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Remarks on Some Growth Functions

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Abstract: In this note, we will focus on the relationships between the growth rates of several functions that describe the topological shape at infinity of discrete groups. In particular, we will consider, in detail, the notion of the rate of vanishing of the simple connectivity at infinity for some geometric classes of finitely presented groups.

Keywords: Cayley graphs; discrete groups; ends; growth functions; topology at infinity

MSC: 20F65; 20F55; 05C40

1. Introduction

The topology at infinity concerns those geometric and topological asymptotic notions that are important both for non-compact manifolds and for discrete groups. For instance, the question of whether a contractible manifold is simply connected at infinity is central in the study of the topology and geometry at infinity of the manifold because it assures that its "infinity" is as tame as possible, namely that of the Euclidean space (see [1]). Here "simply connected at infinity" (abbreviated sci) roughly means that the infinity of the space is simply connected or, more precisely, that any loop "at infinity" bounds a disk that is close to infinity. This notion can be defined and studied for finitely presented groups, too (see, e.g., [2]). To better understand the asymptotic topological behaviour of those finitely presented groups that are simply connected at infinity, in [3], a function was defined that, in some sense, measures how a group is sci, and the growth of this function, called the sci-growth, is another example of a geometric property of discrete groups (i.e., a quasi-isometry invariant for finitely presented groups, following Gromov's viewpoint [4]). A very similar function was defined and studied in [5,6]. This is homologous in dimension 0 of the sci-growth for one-ended groups, and it measures the "depth" of the bounded components of the complement of metric balls in the Cayley graph of the group. The growth of this function is also a geometric invariant of finitely presented groups, called the end-depth (see [5,6]).

In this paper, we will study these notions, together with similarly related ones, for some specific geometric classes of discrete groups.

2. Definitions

Let (X, d) be a metric space and D, E two subsets of it. The distance between D and E, denoted $d(D, E)$, is defined as $\inf\{d(x, y) \mid x \in D, y \in E\}$. For any point $x \in X$, we will write $S(x, r)$ and $B(x, r)$ to denote the sphere and the ball in X of radius r, with centre x.

We now recall the all-important definition of the number of *ends* of a metric space. Let (X, d) be a locally compact and connected metric space. For any compact subset K of X, we can consider the number of unbounded connected components of the complement $X - K$ and write it as $e(X, K)$. The number of ends $e(X)$ of space X is the supremum of $e(X, K)$ for all compacts $K \subset X$.

If $e(X) = 1$, we say that X is a one-ended metric space. This roughly means that there is only "one way to go to infinity" (at least outside large compacts).

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Now, consider a finitely generated group $G = \langle S \rangle$ (this simply means that G has a finite generating set S). The *Cayley graph* associated with the pair (G, S), denoted by $\Gamma(G, S)$, can be defined as the directed graph with one vertex associated with each element of the group and edges (g, h) from g to h, with $g, h \in G$, whenever $gh^{-1} \in S$. The Cayley graph strictly depends on the choice of a generating set, and, for instance, it is connected if and only if S generates the whole group G. If d_S is the word metric in G (where the distance between two elements g and h is the minimum possible length of a word using elements from $S \cup S^{-1}$), the Cayley graph $\Gamma(G, S)$ with the metric d_S also becomes a locally compact and connected metric space. In this way, one can define the number of ends of the Cayley graph, $e(\Gamma(G, S))$, and it turns out that the number of ends is actually independent of the finite generating set chosen [7].

Thus, the number of ends of the finitely generated group G, $e(G)$, can be (well-)defined as the number of ends of (one of) its Cayley graph $\Gamma(G, S)$, for some finite generating set S.

A finitely generated group is *one-ended* whenever its Cayley graph, with respect to one of its finite generating sets, is a one-ended metric space. Note also that, for finitely generated groups, the number of ends is a quasi-isometry invariant [7]. Here we meet the viewpoint of Gromov [4]: two metric spaces are quasi-isometric if they are undistinguishable when looked at from far away. The interesting geometric conditions of groups are those properties that are invariant under quasi-isometry. This is the right way in order to understand the global, asymptotic, geometric behaviour of discrete groups.

