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Exact expressions of survival probabilities of homogeneous discrete-time risk model and Vandermonde matrices

Master's Thesis

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Tikslios homogeninio diskretaus laiko rizikos modelio išgyvenamumo tikimybės išraiškos ir Vandermonde matricos

Santrauka

Šiame baigiamajame magistro darbe apžvelgiamas begalinio laiko išgyvenimo tikimybės $\varphi(u)$ generuojančiosios funkcijos taikymas homogeniniam diskretaus laiko rizikos modeliui. Pristatomas pats modelis bei pagrindiniai apibrėžimai kartu su išgyvenimo tikimybėmis. Suformuluojamos bei įrodomos teoremos, kurių pagalba randami tikimybių skaičiavimo algoritmai. Naudojantis Wolfram Mathematica pagalba, pateiktuose pavyzdžiuose randame išgyvenimo tikimybių išraiškas bei reikšmes skirtingiems žalų skirstiniams ir keliems vienetinio apskritimo šaknų kartotinumo atvejams.

Raktiniai žodžiai : homogeninis diskretaus laiko rizikos modelis, bankroto tikimybė, išgyvenimo tikimybė, rekursinis skaičiavimas, atsitiktinis kintamasis, pasiskirstymo funkcija, matrica, lygčių sistemos.

Exact expressions of survival probabilities of homogeneous discrete-time risk model and Vandermonde matrices

Abstract

This master's thesis reviews application of ultimate time survival probability $\varphi(u)$ generating function for a homogeneous discrete time risk model. The model itself is presented with main definitions and survival probabilities. Theorems required for algorithms of probabilities calculations are formulated and proved. In presented examples, with the help of Wolfram Mathematica expressions and numerical values of survival probabilities are found with different claim amount distributions and cases when roots inside unit circle are multiple.

Key words : homogeneous discrete-time risk model, ruin probability, survival probability, recursive calculation, random variable, probability generating function, matrix, equation systems.

Contents

1 Introduction

Risk theory encounters many versions of surplus processes which define transition from profit to loss and vice versa. These transitions or sequence of sum $\sum_{i=1}^{n} Z_i$ are actually a random walk, where $n \in \mathbb{N}$ and Z_i are some random variables. Considering insurance theory, random walk appears in the insurers' surplus model which was originally introduced as collective risk model by E. Sparre Andersen [1]. One of the modified model versions is known as classical risk process or a discrete version of the more general continuous time Cramér-Lundberg model is called the discrete time risk model. It is based mainly on parameters such as initial insurer's surplus, premium income rate and incurred losses which are interpreted as claim amounts or already mentioned random walk. The goal of discrete time risk model is to evaluate insurer's probability to survive or ruin throughout some finite or infinite time.

In actuarial mathematics the problem of calculation of ultimate time survival probabilities have become more relevant in the recent years. In this master's thesis probability generating functions (p.g.f.) are used to find the ultimate time survival probabilities. Main goal of this paper is to review homogeneous discrete time risk model (1) and based on theoretical definitions and proofs find survival probability and its generating function calculation algorithms and expressions. Also, several numerical examples are given to examine how those expressions change for different claim distributions.

2 Homogeneous Discrete-Time Risk Model

The discrete time risk model describes the behaviour of the insurer's surplus in time. In this master's thesis, let us consider the special case of the general Sparre Andersen's model [1]. Let's define a risk model for discrete non-negative random variable Z and the behaviour of the process U can be considered only for natural time moments $t \in N_0$.

Definition 2.1. *It is said that insurer's surplus $U(t)$ varies according to a homogeneous discrete-time risk model if for each t*

$$U(t) = u + ct - \sum_{i=1}^{t} Z_i \tag{1}$$

with these restrictions:

- *$c \in \mathbb{N}$ denotes premium rate per time unit;*
- *$u = U(0) \in \mathbb{N}_0$ is non-negative integer - the initial insurer's surplus (at the moment $t = 0$);*
- *$Z_i \stackrel{d}{=} Z$, i.e. independently distributed sequence of claim amounts $(Z_i)_{i=1}^{\infty}$ is independent copies of a non-negative random variable Z.*

The fact that claim amounts are identically distributed allows this discrete-time risk model to be considered homogeneous. Integer-valued and non-negative random variables $Z_1, Z_2, \ldots$ are independent and sum $\sum_{i=1}^{n} Z_i$ is considered as random walk. In addition, r.v. Z can be characterized by probability mass function

$$P(Z = k) = h_k.$$

We can also define value of the cumulative distribution function of Z by

$$H(x) = \mathbb{P}(Z \leqslant x) = \sum_{k=0}^{x} h_k.$$

The time of ruin and the ruin probability are the main extremal characteristics of insurance risk models. The time of ruin can be defined as

$$T_u = \begin{cases} min\{t \geqslant 1 : U(t) \leqslant 0\}, \\ \infty, \;\; if \;\; U(t) > 0 \;\; \forall t \in \mathbb{N}. \end{cases}$$

Definition 2.2. *The finite time ruin probability for a discrete-time risk model described in* (1) *is such probability*

$$\psi(u,T) = \mathbb{P}(T_u \leqslant T) = \mathbb{P}\left(\bigcup_{t=1}^{T}\left\{u + ct - \sum_{i=1}^{t} Z_i \leqslant 0\right\}\right) = \mathbb{P}\left(\max_{1\leqslant t\leqslant T}\sum_{i=1}^{t}(Z_i - c) \geqslant u\right).$$

The main concern is whether the initial savings and the subsequent income is always sufficient to cover incurred expenses. For that we can define the opposite probability, known as the survival probability

$$\begin{aligned}\varphi(u,T) &= \mathbb{P}(T_u > T) = 1 - \psi(u,T) = 1 - \mathbb{P}(T_u \leqslant T)\\ &= \mathbb{P}\left(\bigcap_{t=1}^{T}\{U(t) > 0\}\right).\end{aligned}$$

Let us present the algorithm to calculate the values of the finite time survival probability by using the Definition 2.2 above and distribution function of Z. Therefore

$$\varphi(u,1) = \mathbb{P}(u + c - Z_1 > 0) = H(u + c - 1); \tag{2}$$

$$\begin{aligned}\varphi(u,T) &= \mathbb{P}\left(\bigcap_{t=2}^{T}\left\{u + ct - \sum_{i=1}^{t} Z_i > 0\right\}, Z_1 \leqslant u + c - 1\right)\\ &= \sum_{i=0}^{u+c-1}\mathbb{P}\left(\bigcap_{t=1}^{T-1}\left\{u + ct + c - i - \sum_{i=1}^{t} Z_i > 0\right\}, Z_1 = i\right)\\ &= \sum_{i=0}^{u+c-1}\varphi(u + c - i,\ T - 1)h_i, \qquad \text{where } u \geqslant 0 \text{ and } T \geqslant 2.\end{aligned} \tag{3}$$

Both ruin and survival probabilities can be defined for infinite time, where the initial insurer's surplus is considered as the main variable.

Theorem 2.1. *The survival probability of a homogeneous discrete-time risk mode for ultimate time satisfies the expression*

$$\varphi(u) = \sum_{r=1}^{u+c} h_{u+c-r}\varphi(r) \tag{4}$$

Proof. For proof we will use definition of ultimate time ruin probability for the model considered in this thesis.

Therefore

$$\begin{aligned}
\psi(u) &= \mathbb{P}\left(\bigcup_{t=1}^{\infty}\left\{u+ct-\sum_{i=1}^{t} Z_i \leqslant 0\right\}\right)\\
&= \mathbb{P}\left(\bigcup_{t=1}^{\infty}\left\{u+ct-\sum_{i=1}^{t} Z_i \leqslant 0\right\},\ \ Z_1 \leqslant u+c-1\right)\\
&+ \mathbb{P}\left(\bigcup_{t=1}^{\infty}\left\{u+ct-\sum_{i=1}^{t} Z_i \leqslant 0\right\},\ \ Z_1 > u+c-1\right)\\
&= \sum_{j=0}^{u+c-1} \mathbb{P}\left(\bigcup_{t-1=1}^{\infty}\left\{u+c(t-1)+c-Z_1-\sum_{i=1}^{t-1} Z_i \leqslant 0\right\},\ \ Z_1 = j\right) + \mathbb{P}(Z_1 > u)\\
&= \sum_{j=0}^{u+c-1} \mathbb{P}\left(\bigcup_{t=1}^{\infty}\left\{u+ct+c-j-\sum_{i=1}^{t-1} Z_i \leqslant 0\right\}\right) h_j + 1 - H(u)\\
&= \sum_{j=0}^{u+c-1} \psi(u+c-j)h_j + 1 - H(u)
\end{aligned}$$

