mathematics

Article

On Construction of Partially Dimension-Reduced Approximations for Nonstationary Nonlocal Problems of a Parabolic Type

Raimondas Čiegis [1,*], Vadimas Starikovičius [1], Olga Suboč [1] and Remigijus Čiegis [2]

1 Department of Mathematical Modelling, Vilnius Gediminas Technical University, Saulėtekio al. 11, LT-10223 Vilnius, Lithuania; vadimas.starikovicius@vgtu.lt (V.S.); olga.suboc@vgtu.lt (O.S.)
2 Kaunas Faculty, Vilnius University, Muitinės St. 8, LT-44280 Kaunas, Lithuania; remigijus.ciegis@knf.vu.lt
* Correspondence: raimondas.ciegis@vgtu.lt

Abstract: The main aim of this article is to propose an adaptive method to solve multidimensional parabolic problems with fractional power elliptic operators. The adaptivity technique is based on a very efficient method when the multidimensional problem is approximated by a partially dimension-reduced mathematical model. Then in the greater part of the domain, only one-dimensional problems are solved. For the first time such a technique is applied for problems with nonlocal diffusion operators. It is well known that, even for classical local diffusion operators, the averaged flux conjugation conditions become nonlocal. Efficient finite volume type discrete schemes are constructed and analysed. The stability and accuracy of obtained local discrete schemes is investigated. The results of computational experiments are presented and compared with theoretical results.

Keywords: fractional power elliptic operators; partially dimension-reduced models; parabolic problems; stability; convergence analysis

MSC: 65N12

Citation: Čiegis, R.; Starikovičius, V.; Suboč, O.; Čiegis, R. On Construction of Partially Dimension-Reduced Approximations for Nonstationary Nonlocal Problems of a Parabolic Type. *Mathematics* **2023**, *11*, 1984. https://doi.org/10.3390/math11091984

Academic Editor: Nikolai A. Kudryashov

Received: 3 April 2023
Revised: 21 April 2023
Accepted: 21 April 2023
Published: 22 April 2023

1. Introduction

In this paper, we consider two important questions of applied mathematical modelling.

First, a lot of work is done to investigate nonlocal differential equations [1]. Here, we mention an approach when various fractional derivatives are used to describe anomalous transport processes (see, e.g., [2,3] and references given therein), porous media [4,5], and a method for when nonlocal fractional powers of elliptic operators are defined to simulate nonstandard physical processes (see [6–8] and references given therein).

As it is noted in [9], fractional models have attracted a lot of attention due to their ability to simulate a nonlocal behaviour with a relatively small number of parameters. In many cases, this technique is much more effective than the application of nonlinear PDE models or models based on nonlocal discrete neural networks.

In this paper, we use the spectral definition of the fractional power of an elliptic operator A_h^α (see the details in Section 2). Some well-known facts can be mentioned here. It is important to note that the spectral algorithm in combination with FFT is used for practical computations to solve parabolic-type problems with nonlocal diffusion operators. If this approach is applicable, then the complexity of the constructed algorithms is the same as we have in the case of classical diffusion operators. However, clearly, this strategy is computationally efficient only if the FFT technique can be applied for the given discrete problem. A more general approach is based on transformations of nonlocal problems to the local classical differential problems. A very good review of these methods is given in [10]. Second, we are interested in the application of a new technique which is proposed to solve multidimensional applied problems more efficiently. Both 2D and 3D models are reduced to hybrid dimension models, keeping the initial full dimension only small parts of the domain and reducing it to a one-dimensional equation within the remaining parts of the domain (see, [11,12]). The main driving idea of this method is based on the asymptotic analysis of solutions of the given partial differential equation. Then, the regular and singular behaviours of solutions are described by new mathematical models of different complexities and dimensions. A very interesting application for solving the diffusion equation in domains containing thin tubes is described in [13].

Many efficient algorithms are proposed to solve the obtained discrete problems, here we mention our papers [14–16], where non-iterative ADI-type schemes are constructed to solve hybrid dimension approximations of 3D parabolic problems. Some efficient parallel algorithms are investigated in [16]. Still, no results are known about the application of reduced dimension techniques for the parabolic problems with fractional power elliptic operators.

Our main aim is to generalize existing algorithms to this non-trivial nonlocal problem. We describe all steps in the construction of these new algorithms and theoretically justify the proposed discrete schemes. In all cases, we present the results of computational experiments and compare them with theoretical predictions.

The rest of the paper is organized in the following way. In Section 2 the problem is formulated. First, we define the classical 2D parabolic equation and give specific boundary conditions.

Next, the non-stationary parabolic equation with a fractional power elliptic operator is formulated. As was mentioned above, the spectral definition is used to define fractional power elliptic operators.

The boundary conditions for PDE with a classical diffusion operator are selected, such that a solution satisfies the required asymptotic behaviour properties and, therefore, a dimension reduction technique can be used to approximate the full multidimensional problem with a hybrid dimension model. We define a differential equation in the partially dimension-reduced domain and formulate special nonlocal conjugation conditions at the boundary of 2D and 1D domains. This analysis follows techniques described in detail in [12,14].

In Section 3 two classical finite difference schemes are constructed. Both schemes are used as basic parts in the discretization of parabolic problems with fractional power elliptic operators and problems describing partially dimension-reduced mathematical models. Our aim is to test the accuracy of these schemes for specific boundary conditions. The first scheme is based on the Crank–Nicolson method. The FFT algorithm is used to implement it. The second ADI-type scheme is constructed in order to have the possibility of implementing it without iterative solvers. Such a property is very useful in the construction of efficient discrete schemes for partially dimension0reduced mathematical models.

The stability and convergence analysis is presented for both schemes. Then, a special test problem is formulated. It is shown that experimental convergence order of discrete solutions is equal to two in space and time.

In Section 4, discrete finite difference schemes are constructed for the parabolic problem with a fractional power of elliptic operators. A short review on the stability and accuracy of symmetrical difference schemes is presented. It is noted that the singularity of solutions can reduce the convergence rate even in the L_2 norm. Such theoretical estimates are illustrated by the results of computational experiments.

In Section 5, we consider the BURA-BRASIL type approximation of the Crank–Nicolson scheme for the parabolic problem with fractional power of elliptic operators. This scheme defines an important part of the main algorithm to construct discrete models of partially dimension-reduced models. The results of computational experiments illustrate the theoretical stability and accuracy results.

In Section 6, the ADI scheme is constructed for a partially dimension-reduced parabolic problem with a classical discrete elliptic operator. It serves as a benchmark for this type scheme and the results of computational experiments define a level of error expected for approximations of nonlocal problems with fractional power elliptic operators. It is shown that the constructed discrete operators are symmetric and positive definite operators.

These properties enable us to prove the unconditional stability of the ADI scheme within a particular energy norm.

In Section 7, we construct a partially dimension-reduced model for the parabolic problem with a fractional power elliptic operator. The main idea is to start from the BURA-BRASIL type scheme and to apply techniques from the previous section for the obtained discrete classical elliptic operators. All of the details of the algorithm are presented and the results of computations experiments are given.