One of the oldest results concerning the ends of groups is the well-known result of Hopf [7,8], which says that a finitely generated group can have only 0, 1, 2, or infinitely many ends (a Cantor set). Obviously, a finitely generated group is finite if and only if $e(G) = 0$. On the other hand, Stallings' Theorem on the ends of groups [9] states that G has more than one end if and only if it admits a nontrivial decomposition as an amalgamated free product or an HNN extension over a finite subgroup. Nowadays, this is considered the first result in geometric group theory (for proof, see [7,10]). Finally, in 1983, M. Gromov gave a very fast proof for this theorem using minimal surfaces [4].

2.1. Growth Function of the End

We now need a digression on growth functions. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two real functions. One says that the growth of the function f is at most the growth of the function g, and one writes $f \prec g$ if there exist three constants $a_1 > 0, a_2 > 0, a_3$, with $a_i \in \mathbb{R}$, such that, for any $x \in \mathbb{R}_+$, the following holds: $f(x) \leq a_1 g(a_2 x) + a_3$. If $f \prec g$ and $g \prec f$, then one says that the functions f and g have the *same growth*, and one writes $f \sim g$.

It is easy to see that $f \sim g$ is an equivalence relation. The *growth rate* of a function f is then defined as the growth of the equivalence class of the function f. For instance, we will say that f has *linear growth* whenever $f(x) \sim x$.

With all these notions, we are able to properly define the end-depth of a group.

Definition 1. *Let $G = \langle S \rangle$ be a finitely generated one-ended group and let $X = \Gamma(G, S)$ be its Cayley graph. For any real number $r > 0$, let $N(r)$ be the set of all $k \in \mathbb{R}$ such that any two points in $X - B(k)$ can be joined by a path lying outside the ball $B(r)$. The function $V_0^X(r) = \inf N(r)$ is called the end-depth function of X.*

Remark that the end-depth function itself depends on the generating set of the group, but its growth rate does not, as shown in [6]. Actually, the end-depth growth is even linear for any finitely presented group [5].

Here we want to present a more precise and clearer description for the understanding of the end-depth function following [11]. If we consider a finitely generated group $G = \langle S \rangle$, with a finite generating set S and its Cayley graph $X = \Gamma(G, S)$ associated with S, then for any positive real number r, the set of connected components of the complement of the r-ball is finite, i.e., $\pi_0(X - B(e, r)) = m$, for some integer number m. Whenever group G is one-ended, then in the set $X - B(r, e)$, there is only one unbounded connected component.

Let U_r be this unique unbounded connected component and denote by B_r, the union of the bounded components of $X - B(e,r)$. We can make the following useful observations:

- The set B_r is empty if and only if $V_0^X(r) = r$.
- Otherwise, whenever $X - B(r)$ has at least one bounded connected component, then, for any $x \in B_r$, any path joining x to an element $y \in U_r$ must go through the ball $B(e,r)$. Thus, for any $x \in B_r$, one has: $V_0^X \geq d(e,x)$. This implies that $V_0^X(r) \geq \max\{d(e,x) \mid x \in B_r\}$.
- On the other hand, for any $y, z \in X$ with $d(e,y), d(e,z) > \ max\{d(e,x) \mid x \in B_r\}$, we have that $y, z \in U_r$. This implies that the two pints y and z can be joined by a path lying outside the ball $B(e,r)$, and so we find that $y, z \in X - B(e, V_0^X(r))$. Hence, we can obtain an explicit formula for the function V_0:

$$V_0^X(r) = \max\{d(e,x) \mid x \in B_r\}. \tag{1}$$

  This equation makes us understand in what sense the end-depth function gives a measure of the depth of the bounded connected components of $X - B(e,r)$.
- Furthermore, from Equation (1), we can also deduce that there exists a bounded connected component P of $X - B(e,r)$ and at least one of its elements $p \in P$ such that $V_0^X(r) = d(e,p) = \max\{d(e,x) \mid x \in B_r\}$.

2.2. Other End-Topological Notions

Before stating our results, we need to introduce some more topological tameness conditions at infinity for manifolds and discrete groups.