After changing the summation variable j to $r = u + c - j$ we get

$$\psi(u) = \sum_{r=1}^{u+c} h_{u+c-r}\psi(r) + 1 - H(u).$$

Further, let us use equality $\psi(u) = 1 - \varphi(u)$. Then

$$\begin{aligned}
\varphi(u) &= 1 - \left(\sum_{r=1}^{u+c} h_{u+c-r}\psi(r) + 1 - H(u)\right),\\
\varphi(u) &= 1 - \sum_{r=1}^{u+c} h_{u+c-r}(1-\varphi(r)) - 1 + H(u),\\
\varphi(u) &= \sum_{r=1}^{u+c} h_{u+c-r}\varphi(r) - \sum_{r=1}^{u+c} h_{u+c-r} + H(u).
\end{aligned}$$

Let us note that $\sum\limits_{r=1}^{u+c} h_{u+c-r}$ is none other than $H(u)$, so finally ultimate time survival probability

$$\varphi(u) = \sum_{r=1}^{u+c} h_{u+c-r}\varphi(r). \tag{5}$$

■

3 Expression of survival probabilities for a homogeneous discrete-time risk model

To calculate the ultimate time survival probability $\varphi(u),\ u = c,\ c+1,\ \ldots,$ by the derived formula (5) we can notice that it is required to know the initial values of survival probabilities $\varphi(0),\ \varphi(1),\ \ldots,\ \varphi(c-1)$. Initial values of survival probabilities can be found not only by the method of recurrent sequences which were presented and derived in [6, see Section 3], but also by using probability generating functions.

3.1 Several basic notations and net profit condition

The net profit condition $\mathbb{E}Z < c$ for the model described in (1) means that the insurer's activity can continue to develop without experiencing guaranteed bankruptcy. Otherwise, if claim amounts on average are greater or equal to the collected premiums, long term survival is not possible.

Furthermore, let us rewrite the homogeneous discrete-time risk model from (1) as

$$U(t) = u - \sum_{i=1}^{t}(Z_i - c).$$

Recalling that the positive part of a function $f^+ := max\{0, f\},\ f \in \mathbb{R}$ and by using ultimate time survival probability definition

$$\varphi(u) = \mathbb{P}\left(\bigcap_{t=1}^{\infty}\left\{u + ct - \sum_{i=1}^{t} Z_i > 0\right\}\right) = \mathbb{P}\left(\sup_{n\geqslant 1}\sum_{i=1}^{n}(Z_i - c) < u\right)$$

a new random variable is defined

$$\mathcal{M} := \sup_{n\geqslant 1}\left(\sum_{i=1}^{n}(Z_i - c)\right)^{+}. \tag{6}$$

Let us also denote the probability mass function of the new random variable such as was done for r.v. Z by

$$\pi_i := \mathbb{P}(\mathcal{M} = i),\ i \in \mathbb{N}_0.$$

Then from the ultimate time survival probability definition

$$\varphi(u+1)=\sum_{i=0}^{u}\pi_i=\mathbb{P}(\mathcal{M}\leqslant u) \text{ for all } u\in\mathbb{N}_0.$$

By using the definition of generating function of ultimate time ruin probability from [10], let us denote the generating function of the sequence of survival probabilities $\{\varphi(1),\varphi(2),\dots\}$ inside the unit circle $\{s\in\mathbb{C},\ |s|<1\}$

$$\Gamma_\varphi(s):=\sum_{i=0}^{\infty}\varphi(i+1)s^i. \tag{7}$$

Following, probability generating function of some non-negative and integer-valued random variable Z

$$\Gamma_Z(s)=\sum_{i=0}^{\infty}s^i\mathbb{P}(Z=i)=\sum_{i=0}^{\infty}h_is^i=\mathbb{E}s^Z,\ |s|\leqslant 1,$$

as well as of r.v. $\mathcal{M}$

$$\Gamma_{\mathcal{M}}(s)=\sum_{i=0}^{\infty}\pi_is^i,\ |s|\leq 1.$$

Finally, by using notation of $\varphi(u+1)$ above and (7) we get the following relation:

$$\begin{aligned}\Gamma_\varphi(s)&=\sum_{i=0}^{\infty}\varphi(i+1)s^i\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\pi_js^i=\sum_{j=0}^{\infty}\pi_j\sum_{i=j}^{\infty}s^i\\&=\frac{\sum_{j=0}^{\infty}\pi_js^j}{1-s}=\frac{\Gamma_{\mathcal{M}}(s)}{1-s}.\end{aligned} \tag{8}$$

Remark. *Notations in this section were used as in [3, see Section 2].*

3.2 Survival probability calculations using Vandermonde-like matrices

In order to formulate main theorems used in this work, notations from section above are used. Also, several auxiliary statements which similarly recently are used in the works of [3] and [4] and are needed to help prove main results.

Lemma 3.1. *If the net profit condition is satisfied then the random variable* $\mathcal{M}$ *admits the following distribution property*

$$(\mathcal{M}+Z-c)^{+} \stackrel{d}{=} \mathcal{M}.$$

Proof. The equation states that random variables are distributed identically. Actually by [2, page 198] and maximum properties,

$$\begin{aligned}
(\mathcal{M}+Z-c)^{+} &= \max\{0,\, \mathcal{M}+Z-c\} = \max\left\{0,\, \sup_{n\geqslant 1}\left(\sum_{i=1}^{n}(Z_i-c)\right)^{+} + Z - c\right\} \\
&= \max\left\{0,\, \max\left\{0,\, \sup_{n\geqslant 1}\sum_{i=1}^{n}(Z_i-c)\right\} + Z - c\right\} \\
&\stackrel{\mathrm{d}}{=} \max\left\{0,\, \max\left\{Z_1-c,\, \sup_{n\geqslant 2}\sum_{i=1}^{n}(Z_i-c)\right\}\right\} \\
&\stackrel{\mathrm{d}}{=} \max\left\{0,\, \sup_{n\geqslant 1}\sum_{i=1}^{n}(Z_i-c)\right\} = \sup_{n\geqslant 1}\left(\sum_{i=1}^{n}(Z_i-c)\right)^{+} = \mathcal{M}.
\end{aligned}$$

■

Further let us construct a theorem that defines the equalities required to obtain the expression of the probability generating function.

Theorem 3.1. *Suppose that the net profit condition* $\mathbb{E}Z < c$ *for the homogeneous discrete-time risk model* (1) *is satisfied. Then the following equalities are correct:*

$$\Gamma_{\mathcal{M}}(s)(s^{c}-\Gamma_{Z}(s)) = \sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}h_j(s^{c}-s^{i+j}),\ |s|\leqslant 1, \tag{9}$$

$$c-\mathbb{E}Z = \sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}h_j(c-i-j). \tag{10}$$

Proof. By the law of total expectations $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$ and by Lemma 3.1

$$\begin{aligned}
\Gamma_{\mathcal{M}}(s) &= \mathbb{E}s^{(\mathcal{M}+Z-c)^+} = \mathbb{E}\left(\mathbb{E}\left(s^{(\mathcal{M}+Z-c)^+}|\mathcal{M}\right)\right) \\
&= \sum_{i=0}^{c-1} \pi_i \mathbb{E}s^{(i+Z-c)^+} + \sum_{i=c}^{\infty} \pi_i \mathbb{E}s^{(i+Z-c)} \\
&= \pi_0 \mathbb{E}s^{(Z-c)^+} + \cdots + \pi_{c-1}\mathbb{E}s^{(Z-1)^+} + \sum_{i=c}^{\infty} \pi_i \mathbb{E}s^{(i+Z-c)} \\
&= \pi_0 \left(h_0\left(1-s^{-c}\right) + \cdots + h_{c-1}\left(1-s^{-1}\right)\right) + \cdots + \pi_{c-1}\left(h_0\left(1-s^{-1}\right)\right) \\
&+ \mathbb{E}s^Z \left(\pi_0 s^{-c} + \cdots + \pi_{c-1}s^{-1} + \sum_{i=c}^{\infty} \pi_i s^{i-c}\right) \\
&= \sum_{i=0}^{c-1} \pi_i \left(\mathbb{E}s^{(Z+i-c)^+} - s^{i-c}\,\Gamma_Z(s)\right) + s^{-c}\,\Gamma_X(s)\Gamma_{\mathcal{M}}(s).
\end{aligned}$$

By making rearrangements to the euqality (9) we get

$$\begin{aligned}
\Gamma_{\mathcal{M}}(s)(s^c - \Gamma_Z(s)) &= \sum_{i=0}^{c-1} \pi_i \left(\mathbb{E}s^{(Z+i-c)^+ + c} - s^i\,\Gamma_Z(s)\right) \\
&= \pi_0 \mathbb{E}\left(h_0\left(s^c - 1\right) + \cdots + h_{c-1}\left(s^c - s^{c-1}\right)\right) \\
&+ \cdots + \pi_{c-1}\mathbb{E}\left(h_0\left(s^c - s^{c-1}\right)\right) \\
&= \sum_{i=0}^{c-1} \pi_i \sum_{j=0}^{c-1-i} h_j(s^c - s^{i+j}). \qquad (11)
\end{aligned}$$

