Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13066–13072.
DOI:10.3934/math.2023658
Received: 28 October 2022
Revised: 08 March 2023
Accepted: 17 March 2023
Published: 03 April 2023

Research article

Minimum of heavy-tailed random variables is not heavy tailed

Remigijus Leipus[1], **Jonas Šiaulys**[2,*] **and Dimitrios Konstantinides**[3]

[1] Institute of Applied Mathematics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania
[2] Institute of Mathematics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania
[3] Department of Statistics and Actuarial-Financial Mathematics, University of Aegean, Karlovasi, GR-83 200 Samos, Greece

* **Correspondence:** Email: jonas.siaulys@mif.vu.lt; Tel: +37068878493.

Abstract: By constructing an appropriate example, we show that the class of heavy-tailed distributions is not closed under minimum. We provide two independent heavy-tailed random variables, such that their minimum is not heavy tailed. In addition, we establish a few properties of the distributions considered in the example.

Keywords: heavy-tailed distribution; closure properties; minimum of random variables; closure under minimum; generalized long-tailed distribution
Mathematics Subject Classification: 26E40, 46F10, 60E05

1. Introduction

We say that distribution F is *heavy-tailed* and write $F \in \mathscr{H}$ if

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x} \mathrm{d}F(x) = \infty \quad \text{for any} \quad \lambda > 0.$$

If $F(x) = \mathbb{P}(X \leqslant x)$, then random variable X is called heavy-tailed. It is well known (see, for instance, Theorem 2.6 in [10]) that $F \in \mathscr{H}$ if and only if

$$\limsup_{x\to\infty} \mathrm{e}^{\delta x}\overline{F}(x) = \infty \quad \text{for any} \quad \delta > 0.$$

Here $\overline{F}(x) = 1 - F(x)$ denotes the right tail of $F(x)$. We say that distribution F is *strongly heavy-tailed* and write $F \in \mathscr{H}^*$ if

$$\lim_{x\to\infty} \mathrm{e}^{\delta x}\overline{F}(x) = \infty \quad \text{for any} \quad \delta > 0.$$

Obviously, $\mathscr{H}^* \subset \mathscr{H}$ and one can check that $\mathscr{H}\backslash\mathscr{H}^* \neq \emptyset$. For discussion on classes $\mathscr{H}$, $\mathscr{H}^*$ and examples $F \in \mathscr{H} \setminus \mathscr{H}^*$ see [2, 15, 16] among others.

Concerning other properties of heavy-tailed distribution class, it is easy to see that $\mathscr{H}$ is closed under convolution, mixing, maximum and product-convolution.

Let us denote the convolution of distributions F_1 and F_2 by

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y).$$

We say that some class of distributions $\mathscr{B}$ is closed under convolution if for any two distributions F_1 and F_2 it holds that

$$F_1 \in \mathscr{B}, F_2 \in \mathscr{B} \;\Rightarrow\; F_1 * F_2 \in \mathscr{B}. \tag{1.1}$$

The relation (1.1) for class of distributions $\mathscr{B} = \mathscr{H}$ follows immediately from definition of $\mathscr{H}$. Namely, by supposing that F_1, F_2 are distributions of independent random variable X_1 and X_2, we get

$$\begin{aligned} F_1 * F_2 \in \mathscr{H} \;&\Leftrightarrow\; \mathbb{E}\mathrm{e}^{\lambda(X_1+X_2)} = \mathbb{E}\mathrm{e}^{\lambda X_1}\mathbb{E}\mathrm{e}^{\lambda X_2} = \infty \text{ for any } \lambda > 0 \\ &\Leftrightarrow\; F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H}. \end{aligned}$$

Similarly, we say that a class of distributions $\mathscr{B}$ is closed under mixing if for $p \in (0, 1)$

$$F_1 \in \mathscr{B}, F_2 \in \mathscr{B} \;\Rightarrow\; pF_1 + (1-p)F_2 \in \mathscr{B}.$$

Since for any $\lambda > 0$

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}(pF_1 + (1-p)F_2)(x) = p\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}F_1(x) + (1-p)\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\mathrm{d}F_2(x),$$

we get a stronger assertion

$$F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H} \;\Leftrightarrow\; pF_1 + (1-p)F_2 \in \mathscr{H} \text{ for } p \in (0, 1).$$