Recall that a *ray* in a non-compact topological space X is just a proper map $r : [0,\infty) \to X$ (where "proper" means that the inverse image of a compact subset is compact too). Two rays r_1 and r_2 are said to converge to the same end of X if, for any compact subset $C \subset X$, there exists a real number T such that $r_1([T,\infty))$ and $r_2([T,\infty))$ are contained in the same component of $X - C$. (Note that the set of rays modulo this relation actually coincides with is the set of ends of X, as defined above).

The next notion, called *semistability at infinity*, is an old topological property much used in the study of the topological shape of ends of finitely presented groups (see [12]). If a finitely presented group is semistable at infinity, then subtle invariants of the group can be defined (such as the so-called fundamental group at infinity, see [13]).

An end of X is said to be *semistable* if any two rays of X converging to this end are properly homotopic. This notion is equivalent to the following fact: for any ray r that converges to the end and for any natural number n, there exists a $N \geq n$ such that any loop at r with the image outside $B(N)$ can be pushed (relative to r) to infinity by a homotopy within $X - B(n)$. Finally, a finitely presented group is semistable at infinity if all of its ends are semistable.

Definition 2. *Let X be a metric space, e an end of X and r a ray converging to e. The semistability function $S_e(r)$ is defined as the smallest N such that, for any $R \geq N$ and any loop l based on r lying in $X - B(N)$ there exists a homotopy relative to r and supported within $X - B(n)$, which pushes l to a loop in $X - B(R)$.*

Remark 1. *Notice that a semistable end has a well-defined semistability function.*

Let G be a one-ended, finitely presented group. Denote by $S_G = S_{\tilde{X}}$ the semistability function of the space $\tilde{X}$ with respect to its only end, where X is a compact space with fundamental group G and $\tilde{X}$ its universal cover. It is easy to see that the growth of S_G is independent of the space X (i.e., if Y is another compact space with $\pi_1 Y = G$, then one has $S_{\tilde{X}} \sim S_{\tilde{Y}}$).

Remark 2. *The proof of the quasi-isometry invariance of the end-depth [6] (and that of the sci-growth [3]) can be used to show that the semistability function S_G for semistable infinity groups is also a geometric property (i.e., invariant under quasi-isometries).*

Lemma 1. *For a one-ended, semistable infinity group G, the end-depth V_0 is bounded by the semistability function S_G.*

Proof. Let x be a point in $\tilde{X} - B(S_G(n))$. The semistability condition implies that the constant loop x is a homotope of loop out of $B(N)$ for any large enough N, with a homotopy in $\tilde{X} - B(n)$. Thus x can be joined to a point x_1 of $\tilde{X} - B(N)$ by a path in $\tilde{X} - B(n)$. Since G is one-ended, there exists $N(n)$ such that any two points out of $B(N(n))$ can be joined out of $B(n)$. Hence any $x, y \in \tilde{X} - B(n)$ is joined by the path $[x, x_1][x_1, y_1][y_1, y]$, which sits outside $B(n)$. This means that $V_0(G) \leq S_G$. □

Another all-important topological notion at infinity, possibly the most powerful one (see, e.g., [1,14]), is the simple connectivity at infinity.

Definition 3. *A connected, locally compact, topological space X with $\pi_1 X = 0$ is simply connected at infinity (abbreviated sci, or $\pi_1^\infty X = 0$) if for each compact $k \subseteq X$ there exists a larger compact $k \subseteq K \subseteq X$ such that any closed loop in $X - K$ is null homotopic in $X - k$.*

This notion can be extended to a group-theoretical framework as follows (for more details, see [2]):

Definition 4. *A finitely generated group G is simply connected at infinity if, for some (equivalently for any) finite complex X such that $\pi_1 X = G$, one has that $\pi_1^\infty \tilde{X} = 0$ (where $\tilde{X}$ is the universal covering space of X).*

In order to measure the growth of the simple connectivity at infinity, the following function was introduced in [3]:

Definition 5. *Let X be a simply connected metric space. The rate of vanishing of π_1^∞, denoted by $V_X(r)$, is the smallest $N(r)$ such that any loop lying outside the ball $B(N(r))$ of radius $N(r)$ bounds a 2-disk out of $B(r)$.*

It was proven in [3] that the growth rate (namely the equivalence class) of $V_X(r)$ is a geometric invariant of discrete groups. More precisely, if G is a finitely presented group, $\tilde{X}_G$ is the universal covering space of a compact simplicial complex X_G, with $\pi_1(X_G) = G$, the function $V_G = V_{\tilde{X}_G}$ is a quasi-isometry invariant of G.