Moving forward, let us calculate the derivative of (11) with respect to s

$$\begin{aligned}
\frac{d}{ds}\Big(\Gamma_{\mathcal{M}}(s)(s^c - \Gamma_Z(s))\Big) &= \Big(s^c - \Gamma_Z(s)\Big)\frac{d}{ds}\Big(\Gamma_{\mathcal{M}}(s)\Big) + \Gamma_{\mathcal{M}}(s)\frac{d}{ds}\Big(s^c - \Gamma_Z(s)\Big) \\
&= \sum_{i=0}^{c-1} \pi_i \sum_{j=0}^{c-1-i} h_j(cs^{c-1} - (i+j)s^{i+j-1}). \qquad (12)
\end{aligned}$$

Furthermore, let s approach 1 from the left, that is $s \to 1^-$. To find the limit as $\mathbb{E}\mathcal{M} = \infty$, let us divide formula (12) into three parts. For the first part of the left-hand side, by using Lemma 9 from [5]

$$\lim_{s\to 1^-}\left(\Big(s^c - \Gamma_Z(s)\Big)\frac{d}{ds}\Big(\Gamma_{\mathcal{M}}(s)\Big)\right) = \lim_{s\to 1^-}\frac{s^c - \Gamma_Z(s)}{1/\frac{d}{ds}\Gamma_{\mathcal{M}}(s)} = \lim_{s\to 1^-}\frac{cs^{c-1} - \frac{d}{ds}\Gamma_Z(s)}{-\frac{d^2}{ds^2}\Gamma_{\mathcal{M}}(s)/\left(\frac{d}{ds}\Gamma_{\mathcal{M}}(s)\right)^2},$$

where

$$\limsup_{s\to 1^-} \frac{\left(\frac{d}{ds}\Gamma_{\mathcal{M}}(s)\right)^2}{\frac{d^2}{ds^2}\Gamma_{\mathcal{M}}(s)} \leqslant \frac{N}{N-1}\sum_{i=N}^{\infty}\pi_i \text{ for any } N \in \{2,\, 3,\, \ldots\}.$$

Therefore, by observing that $\mathbb{E}\mathcal{M} < \infty$ [3, see proof of Theorem 3.1] this limit equals to zero in spite $\mathbb{E}\mathcal{M} = \infty$. Next, the remaining limit on the left-hand side of (12) equation

$$\lim_{s\to 1^-} \Gamma_{\mathcal{M}}(s)\frac{d}{ds}\Big(s^c - \Gamma_Z(s)\Big) = \lim_{s\to 1^-} \Gamma_{\mathcal{M}}(s)\Big(cs^{c-1} - \frac{d}{ds}\Gamma_Z(s)\Big) = c - \mathbb{E}Z.$$

Finally,

$$\lim_{s\to 1^-}\left(\frac{d}{ds}\Big(\Gamma_{\mathcal{M}}(s)(s^c - \Gamma_Z(s))\Big)\right) = \sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}h_j(c-i-j).$$

Then, after inserting the found limits into the formula above, we get equality (10)

$$c - \mathbb{E}Z = \sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}h_j(c-i-j).$$

■

It is worth mentioning that equalities from Theorem 3.1 for survival probabilities generating function $\Gamma_\varphi(s)$ imply

$$\Gamma_\varphi(s) = \frac{\Gamma_{\mathcal{M}}(s)}{1-s} = \frac{\sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}h_j(s^c - s^{i+j})}{(1-s)(s^c - \Gamma_Z(s))}. \tag{13}$$

Of course, this generating function requires initial values of $\pi_i,\ i = 0, 1, \ldots, c-1$. In order to find them relations (9) and (10) can be used and another theorem with Vandermonde-like matrices can be constructed. First of all, let us notice that such $|s| \leqslant 1$ can be chosen to find the roots by solving the equation $s^c = \Gamma_Z(s)$, which means that the left-hand side of (9) vanishes.

Theorem 3.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ be the simple roots of $s^c = \Gamma_Z(s)$ inside unit circle $|s| \leqslant 1$. Then by Theorem 3.1,*

$$\begin{pmatrix} \sum_{j=0}^{c-1}\alpha_1^j H(j) & \sum_{j=0}^{c-2}\alpha_1^{j+1}H(j) & \ldots & \alpha_1^{c-1}h_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{c-1}\alpha_{c-1}^j H(j) & \sum_{j=0}^{c-2}\alpha_{c-1}^{j+1}H(j) & \ldots & \alpha_{c-1}^{c-1}h_0 \\ \sum_{j=0}^{c-1}h_j(c-j) & \sum_{j=0}^{c-2}h_j(c-j-1) & \ldots & h_0 \end{pmatrix}\begin{pmatrix}\pi_0\\ \vdots \\ \pi_{c-2}\\ \pi_{c-1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\\ c-\mathbb{E}Z\end{pmatrix}. \tag{14}$$

Proof. By replacing $s = \alpha$ from equation (9) we get system of equations for $\alpha_i,\ i = 1, 2, \ldots, c-1$.

$$\begin{cases}
\pi_0 \sum\limits_{j=0}^{c-1} h_j \left(\alpha_1^c - \alpha_1^j\right) + \pi_1 \sum\limits_{j=0}^{c-2} h_j \left(\alpha_1^c - \alpha_1^{1+j}\right) + \ldots + \pi_{c-1} h_0 \left(\alpha_1^c - \alpha_1^{c-1}\right) = 0 \\
\pi_0 \sum\limits_{j=0}^{c-1} h_j \left(\alpha_2^c - \alpha_2^j\right) + \pi_1 \sum\limits_{j=0}^{c-2} h_j \left(\alpha_2^c - \alpha_2^{1+j}\right) + \ldots + \pi_{c-1} h_0 \left(\alpha_2^c - \alpha_2^{c-1}\right) = 0 \\
\vdots \\
\pi_0 \sum\limits_{j=0}^{c-1} h_j \left(\alpha_{c-1}^c - \alpha_{c-1}^j\right) + \pi_1 \sum\limits_{j=0}^{c-2} h_j \left(\alpha_{c-1}^c - \alpha_{c-1}^{1+j}\right) + \ldots + \pi_{c-1} h_0 \left(\alpha_{c-1}^c - \alpha_{c-1}^{c-1}\right) = 0 \\
\pi_0 \sum\limits_{j=0}^{c-1} h_j (c-j) + \pi_1 \sum\limits_{j=0}^{c-2} h_j (c-1-j) + \ldots + \pi_{c-1} h_0 = c - \mathbb{E}Z.
\end{cases}$$

Let us notice that the last equation of the system is none other than (10) presented in Theorem 3.1. By rewriting this system in matrix form the following is

$$\begin{pmatrix}
\sum\limits_{j=0}^{c-1} h_j \left(\alpha_1^c - \alpha_1^j\right) & \sum\limits_{j=0}^{c-2} h_j \left(\alpha_1^c - \alpha_1^{1+j}\right) & \ldots & h_0 \left(\alpha_1^c - \alpha_1^{c-1}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\sum\limits_{j=0}^{c-1} h_j \left(\alpha_{c-1}^c - \alpha_{c-1}^j\right) & \sum\limits_{j=0}^{c-2} h_j \left(\alpha_{c-1}^c - \alpha_{c-1}^{1+j}\right) & \ldots & h_0 \left(\alpha_{c-1}^c - \alpha_{c-1}^{c-1}\right) \\
\sum\limits_{j=0}^{c-1} h_j (c-j) & \sum\limits_{j=0}^{c-2} h_j (c-1-j) & \ldots & h_0
\end{pmatrix}
\begin{pmatrix} \pi_0 \\ \vdots \\ \pi_{c-2} \\ \pi_{c-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c - \mathbb{E}Z \end{pmatrix}.$$

Now the first $c-1$ rows of the matrix above can be divided by $(\alpha_i - 1),\ i = 1, 2, \ldots, c-1$. Then it leads to

$$\begin{pmatrix}
\sum\limits_{j=0}^{c-1} h_j \sum\limits_{k=0}^{c-1-j} \alpha_1^{c-1-k} & \sum\limits_{j=0}^{c-2} h_j \sum\limits_{k=0}^{c-2-j} \alpha_1^{c-1-k} & \ldots & h_0 \alpha_1^{c-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum\limits_{j=0}^{c-1} h_j \sum\limits_{k=0}^{c-1-j} \alpha_{c-1}^{c-1-k} & \sum\limits_{j=0}^{c-2} h_j \sum\limits_{k=0}^{c-2-j} \alpha_{c-1}^{c-1-k} & \ldots & h_0 \alpha_{c-1}^{c-1} \\
\sum\limits_{j=0}^{c-1} h_j (c-j) & \sum\limits_{j=0}^{c-2} h_j (c-1-j) & \ldots & h_0
\end{pmatrix}
\begin{pmatrix} \pi_0 \\ \vdots \\ \pi_{c-2} \\ \pi_{c-1} \end{pmatrix}
= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c - \mathbb{E}Z \end{pmatrix}.$$

By making rearrangements using definition of r.v. Z accumulated distribution function $H(x) = \mathbb{P}(Z \leqslant x) = \sum\limits_{k=0}^{x} h_k$ the form presented in (14) is obtained. ■

It is worth mentioning that this matrix system of finding local probabilities π_i of random variable $\mathcal{M}$, which is actually a Vandermonde-like matrix [8, see p.27], can be used if roots of $s^c = \Gamma_Z(s)$ has multiplicity higher than one. Then another theorem can be formulated.