Some final conclusions are discussed in Section 8.

2. Problem Formulation

Let $D = (0, X) \times (0, Y)$ be a two-dimensional rectangular bounded domain and ∂D is a boundary of it. We define the diffusion operator

$$Au = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (x, y) \in D.$$

We start by formulating the following two-dimensional parabolic problem:

$$\frac{\partial u}{\partial t} + Au = 0, \quad (x, y) \in D, \ \ 0 < t \leqslant T, \tag{1}$$

$$u(0, y, t) = g_1(y, t), \quad u(X, y, t) = g_2(y, t), \quad y \in [0, Y], \ t \in [0, T], \tag{2}$$

$$\frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, Y, t) = 0, \quad x \in (0, X), \ t \in [0, T], \tag{3}$$

$$u(x, y, 0) = 0, \quad (x, y) \in D. \tag{4}$$

Homogeneous initial conditions and source terms are selected in order to concentrate on the main topic of this paper, i.e., the application of the dimension reduction method to parabolic problems with fractional power elliptic operators. A generalization of all of the obtained results for non-homogeneous conditions is quite straightforward.

Next, we give a spectral definition of the fractional power of elliptic operators. It will be shown that operator A satisfies all assumptions on a class of operators considered in this definition.

At the beginning, we assume that boundary conditions (2) are homogeneous. A general case of non-homogeneous boundary conditions is considered later.

Let H be a Hilbert space with a scalar product (u, v) for $u, v \in H$. Then, the L_2 norm is defined as $\|u\| = (u, u)^{1/2}$. Let A be a self-adjoint positive definite operator

$$A: \ \ H \to H, \quad A = A^*, \quad A \geqslant cI, \ \ c > 0,$$

where I is the identity operator.

A definition of the fractional power of this operator A^α with a fractional parameter $0 < \alpha < 1$ can be done in a non-unique way, here we apply the spectral approach [17]. Let us solve the standard eigen-problem

$$A\psi_j = \lambda_j \psi_j, \quad j = 1, 2, \ldots.$$

All eigenvalues are positive

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m \leqslant \cdots$$

and the set of eigenfunctions $\{\psi_j\}$ make an orthonormal basis for H. Then, any function $u \in H$ can be expressed as

$$u = \sum_{j=1}^{\infty} (u, \psi_j)\psi_j.$$

A nonlocal operator A^α with fractional parameter $0 < \alpha < 1$ is defined as

$$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha \, (u, \psi_j)\psi_j.$$

Similar to problem (1)–(4) we solve the Cauchy problem:

$$\frac{\partial u}{\partial t} + A^\alpha u = 0, \quad 0 < t \leqslant T, \tag{5}$$

$$u(0) = u_0, \quad u_0 \in H. \tag{6}$$

By using the Fourier method it is possible to write the solution of this problem in an explicit form. In practical applications, this technique is applicable only for simple cases of operators A in standard domains. In this paper, we are interested in constructing general discrete schemes, which can be used for problems with variable coefficients of the elliptic operators and for non-uniform space grids. Still, we want to test the efficiency of new algorithms when non-iterative linear algebra methods are used to solve partial dimension-reduced problems with the fractional power of elliptic operators. Thus, the analysis is restricted to rectangular domains.

For us, it is important that these schemes give us the possibility of reducing the complexity of discrete algorithms by formulating a partially dimension-reduced approximations of the given 2D space problem (5).

Next, we explain the main idea of partial dimension reduction technique by applying it to a classical parabolic problem (1)–(4) (see also [12,14]). The model reduction is a popular procedure which enables the users to construct a reduced-complexity model that preserves some important properties of the full model. It is important to guarantee that the approximate solution is close to the solution of the original full order model. In this paper we apply the method, which is based on the partial dimension reduction method, when the dimension of the full mathematical model is reduced in a big part of the initial domain. Such a reduction is justified by the asymptotic analysis of solutions of the partial differential equations and a projection of the solution can be done to a subspace of functions having a form of the asymptotic expansion in the zones of regular behaviour of the solution. Thus, we can reduce the dimension within the main part of the domain and keep the full dimension description only in small zones of a singular behaviour of the solution. It is clear that nonlocal junctions of one-dimensional and full-dimensional equations should be formulated in the new model.

Now, we define a partially dimension-reduced approximation of the 2D mathematical model (1)–(4). The problem is solved in the domain

$$D_\delta = \big((0,\delta) \times (0,Y)\big) \cup [\delta, X-\delta] \cup \big((X-\delta, X) \times (0,Y)\big), \quad \delta > 0.$$

Let

$$S(u) = \frac{1}{Y} \int_0^Y u(x,y,t) \, dy$$

denote the averaging operator with respect to y dimension. Function U is called an approximate solution to problem (1)–(4) if it satisfies the following differential problem [12]

$$\frac{\partial U}{\partial t} + AU = 0, \quad (x,y,t) \in (0,\delta) \cup (X-\delta, X) \times (0,Y) \times (0,T], \tag{7}$$
$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad (x,t) \in (\delta, X-\delta) \times (0,T], \tag{8}$$
$$U(0,y,t) = g_1(y,t), \;\; U(X,y,t) = g_2(y,t), \quad (y,t) \in [0,Y] \times (0,T], \tag{9}$$
$$\frac{\partial U}{\partial y}(x,0,t) = 0, \quad \frac{\partial U}{\partial y}(x,Y,t) = 0, \qquad (x,t) \in (0,\delta) \cup (X-\delta, X) \times (0,T], \tag{10}$$
$$U(x,y,0) = 0, \quad (x,y) \in D_\delta. \tag{11}$$

The following two pairs of conjugation conditions are valid at the truncation lines:

$$U\big|_{x=\delta-0} = U\big|_{x=\delta+0}, \quad U\big|_{x=X-\delta-0} = U\big|_{x=X-\delta+0}, \tag{12}$$
$$\frac{\partial S(U)}{\partial x}\bigg|_{x=\delta-0} = \frac{\partial U}{\partial x}\bigg|_{x=\delta+0}, \quad \frac{\partial U}{\partial x}\bigg|_{x=X-\delta-0} = \frac{\partial S(U)}{\partial x}\bigg|_{x=X-\delta+0}. \tag{13}$$

The first two conditions (12) are classical and state that U is continuous at the separation interface. The last two conditions (13) are nonlocal and they follow from the conservation law of fluxes along the separation lines.

Our main aim is to also construct a similar partial dimension reduction for the parabolic problem with the fractional power of elliptic operators (5) and (6).