If V_G is well-defined and linear, one says that group G has linear sci-growth.

Proposition 1. *If a finitely presented group G is simply connected at infinity, then $V_G \sim S_G$.*

Proof. This is proven in [5] (Proposition 2.5). □

Remark 3.

- *For topological spaces that are k-connected at infinity, we can define the function $V_k(X) = \inf(N(r))$ such that any k-sphere outside $B(N(r))$ bounds a $(k+1)$-sphere out of $B(r)$.*
- *For general metric spaces that are not Cayley complexes, these functions can have arbitrary large growth. In fact, let $f : \mathbb{N} \to \mathbb{R}^+$ be a real function and let $C_k(n)$ be the set $[0, f(n)] \times S^k/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation $(0, x) \sim (0, y)$ (here $C_k(n)$ is the cone of the k-dimensional sphere $S^k \subset \mathbb{R}^k$ of height $f(n)$). Now, consider the real half-line $[0, \infty)$ and attach to any n the cone $C_k(n)$. Obviously, the resulting space X_k is one-ended,*

k-connected at infinity, and whose function $V_k(X_k)(n)$ is equal to $n + f(n)$. Hence the space X_k has the same growth as any arbitrarily chosen function f.

3. Results

In this section, we will study the growth of the simple connectivity at infinity for some classes of groups, and we will enlarge the set of sci groups with a linear sci-growth.

3.1. Coxeter Groups

Coxeter groups were introduced long ago as generalisations of reflection groups (see [7,14,15]). A *Coxeter group* is an abstract group that admits a formal description in terms of reflections; more precisely, it is a finitely presented group W with particular presentations of the following form: $\langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1$ for $i \in \{1, 2, \ldots, n\}, (s_i s_j)^{m_{ij}} = 1$ where $i < j$ ranges over some subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and $m_{ij} \geq 2\rangle$. Not all Coxeter groups are finite, and not all can be described in terms of Euclidean symmetries and reflections.

Here we will provide direct proof, using the semistability condition, of the fact that simply connected at infinity (infinite) Coxeter groups have a linear sci-growth.

Lemma 2. *Let W be a Coxeter group and Γ_W be its Cayley graph. If $[v, w]$ is an edge of Γ_W, then $d(e, v) \neq d(e, w)$.*

Proof. Consider the homomorphism $f : W \to \{+1, -1\}$ defined by $f(s_i) = -1$ for all i. Let p be an edge path joining the origin e with the point v. Then $f(v) = (-1)^{n(p)}$, where $n(p)$ is the number of edges in p. If $[v, w]$ is an edge in Γ_W from v to w, then $d(e, w) = f(w) = (-1)^{n(p)+1} \neq d(e, v)$. □

Theorem 1. *Coxeter groups have linear semistability.*

Proof. We will show that S_W is linear for any Coxeter group W. Let $W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1\rangle$ be the standard presentation of a Coxeter group, where $m_{ij} \in \{2, 3, \ldots, \infty\}$.

Lemma 3. *Assume that $m_{ij} \neq \infty$ for all i, j. Then S_W is linear.*

Proof. Let X be the standard two-complex associated with the presentation of above. In what follows, all metric balls are centred at the origin of the Cayley complex $\tilde{X}$.

Consider a loop l in $\tilde{X} - B(r+1)$. Let v be a vertex of l that realizes the minimum distance from the origin. If it is not unique, Lemma 2 implies that two such vertices cannot be adjacent.