It is said that class of distributions $\mathscr{B}$ is closed under maximum if $F_1, F_2 \in \mathscr{B}$ implies

$$F_{X_1 \vee X_2} = F_1 F_2 \in \mathscr{B}.$$

Like in the case of convolution, a stronger assertion on closure under maximum follows

$$F_1 \in \mathscr{H} \text{ or } F_2 \in \mathscr{H} \;\Leftrightarrow\; F_1F_2 \in \mathscr{H}$$

because

$$\begin{aligned} \overline{F_1F_2}(x) &= \overline{F}_1(x) + \overline{F}_2(x) - \overline{F}_1(x)\overline{F}_2(x) \\ &\underset{x\to\infty}{\sim} \overline{F}_1(x) + \overline{F}_2(x). \end{aligned}$$

Considering the closure under the product-convolution, we present the following result:

$$F_1 \in \mathscr{H}, F_2(-0) = 0, F_2(0) < 1 \;\Rightarrow\; F_1 \otimes F_2 \in \mathscr{H}, \tag{1.2}$$

where symbol $\otimes$ denotes the product-convolution, i.e., $F_1 \otimes F_2(x) = \mathbb{P}(X_1X_2 \leqslant x)$ for independent random variables X_1 and X_2 with distributions F_1 and F_2. For the proof of (1.2) it suffices to observe that

$$\mathbb{E}\,\mathrm{e}^{\lambda X_1X_2} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda X_1^+X_2} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda X_1^+X_2}\mathbb{1}_{\{X_2>a\}} \geqslant \mathbb{E}\,\mathrm{e}^{\lambda aX_1^+}\mathbb{P}(X_2 > a),$$

where $\lambda > 0$ is an arbitrary constant, and $a > 0$ is such that $\mathbb{P}(X_2 > a) > 0$.

Studies of other interesting properties of heavy-tailed distributions can be found in [2–4, 7–10] among others.

The problem whether class $\mathscr{H}$ is closed with respect to minimum is much more difficult and, to our knowledge, was not solved. In this paper, we prove that class $\mathscr{H}$ is not closed under minimum. We construct two independent random variables X and Y with the corresponding distributions $F \in \mathscr{H}$ and $G \in \mathscr{H}$, such that their minimum $X \wedge Y = \min\{X, Y\}$ is not heavy tailed, i.e., $F_{X\wedge Y} = 1 - \overline{F}\,\overline{G} = F + G - FG \notin \mathscr{H}$.

2. Main results

Consider the distribution tail $\overline{F}(x)$ of the following form:

$$\overline{F}(x) = \mathbb{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x}\mathbb{1}_{[0,1)}(x) + \sum_{n=1}^{\infty}\mathrm{e}^{-x}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!}\mathbb{1}_{[(2n)!,(2n+1)!)}(x)$$
$$+ \sum_{n=1}^{\infty}\mathrm{e}^{-(2n-1)!}\prod_{j=1}^{n-1}\mathrm{e}^{(2j)!-(2j-1)!}\mathbb{1}_{[(2n-1)!,(2n)!)}(x). \tag{2.1}$$

This distribution and distribution in (2.5) below will be used for the main result on the minimum of heavy-tailed r.v.s. Our first result yields several properties of the distribution F.

Theorem 2.1. *Assume that F is defined in* (2.1). *Then $F \in \mathscr{H}$, $F \notin \mathscr{H}^*$ and*

$$\limsup_{x\to\infty}\frac{\overline{F}(x-1)}{\overline{F}(x)} < \infty. \tag{2.2}$$

The property in (2.2) defines the class of *generalized long-tailed distributions*, $\mathscr{OL}$, introduced in [13]. Recall that a distribution F on $\mathbb{R}$ belongs to the class $\mathscr{OL}$, if for any (or some) $y > 0$

$$\limsup_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty. \tag{2.3}$$

Thus, Theorem 2.1 says that

$$(\mathscr{H} \cap \mathscr{OL}) \setminus \mathscr{H}^* \neq \varnothing. \tag{2.4}$$