3. Discrete Schemes

A discrete approximation of the 2D diffusion operator A is constructed by using the finite volume method. First, a uniform space mesh is defined $\overline{\Omega}_h = \bar{\omega}_x \times \bar{\omega}_y$:

$$\bar{\omega}_x = \{x_i : \;\; x_i = ih, \;\; i = 0,\dots,J_x, \;\; h_x = X/J_x\},$$
$$\bar{\omega}_y = \{y_j : \;\; y_j = jh, \;\; j = 0,\dots,J_y, \;\; h_y = Y/J_y\}.$$

Next, for simplicity of notations we consider a uniform time mesh:

$$\bar{\omega}_t = \{t^n : \;\; t^n = n\tau, \qquad n = 0,\dots,N\}, \quad t^N = T.$$

The following operators are defined for discrete functions:

$$\partial_x U_{ij}^n := \frac{U_{ij}^n - U_{i-1,j}^n}{h_x}, \quad \partial_y U_{ij}^n := \frac{U_{ij}^n - U_{i,j-1}^n}{h_y},$$
$$\partial_t U_{ij}^n := \frac{U_{ij}^{n+1} - U_{ij}^n}{\tau}, \quad U_{ij}^{n+\frac{1}{2}} = \frac{U_{ij}^{n+1} + U_{ij}^n}{2}.$$

The discrete diffusion operators are constructed as:

$$A_{hx} U_{ij}^n := -\frac{1}{h_x}\left(\partial_x U_{i+1,j}^n - \partial_x U_{ij}^n\right), \;\; 0 < i < J_x, \;\; 0 \leqslant j \leqslant J_y, \tag{14}$$
$$A_{hy} U_{ij}^n := \begin{cases} -\frac{2}{h_y}\partial_y U_{i1}^n, \;\; j = 0, \\ -\frac{1}{h_y}\left(\partial_y U_{i,j+1}^n - \partial_y U_{ij}^n\right), \;\; 0 < j < J_y, \\ \frac{2}{h_y}\partial_y U_{i,J_y}^n, \;\; j = J_y, \end{cases} \quad 0 < i < J_x. \tag{15}$$

By applying the symmetrical approximation in time and the finite volume method for approximation of space derivatives, we get the Crank–Nicolson discrete scheme

$$\partial_t U_{ij}^n + (A_{hx} + A_{hy}) U_{ij}^{n+\frac{1}{2}} = \frac{1}{h_x^2}\Big(\delta_{i1} g_1^{n+\frac{1}{2}}(y_j) + \delta_{i,J_x-1} g_2^{n+\frac{1}{2}}(y_j)\Big), \ \ (x_i, y_j) \in \omega_x \times \bar{\omega}_y, \tag{16}$$

where δ_{ik} is the Kronecker delta function. Non-homogeneous boundary conditions are included into the discrete equation as additional source terms.

We assume that vectors U_{ij}^n satisfy homogeneous boundary conditions for $i = 0$ and $i = J_x$. Thus, the exact solution of the Crank–Nicolson scheme $\widetilde{U}^n$ is defined as

$$\widetilde{U}_{ij}^n = U_{ij}^n, \quad 0 < i < J_x, \ \ 0 \leqslant j \leqslant J_y,$$
$$\widetilde{U}_{0j}^n = g_1^n(y_j), \quad \widetilde{U}_{J_x,j}^n = g_2^n(y_j), \ \ 0 \leqslant j \leqslant J_y,$$

This assumption is always used in this paper when some spectral algorithms are applied, including the FFT algorithm.

Lemma 1. *The Crank–Nicolson discrete scheme* (16) *is unconditionally stable in the L_2 norm. If a solution of the problem* (1)–(4) *is sufficiently smooth, then the approximation error of this scheme is of order $O(h_t^2 + h_x^2 + h_y^2)$.*

The proof of the stability follows from the spectral Fourier analysis and the estimate of the approximation accuracy is based on the Taylor expansion technique. Then, it follows from the well known Lax theorem (see e.g., [18]) that the discrete solution converges with the second-order in space and time.

The implementation of the Crank–Nicolson discrete scheme (16) is done efficiently using the FFT algorithm. An explicit definition of eigenvectors and eigenvalues of the operators A_{hx} and A_{hy} will be given in the section on numerical experiments.

In all numerical experiments, we use solutions of the symmetrical scheme (16), when they are computed for sufficiently small time and space steps of discrete meshes, as virtual benchmarks of exact solutions. Thus, it is important to test the convergence rates of solutions of scheme (16) and to have a a posteriori error estimates of the selected "exact" solutions.

Example 1. *We solve a linear two-dimensional parabolic problem* (1)–(4) *defined on the domain $\overline{D} = [0,2] \times [0,1]$. The boundary conditions are selected as*

$$g_1(y,t) = t\exp\Big(-25(y-0.5)^2\Big), \ g_2(y.t) = 2t\exp\Big(-36(y-0.5)^2\Big), \ y \in [0,1], \ t \in [0,1].$$

Let us solve the standard eigen-problem

$$A_h \Psi_{lk} = \lambda_{lk} \Psi_{lk}, \quad 0 < l < J_x, \ \ 0 \leqslant k \leqslant J_y, \tag{17}$$

where $\Psi_{lk} = \Phi_l^x \Phi_k^y$, $\lambda_{lk} = \mu_l^x + \mu_k^y$ and Φ_l^x, Φ_k^y are solutions of eigen-problems

$$A_{hx} \Phi_l^x = \mu_l^x \Phi_l^x, \quad 0 < l < J_x,$$
$$A_{hy} \Phi_k^y = \mu_k^y \Phi_k^y, \quad 0 < k < J_y.$$

For the given test problem, the solutions of eigen-problems can be written explicitly

$$\Phi_l^x(x_i) = \sqrt{2/X}\sin\left(\pi l \frac{x_i}{X}\right), \quad \mu_l^x = \frac{4}{h_x^2}\sin^2\left(\frac{\pi l}{2J_x}\right), \quad 0 < l < J_x, \ \ x_i \in \bar{\omega}_x, \tag{18}$$

$$\Phi_k^y(y_j) = \sqrt{2/Y}\cos\left(\pi k \frac{y_j}{Y}\right), \quad \mu_k^y = \frac{4}{h_y^2}\sin^2\left(\frac{\pi k}{2J_y}\right), \quad 0 \leqslant k \leqslant J_y, \ \ y_j \in \bar{\omega}_y. \tag{19}$$

Table 1 provides a sequence of decreasing time step widths τ and space mesh steps h_x, h_y, the errors $E(\tau)$, and the experimental convergence rates $\rho(\tau)$ of the discrete solution for scheme (16) in the maximum norm:

$$E(\tau, h_x, h_y) = \max_{(x_i, y_j) \in \overline{\Omega}_h} \left| U_{ij}^N - u(x_i, y_j, 1) \right|, \quad \rho(\tau, h_x, h_y) = \log_2 \left(\frac{E(2\tau, 2h_x, 2h_y)}{E(\tau, h_x, h_y)} \right).$$

Uniform space grids Ω_h are used in numerical experiments. The "exact" solution $u(x_i, y_j, 1)$ is computed on a grid with $J_x = 1024$, $J_y = 512$, $N = 500$.

Table 1. Errors $E(\tau, h)$ and experimental convergence rates $\rho(\tau, h)$ for the discrete solution of scheme (16) for a sequence of time steps τ and space steps $h_x = h_y = h$.