Theorem 3.3. *Suppose $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ are the roots of $s^c = \Gamma_Z(s)$ in $s \leqslant 1$ are of multiplicity $l \in \{2, 3, \ldots, c-1\}$, $c \geqslant 3$. Then modified version of* (14) *can be created by replacing its lines, except last one, with the corresponding derivatives and the main matrix from* (14) *remains non-singular.*

Proof. First of all, let us note that the derivatives

$$\frac{d^m}{d\alpha^m}\left(\sum_{i=0}^{c-1}\pi_i\sum_{j=0}^{c-1-i}\alpha^{j+i}H(j)\right) = 0 \text{ for all } m \in \{0, 1, \ldots, l-1\},$$

when $l \in \{2, 3, \ldots, c-1\}$, $c \geqslant 3$ is the multiplicity of the roots of $s^c = \Gamma_Z(s)$ in $s \leqslant 1$.

Then for the simplicity let us state that root α_1 has multiplicity two and there exists $\delta \in \mathbb{R} \setminus \{0\}$ which is a number close to zero. Then matrix (14) with replaced second line by derivative

$$\begin{pmatrix} \sum_{j=0}^{c-1}\alpha_1^j H(j) & \sum_{j=0}^{c-2}\alpha_1^{j+1}H(j) & \ldots & \alpha_1^{c-1}h_0 \\ \sum_{j=0}^{c-1}(\alpha_1+\delta)^j H(j) & \sum_{j=0}^{c-2}(\alpha_1+\delta)^{j+1}H(j) & \ldots & (\alpha_1+\delta)^{c-1}h_0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^{c-1}\alpha_{c-1}^j H(j) & \sum_{j=0}^{c-2}\alpha_{c-1}^{j+1}H(j) & \ldots & \alpha_{c-1}^{c-1}h_0 \\ \sum_{j=0}^{c-1}h_j(c-j) & \sum_{j=0}^{c-2}h_j(c-j-1) & \ldots & h_0 \end{pmatrix}. \tag{15}$$

By substracting second line of the matrix from the first and then dividing it by δ, when $\delta \to 0$, it would lead to desired line replacement by derivative. ■

4 Particular examples

In this section the already defined expressions of survival probabilities and probability generating functions will be used for practical calculations. Wolfram Mathematica 13.1 was used for various claim distributions, graphs and calculations of exact values of probabilities. Probability values that are presented in the below tables and are less than 1 are rounded to four decimal places.

Example 4.1. *Suppose the random claim amount Z is distributed according to the displaced Poisson distribution $\mathcal{P}(\lambda, r)$ with parameters $\lambda > 0$ and $r \in \mathbb{N}_0$, if*

$$\mathbb{P}(Z = m) = e^{-\lambda}\frac{\lambda^{m-r}}{(m-r)!}, \qquad m \in \{r, r+1, \dots\}.$$

Let $c = 3$ and $Z \sim \mathcal{P}(\frac{4}{5}, 1)$.

First of all, it needs to be checked if in this particular example net profit condition is satisfied, actually

$$\mathbb{E}Z = \lambda + r = \frac{4}{5} + 1 = 1.8 \;\; < \;\; 3.$$

Further, the expression of probability generating function of random variable Z for displaced Poisson distribution

$$\begin{aligned}\Gamma_Z(s) &= \sum_{i=r}^{\infty} e^{-\lambda}\frac{\lambda^{i-r}}{(i-r)!}s^i = e^{-\lambda}\sum_{i=r}^{\infty}\frac{\lambda^{i-r}}{(i-r)!}s^i \\ &= e^{-\lambda}s^r\left(s^0\frac{\lambda^0}{0!} + s^1\frac{\lambda^1}{1!} + s^2\frac{\lambda^2}{2!} + \dots\right) \\ &= e^{-\lambda}s^r\sum_{n=0}^{\infty}\frac{(s\lambda)^n}{n!} = e^{-\lambda}s^r e^{s\lambda} \\ &= s^r e^{\lambda(s-1)}. \end{aligned} \tag{16}$$

As it can be seen, applying Taylor series of the exponential function e^x in (16) gives a simplified expression of p.g.f for this example. By the statement provided in Theorem 3.1 the equation looks like this:

$$\Gamma_Z(s) = se^{0.8(s-1)} = s^3.$$

By solving it for $|s| \leqslant 1$, $s \neq 1$, one non-zero solution is found inside the unit circle $\alpha =$ -0.54008. Also from (13) the survival probabilities generating function

$$\Gamma_\varphi(s) = \frac{\sum_{i=0}^{2}\pi_i\sum_{j=0}^{2-i}h_j(s^3 - s^{i+j})}{(1-s)(s^3 - se^{0.8(s-1)})}.$$

Now using formulas (9) and (10) let us construct a matrix system

$$\begin{pmatrix} h_1(\alpha^3-\alpha)+h_2(\alpha^3-\alpha^2) & h_1(\alpha^3-\alpha) \\ 2h_1+h_2 & h_1 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3-\mathbb{E}Z \end{pmatrix}.$$

Solving this system gives the unique solution of $(\pi_0,\, \pi_1) = (0.93656,\, 0.04829)$, where

$$\pi_0 = \frac{(3-\mathbb{E}Z)\,\alpha}{h_1(\alpha-1)},$$
$$\pi_1 = \frac{(3-\mathbb{E}Z)\,(h_1+\alpha\,(h_1+h_2))}{h_1^2(1-\alpha)}.$$

Then, finally by (5) and relation $\varphi(u+1) = \sum_{i=0}^{u} \pi_i$ the ultimate time survival probabilities

$$\varphi(0) = \sum_{r=1}^{2} h_{3-r}\varphi(r) = h_2\varphi(1) + h_1\varphi(2) = \frac{3-\mathbb{E}Z}{1-\alpha} = 0.77917\ldots,$$
$$\varphi(1) = \pi_0 = \frac{(3-\mathbb{E}Z)\,\alpha}{h_1(\alpha-1)} = 0.93656\ldots,$$
$$\varphi(2) = \pi_0+\pi_1 = \frac{(3-\mathbb{E}Z)\,(h_1+h_2\alpha)}{h_1^2(1-\alpha)} = 0.98485\ldots,$$
$$\varphi(u) = \frac{1}{h_1}\left(\varphi(u-2) - \sum_{i=0}^{u-2} h_{u-i}\varphi(i+1)\right), \quad u \geqslant 3.$$

As for the finite time survival probabilities, formulas (2) and (3) are used. Calculated results are presented in the Table 1.

T	u								
	0	1	2	3	4	5	10	20	30
1	0.8088	0.9526	0.9909	0.9986	0.9998	1	1	1	1
2	0.7877	0.9419	0.9872	0.9976	0.9996	0.9999	1	1	1
3	0.7821	0.9385	0.9858	0.9971	0.9995	0.9999	1	1	1
4	0.7803	0.9374	0.9853	0.9969	0.9994	0.9999	1	1	1
5	0.7796	0.9369	0.9850	0.9968	0.9994	0.9999	1	1	1
10	0.7792	0.9366	0.9849	0.9968	0.9993	0.9999	1	1	1
20	0.7792	0.9366	0.9848	0.9967	0.9993	0.9998	1	1	1
30	0.7792	0.9366	0.9848	0.9967	0.9993	0.9998	1	1	1
∞	0.7791	0.9366	0.9848	0.9967	0.9993	0.9998	0.9999	1	1

Table 1: Finite and ultimate time survival probabilities for the model in Example 4.1.

