frac{k}{2} - \frac{n+1}{2}}}{2^{n(\frac{k}{2}-1)+\frac{n-1}{2}}\ \Gamma^n\left(\frac{k}{2}\right)} \exp\left\{-\frac{1}{2} n x^{\frac{1}{n}}\right\}\left(1 + O_n\left(x^{-\frac{1}{n}}\right)\right),$$

kur konstanta simbolyje O_n priklauso nuo n.

4. Išvados

Iš pagrindinių darbo rezultatų išplaukia, kad dauginant skirstinius keičiasi skirstinio reguliarumo klasė. Jeigu ξ yra pasiskirstęs pagal gama skirstinį, tai iš asimptotinės formulės (1) išplaukia, kad $F_\xi \in \mathcal{L}(\beta)$, t.y.

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\beta y}$$

bet kokiam $y > 0$. Visi skirstiniai iš klasės $\mathcal{L}(\beta)$ turi lengvas uodegas.

Sudauginus $n \geq 2$ gama skirstinius gavome, kad sandaugos Π pasiskirstymo funkcija turi sunkią uodegą, t.y.

$$\mathbb{E}e^{\delta\Pi} = \infty, \ \forall\delta > 0.$$

Taigi darbo rezultatai parodo vieną iš galimų būdų konstruoti skirstinius su sunkiomis uodegomis.

Darbo rezultatus galima būtų praplėsti taikant minėtąją teoremą bei balno taško metodą apibendrintam gama skirstiniui arba kitiems absoliučiai tolydiems atsitiktiniams dydžiams.

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Priedai

Hipotezė

$$\overline{F}(x) = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left(1 + O\left(x^{-1}\right)\right) \tag{26}$$

(i) Jeigu $\alpha = 1$, tai

$$\overline{F}(x) = \int_x^\infty \beta e^{-\beta y} dy = e^{-\beta x}$$

(ii) Jeigu $\alpha < 1$, tai $\forall x > 0$

$$\begin{aligned}
\overline{F}(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} d\left(\frac{e^{-\beta g}}{-\beta}\right) \\
&= -\frac{\beta^{\alpha-1}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}\Big|_x^\infty + \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} (\alpha - 1) \int_x^\infty y^{\alpha-2} e^{-\beta y} dy \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-1}(\alpha-1)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-2} e^{-\beta y} dy
\end{aligned}$$

Iš čia gauname, kad

$$\overline{F}(x) \leq \frac{x^{\alpha-1} e^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)}.$$

Antra vertus,

$$\begin{aligned}
\overline{F}(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty \frac{1}{y^{1-\alpha}} e^{-\beta y} dy \leq \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty x^{\alpha-1} e^{-\beta y} \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \\
&\text{nes jei } \frac{1}{y} \leq \frac{1}{x}, \text{ tai } \left(\frac{1}{y}\right)^{1-\alpha} \leq \left(\frac{1}{x}\right)^{1-\alpha}.
\end{aligned}$$

Šiuo atveju vėl gavome tą patį įvertį. Bandome gauti įvertį iš apačios

$$\begin{aligned}
\overline{F}(x) &= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-1}(\alpha-1)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-2} d\left(\frac{e^{-\beta y}}{-(\beta)}\right) \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} - \frac{\beta^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} y^{\alpha-2} e^{-\beta y}\Big|_x^\infty \\
&\quad + \underbrace{\frac{\beta^{\alpha-2}(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-3} e^{-\beta y} dy}_{\geq 0} \\
&\geq \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} x^{\alpha-2} e^{-\beta x} \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left(1 + \frac{\alpha-1}{\beta x}\right) \geq \frac{\beta}{\Gamma(\alpha)} \left(1 - \frac{1}{\beta x}\right).
\end{aligned}$$

Matome, jog taip pat šiuo atveju mūsų hipotezė tenkinama.

(iii) Tegul dabar $\alpha > 1, \quad \alpha \notin \mathbb{N}$

$$\overline{F}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\beta y} dy \geq \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \int_x^\infty e^{-\beta y} dy = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

nes jei $y \geq x$, tai $y^{\alpha-1} \geq x^{\alpha-1}$.

Antra vertus,

$$\begin{aligned}
\overline{F}(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} d\left(\frac{e^{-\beta y}}{-\beta}\right) \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-1}(\alpha-1)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-2} e^{-\beta y} dy \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-1}(\alpha-1)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-2} d\left(\frac{e^{-\beta y}}{-\beta}\right) \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \frac{\beta^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} x^{\alpha-2} e^{-\beta x} + \frac{\beta^{\alpha-2}(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-3} e^{-\beta y} dy \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} + \sum_{k=0}^{[\alpha]} \beta^{\alpha-k-1} \frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)} x^{\alpha-k-2} e^{-\beta x} \\
&\quad + \frac{\beta^{\alpha-[\alpha]-1}(\alpha-1)\cdots(\alpha-[\alpha]-1)}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-[\alpha]-2} e^{-\beta y} dy \\
&\leq \frac{\beta^{\alpha-1} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} + \sum_{k=0}^{[\alpha]} \beta^{\alpha-k-1} \frac{(\alpha-\beta)\cdots(\alpha-k-1)}{\Gamma(\alpha)} x^{\alpha-k-2} e^{-\beta x} \\
&= \frac{\beta^{\alpha-1} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \left(1 + \sum_{k=0}^{[\alpha]} \beta^{-k} \frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)} x^{k-1}\right) \\
&\leq \frac{\beta^{\alpha-1} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \left(1 + \sum_{k=0}^{[\alpha]-1} \beta^{-k} \frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)} x^{k-1}\right) \\
&\leq \frac{\beta^{\alpha-1} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \left(1 + c_{\alpha,\beta} x^{-1}\right).
\end{aligned}$$

Taigi šis atvejis tenkina hipotezę.

(iv) Jeigu $\alpha \in \mathbb{N}$, tai tada gauname

$$\begin{aligned}
\overline{F}(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_x^\infty y^{\alpha-1} e^{-\beta y} dy = \\
&= e^{-\beta x} \left(1 + \beta x + \frac{(\beta x)^2}{2!} + \ldots + \frac{(\beta x)^{\alpha-1}}{(\alpha-1)!}\right) \\
&= e^{-\beta x} \frac{(\beta x)^{\alpha-1}}{(\alpha^{-}1)!} \left(1 + O\left(x^{-1}\right)\right) \\
&= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \left(1 + O\left(x^{-1}\right)\right).
\end{aligned}$$

Taigi formulė (26) teisinga.