$\tau_0 = \frac{1}{10}, h_0 = \frac{1}{16}$	τ_0, h_0	$\frac{\tau_0}{2}, \frac{h_0}{2}$	$\frac{\tau_0}{4}, \frac{h_0}{4}$	$\frac{\tau_0}{8}, \frac{h_0}{8}$
$E(\tau, h)$	1.557×10^{-2}	3.915×10^{-3}	9.558×10^{-4}	2.316×10^{-4}
$\rho(\tau, h)$	—	1.992	2.034	2.045

It follows from the presented results that the convergence order of the discrete solution is close to the second, as it is predicted by the theoretical error estimates.

In the implementation of the partial dimension reduction method for the parabolic problem (1)–(4), we will use an ADI-type scheme. Thus, here we test the accuracy of the classical ADI scheme [16,18]. The differential problem is approximated by the discrete scheme:

$$\frac{\widehat{U}_{ij}^n - U_{ij}^n}{\frac{1}{2}\tau} + A_{hx} U_{ij}^n + A_{hy} \widehat{U}_{ij}^n = 0, \quad (x_i, y_j) \in \omega_x \times \bar{\omega}_y, \tag{20}$$

$$\frac{U_{ij}^{n+1} - \widehat{U}_{ij}^n}{\frac{1}{2}\tau} + A_{hx} U_{ij}^{n+1} + A_{hy} \widehat{U}_{ij}^n = 0, \quad (x_i, y_j) \in \omega_x \times \bar{\omega}_y. \tag{21}$$

As for most splitting type schemes, the implementation of the given algorithm is reduced to a solution of one-dimensional systems of linear equations and the efficient factorization algorithm can be used. As an additional bonus, we note that independent subproblems can be solved in parallel (see [16]).

Lemma 2. *The ADI discrete scheme (20) and (21) is unconditionally stable in the L_2 norm. If a solution of the problem (1)–(4) is sufficiently smooth, then the approximation error of this scheme is of order $O(h_t^2 + h_x^2 + h_y^2)$.*

Again, the proof of the stability follows from the spectral Fourier analysis. The estimate of the approximation accuracy is obtained by transforming the ADI scheme to a perturbed Crank–Nicolson discrete scheme (16). It follows from the well known Lax theorem (see e.g., [18]) that the discrete solution of scheme (20) and (21) converge with the second-order in space and time.

Table 2 provides a sequence of decreasing time step widths τ and space mesh steps h_x, h_y, the errors $E(\tau)$, and the experimental convergence rates $\rho(\tau)$ of the discrete solution for the ADI scheme (20) and (21) in the maximum norm.

Table 2. Errors $E(\tau, h)$ and experimental convergence rates $\rho(\tau, h)$ for the discrete solution of the ADI scheme (20) and (21) for a sequence of time steps τ and space steps $h_x = h_y = h$.

$\tau_0 = \frac{1}{10}, h_0 = \frac{1}{16}$	τ_0, h_0	$\frac{\tau_0}{2}, \frac{h_0}{2}$	$\frac{\tau_0}{4}, \frac{h_0}{4}$	$\frac{\tau_0}{8}, \frac{h_0}{8}$
$E(\tau, h)$	6.617×10^{-2}	1.837×10^{-2}	4.721×10^{-3}	1.176×10^{-3}
$\rho(\tau, h)$	—	1.849	1.960	2.005

It follows from the presented results that the convergence order of the discrete solution of the ADI scheme (20) and (21) is close to the second, as it is predicted by the theoretical error estimates. Still, comparing these results with the results presented in Table 1, we conclude that that the solutions of of the Crank–Nicolson scheme (16) are more accurate.

4. Discrete Schemes for the Parabolic Problem with a Fractional Power of Elliptic Operators

In this section, we approximate operator A^α by considering finite-dimensional Hilbert space H_h. A scalar product of $U, V \in H_h$ is denoted as (U, V) and defined by

$$(U, V) = \sum_{i=1}^{J_x-1} \sum_{j=0}^{J_y} U_{ij} V_{ij} h_x h_y c_j, \quad c_j = \begin{cases} 1, & 0 < j < J_y, \\ 0.5, & j = 0, J_y. \end{cases} .$$

Here, it is assumed that boundary conditions for U are homogeneous

$$U_{0j} = 0, \quad U_{J_x,j} = 0, \quad 0 \leqslant j \leqslant J_y.$$

It is easy to check that $A_h = A_{hx} + A_{hy}$, where A_{hx}, A_{hy} are defined by (14), (15), is a self-adjoint positive definite operator

$$A_h : \ H_h \to H_h, \quad A_h = A_h^*, \quad A_h \geqslant c I_h, \ \ c > 0,$$

where I_h is the discrete identity operator.

Let us solve the standard eigen-problem

$$A_h \Psi_{lk} = \lambda_{lk} \Psi_{lk}, \quad 0 < l < J_x, \ \ 0 \leqslant k \leqslant J_y,$$

where $\Psi_{lk} = \Phi_l^x \Phi_k^y$, $\lambda_{lk} = \mu_l^x + \mu_k^y$ and Φ_l^x, Φ_k^y are solutions of eigen-problems

$$\begin{aligned} A_{hx} \Phi_l^x &= \mu_l^x \Phi_l^x, \quad 0 < l < J_x, \\ A_{hy} \Phi_k^y &= \mu_k^y \Phi_k^y, \quad 0 < k < J_y. \end{aligned}$$

All eigenvalues of A_h are positive

$$0 < \lambda_{10} \leqslant \cdots \leqslant \lambda_{J_x-1,J_y}$$

and the set of eigenfunctions $\{\Psi_{lk}\}$ make an orthonormal basis for H_h.

Then, any function $U \in H_h$ can be written as

$$U = \sum_{k=1}^{J_x-1} \sum_{l=0}^{J_y} (U, \Psi_{kl}) \Psi_{kl}.$$

The fractional operator A^α is replaced by its discrete approximation

$$A_h^\alpha U = \sum_{k=1}^{J_x-1} \sum_{l=0}^{J_y} \lambda_{kl}^\alpha \, (U, \Psi_{kl}) \Psi_{kl}.$$

Next, we approximate the nonlocal differential problem (5) and (6) with the discrete scheme which is obtained by applying a symmetrical approximation in time and the discrete operator A_h^α:

$$\partial_t U^n + A_h^\alpha U^{n+\frac{1}{2}} = A_h^{\alpha-1} F^{n+\frac{1}{2}}, \tag{22}$$

where $F^{n+\frac{1}{2}} = \left(f_{ij}^{n+\frac{1}{2}}, 0 < i < J_x, 0 \leqslant j \leqslant J_y\right)$ and components of this vector $f_{ij}^{n+\frac{1}{2}} = \frac{1}{h_x^2}\left[\delta_{i1} g_1^{n+\frac{1}{2}}(y_j) + \delta_{i,J_x-1} g_2^{n+\frac{1}{2}}(y_j)\right]$ take into account the non-homogeneous boundary conditions

$$u(0,y,t) = g_1(y,t), \quad u(X,y,t) = g_2(y,t), \quad y \in [0,Y], \ \ 0 < t \leqslant T$$

For more details on this technique see [8,19].