Example 4.2. *Let us consider the model* (1) *when* $c = 8$ *and random claim amount* Z *is distributed by Poisson distribution which is displaced Poisson distribution with* $r = 0$*, i.e.* $Z \sim \mathcal{P}(\lambda, 0)$ *with parameter* $\lambda = 7$ *and p.g.f.*

$$\mathbb{P}(Z = m) = e^{-\lambda}\frac{\lambda^m}{m!}, \qquad m = 0\,,1\,,2\,,\ldots\,.$$

Starting with an observation on the net profit condition one can notice that

$$\mathbb{E}Z = \lambda = 7 \;\;<\;\; 8.$$

By using Taylor series of the exponential function e^x as in the example above, we find probability generating function of r.v. Z

$$\Gamma_Z(s) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}s^i = e^{\lambda(s-1)}.$$

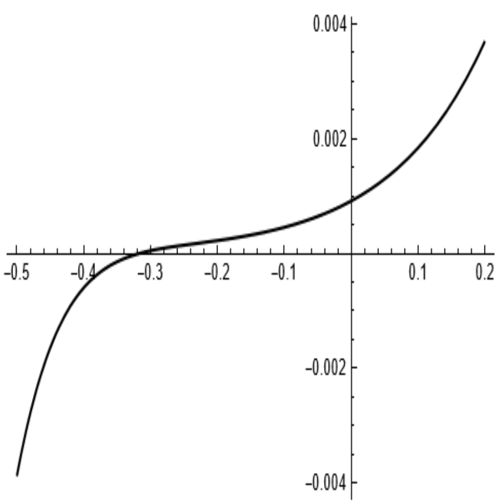

Figure 1: $\Gamma_Z(s) = e^{7(s-1)} = s^8$ for $\lambda = 7$.

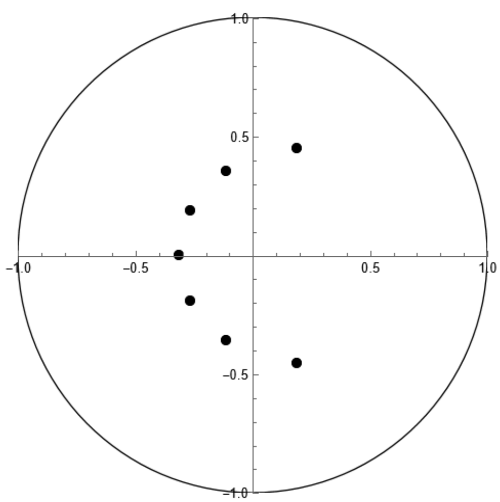

Figure 2: Roots of $\Gamma_Z(s) = e^{7(s-1)} = s^8$.

The function shown in Figure 1 has seven complex solutions inside the unit circle $|s| < 1$ which are shown in Figure 2 figure and their complex values are:

$$\alpha_1 = -0.3161 + 0.0000i,\;\; \alpha_2 = -0.2686 - 0.1910i,\;\; \alpha_3 = -0.26856 + 0.1910i,$$
$$\alpha_4 = -0.1161 - 0.3582i,\;\; \alpha_5 = -0.1161 + 0.3582i,\;\; \alpha_6 = 0.1856 - 0.4538i,$$
$$\alpha_7 = 0.1856 + 0.4538i.$$

Then by Theorem 3.2

$$\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\\ \pi_6\\ \pi_7\end{pmatrix} = \begin{pmatrix}0.496398\\ 0.100381\\ 0.0860687\\ 0.0704941\\ 0.0560149\\ 0.043717\\ 0.0338064\\ 0.0260431\end{pmatrix},$$

is the unique solution of

$$\begin{pmatrix}
\sum\limits_{j=0}^{7}\alpha_1^j H(j) & \sum\limits_{j=0}^{6}\alpha_1^{j+1}H(j) & \dots & \sum\limits_{j=0}^{1}\alpha_1^{j+6}H(j) & \alpha_1^7 h_0\\
\sum\limits_{j=0}^{7}\alpha_2^j H(j) & \sum\limits_{j=0}^{6}\alpha_2^{j+1}H(j) & \dots & \sum\limits_{j=0}^{1}\alpha_2^{j+6}H(j) & \alpha_2^7 h_0\\
\vdots & \vdots & \ddots & \vdots & \\
\sum\limits_{j=0}^{7}\alpha_7^j H(j) & \sum\limits_{j=0}^{6}\alpha_7^{j+1}H(j) & \dots & \sum\limits_{j=0}^{1}\alpha_7^{j+6}H(j) & \alpha_7^7 h_0\\
\sum\limits_{j=0}^{7} h_j(8-j) & \sum\limits_{j=0}^{6} h_j(7-j) & \dots & \sum\limits_{j=0}^{6} h_j(2-j) & h_0
\end{pmatrix}
\begin{pmatrix}\pi_0\\ \pi_1\\ \vdots\\ \pi_6\\ \pi_7\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ 8-\mathbb{E}Z\end{pmatrix}.$$

Once again, the ultimate time survival probability definition imply

$$\begin{aligned}
\varphi(0) &= \sum_{r=1}^{8} h_{8-r}\varphi(r) = h_7\varphi(1) + h_6\varphi(2) + \ldots + h_0\varphi(0) = 0.3866\ldots,\\
\varphi(1) &= \pi_0 = 0.4964\ldots,\\
\varphi(2) &= \pi_0 + \pi_1 = 0.59678\ldots,\\
\varphi(3) &= \pi_0 + \pi_1 + \pi_2 = 0.6828\ldots,\\
\varphi(4) &= \pi_0 + \pi_1 + \pi_2 + \pi_3 = 0.7533\ldots,\\
\varphi(5) &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 0.8094\ldots,\\
\varphi(6) &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 0.8531\ldots,\\
\varphi(7) &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 0.8869\ldots,\\
\varphi(8) &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 = 0.9129\ldots,\\
\varphi(u) &= \frac{1}{h_0}\left(\varphi(u-8) - \sum_{i=0}^{7} h_{8-i}\varphi(i)\right), \quad u \geqslant 8.
\end{aligned}$$

If by Theorem 3.2survival probabilities generating function $\Gamma_\varphi(s)$ expression in (13) are divided by $(1-s)$ then

$$\Gamma_\varphi(s) = \frac{\sum_{i=0}^{7}\pi_i\sum_{j=0}^{7-i}s^{i+j}H(j)}{s^8 - \Gamma_Z(s)}.$$

The results of finite and ultimate time survival probabilities for $c = 8$ are presented in Table 2.

T	u									
	0	1	2	3	4	5	10	20	30	40
1	0.5987	0.7291	0.8305	0.9015	0.9467	0.9730	0.9996	1	1	1
2	0.5139	0.6437	0.7533	0.8379	0.8985	0.9391	0.9972	1	1	1
3	0.4748	0.6009	0.7107	0.7989	0.8652	0.9125	0.9931	1	1	1
4	0.4521	0.5750	0.6837	0.7730	0.8419	0.8926	0.9884	1	1	1
5	0.4373	0.5578	0.6653	0.7547	0.8248	0.8775	0.9839	0.9999	1	1
10	0.4056	0.5199	0.6236	0.7117	0.7829	0.8385	0.9671	0.9992	1	1
20	0.3912	0.5022	0.6035	0.6902	0.7610	0.8171	0.9541	0.9976	0.9999	1
30	0.3881	0.4983	0.5989	0.6852	0.7558	0.8119	0.9504	0.9968	0.9998	1
∞	0.3866	0.4964	0.5968	0.6828	0.7533	0.8094	0.9484	0.9962	0.9996	0.9999

Table 2: Survival probabilities for $c = 8$ when claim amount $Z \sim \mathcal{P}(7, 0)$.

Example 4.3. *Let us suppose that discrete-time risk model random claim amount's Z distribution is:*

Z	0	1	2	3
$\mathbb{P}$	$1/8$	$1/2$	$5/32$	$7/32$

One can check that net profit condition for premium income rate $c = 3$ is satisfied

$$\mathbb{E}Z = 0 \cdot \frac{1}{8} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{32} + 3 \cdot \frac{7}{32} = \frac{47}{32} < 3.$$

Now the equation $\Gamma_Z(s) = \sum\limits_{i=0}^{\infty} h_i s_i = s^c$ is:

$$\frac{1}{8} + \frac{1}{2}s + \frac{5}{32}s^2 + \frac{7}{32}s^3 = s^3,$$

and has one root $s := \alpha = -0.4$ of multiplicity two. Continuing forward, (15) from Theorem 3.3 is used. Then the second line of matrix is derivative with respect to α

$$\begin{pmatrix} h_0 + \alpha(h_0 + h_1) + \alpha^2(h_0 + h_1 + h_2) & \alpha h_0 + \alpha^2(h_0 + h_1) & \alpha^2 h_0 \\ (h_0 + h_1) + 2\alpha(h_0 + h_1 + h_2) & h_0 + 2\alpha(h_0 + h_1) & 2\alpha h_0 \\ 3h_0 + 2h_1 + h_2 & 2h_0 + h_1 & h_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 - \mathbb{E}Z \end{pmatrix}.$$

By solving matrix system above given result $(\pi_0,\, \pi_1,\, \pi_2) = (1,\, 0,\, 0)$ with expressions

$$\pi_0 = \frac{(3 - \mathbb{E}Z)\alpha^2}{h_0(\alpha - 1)^2}, \qquad \pi_1 = \frac{(3 - \mathbb{E}Z)\alpha(h_1\alpha + h_0(2 + \alpha))}{h_0^2(1 - \alpha)^2},$$

$$\pi_2 = \frac{(3 - \mathbb{E}Z)(h_1^2\alpha^2 + h_0^2(1 + 2\alpha) + h_0\alpha(h_1(2 + \alpha) - h_2\alpha))}{h_0^3(\alpha - 1)^2}.$$