By Proposition 2.2(ii) in [13], $F \in \mathscr{OL}$ implies that $\lim_{x\to\infty}\mathrm{e}^{\delta x}\overline{F}(x) = \infty$ for some $\delta > 0$, and $\mathscr{OL}$ also admits some light-tailed distributions. Various results related to class $\mathscr{OL}$ can be found in [1, 5, 6, 19, 20]. In particular, authors of [20] showed that $\mathscr{H}^*\setminus\mathscr{OL} \neq \varnothing$, cf. (2.4). Note that class $\mathscr{OL}$ was also introduced in [14], where it was called a Semi-$\mathscr{L}$ class of distributions.

Consider now another distribution with the tail $\overline{G}(x)$ of the following form:

$$\overline{G}(x) = \mathbb{1}_{(-\infty,1)}(x) + \sum_{n=1}^{\infty} \mathrm{e}^{-x+1} \prod_{j=2}^{n} \mathrm{e}^{(2j-1)!-(2j-2)!} \mathbb{1}_{[(2n-1)!,(2n)!)}(x)$$
$$+ \sum_{n=1}^{\infty} \mathrm{e}^{-(2n)!+1} \prod_{j=2}^{n} \mathrm{e}^{(2j-1)!-(2j-2)!} \mathbb{1}_{[(2n)!,(2n+1)!)}(x). \tag{2.5}$$

Analogously to the result in Theorem 2.1, it holds that $G \in \mathscr{H}$, $G \notin \mathscr{H}^*$ and $G \in \mathscr{OL}$.

The main result of the paper says that the distribution $F_{X\wedge Y}(x) = 1-\overline{F}(x)\overline{G}(x)$ is light-tailed. Indeed, by construction of $\overline{F}$ and $\overline{G}$, we have

$$\overline{F}(x)\overline{G}(x) = \mathbb{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x}\mathbb{1}_{[0,\infty,0)}(x)$$

and we obtain the following assertion.

Theorem 2.2. *Assume that X and Y are independent r.v.s with distribution tails $\overline{F}$ in* (2.1) *and $\overline{G}$ in* (2.5)*, respectively. Then*

$$F_{X\wedge Y} \notin \mathscr{H}.$$

Remark 2.1. We mention two related results, which follow easily from definitions. First result says that, although class $\mathscr{H}$ is not closed under minimum, it is closed in the class $\mathscr{H}^*$, i.e.,

$$F_1 \in \mathscr{H}, F_2 \in \mathscr{H}^* \ \Rightarrow\ F_{X_1\wedge X_2} \in \mathscr{H},$$

where X_1 and X_2 are random variables with corresponding distributions F_1 and F_2. Second result says that class $\mathscr{OL}$ is closed under minimum:

$$F_1 \in \mathscr{OL}, F_2 \in \mathscr{OL} \Rightarrow F_{X_1\wedge X_2} \in \mathscr{OL}.$$

The study of the minimum of random variables is important for problems related to various stochastic models. For example it concerns the order statistics $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ of random variables $X_1, X_2, \ldots, X_n$. It is obvious that

$$F_{k:n}(x) = \mathbb{P}(X_{k:n} \leqslant x) = \sum_{j=0}^{k-1} \binom{n}{j} (F_X(x))^j (\overline{F}_X(x))^{n-j}$$

in the case of independent and identically distributed random variables with common distribution F_X. We can see from this expression that properties of order statistics are related to the closure property of random variables under minimum. The order statistics properties for various subclasses of $\mathscr{H}$ were considered in [11, 12, 17, 18], for instance. The definition of the class $\mathscr{H}$ implies immediately the following assertion.

Theorem 2.3. *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common distribution F_X. Then $F_{X_{k,n}} \in \mathscr{H}$ for $k \in \{1, 2, \ldots, n\}$ if and only if $F_X \in \mathscr{H}$.*

While, it follows from Theorem 2.2 that the analogous statement to Theorem 2.3 fails even in the case $n = 2$ if the random variables $X_1, X_2, \ldots, X_n$ are independent but possibly differently distributed.