The solution of (22) can be computed directly by using the Fourier method. Still, this technique is restricted to specific elliptic operators in rectangular domains and uniform space meshes.

Next, we stop briefly on the stability and accuracy of the constructed discrete scheme (22). A detailed analysis of such schemes is given in [20]. The stability of the scheme is proved exactly as it was done in Lemma 1 for the Crank–Nicolson scheme (16).

The analysis of the approximation accuracy is more complicated. Following techniques used in [20], it is possible to estimate the approximation error in the case of homogeneous boundary conditions and an assumption that the solution of the differential problem (5) and (6) is a sufficiently smooth function. First we write the approximation error in the standard form [18]

$$\Psi_h(t^{n+\frac{1}{2}}) := \partial_t u_h^n + A_h^\alpha u_h^{n+\frac{1}{2}} = A_h^\alpha u_h(t^{n+\frac{1}{2}}) - \left(A^\alpha u(t^{n+\frac{1}{2}})\right)_h + O(\tau^2). \tag{23}$$

It is clear that the classical technique of the Taylor expansion cannot be used for the fractional power elliptic operators. Let us apply the spectral definition of nonlocal operators A^α and A_h^α, then we obtain

$$A_h^\alpha u_h(t^n) - \left(A^\alpha u(t^n)\right)_h = \sum_{j=1}^{J} \left(u_{hj}^n \mu_j^\alpha \psi_j^h - u_j(t^n) \lambda_j^\alpha \psi_j\right) - \sum_{j=J+1}^{\infty} \lambda_j^\alpha u_j \psi_j.$$

The second term can be bounded by $O(h^m)$ depending on the smoothness of the solution. Additionally, the estimates for the accuracy of discrete eigenvalues and eigenvectors are well-known for many popular approximations of elliptic problems [18,21]:

$$\left\|\psi_j^h - \psi_{jh}\right\| \leqslant Ch^2, \quad \left|\mu_j^\alpha - \lambda_j^\alpha\right| \leqslant Ch^2, \quad j = 1, \dots, J.$$

Still, due to inclusion of non-homogeneous boundary conditions, the smoothness of the solution of differential problem (5) and (6) is reduced. In this case, the convergence rate of the discrete solution depends on a balance of two properties, i.e., the smoothness of the exact solution and on the fractional power parameter α. For standard second-order space approximations it has been shown in [10,17] that the discretization error in the L_2 -norm for similar problems can behave like

$$E_h \leqslant Ch^{\min(2, 2\alpha+0.5)} \log(1/h). \tag{24}$$

In order to illustrate such theoretical results we provide, in Table 3, experimental convergence rates for the discrete solution of symmetrical scheme (22) in the maximum norm $E(\tau, h)$. A sequence of time steps and space steps are used, and $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Table 3. Errors $E(\tau, h)$ and experimental convergence rates $\rho(\tau, h)$ for the discrete solution of the discrete scheme (22) for a sequence of time steps τ and space steps $h_x = h_y = h$ and two different values of the fractional power parameter α.

$\tau_0 = \frac{1}{10}, h_0 = \frac{1}{16}$	τ_0, h_0	$\frac{\tau_0}{2}, \frac{h_0}{2}$	$\frac{\tau_0}{4}, \frac{h_0}{4}$	$\frac{\tau_0}{8}, \frac{h_0}{8}$
$E(\tau, h)$, $\alpha = \frac{3}{4}$	1.528×10^{-2}	3.919×10^{-3}	9.865×10^{-4}	2.365×10^{-4}
$\rho(\tau, h)$	—	1.963	1.990	2.060
$E(\tau, h)$, $\alpha = \frac{1}{2}$	1.696×10^{-2}	4.466×10^{-3}	1.194×10^{-3}	4.038×10^{-4}
$\rho(\tau, h)$	—	1.925	1.903	1.564
$E(\tau, h)$, $\alpha = \frac{1}{4}$	2.213×10^{-2}	1.028×10^{-2}	5.850×10^{-3}	3.682×10^{-3}
$\rho(\tau, h)$	—	1.106	0.8130	0.6680

The presented results show a degradation of the convergence rate predicted in (24). Next, we solved the same test problem, but with homogeneous boundary conditions

$$g_1(y, t) = 0, \quad g_2(y, t) = 0.$$

The initial condition is defined as

$$u_0(x, y) = x(2 - x)\cos(\pi y), \quad 0 \leqslant x \leqslant 2, \ \ 0 \leqslant y \leqslant 1. \tag{25}$$

Our aim is to show that, in this case, the second-order convergence rate is expected for the solution of the symmetrical scheme (22). A sequence of time steps and space steps are used, and $\alpha = \frac{1}{2}, \frac{3}{4}$. Errors $E(\tau, h)$ and experimental convergence rates are presented in Table 4.

Table 4. Errors $E(\tau, h)$ and experimental convergence rates $\rho(\tau, h)$ for the discrete solution of the discrete scheme (22) for a sequence of time steps τ and space steps $h_x = h_y = h$ and two different values of the fractional power parameter α. The homogeneous boundary conditions and initial condition (25) are used.

$\tau_0 = \frac{1}{10}, h_0 = \frac{1}{16}$	τ_0, h_0	$\frac{\tau_0}{2}, \frac{h_0}{2}$	$\frac{\tau_0}{4}, \frac{h_0}{4}$	$\frac{\tau_0}{8}, \frac{h_0}{8}$
$E(\tau, h)$, $\alpha = \frac{3}{4}$	6.657×10^{-3}	1.637×10^{-3}	4.074×10^{-4}	1.015×10^{-4}
$\rho(\tau, h)$	—	2.024	2.006	2.005
$E(\tau, h)$, $\alpha = \frac{1}{2}$	3.027×10^{-3}	7.512×10^{-4}	1.875×10^{-4}	4.694×10^{-5}
$\rho(\tau, h)$	—	2.011	2.002	1.998

The presented results confirm the second order rate of the convergence.

5. BURA-BRASIL Algorithm

In the previous section, the discrete scheme (22) was implemented by using the FFT algorithm. This algorithm is efficient only for special classes of problems and uniform discrete space meshes. In order to construct a universal discrete scheme, we write the Crank–Nicolson scheme (22) in the factorized form

$$\begin{aligned} U^{n+\frac{1}{2}} &= (I_h + 0.5\tau A_h^{\alpha})^{-1}\Big(U^n + 0.5\tau A_h^{\alpha-1}F^{n+\frac{1}{2}}\Big), \\ U^{n+1} &= 2U^{n+\frac{1}{2}} - U^n. \end{aligned} \tag{26}$$

Then, applying a general technique described in [8,20] we approximate both nonlocal operators $(I_h + \tau A_h^{\alpha})^{-1}$ and $A_h^{\alpha-1}$ with two local rational operators

$$(I_h + 0.5\tau A_h^{\alpha})^{-1} \approx r_m(A_h), \quad A_h^{\alpha-1} \approx \widetilde{r}_m(A_h).$$

For example, a function $r_m(z)$ is defined as

$$r_m(\lambda) = \frac{p_m(\lambda)}{q_m(\lambda)}$$

with polynomials p_m and q_m of the same degree m. The constructed rational function r_m can be written in a partial fraction decomposition form [8]

$$r_m(\lambda) = c_0 + \sum_{j=1}^{m} \frac{c_j}{\lambda - d_j} \tag{27}$$

which enables us to reduce the computation of images of nonlocal operators to a solution of systems of linear equations. All m systems can be solved in parallel.