Finally to find ultimate time survival probabilities $\varphi(u)$ we notice that

$$\varphi(0) = \sum_{i=1}^{3} h_{3-i}\varphi(i) = h_2 + h_1 + h_0 = 0.78125,$$

since $\varphi(u) = \sum\limits_{i=0}^{u-1} \pi_i = 1,$ for all $u \geqslant 1$. The expression of $\varphi(u)$ p.g.f after making required arrangements is:

$$\Gamma_\varphi(s) = \frac{\sum\limits_{i=0}^{2} \pi_i \sum\limits_{j=0}^{2-i} h_j(s^3 - s^{i+j})}{(1-s)\left(\frac{25}{32}s^3 - \frac{5}{32}s^2 - \frac{1}{2}s - \frac{1}{8}\right)} = \frac{1}{1-s}, \quad |s| < 1.$$

Example 4.4. *Let us suppose that discrete-time risk model random claim amount's Z distribution is:*

Z	0	1	2	3
$\mathbb{P}$	1/64	11/64	9/16	1/4

In this example let us say that premium income rate is 4. Then

$$\mathbb{E}Z = \frac{131}{64} = 2.046875 < 4,$$

which states that net profit condition is satisfied. Moving forward,

$$\frac{1}{4}s^3 + \frac{9}{16}s^2 + \frac{11}{64}s + \frac{1}{64} = s^4.$$

By solving equation above it needs to be noticed that the only root $s := \alpha = -\frac{1}{4}$ has multiplicity three. By Theorem 3.3 replacing the second and the third lines of a matrix with the corresponding the first and the second order derivatives we get

$$\begin{pmatrix} h_0 + \alpha H(1) + \alpha^2 H(2) + \alpha^3 H(3) & \alpha h_0 + \alpha^2 H(1) + \alpha^3 H(2) & \alpha^2 h_0 + \alpha^3 H(1) & \alpha^3 h_0 \\ H(1) + 2\alpha H(2) + 3\alpha^2 H(3) & h_0 + 2\alpha H(1) + 3\alpha^2 H(2) & 2\alpha h_0 + 3\alpha^2 H(1) & 3\alpha^2 h_0 \\ 2H(2) + 6(3) & 2H(1) + 6\alpha H(2) & 2h_0 + 6\alpha H(1) & 6\alpha h_0 \\ 4h_0 + 3h_1 + 2h_2 + h_1 & 3h_0 + 2h_1 + h_2 & 2h_0 + h_1 & h_0 \end{pmatrix}$$

$$\times \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 - \mathbb{E}Z \end{pmatrix}.$$

Once again we find π_i, $i = 0, 1, 2, 3$. In fact,

$$
\begin{aligned}
\pi_0 =& \frac{(4-\mathbb{E}Z)\alpha^3}{h_0(\alpha-1)^3},\\
\pi_1 =& \frac{(4-\mathbb{E}Z)\alpha^2(h_1\alpha + h_0(3+\alpha))}{h_0^2(1-\alpha)^3},\\
\pi_2 =& \frac{(4-\mathbb{E}Z)\alpha(h_1^2\alpha^2 + 3h_0^2(1+\alpha) + h_0\alpha(h_1(3+\alpha) - h_2\alpha))}{h_0^3(\alpha-1)^3},\\
\pi_3 =& (4-\mathbb{E}Z)(h_1^3\alpha^3 + h_0^3(1+3\alpha) + h_0h_1\alpha^2(h_1(3+\alpha) - 2h_2) + h_0^2\alpha(3h_1(1+\alpha)\\
& + \alpha(h_3\alpha - h_2(3+\alpha))))\frac{1}{h_0^4(1-\alpha)^3}.
\end{aligned}
$$

Even though expressions of π_i, $i = 0,\ 1,\ 2,\ 3$ might seem a bit complex, but calculated values gives $(\pi_0,\ \pi_1,\ \pi_2,\ \pi_3) = (1,\ 0,\ 0,\ 0)$. It is worth noticing that probability generating function's form is

$$\Gamma_\varphi(s) = \frac{1}{1-s},\ |s| < 1.$$

Consequently $\varphi(1) = \Gamma_\varphi(0) = 64\,\pi_0\,h_0$ and $\varphi(u) = 1,\ \forall\, u \in \mathbb{N}$. Coming back to $u = 0$

$$\varphi(0) = \sum_{r=1}^{4} h_{4-i} = 1/4 + 9/16 + 11/64 + 1/64 = 1.$$

This means that ruin will not occur if initial surplus is higher or even equal to zero and then modified expression $\varphi(u) = 1,\ \forall\, u \in \mathbb{N}_0$.

5 Conclusions

In this master's thesis by using theoretical definitions and notations it was reviewed that ultimate time survival probabilities for homogeneous discrete-time risk model (1) can be calculated using the linear equations in the form of Vandermonde-like matrices. Expressions of survival probabilities are based on probability generating functions of random variables, in fact, by solving the equation $s^c = \Gamma_Z(s)$ and finding its roots. Based on corollaries, numerical examples were given to illustrate that different claim amount distributions and even root multiplicity can lead to different numerical results. To avoid complexity of multiple formulas expressions and to find numerical values of ultimate and finite time survival probabilities Wolfram Mathematica programming software and language was used. Eventually from the results given in section 4 it can be noticed and once again ascertained that as the initial surplus u increases, the survival probability approaches 1 at different rates depending on the parameter of the claim distribution and income rate.

The obtained results and used techniques can be adapted in any future's works on discrete-time risk models and its modifications such as when model is multi seasonal (as in [5]) or when the net profit condition is not satisfied. These are only few options because model not only can be dependent on random variable or premium income rate c. A wide variety of possible applications can be used which should not only dwell on ruin theory.

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Appendix

Calculations were performed as well as graphics were drawn using Wolfram Mathematica Desktop 13.1 software. All source codes with their outputs and required comments for the examples in section 4 are provided below.

Example 4.1 source code

```
In[1]= (* 4.1 Example *)
       Z=PoissonDistribution[4/5]

Out[1]= PoissonDistribution[4/5]

In[2]= h[n_]:=PDF[Z,{n-1}]
In[3]= H[n_]:=CDF[Z,{n-1}]

In[4]= (* Solve and find s=α expression *)
       nsol1 = Simplify[Solve[{s*Exp[(8/10)*(s-1)]==s^3, s>-1&&s<0},{s}]]
Out[4]=
```

$$\left\{\left\{s \to -\frac{5}{2}\,\text{ProductLog}\left[\frac{2}{5\,e^{2/5}}\right]\right\}\right\}$$

```
In[5]= sol1 = N[Solve[{s*Exp[(8/10)*(s-1)]==s^3, s>-1&&s<0},s]]
Out[5]= {s → -0.540083}

In[6]= (* Solve and find π expressions *)
       nsol2 = Simplify[Solve[{π0*(x1*(α^3-α)+x2*(α^3-α^2))+π1*x1
               *(α^3-α^2)==0, π0*(2*x1+x2)+π1*x1==3-D}, {π0,π1}]]
Out[6]=
```

