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Article

Hurwitz Zeta Function Is Prime

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Abstract: We proved that the Hurwitz zeta function is prime. In addition, we derived the Nevanlinna characteristic for this function.

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1. Introduction and Preliminaries

The study of prime functions began in the middle of the last century. The first non-trivial example of a prime function $e^s + s$ was stated by Rosenbloom [1] and proved by Gross [2]. This research program was then resurrected in the 2000s. One of the more important results was obtained by Liao and Yang [3]. They proved that the Gamma function and the Riemann zeta function are prime. In this paper, we proved the primeness of the Hurwitz zeta function. To obtain this result, we had to obtain the Nevanlinna characteristic of the Hurwitz zeta function. This could also be considered as a novel fact.

We begin with a few definitions. Let $s = \sigma + it \in \mathbb{C}$.

Definition 1. *Suppose* $0 < a \leq 1$ *is a constant. The* Hurwitz zeta function $\zeta(s,a)$ *is:*

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\sigma > 1) \tag{1}$$

(see Hurwitz [4]).

The Hurwitz zeta function can be continued analytically for the whole complex plane, except for a simple pole at $s = 1$ with the residue 1 (see Apostol [5] (Theorem 12.4)). For $a = 1$, the Hurwitz zeta function becomes the Riemann zeta function $\zeta(s)$, which is of great importance in number theory. The yet-unsolved Riemann hypothesis states that all non-real zeros of $\zeta(s)$ exist on the critical line $\sigma = 1/2$, or equivalently, the non-vanishing of $\zeta(s)$ in the half-plane $\sigma > 1/2$. For $a = 1/2$, we have the following:

$$\zeta(s, 1/2) = (2^s - 1)\zeta(s, 1).$$

Hurwitz zeta functions are closely related to Dirichlet L-functions $L(s,\chi)$, which have been used to investigate the distribution of prime numbers in arithmetical progressions. Namely, we have the following:

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, \frac{r}{k}),$$

where χ is any Dirichlet character $\bmod k$ (see Apostol [5] (Section 12.6)). For recent research concerning the Hurwitz zeta function, see Laurinčikas [6], Sourmelidis and Steuding [7], and Fejzullahu [8].

Definition 2. *Suppose f is meromorphic. Let $n(r,f)$ be the number of poles of f in the disc $|s| \leq r$. The* Nevanlinna counting function *is:*

$$N(r,f) := \int_0^r (n(t,f) - n(0,f))\frac{dt}{t} + n(0,f)\log r.$$

The Nevanlinna proximity function *is:*

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta.$$

Here, $\log^+ x := \max\{\log x, 0\}$*. The* Nevanlinna characteristic function *is:*

$$T(r,f) = N(r,f) + m(r,f)$$

(see Nevanlinna [9]).

Suppose we had a meromorphic function $F(s)$. We then defined various versions of primeness.

Definition 3. *An expression:*

$$F(s) = f(g(s)),$$

where f is meromorphic and g is entire, or g is meromorphic and f is rational. This is called a factorization *of F. The functions f and g are left and right factors, respectively, (see Liao and Yang [3] (Definition 1.3, p. 60)).*

Definition 4. *Suppose we had a factorization $F(s) = f(g(s))$. If f or g was necessarily bi-linear, then F would be called* prime*. If f was rational or g was a polynomial, then F would be called* pseudo-prime *(see Liao and Yang [3] (Definition 1.4, p. 60)).*

See Chuang and Yang [10] for further results in Nevanlinna theory and factorization of meromorphic functions. A recent paper by Saoud, Boutabaa, and Zerzaihi [11] investigated the factorization of p-adic meromorphic functions.

The main results of this paper were the following two theorems.

Theorem 1. *The Nevanlinna characteristic function for the Hurwitz zeta function satisfied the following:*

$$T(r,\zeta) = \frac{r\log r}{\pi} + O(r).$$

Consequently, the order of ζ was 1.

The Nevanlinna characteristic function for the Riemann zeta function has also been calculated by Ye [12], as have the elements of the Selberg class by Steuding [13] (Section 7.3).

Theorem 2. *The Hurwitz zeta function is prime.*

In [14,15], it was proved that the Selberg zeta function for a compact Riemann surface is prime.

The next section is devoted to the proofs of Theorems 1 and 2.

2. Proofs of Theorems

The formulation of the lemma that follows required the definition of an accumulation line.

Definition 5. *Let $E \subset \mathbb{C}$. If $\theta \in [0, 2\pi]$ is an accumulation point of $S = \{\arg s : s \in E\}$, then $\theta = \arg s$ is called an* accumulation line *of E (see Liao and Yang [3] (Definition 2.3)).*

We employed this lemma.