In this work, the required rational functions are computed by using the BURA-BRASIL algorithm, which is based on the barycentric rational formula [22]. The main aim of this algorithm is to construct BURA (best uniform rational approximation) type approximations [7]. A free and open-source Python implementation of the BRASIL algorithm is used in computations [22].

Thus, the basic nonlocal discrete scheme (26) is approximated using the following local discrete scheme

$$\begin{aligned}
\widetilde{F}^{n+\frac{1}{2}} &= \widetilde{r}_m(A_h)F^{n+\frac{1}{2}}, \\
V^{n+\frac{1}{2}} &= r_m(A_h)\Big(V^n + 0.5\tau\widetilde{F}^{n+\frac{1}{2}}\Big), \\
V^{n+1} &= 2V^{n+\frac{1}{2}} - V^n.
\end{aligned} \tag{28}$$

Here, $\widetilde{F}^{n+\frac{1}{2}}$ is computed as

$$\begin{aligned}
\widetilde{F}^{n+\frac{1}{2}} &= \tilde{c}_0 F^{n+\frac{1}{2}} + \sum_{k=1}^{m} \tilde{c}_k G_k^{n+\frac{1}{2}}, \\
(A_h - \tilde{d}_k I_h)G_k^{n+\frac{1}{2}} &= F^{n+\frac{1}{2}}, \ \ k = 1, \dots, m
\end{aligned} \tag{29}$$

and similarly $V^{n+\frac{1}{2}}$ is computed as

$$\begin{aligned}
V^{n+\frac{1}{2}} &= c_0\Big(V^n + 0.5\tau\tilde{F}^{n+\frac{1}{2}}\Big) + \sum_{k=1}^{m} c_k H_k^{n+\frac{1}{2}}, \\
(A_h - d_k I_h)H_k^{n+\frac{1}{2}} &= \Big(V^n + 0.5\tau\tilde{F}^{n+\frac{1}{2}}\Big), \ \ k = 1, \dots, m
\end{aligned} \tag{30}$$

The stability analysis of such discrete schemes is done in [20]. According to it, we should estimate the norm $\|2r_m(A_h) - I_h\|$. We investigated the stability by computing the stability factor R of the scheme (28) for a set of eigenvalues of the discrete operator A_h and selected values of fractional power parameters α:

$$R = \max_{0 \leqslant j \leqslant K} |2r_m(z_j) - 1|, \quad z_j = \mu_1 + \frac{j}{K}(\mu_J - \mu_1).$$

In the case of the parameters of this paper, it was ensured that the stability requirement $R \leqslant 1$ was satisfied.

The accuracy of the discrete solution depends on the parameter m of the BURA-BRASIL algorithm. In Table 5, we present the error in the maximum norm $E(\tau, h)$ with which the solution of BURA-BRASIL scheme (28) approximates the solution of the Crank–Nicolson scheme (22). Results are given for a sequence of time steps τ and space steps $h_x = h_y = h$ and two different values of the parameter m. The value of the fractional power parameter is fixed to $\alpha = \frac{3}{4}$.

Table 5. Errors $E(\tau, h)$ for the discrete solution of the BURA-BRASIL scheme (28) for a sequence of time steps τ and space steps $h_x = h_y = h$ and two different values of m.

$\tau_0 = \frac{1}{10}, h_0 = \frac{1}{16}$	τ_0, h_0	$\frac{\tau_0}{2}, \frac{h_0}{2}$	$\frac{\tau_0}{4}, \frac{h_0}{4}$
$E(\tau, h), \quad m = 5$	1.044×10^{-2}	1.894×10^{-2}	2.997×10^{-2}
$E(\tau, h), \quad m = 10$	2.611×10^{-4}	8.230×10^{-4}	1.953×10^{-3}

6. ADI Scheme for Partially Dimension-Reduced Problem

As it was stated above, our main aim is to construct partially dimension-reduced models for parabolic problems with the fraction power of elliptic operators.

First, we note that this possibility is based on the properties of solutions for both types of parabolic problems, when classical diffusion operators and fractional powers of such operators are used to define PDEs. For the selected test problem, we see a typical asymptotical behaviour of the solution $U^n(x_i, y_j)$, when in a big part of the domain function U is close to a constant with respect to one space direction y (a regular part of the solution). The full two dimension model should be only be used in a small part of the domain (a singular part of the solution). As an example, in Table 6 we present values of

$$\tilde{\delta}(x_i) = \max_{j \in \bar{\omega}_y} |U_{ij}^N - U_{i,J_y/2}^N|$$

at $t = 1$, for mesh steps $h_x = h_y = h_t = 0.01$ and two values of the fractional power parameters: $\alpha = 1$ (the classical diffusion operator) and $\alpha = \frac{3}{4}$ (a fractional power of the elliptic operator).

Table 6. Values of the variance function $\tilde{\delta}(x_i)$ for the discrete solutions of ADI scheme (16) and of the BURA-BRASIL scheme (28) for a sequence of points x_i and two different values of the fractional poser parameter $\alpha = 1, 0.75$.

	$x_i = 0.7$	$x_i = 0.9$	$x_i = 1$	$x_i = 1.1$	$x_i = 1.3$
$\tilde{\delta}(x_i), \alpha = 1$	0.0118	0.00475	0.00474	0.00672	0.0211
$\tilde{\delta}(x_i), \alpha = \frac{3}{4}$	0.00906	0.00358	0.00356	0.00509	0.0166

Here, we note that adaptive meshes can be used as an alternative for the partial dimension reducing technique. As an example, a two-scale solver can be constructed. The implementation of the solver is done in two steps: first, the global approximation is done by using classical bases functions uniformly distributed in the whole domain, and second, enriched bases functions are used locally to represent a microstructure in singular zones (see, e.g., [23]) A comparison of both approaches is planed for a future work.