3. Proof of Theorem 2.1

Take the sequence $x_n = (2n)!$, $n \geq 1$. For any $\lambda > 0$ we have

$$\begin{aligned} e^{\lambda x_n}\overline{F}(x_n) &= e^{\lambda(2n)!}\exp\{-(2n)! + (2n)! - (2n-1)! + \cdots + 2! - 1!\} \\ &= \exp\{\lambda(2n)! - (2n-1)! + (2n-2)! - (2n-3)! + \cdots + 2! - 1!\} \\ &\geqslant \exp\{(2n-1)!(2n\lambda - 1)\} \ \to\ \infty \end{aligned}$$

as $n \to \infty$. Hence,

$$\limsup_{x\to\infty} e^{\lambda x}\overline{F}(x) \geqslant \lim_{n\to\infty} e^{\lambda x_n}\overline{F}(x_n) \ =\ \infty,$$

implying $F \in \mathscr{H}$.

To show that $F \notin \mathscr{H}^*$, define the sequence $y_n = ((2n)! + (2n+1)!)/2$, $n \geqslant 1$. Then

$$\begin{aligned} e^{\lambda y_n}\overline{F}(y_n) &= \exp\Big\{\lambda\frac{(2n)! + (2n+1)!}{2} - \frac{(2n)! + (2n+1)!}{2} \\ &\qquad +\ (2n)! - (2n-1)! + \cdots + 2! - 1!\Big\} \\ &= \exp\{(2n)!(n(\lambda-1) + \lambda) - ((2n-1)! - (2n-2)!) - \cdots - (3! - 2!) - 1\} \\ &\leqslant \exp\{(2n)!(n(\lambda-1) + \lambda)\} \ \to\ 0 \end{aligned}$$

as $n \to \infty$ for $0 < \lambda < 1$. Hence, for such λ,

$$\liminf_{x\to\infty} e^{\lambda x}\overline{F}(x) \leqslant \lim_{n\to\infty} e^{\lambda y_n}\overline{F}(y_n) \ =\ 0.$$

It remains to prove that $F \in \mathscr{OL}$. Take $x \in [(2n)!, (2n+2)!)$ and consider the following four cases:

$$\text{(a)} \begin{cases} x & \in [(2n+1)!, (2n+2)!), \\ x-1 & \in [(2n+1)!, (2n+2)!), \end{cases} \quad \text{(b)} \begin{cases} x & \in [(2n+1)!, (2n+2)!), \\ x-1 & \in [(2n)!, (2n+1)!), \end{cases}$$

$$\text{(c)} \begin{cases} x & \in [(2n)!, (2n+1)!), \\ x-1 & \in [(2n)!, (2n+1)!), \end{cases} \quad \text{(d)} \begin{cases} x & \in [(2n)!, (2n+1)!), \\ x-1 & \in [(2n-1)!, (2n)!). \end{cases}$$

In case (a) we have

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = 1.$$

In case (b),

$$\overline{F}(x-1) = e^{-(x-1)}\prod_{j=1}^{n} e^{(2j)!-(2j-1)!}, \ \ \overline{F}(x) = e^{-(2n+1)!}\prod_{j=1}^{n} e^{(2j)!-(2j-1)!},$$

and, therefore,

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = e^{-(x-(2n+1)!)+1} \leqslant e.$$

In case (c),

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}.$$

In case (d),

$$\overline{F}(x-1) = \mathrm{e}^{-(2n-1)!}\prod_{j=1}^{n-1}\mathrm{e}^{(2j)!-(2j-1)!},\quad \overline{F}(x) = \mathrm{e}^{-x}\prod_{j=1}^{n}\mathrm{e}^{(2j)!-(2j-1)!}$$

and, because $x < (2n)! + 1$, it holds

$$\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}^{x-(2n)!} \; < \; \mathrm{e}.$$