$$\left\{\left\{\pi 0 \to -\frac{(-3+D)\,\alpha}{x1\,(-1+\alpha)},\ \pi 1 \to \frac{(-3+D)\,(x1+x1\,\alpha+x2\,\alpha)}{x1^2\,(-1+\alpha)}\right\}\right\}$$

```
In[7]= (* π numerical values *)
       sol2 = NSolveValues[{π0*(h[1]*((-0.540083)^3-(-0.540083))
              +h[2]*((-0.540083)^3-(-0.540083)^2))
              +π1*h[1]*((-0.540083)^3-(-0.540083)^2)==0,
              π0*(2*h[1]+h[2])+π1*h[1]==3-(1.8)}, {π0,π1}]
Out[7]= {{0.936555, 0.0482956}}

In[8]= (* Initial φ(u) values *)
       φ[1]=sol2[[1,1]]
Out[8]= 0.936555

In[9]= φ[2]=sol2[[1,1]] + sol2[[1,2]]
Out[9]= 0.98485

In[10]= φ[0]=h[2]*φ[1] + h[1]*φ[2]
Out[10]= {0.779179}

In[11]= Do[Print[{N[φ[n]=(1/h[1])*(φ[n-2]-(Sum[h[n+1-i]*φ[i],
        {i,1,n-1}])), 4], n}], {n,3,5}]
```

```
Out[11]= {{0.9967}, 3}
         {{0.9993}, 4}
         {{0.9998}, 5}

In[12]= Do[Print[{N[φ[n]=(1/h[1])*(φ[n-2]-(Sum[h[n+1-i]*φ[i],
        {i,1,n-1}])), 4], n}], {n,10,30,10}]
Out[12]= {{0.9999}, 10}
         {{1.0000}, 20}
         {{1.0000}, 30}

In[13]= (* φ(u,T) values when T = 1 and u = 0,1,2,3,4,5,10,20,30 *)
        Do[Print[{N[φ[n,1]=H[n+2],5],n}], {n,0,5}]
Out[13]= {{0.8088}, 0}
         {{0.9526}, 1}
         {{0.9909}, 2}
         {{0.9986}, 3}
         {{0.9998}, 4}
         {{1.0000}, 5}

In[14]= Do[Print[{N[φ[n,1]=H[n+2],5],n}], {n,10,30,10}]
Out[14]= {{1.0000}, 10}
         {{1.0000}, 20}
         {{1.0000}, 30}

In[15]= (* φ(u,T) values when T=2,3,4,5,10,20,30 and u = 0,1,2,3,4,5,10,20,30 *)
        Do[Print[{N[φ[n,T]=Sum[φ[n+3-i,T-1]*h[i], {i,0,n+2}], 4], n,T}],
        {n,0,5}, {T,2,5}]
Out[15]= {{0.7877}, 0, 2}
         {{0.9419}, 1, 2}
         {{0.9872}, 2, 2}
         {{0.9976}, 3, 2}
         {{0.9996}, 4, 2}
         {{0.9999}, 5, 2}
         {{0.7821}, 0, 3}
         {{0.9385}, 1, 3}
         {{0.9858}, 2, 3}
         {{0.9971}, 3, 3}
         {{0.9995}, 4, 3}
         {{0.9999}, 5, 3}
         {{0.7803}, 0, 4}
         {{0.9374}, 1, 4}
         {{0.9853}, 2, 4}
         {{0.9969}, 3, 4}
         {{0.9994}, 4, 4}
         {{0.9999}, 5, 4}
         {{0.7796}, 0, 5}
         {{0.9369}, 1, 5}
         {{0.9850}, 2, 5}
         {{0.9968}, 3, 5}
         {{0.9994}, 4, 5}
         {{0.9999}, 5, 5}
```

Example 4.2 source code

```
In[1]= (* 4.2 Example *)
       Z=PoissonDistribution[7]

Out[1]= PoissonDistribution[7]

In[2]= h[n_]:=PDF[Z,n]
In[3]= H[n_]:=CDF[Z,n]

In[4]= EZ = N[Mean[Z]]
Out[4]= 7.

In[5]= c=8
Out[5]= 8

In[6]= (* Solve and find s=α expressions *)
       sol1 = {s}/.NSolve[{Exp[(7)*(s - 1)]==s^c, Abs[s]<1}, {s}]
Out[6]=
       {{-0.316127 + 0. i}, {-0.268558 - 0.19102 i},
        {-0.268558 + 0.19102 i}, {-0.116121 - 0.358237 i},
        {-0.116121 + 0.358237 i}, {0.185643 - 0.453889 i}, {0.185643 + 0.453889 i}}

In[7]= Do[α[i]=sol1[[i]], {i,1,7}]

In[8]= Plot[Exp[7(-1+s)]-s^8, {s,-1,0}, PlotRange->{{-1,0},{-.5,.2}}]

In[9]= ComplexListPlot[Table[α[i], {i,1,7}],
       PlotStyle->{Directive[Black,PointSize[0.02]]},
       PlotRange->{{-.5,.5},{-.5,.5}}]

In[10]= (*π numerical values*)
       sol2 = Chop[NSolveValues[{Sum[p[i]*Sum[h[j]*(α[1]^(c)-α[1]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[2]^(c)-α[2]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[3]^(c)-α[3]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[4]^(c)-α[4]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[5]^(c)-α[5]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[6]^(c)-α[6]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(α[7]^(c)-α[7]^(i+j)),
              {j,0,c-1-i}],{i,0,c-1}]==0,
              Sum[p[i]*Sum[h[j]*(c-i-j), {j,0,c-1-i}],{i,0,c-1}]==8-EZ},
              {p[0],p[1],p[2],p[3],p[4],p[5],p[6],p[7]}]]
Out[10]= {{0.496398,0.100381,0.0860687,0.0704941,0.0560149,0.043717,0.0338064,0.0260431}}

In[11]= π0 = sol2[[1,1]]
        π1 = sol2[[1,2]]
        π2 = sol2[[1,3]]
        π3 = sol2[[1,4]]
```

```
        π4 = sol2[[1,5]]
        π5 = sol2[[1,6]]
        π6 = sol2[[1,7]]
        π7 = sol2[[1,8]]
Out[11]= {0.4964}
Out[12]= {0.1004}
Out[13]= {0.0861}
Out[14]= {0.0705}
Out[15]= {0.0560}
Out[16]= {0.0437}
Out[17]= {0.0338}
Out[18]= {0.0260}

In[19]= φ[1]=π0
        φ[2]=π0+π1
        φ[3]=π0+π1+π2
        φ[4]=π0+π1+π2+π3
        φ[5]=π0+π1+π2+π3+π4
        φ[6]=π0+π1+π2+π3+π4+π5
        φ[7]=π0+π1+π2+π3+π4+π5+π6
        φ[8]=π0+π1+π2+π3+π4+π5+π6+π7
Out[19]= 0.4964
Out[20]= 0.5968
Out[21]= 0.6828
Out[22]= 0.7533
Out[23]= 0.8094
Out[24]= 0.8531
Out[25]= 0.8869
Out[26]= 0.9130

In[27]= φ[0]=Sum[h[8-i]*φ[i], {i,1,8}]
Out[27]= 0.3866

In[28]= Do[Print[{N[φ[n]=((1/h[0])*(φ[n-c]-Sum[h[n-i]*φ[i],
       {i,1,n-1}])), 4], n}], {n,10,40,10}]
Out[28]= {{0.9484}, 10}
         {{0.9962}, 20}
         {{0.9996}, 30}
         {{0.9999}, 40}

In[29]= (* φ(u,T) values when T = 1 and u = 0,1,2,3,4,5,10,20,30,40 *)
       Do[Print[{N[φ[n,1]=H[n+7],4],n}], {n,0,5}]

Out[29]= {{0.5987}, 0}
         {{0.7291}, 1}
         {{0.8305}, 2}
         {{0.9015}, 3}
         {{0.9467}, 4}
         {{0.9730}, 5}

In[30]= Do[Print[{N[φ[n,1]=H[n+7],4],n}], {n,10,40,10}]
Out[30]= {{0.9996}, 10}
         {{1.0000}, 20}
         {{1.0000}, 30}
```

```
{{1.0000}, 40}

In[31]= (* φ(u,T) values when T=2,3,4,5,10,20,30,40 and u = 0,1,2,3,4,5,10,20,30 *)
        Do[Print[{N[φ[n,T]=Sum[φ[n+8-i,T-1]*h[i], {i,0,n+7}], 4], n,T}],
        {n,0,5}, {T,2,5}]
Out[31]= {{0.5139}, 0, 2}
         {{0.6437}, 1, 2}
         {{0.7533}, 2, 2}
         {{0.8379}, 3, 2}
         {{0.8985}, 4, 2}
         {{0.9391}, 5, 2}
         {{0.4748}, 0, 3}
         {{0.6009}, 1, 3}
         {{0.7107}, 2, 3}
         {{0.7989}, 3, 3}
         {{0.8652}, 4, 3}
         {{0.9125}, 5, 3}
         {{0.4521}, 0, 4}
         {{0.5750}, 1, 4}
         {{0.6837}, 2, 4}
         {{0.7730}, 3, 4}
         {{0.8419}, 4, 4}
         {{0.8926}, 5, 4}
         {{0.4373}, 0, 5}
         {{0.5578}, 1, 5}
         {{0.6653}, 2, 5}
         {{0.7547}, 3, 5}
         {{0.8248}, 4, 5}
         {{0.8775}, 5, 5}

In[32]= Do[Print[{N[φ[n,T]=Sum[φ[n+8-i,T-1]*h[i], {i,0,n+7}], 4], n,T}],
        {n,0,5}, {T,10,30,10}]
Out[32]= {{0.4056}, 0, 10}
         {{0.5199}, 1, 10}
         {{0.6236}, 2, 10}
         {{0.7117}, 3, 10}
         {{0.7829}, 4, 10}
         {{0.8385}, 5, 10}
         {{0.3912}, 0, 20}
         {{0.5022}, 1, 20}
         {{0.6035}, 2, 20}
         {{0.6902}, 3, 20}
         {{0.7610}, 4, 20}
         {{0.8171}, 5, 20}
         {{0.3881}, 0, 30}
         {{0.4983}, 1, 30}
         {{0.5989}, 2, 30}
         {{0.6852}, 3, 30}
         {{0.7558}, 4, 30}
         {{0.8119}, 5, 30}

In[33]= Do[Print[{N[φ[n,T]=Sum[φ[n+8-i,T-1]*h[i], {i,0,n+7}], 4], n,T}],
        {n,10,40,10}, {T,2,5}]
Out[33]= {{0.9972}, 10, 2}
```

```
{{1.0000}, 20, 2}
{{1.0000}, 30, 2}
{{1.0000}, 40, 2}
{{0.9931}, 10, 3}
{{1.0000}, 20, 3}
{{1.0000}, 30, 3}
{{1.0000}, 40, 3}
{{0.9884}, 10, 4}
{{1.0000}, 20, 4}
{{1.0000}, 30, 4}
{{1.0000}, 40, 4}
{{0.9839}, 10, 5}
{{0.9999}, 20, 5}
{{1.0000}, 30, 5}
{{1.0000}, 40, 5}
```

```
In[34]= Do[Print[{N[φ[n,T]=Sum[φ[n+8-i,T-1]*h[i], {i,0,n+7}], 4], n,T}],
        {n,10,40,10}, {T,10,30,10}]
Out[34]= {{0.9671}, 10, 10}
         {{0.9992}, 20, 10}
         {{1.0000}, 30, 10}
         {{1.0000}, 40, 10}
         {{0.9541}, 10, 20}
         {{0.9976}, 20, 20}
         {{0.9999}, 30, 20}
         {{1.0000}, 40, 20}
         {{0.9504}, 10, 30}
         {{0.9968}, 20, 30}
         {{0.9998}, 30, 30}
         {{1.0000}, 40, 30}
```