First we define discrete meshes on the reduced dimension domain

$$\begin{aligned}
&\omega_{x1} = \{x_i : \ 0 < i < J_{x1}\}, \ \ \omega_{x2} = \{x_i : \ J_{x2} < i < J_x\}, \ \ \omega_{x3} = \{x_i : \ J_{x1} < i < J_{x2}\}, \\
&\bar{\omega}_{x1} = \{x_i : \ 0 \leqslant i < J_{x1}\}, \ \ \bar{\omega}_{x2} = \{x_i : \ J_{x2} < i \leqslant J_x\}, \ \ \bar{\omega}_{x3} = \{x_i : \ J_{x1} \leqslant i \leqslant J_{x2}\}, \\
&x_{J_{x1}} = \delta, \quad x_{J_{x2}} = X - \delta.
\end{aligned}$$

The discrete solution U_{ij} is defined on the reduced dimension (RD) mesh:

$$\overline{\Omega}_h^{RD} = \left(\bar{\omega}_{x1} \times \bar{\omega}_y\right) \cup \bar{\omega}_{x3} \cup \left(\bar{\omega}_{x2} \times \bar{\omega}_y\right).$$

A set of such discrete functions U is denoted by D_h.

Let $U, V \in D_h$ and assume that they satisfy the following boundary conditions

$$U_{0j} = 0, \ \ U_{J_x,j} = 0, \quad V_{0j} = 0, \ \ V_{J_x,j} = 0, \quad 0 \leqslant j \leqslant J_y.$$

Then the formulae

$$\begin{aligned}
&(U,V) = \sum_{j=0}^{J_y} c_j \Big(\sum_{i=1}^{J_{x1-1}} U_{ij} V_{ij} h_x + \sum_{i=J_{x2+1}}^{J_x-1} U_{ij} V_{ij} h_x \Big) h_y + Y \sum_{k=K_1}^{K_2} U_{i0} V_{i0} h_x, \\
&c_0 = \frac{1}{2}, \quad c_{J_y} = \frac{1}{2}, \quad c_j = 1, \; 0 < j < J_y, \\
&\|U\| = (U,U)^{1/2}
\end{aligned}$$

define a scalar product and a norm in this vector space.

The discrete averaging operator S_h is defined as:

$$S_h(U_i^n) = \frac{h_y}{Y} \left(\frac{1}{2} U_{i0}^n + \sum_{j=1}^{J_y-1} U_{ij}^n + \frac{1}{2} U_{i,J_y}^n \right).$$

In order to approximate problem (7)–(11) we construct a modified version of the ADI scheme (see also [14]):

$$\frac{\widehat{U}_{ij}^n - U_{ij}^n}{\frac{1}{2}\tau} + A_{hx} U_{ij}^n + A_{hy} \widehat{U}_{ij}^n = 0, \quad (x_i, y_j) \in (\omega_{x1} \cup \omega_{x2}) \times \bar{\omega}_y, \tag{31}$$

$$\frac{\widehat{U}_{i0}^n - U_{i0}^n}{\frac{1}{2}\tau} + A_{hx} U_{i0}^n = 0, \quad x_i \in \omega_{x3}, \tag{32}$$

$$\frac{\widehat{U}_{J_{x1},0}^n - U_{J_{x1},0}^n}{\frac{1}{2}\tau} + \frac{1}{h_x^2}\big(- S_h(U_{J_{x1}-1}^n) + 2U_{J_{x1},0}^n - U_{J_{x1}+1,0}^n \big) = 0, \tag{33}$$

$$\frac{\widehat{U}_{J_{x2},0}^n - U_{J_{x2},0}^n}{\frac{1}{2}\tau} + \frac{1}{h_x^2}\big(- U_{J_{x2}-1,0}^n + 2U_{J_{x2},0}^n - S_h(U_{J_{x2}+1}^n) \big) = 0, \tag{34}$$

$$\frac{U_{ij}^{n+1} - \widehat{U}_{ij}^n}{\frac{1}{2}\tau} + A_{hx} U_{ij}^{n+1} + A_{hy} \widehat{U}_{ij}^n = 0, \quad (x_i, y_j) \in (\omega_{x1} \cup \omega_{x2}) \times \bar{\omega}_y, \tag{35}$$

$$\frac{U_{i0}^{n+1} - \widehat{U}_{i0}^n}{\frac{1}{2}\tau} + A_{hx} U_{i0}^{n+1} = 0, \quad x_i \in \omega_{x3}, \tag{36}$$

$$\frac{U_{J_{x1},0}^{n+1} - \widehat{U}_{J_{x1},0}^n}{\frac{1}{2}\tau} + \frac{1}{h_x^2}\big(- S_h(U_{J_{x1}-1}^{n+1}) + 2U_{J_{x1},0}^{n+1} - U_{J_{x1}+1,0}^{n+1} \big) = 0, \tag{37}$$

$$\frac{U_{J_{x2},0}^{n+1} - \widehat{U}_{J_{x2},0}^n}{\frac{1}{2}\tau} + \frac{1}{h_x^2}\big(- U_{J_{x2}-1,0}^{n+1} + 2U_{J_{x2},0}^{n+1} - S_h(U_{J_{x2}+1}^{n+1}) \big) = 0. \tag{38}$$

Let us define two operators for $U \in D_h$:

$$\mathcal{A}_{hy} U = \begin{cases} A_{hy} U_{ij}, & (x_i, y_j) \in (\omega_{x1} \cup \omega_{x2}) \times \bar{\omega}_y, \\ 0, & x_i \in \bar{\omega}_{x3}, \end{cases}$$

$$\mathcal{A}_{hx} U = \begin{cases} A_{hx} U_{ij}, & (x_i, y_j) \in (\omega_{x1} \cup \omega_{x2}) \times \bar{\omega}_y, \\ A_{hx} U_{i0}, & x_i \in \omega_{x3}, \\ \frac{1}{h_x^2}\big(- S_h(U_{i-1}) + 2U_{i0} - U_{i+1,0} \big), & i = J_{x1}, \\ \frac{1}{h_x^2}\big(- U_{i-1,0} + 2U_{i0} - S_h(U_{i+1}) \big), & i = J_{x2}, \end{cases}$$

Then we can write the ADI scheme in operator form:

$$\frac{\widehat{U}^n - U^n}{\frac{1}{2}\tau} + \mathcal{A}_{hx} U^n + \mathcal{A}_{hy} \widehat{U}^n = 0, \quad (39)$$

$$\frac{U^{n+1} - \widehat{U}^n}{\frac{1}{2}\tau} + \mathcal{A}_{hx} U^{n+1} + \mathcal{A}_{hy} \widehat{U}^n = 0. \quad (40)$$

The implementation of the first step of the ADI scheme (39) is straightforward—the systems with three diagonal matrices are solved by using the classical factorization method. It is important to note, that all systems can be solved in parallel.