Lemma 1. *Let $f(s)$ be a meromoprhic function of finite order, and the number of accumulation lines of $E = \{s : f(s) = a_j \in \overline{\mathbb{C}}, j = 1, 2\}$ is finite. Therefore, $\overline{\mathbb{C}}$ is the extended complex plane. Then, $f(s)$ is pseudo-prime.*

Proof. See [16] (p. 141). □

Definition 6. *Suppose $0 < a, \lambda \leq 1$ are constants. The* Lerch zeta function $L(\lambda, a, s)$ *is:*

$$L(\lambda, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+a)^s} \quad (\Re s > 1),$$

(see Lerch [17]).

The Lerch zeta function could be continued analytically for the whole complex plane, except for a possible simple pole at $s = 1$ with the residue 1 (see Lerch [17]). We applied the following lemma to the growth of the Lerch zeta function, of which Hurwitz zeta function was a special case.

Lemma 2. *Let $0 < \lambda, a \leq 1$, $\sigma \geq 0$, and $t \geq 1$. Then,*

$$\begin{aligned} &L(\lambda, a, s) - \frac{1}{a^{1/2+it}} \\ &- \left(\frac{2\pi}{t}\right)^{\sigma-1/2+it} \exp(it + \pi/4 - 2\pi i \lambda a)\left(\frac{1}{\lambda^{1-s}} - \frac{\exp(\pi i + \pi i \sigma + 2\pi i a)}{(1-\lambda)^{1-s}}\right) \\ &\ll \begin{cases} \frac{1}{(1-\sigma)} t^{(1-\sigma)/2} \text{ if } 0 < \sigma \leq 1 - \frac{1}{\log t}, \\ \log t \text{ if } \sigma > 1 - \frac{1}{\log t}, \end{cases} \end{aligned}$$

Proof. See Garunkštis [18] (Corollary 2). □

To prove our result concerning the Hurwitz zeta function, we needed the asymptotics for its Nevanlinna characteristic. We used these lemmas, as follows:

Lemma 3 (Stirling formula)**.** *Let $\delta > 0$. Then,*

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O\left(\frac{1}{|s|}\right),$$

as $|s| \to \infty$, uniformly for $-\pi + \delta \leq \arg s \leq \pi - \delta$.

Proof. See Titchmarsh [19] (Section 4.42). □

Let $A \leq B$. The Stirling formula produces (see [19] (formula (4.12.3))), as follows:

$$\Gamma(s) = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2} + it} e^{-\frac{\pi|t|}{2} - it \pm \frac{i\pi}{2}(\sigma - \frac{1}{2})} \left(1 + O\left(\frac{1}{|t|}\right)\right) \tag{2}$$

as $t \to \pm\infty$ and uniformly for $A \leq \sigma \leq B$.

We needed one more statement from the Nevanlinna theory. Let $c \in \mathbb{C}$. The definition of the Nevanlinna counting function could then be extended to the finite c-points of the meromorphic function f. Therefore,

$$N(r, f, c) = N\left(r, \frac{1}{f-c}\right), \quad m(r, f, c) = m\left(r, \frac{1}{f-c}\right),$$

and

$$T(r, f, c) = N(r, f, c) + m(r, f, c).$$

Next, we formulated the first fundamental theorem of the Nevanlinna theory.

Lemma 4. *Let f be a meromorphic function, and let c be any complex number. Then,*

$$T(r, f, c) = T(r, f) + O(1).$$

Proof. See Hayman [20] (Theorem 1.2). □

The last lemma was used for $c = 0$ in the case of $f(s) = \zeta(s, a)$. Thereby, we obtained the zero distribution results for the Hurwitz zeta function.

Lemma 5. *If $\sigma \geq 1 + a$ or $|t| \geq 1$ and $\sigma \leq -1$, then $\zeta(s, a) \neq 0$.*

If $\sigma \leq -(4a + 1 + 2[1 - 2a])$ and $|t| \leq 1$, then $\zeta(s, a) \neq 0$, except for the zeros on the negative real axis, one in each interval $(-2n - 4a - 1, -2n - 4a + 1)$, $n \in \mathbb{Z}$, $n \geq 1 - 2a$.

Moreover, the region $0 < \Im s \leq T$ contained the following:

$$\frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

zeros of $\zeta(s, a)$.