Assume that the edges of l adjacent to v are $[wv]$, labelled s_i, and $[v, u]$, labelled s_j. Consider the null homotopic loop $(s_i s_j)^{m_{ij}}$, which starts at v. This loop is made of wvu and its complementary part $wy_1 \ldots y_N u$.

First, we have to note that all points y_j are at a distance of at least $r + 2$ from the identity element e. Then, since loop $(s_i s_j)^{m_{ij}} = uvwy_1 \ldots y_N u$, in order to replace wvu by $wy_1 \ldots y_N u$, it is a sufficient homotope supported in $\tilde{X} - B(r)$.

Continuing this procedure, we can move loop l as far as we wish. □

Lemma 4. *Assume that A, B are two finitely presented semistable at infinity groups with a common finitely generated subgroup C. If S_A and S_B are linear, then $S_{A*_C B}$ is linear too.*

Proof. This is proven in [12] (one can find another proof in [5] for the sci-growth case). □

Now, let W be a Coxeter group. If in its presentation there exist i, j such that $m_{ij} = \infty$, then one can split W into an amalgamated free product $W = A *_C B$ where C is finitely generated, and A and B are presented with $m_{ij} \neq \infty$ (see [12]). Continued reduction of this kind, and the application of the above two lemmas, yield the proof of Theorem 1. □

From Theorem 1 and Proposition 1 we obtain a new proof of [5] (Proposition 5.1):

Corollary 1. *Simply connected at infinity, Coxeter groups have a linear sci-growth.*

3.2. Buildings

A *building*, first defined by J. Tits, is a special geometrical and combinatorial structure on which (semi-simple) groups act (for more details on this interesting subject, see [14]). Later, the definition evolved, and buildings are nowadays defined as particular metric spaces with a special collection of subspaces satisfying several conditions. Associated with any building B there is a corresponding Coxeter group W_B. For classical buildings, the associated group W_B can be finite (and, in such a case, the building is called "spherical") or a Euclidean reflection group (and, in such a case, the building is said to be "affine"). There is also a geometric realisation of spherical or affine buildings, and it is defined as a particular simplicial complex in which any simplex of the highest dimension is called a *chamber*. In such a geometric realisation there are many embedded copies of the Coxeter complex of W_B, which are called *apartments*, and, in fact, the whole building can be defined and constructed as the union of all the apartments. When the building is spherical, any apartment is just a sphere (hence the name), whereas, for (irreducible) affine buildings, each apartment is a copy of the Euclidean space.

As shown in [15] (where the authors study the end-topology of Coxeter groups and buildings in depth), for a better understanding of the shape at infinity of them, it is more convenient to slightly change the viewpoint and to define the geometric realisation of a building B in such a way that each apartment is isomorphic to the so-called "Davis complex" of W_B (for its construction see [1]).

Theorem 2. *Buildings that are simply connected at infinity have a linear sci-growth.*

Proof. Davis and Meyer have shown that the geometric realisation of a building B constructed out of Davis complexes is simply connected at infinity if and only if the associated Coxeter group W_B is sci [15]. This result, together with Corollary 1, leads to the proof. □

3.3. An Open Question

It is a standard fact that a lattice in a solvable Lie group is uniform and finitely presented. Furthermore, one can prove that such a lattice is polycyclic and also a virtually strongly polycyclic group [14]. Hence, the problem of the linearity of the sci-growth for all solvable Lie groups can be reduced to the linearity of the sci-growth of strongly polycyclic sci groups, which is still not proven.

This is a concrete and interesting problem to work on in the next future.

4. Conclusions

In this paper, we continued the investigations on the metric refinements of some known topological tameness conditions at infinity of discrete groups, a subject that is not very well-developed and studied. There are several interesting open questions waiting to be considered. Our work on this topic leads to the conclusion that for most geometric classes of groups, the sci-growth is just linear. On the other hand, the question of whether (or not) there are finitely presented groups with a super-linear sci-growth that act freely and co-compactly on the Euclidean space $\mathbb{R}^n$ is the main open problem in this research line. The existence of such a strangely behaved group should have a strong impact on the fields of geometric group theory, topology at infinity and shape theory.

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