These four estimates yield

$$\limsup_{x\to\infty}\frac{\overline{F}(x-1)}{\overline{F}(x)} = \mathrm{e}.$$

Thus, $F \in \mathscr{OL}$. □

Acknowledgments

The authors would like to thank the anonymous reviewers for the careful reading of the manuscript, and for all the suggestions which contributed to improve the quality and presentation of the paper.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. J. M. P. Albin, M. Sundén, On the asymptotic behavior of Lévy processes, Part I: subexponential and exponential processes, *Stoch. Proc. Appl.*, **119** (2009), 281–304. https://doi.org/10.1016/j.spa.2008.02.004
2. S. Beck, J. Blath, M. Sheutzow, A new class of large claim sizes distributions: definition, properties and ruin theory, *Bernoulli*, **21** (2015), 2457–2483. https://doi.org/10.3150/14-BEJ651
3. L. Breiman, On some limit theorems similar to the arc-sin law, *Theor. Probab. Appl.*, **10** (1965), 323–331. https://doi.org/10.1137/1110037
4. D. Cheng, Y. Wang, Asymptotic behavior of the ratio of tail probabilities of sum and maximum of independent random variables, *Lith. Math. J.*, **52** (2012), 29–39. https://doi.org/10.1007/s10986-012-9153-9
5. Z. Cui, Y. Wang, On the long tail property of product convolution, *Lith. Math. J.*, **60** (2020), 315–329. https://doi.org/10.1007/s10986-020-09482-w

6. S. Danilenko, J. Šiaulys, G. Stepanauskas, Closure properties of O-exponential distributions, *Stat. Probabil. Lett.*, **140** (2018), 63–70. https://doi.org/10.1016/j.spl.2018.04.012
7. L. Dindienė, R. Leipus, Weak max-sum equivalence for dependent heavy-tailed random variables, *Lith. Math. J.*, **56** (2016), 49–59. https://doi.org/10.1007/s10986-016-9303-6
8. P. Embrechts, A property of the generalized inverse Gaussian distribution with some applications, *J. Appl. Probab.*, **20** (1983), 537–544. https://doi.org/10.2307/3213890
9. P. Embrechts, C. M. Goldie, On closure and factorization properties of subexponential and related distributions, *J. Aust. Math. Soc.*, **29** (1980), 243–256. https://doi.org/10.1017/S1446788700021224
10. S. Foss, D. Korshunov, S. Zachary, *An introduction to heavy-tailed and subexponential distributions*, 2 Eds., New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-7101-1
11. J. L. Geluk, Some closure properties for subexponential distributions, *Stat. Probabil. Lett.*, **79** (2009), 1108–1111. https://doi.org/10.1016/j.spl.2008.12.020
12. J. L. Geluk, J. B. G. Frenk, Renewal theory for random variables with a heavy tailed distribution and finite variance, *Stat. Probabil. Lett.*, **81** (2011), 77–82. https://doi.org/10.1016/j.spl.2010.09.021
13. T. Shimura, T. Watanabe, Infinite divisibility and generalized subexponentiality, *Bernoulli*, **11** (2005), 445–469. https://doi.org/10.3150/bj/1120591184
14. C. Su, Y. Chen, Behaviors of the product of independent random variables, *Int. J. Math. Anal.*, **1** (2007), 21–35.
15. Z. Su, C. Su, Z. Hu, J. Liu, On domination problem of non-negative distributions, *Front. Math. China*, **4** (2009), 681–696. https://doi.org/10.1007/s11464-009-0040-6
16. Y. B. Wang, F. Y. Cheng, Y. Yang, Dominant relations on some subclasses of heavy-tailed distributions and their applications, (Chinese), *Chinese J. Appl. Probab.*, **21** (2005), 21–30.
17. Y. Wang, C. Yin, Minimum of dependent random variables with convolution-equivalent distributions, *Commun. Stat.-Theor. Meth.*, **40** (2011), 3245–3251. https://doi.org/10.1080/03610926.2010.498649
18. E. Willekens, The structure of the class of subexponential distributions, *Probab. Theory Relat. Fields*, **77** (1988), 567–581. https://doi.org/10.1007/BF00959618
19. H. Xu, M. Scheutzow, Y. Wang, On a transformation between distributions obeying the principle of a single big jump, *J. Math. Anal. Appl.*, **430** (2015), 672–684. https://doi.org/10.1016/j.jmaa.2015.05.011
20. H. Xu, M. Scheutzow, Y. Wang, Z. Cui, On the structure of a class of distributions obeying the principle of a single big jump, *Probab. Math. Stat.*, **36** (2016), 121–135.