Proof. See Spira [21]. □

In Lemma 5, the zeros were counted according to multiplicities. Clearly, the zeros of $\zeta(s, a)$ were symmetrically distributed with respect to the real axis. Therefore, Lemma 5 implied the following:

$$N(r, \zeta, 0) = \frac{r}{\pi} \log r + O(r). \tag{3}$$

Proof of Theorem 1. By (3) and Lemma 4, we found the following:

$$\frac{r}{\pi} \log r + O(r) = N(r, \zeta, 0) \leq T(r, f). \tag{4}$$

The Nevanlinna counting function for the Hurwitz zeta function was, as follows:

$$N(r, \zeta) = \log^{+} r, \tag{5}$$

based on the fact that $\zeta(s, a)$ had a single simple pole at $s = 1$. Since the Nevanlinna characteristic function $T(r, f)$ for f was given by the following:

$$T(r, f) = N(r, f) + m(r, f), \tag{6}$$

and all that was needed was a good upper bound for $m(r, f)$.

Divide the parameter range $0 \le \theta < 2\pi$ into the following parts:

$$E_1(r) = \{\theta : -1 \le \Re re^{i\theta} \le 2\},$$
$$E_2(r) = \{\theta : \Re re^{i\theta} < -1\},$$
$$E_3(r) = \{\theta : \Re re^{i\theta} > 2\}.$$

Then,

$$2\pi m(r,\zeta) = \left\{ \int_{E_1(r)} + \int_{E_2(r)} + \int_{E_3(r)} \right\} \log^+ |\zeta(re^{i\theta}, a)| \, d\theta. \quad (7)$$

By Lemma 2, and since we had $\zeta(s,a) = L(1,a,s)$, the Hurwitz zeta function satisfied these growth conditions for $\sigma \ge 0$:

$$\zeta(s,a) \ll \begin{cases} \frac{1}{1-\sigma} t^{(1-\sigma)/2}, \text{ if } 0 \le \sigma \le 1 - \frac{1}{\log t}, \\ \log t, \text{ if } \sigma \ge 1 - \frac{1}{\log t}, \quad (t \to \infty). \end{cases} \quad (8)$$

Therefore, for a sufficiently large r, $|s| = r$, $\sigma \ge 0$, we used the following estimate:

$$\zeta(s,a) \ll t^{1/2}.$$

We also needed the growth estimate of the Hurwitz zeta function for $\Re s < 0$. Therefore, we defined the following function:

$$L(s,a) := \sum_{n=1}^{\infty} \frac{e^{2\pi ian}}{n^s}.$$

This function was a special case of the Lerch zeta function. Namely, $L(s,a) = L(a,1,s)$. The Hurwitz zeta function satisfied the following functional equation:

$$\zeta(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} (\exp(\pi i(1-s)/2) L(1-s,-a) \quad (9)$$
$$+ \exp(-\pi i(1-s)/2) L(1-s,a))$$

(see Garunkštis and Steuding [22] (Formula (12))). Then, (2), (9) and Lemma 2 indicated that for $A \le \sigma \le 0$,

$$\zeta(s,a) \ll t^{1/2-\sigma} \log t. \quad (10)$$

We had meas $E_1(r) = O(1/r)$. By (10),

$$\int_{E_1(r)} \log^+ |\zeta(re^{i\theta})| \, d\theta \ll 1.$$

The Dirichlet series (1) of $\zeta(s)$ converged absolutely if $\Re s \ge 2$. Therefore,

$$\int_{E_3(r)} \log^+ |\zeta(re^{i\theta})| \, d\theta \ll 1.$$

For the remaining region $E_2(r)$, the functional Equation (9) and the growth properties (8) lead to the following:

$$\int_{E_2(r)} \log^+ |\zeta(re^{i\theta})| \, d\theta \le \int_{E_2(r)} \log^+ |\Gamma(1 - re^{i\theta})| \, d\theta + O(r).$$

Ye [12] (p. 429) proved the following:

$$\int_{E_2(r)} \log^+ |\Gamma(1 - re^{i\theta})|\, d\theta \leq 2r \log r + O(r). \tag{11}$$

Lastly, the bounds, together with (4)–(7), proved Theorem 1. □

The next two statements were useful for the proof of Theorem 2.

Lemma 6 (Picard's theorem)**.** *Suppose f is a meromorphic function. Then its range consists of all $\mathbb{C}$, with the possible exception of a single point.*

Proof. See Conway [23] (Chapter XII, paragraph 4, p. 302). □

Lemma 7 (Hadamard's theorem)**.** *Define*

$$E_n(s) := \begin{cases} 1 - s, \; n = 0, \\ (1 - s)\exp\left(\frac{s^1}{1} + \frac{s^2}{2} + \dots \frac{s_n}{n}\right), \text{ otherwise.} \end{cases}$$

If f was an entire function of finite order ρ and m was the order of zero of f at $s = 0$, then it admitted a factorization, as follows:

$$f(s) = s^m e^{g(s)} \prod_{n=1}^{\infty} E_p\left(\frac{s}{a_n}\right),$$

where $g(s)$ is a polynomial of degree $q \leq \rho$, and p is the integer part of ρ.