Example 4.3 source code

```
In[1]= (* 4.3 Example *)
       h[0]=1/8
       h[1]=1/2
       h[2]=5/32
       h[3]=7/32
Out[1]= 1/8

Out[2]= 1/2

Out[3]= 5/32

Out[4]= 7/32


In[5]= EZ = 0*h[0]+1*h[1]+2*h[2]+3*h[3]

Out[5]= 47/32

In[6]= c=3
Out[6]= 3
```

```
In[7]= (* Solve and find s=α value *)
      sol1 = {s}/.NSolve[{(4/32)+(16/32)s+(5/32)s^2-(25/32)s^3==0,
              Abs[s]<1}, {s}]
Out[7]= {{-0.4}, {-0.4}}

In[8]= (*Solve and find π expressions*)
      nsol2 = Simplify[Solve[{p0*(h0+α*(h0+h1)+α^2*(h0+h1+h2))+
              p1*(α*h0+α^2*(h0+h1))+p2*α^2*h0==0,
              p0*((h0+h1)+2*α*(h0+h1+h2))+p1*(h0+2*α*(h0+h1))+
              p2*2*α*h0==0,
              p0*(3*h0+2*h1+h2)+p1*(2*h0+h1)+p2*h0==3-D}, {p0,p1,p2}]]
Out[8]=
```

$$\left\{\left\{p0 \to -\frac{(-3+D)\,\alpha^2}{h0\,(-1+\alpha)^2},\ p1 \to \frac{(-3+D)\,\alpha\,(h1\,\alpha+h0\,(2+\alpha))}{h0^2\,(-1+\alpha)^2},\right.\right.$$
$$\left.\left.p2 \to -\frac{(-3+D)\,\left(h1^2\,\alpha^2+h0^2\,(1+2\,\alpha)+h0\,\alpha\,(-h2\,\alpha+h1\,(2+\alpha))\right)}{h0^3\,(-1+\alpha)^2}\right\}\right\}$$

```
In[9]= (*π numerical values*)
      sol2 = NSolveValues[{π0*(1/8+(-0.4)*(1/8+1/2)+(-0.4)^2*
              (1/8+1/2+5/32))+π1*((-0.4)*1/8+(-0.4)^2*(1/8+1/2))+
              π2*(-0.4)^2*1/8==0, π0*((1/8+1/2)+2*(-0.4)*(1/8+1/2+5/32))+
              π1*(1/8+2*(-0.4)*(1/8+1/2))+π2*2*(-0.4)*1/8==0,
              π0*(3*1/8+2*1/2+5/32)+π1*(2*1/8+1/2)+π2*1/8==3-47/32},
              {π0,π1,π2}]
Out[9]=
```

$$\{\{1.,\ 1.11022\times10^{-15},\ -4.16334\times10^{-15}\}\}$$

Example 4.4 source code

```
In[1]= (* 4.4 Example *)
      h[0]=1/64
      h[1]=11/64
      h[2]=36/64
      h[3]=16/64
```

Out[1]= $\frac{1}{64}$

Out[2]= $\frac{11}{64}$

Out[3]= $\frac{9}{16}$

Out[4]= $\frac{1}{4}$

```
In[5]= EZ = 0*h[0]+1*h[1]+2*h[2]+3*h[3]
```

Out[5]= $\frac{131}{64}$

```
In[6]= c=4
Out[6]= 4

In[7]= (* Solve and find s=α value *)
```

```
sol1 = {s}/.NSolve[{(16/64)*s^3+(36/64)*s^2+(11/64)s+(1/64)==s^4,
       Abs[s]<1}, {s}]
```

Out[7]= $\{\{s \to -\frac{1}{4}\}, \{s \to -\frac{1}{4}\}\}, \{s \to -\frac{1}{4}\}\}$

```
In[8]= (*Solve and find π expressions*)
nsol2 = Simplify[Solve[{p0*(h0+α*(h0+h1)+α^2*(h0+h1+h2)+α^3
        *(h0+h1+h2+h3))+p1*(α*h0+α^2*(h0+h1)+α^3*(h0+h1+h2))
        +p2*(α^2*h0+α^3*(h0+h1))+p3*α^3*h0==0,
        p0*((h0+h1)+2*α*(h0+h1+h2)+3*α^2*(h0+h1+h2+h3))
        +p1*(h0+2*α*(h0+h1)+3*α^2*(h0+h1+h2))
        +p2*(2*α*h0+3*α^2*(h0+h1))+p3*3*α^2*h0==0,
        p0*(2*(h0+h1+h2)+6*α*(h0+h1+h2+h3))
        +p1*(2*(h0+h1)+6*α*(h0+h1+h2))
        +p2*(2*h0+6*α*(h0+h1))+p3*6*α*h0==0,
        p0*(4*h0+3*h1+2*h2+h3)+p1*(3*h0+2*h1+h2)+p2*(2*h0+h1)
        +p3*h0==4-D}, {p0,p1,p2,p3}]]
```

Out[8]=

$$\left\{\left\{p0 \to -\frac{(-4+D)\,\alpha^3}{h0\,(-1+\alpha)^3},\ p1 \to \frac{(-4+D)\,\alpha^2\,(h1\,\alpha+h0\,(3+\alpha))}{h0^2\,(-1+\alpha)^3},\right.\right.$$

$$p2 \to -\frac{(-4+D)\,\alpha\,\left(h1^2\,\alpha^2+3\,h0^2\,(1+\alpha)+h0\,\alpha\,(-h2\,\alpha+h1\,(3+\alpha))\right)}{h0^3\,(-1+\alpha)^3},$$

$$p3 \to \frac{1}{h0^4\,(-1+\alpha)^3}$$

$$\left.\left.(-4+D)\,\left(h1^3\,\alpha^3+h0^3\,(1+3\,\alpha)+h0\,h1\,\alpha^2\,(-2\,h2\,\alpha+h1\,(3+\alpha))+h0^2\,\alpha\,(3\,h1\,(1+\alpha)+\alpha\,(h3\,\alpha-h2\,(3+\alpha)))\right)\right\}\right\}$$

```
In[9]= (*π numerical values*)
sol2 = NSolveValues[{p0*(h0+α*(h0+h1)+α^2*(h0+h1+h2)
       +α^3*(h0+h1+h2+h3))+p1*(α*h0+α^2*(h0+h1)+α^3*(h0+h1+h2))
       +p2*(α^2*h0+α^3*(h0+h1))+p3*α^3*h0==0,
       p0*((h0+h1)+2*α*(h0+h1+h2)+3*α^2*(h0+h1+h2+h3))
       +p1*(h0+2*α*(h0+h1)+3*α^2*(h0+h1+h2))
       +p2*(2*α*h0+3*α^2*(h0+h1))+p3*3*α^2*h0==0,
       p0*(2*(h0+h1+h2)+6*α*(h0+h1+h2+h3))+p1*(2*(h0+h1)
       +6*α*(h0+h1+h2))+p2*(2*h0+6*α*(h0+h1))+p3*6*α*h0==0,
       p0*(4*h0+3*h1+2*h2+h3)+p1*(3*h0+2*h1+h2)+p2*(2*h0+h1)
       +p3*h0==4-EZ}, {p0,p1,p2,p3}]
Out[9]= {{1., 0., 0., 0.}}
```