Proof. See Conway [23] (Chapter XI, paragraph 3, p. 289). □

Proof of Theorem 2. The Hurwitz zeta function $\zeta(s, a)$ had a simple pole at $s = 1$. By Lemma 5, the half-plane $\sigma > 1 + a$ and the region $|t| > 1, \sigma < -1$ were zero-free. Therefore, by Lemma 1, $\zeta(s, a)$ is pseudo-prime.

Let us consider the following decomposition:

$$\zeta(s, a) = f(p(s))$$

where f is a transcendental meromorphic, and p is a polynomial of degree ≥ 2. Arguing similarly to Theorem 3.2 in [3], we could set the following:

$$f(w) = \frac{f_1(w)}{w - w_1},$$

where $f_1(w)$ is entire since $\zeta(s, a)$ has a single simple pole at $s = 1$. Therefore, $p(s) - w_1$ was of the degree $n \geq 2$. It followed that $\zeta(s, a)$ had n poles counting multiplicity, which was a contradiction. Therefore, if we had the decomposition $\zeta(s, a) = f(p(s))$, as mentioned above, then the degree of p would be 1.

Consider the following decomposition:

$$\zeta(s, a) = Q(g(s)),$$

where Q is a rational function with $\deg Q \geq 2$, and g is a transcendental meromorphic. By Picard's theorem (Lemma 6), since $\zeta(s, a)$ had a single simple pole, the function Q could have, at most, two poles, at least one of which must be simple. Our analysis identified three cases.

Case 1. The function $Q(w)$ had no poles. Therefore, Q was a polynomial. It had to be the case that $g(s)$ had a pole at $s = 1$. Then, the degree of Q had to be 1.

Case 2. The function $Q(w)$ had only a simple pole at w_0. Then, we could express it, as follows:

$$Q(w) = \frac{P(w)}{w - w_0},$$

where P is a polynomial of degree n. It followed that $g(s)$ was either (i) an entire function with a simple w_0-point at $s = 1$, or (ii) $g(s)$ did not assume a w_0-point and, thus, had a pole at $s = 1$.

Suppose (i) held. Then, by Hadamard's theorem (Lemma 7) and Theorem 1 (ζ was of the order 1), we found the following:

$$g(s) - w_0 = (s - 1)\exp(as + b).$$

By Hayman [20] (Chapter 1), we defined the following properties of T:

$$T(r, \sum_{\nu=1}^{p} f_\nu(s)) \leq \sum_{\nu=1}^{p} T(r, f_\nu(s)) + \log p,$$
$$T(r, \prod_{\nu=1}^{p} f_\nu(s)) \leq \sum_{\nu=1}^{p} T(r, f_\nu(s)), \quad T(r, f(s)) = T(r, 1/f(s)) + \log|f(0)|$$

for meromorphic f_ν and f. Moreover,

$$T(r, s - 1) = \log r + O(1), \quad T(r, \exp(as + b)) = \frac{|a|}{\pi} r + o(r).$$

Therefore, in this sub-case, we had the following:

$$T(r, \zeta) = O(T(r, g)) = O(r), \; r \to \infty,$$

which contradicted Theorem 1.

Suppose (ii) held. Hadamard's theorem and Theorem 1 would lead to the following:

$$g(s) - w_0 = \frac{\exp(as + b)}{s - 1}.$$

Similarly, we obtained a contradiction.

Case 3. The function $Q(w)$ had two poles, w_1 and w_2. By Picard's theorem, $g(s)$ did not assume one of them, say, w_1. Then, $g(s)$ had one simple w_2-point at $s = 1$. Therefore, we calculated the following:

$$\frac{g(s) - w_2}{g(s) - w_1} = \exp(as + b).$$

Similarly to Case 2, we obtained a contradiction. This yielded the theorem. □

3. Concluding Remarks

Function compositions appear in many branches of pure and applied mathematics, e.g., complex dynamics and the related topic of fixed-points (see Milnor [24], Chuang and Yang [10]). In addition, the propagation of information through a neural network via its layers could be considered a composition of activation functions, according to Kratsios and Papon [25] (first formula in p. 5). Category theory is based on the idea of a composition (for applications, see Fong and Spivak [26]).

Factorization is a basic concept of mathematics that has been reflected in the fundamental theorem of arithmetic, which indicates that any natural number can be uniquely factored into prime numbers. It has had direct applications in cryptography, according to Katz and Lindell [27] (Section 9).

In this paper, by continuing the research of Liao and Yang [3], we considered the factorization of the Hurwitz zeta function based on the aspects of compositions. This function is considered classical in the analytic number theory. Using the value distribution properties of $\zeta(s,a)$, together with Nevanlinna theory, we proved that the Hurwitz zeta function is prime.

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