The second step of the ADI scheme (40) requires a modification of the factorization algorithm. First, the solution of equations on mesh $\omega_{x1} \times \bar{\omega}_y$ is represented in the form

$$U_{ij} = \alpha_{ij} g_1(y_j) + \beta_{ij} U_{J_{x1},0} + \gamma_{ij}, \quad 0 \leqslant i < J_{x1}, \ \ y_j \in \bar{\omega}_y,$$

and similarly on mesh $\omega_{x2} \times \bar{\omega}_y$

$$U_{ij} = \alpha_{ij} U_{J_{x2},0} + \beta_{ij} g_2(y_j) + \gamma_{ij}, \quad J_{x1} < i \leqslant J_{x2}, \ \ y_j \in \bar{\omega}_y.$$

The solution on $\bar{\omega}_{x3}$ is defined by the discrete problem

$$\begin{aligned}
&\left(\frac{2}{\tau} I + A_{hx}\right) U_{i0}^{n+1} = \frac{2}{\tau} \widehat{U}_{i0}^{n}, \quad J_{x1} < i < J_{x2}, \\
&\left(\frac{2}{\tau} + \frac{1}{h_x^2}(2 - S_h(\beta_{i-1}))\right) U_{i0}^{n+1} - \frac{1}{h_x^2} U_{i+1,0}^{n+1} = \frac{2}{\tau} \widehat{U}_{i0}^{n} + \frac{1}{h_x^2} S_h\left(\alpha_{i-1} g_1 + \gamma_{i-1}\right), \ i = J_{x1}, \\
&\left(\frac{2}{\tau} + \frac{1}{h_x^2}(2 - S_h(\beta_{i+1}))\right) U_{i0}^{n+1} - \frac{1}{h_x^2} U_{i-1,0}^{n+1} = \frac{2}{\tau} \widehat{U}_{i0}^{n} + \frac{1}{h_x^2} S_h\left(\alpha_{i+1} g_2 + \gamma_{i+1}\right), \ i = J_{x2}.
\end{aligned} \quad (41)$$

It can be solved efficiently by using the classical factorization algorithm. When $U_{J_{x1},0}^{n+1}$, $U_{J_{x2},0}^{n+1}$ are obtained, the remaining part of the solution U^{n+1} is computed.

Next, we present some basic theoretical results on the stability of the constructed discrete scheme (this analysis uses techniques presented in [14]).

Lemma 3. *The discrete operators $\mathcal{A}_{hx}$ and $\mathcal{A}_{hy}$ are symmetric and positive and non-negative definite operators, respectively.*

Proof. Here, we restrict to the analysis of operator $\mathcal{A}_{hx}$ which includes the most important specific details of the discrete scheme. We also define vectors U, V on the full mesh ω_y:

$$U_{J_{x1},j} = U_{J_{x1},0}, \ U_{J_{x2},j} = U_{J_{x2},0}, \quad V_{J_{x1},j} = V_{J_{x1},0}, \ V_{J_{x2},j} = V_{J_{x2},0}, \ \ y_j \in \bar{\omega}_y.$$

Let us compute $(\mathcal{A}_{hx} U, V)$ for vectors U, V which satisfy homogeneous boundary conditions. Applying the summation of the parts formula and taking into account the boundary and conjugation conditions we get

$$
\begin{aligned}
(\mathcal{A}_{hx}U, V) &= \sum_{j=0}^{J_y} c_j \Big(\sum_{i=1}^{J_{x1}-1} (A_{hx}U)_{ij} V_{ij} h_x + \sum_{i=J_{x2}+1}^{J_x-1} (A_{hx}U)_{ij} V_{ij} h_x \Big) h_y \\
&\quad + Y \Big(\frac{1}{h_x} \big(-S_h(U_{J_{x1}-1,\cdot}) + 2U_{J_{x1},0} - U_{J_{x1+1},0} \big) V_{J_{x1},0} + \sum_{i=J_{x1}+1}^{J_{x2}-1} (A_{hx}U)_{i0} V_{i0} h_x \\
&\quad + \frac{1}{h_x} \big(-S_h(U_{J_{x2}+1,\cdot}) + 2U_{J_{x2},0} - U_{J_{x2}-1,0} \big) V_{J_{x2},0} \Big) \\
&= \sum_{j=0}^{J_y} c_j \Big(\sum_{i=1}^{J_{x1}} \partial_x U_{ij} \, \partial_x V_{ij} h_x + \sum_{i=J_{x2}+1}^{J_x} \partial_x U_{ij} \, \partial_x V_{ij} h_x \Big) h_y \\
&\quad + Y \sum_{i=J_{x1}+1}^{J_{x2}} \partial_x U_{i0} \, \partial_x V_{i0} h_x.
\end{aligned}
$$

Thus, the operator $\mathcal{A}_{hx}$ is symmetric and non-negative definite. The positivity of this operator follows from the positivity of general elliptic operators with the given boundary conditions. □

Taking into account the results of Lemma 3, the stability of the ADI scheme can be shown in a particular energy norm [18].

Lemma 4. *If U^n is the solution of ADI scheme* (39) *and* (40)*, then the following stability estimate is valid*

$$
\|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^n\| \leqslant \|(I + \frac{\tau}{2}\mathcal{A}_2^h)U^0\|. \tag{42}
$$

In Table 7, we present the error $e(\delta)$ with which the solution of discrete scheme (39) and (40) approximates the solution of the Crank–Nicolson scheme (16). Computations are done for a sequence of truncation parameters δ and the discrete mesh steps are fixed to $\tau = 0.004$, $h_x = h_y = 0.004$.

Table 7. Errors $e(\delta)$ with which the solution of the scheme (39) and (40) approximates the solution of the Crank–Nicolson scheme (16) for a sequence of truncation parameters δ.

	$\boldsymbol{\delta = 0.8}$	$\boldsymbol{\delta = 0.6}$	$\boldsymbol{\delta = 0.4}$	$\boldsymbol{\delta = 0.3}$
$e(\delta)$	5.7695×10^{-3}	2.0002×10^{-2}	7.3100×10^{-2}	1.4255×10^{-1}

It follows from the presented results starting from $\delta = 0.3$ that the solution of the truncated problem (partially dimension-reduced model) approximates quite accurately the solution of the Crank–Nicolson scheme (16).

7. Partially Dimension-Reduced Approximation of the BURA-BRASIL-Type Discrete Scheme (28)

The main idea is to solve all elliptic problems in Equations (29) and (30) by applying the technique similar to one presented in a previous section for the classical parabolic problem.

Let us consider the full algorithm in detail. We demonstrate how solve a template problem:

$$
(A_h - dI_h)V = F, \tag{43}
$$

where constant $d \leqslant 0$ and discrete vectors V, F belong to a class of partially dimension-reduced vectors D_h. Note that non-homogeneous boundary conditions of the solution V are included into the source function F.

Our aim is to solve problem (43) with non-iterative algorithms of linear algebra. Let us present the solution V in the following form

$$V_{ij} = \begin{cases} W^L_{ij} + V_{J_{x1},0} U^L_{ij}, \quad 0 < i < J_{x1}, \ 0 \leqslant j \leqslant J_y, \\ V_{i0}, \quad J_{x1} \leqslant i \leqslant J_{x2}, \\ W^R_{ij} + V_{J_{x2},0} U^R_{ij}, \quad J_{x2} < i < J_x, \ 0 \leqslant j \leqslant J_y, \end{cases} \tag{44}$$

where $W^{L,R}$ are solutions of the problems

$$(A_h - dI_h) W^{L,R} = F^{L,R}. \tag{45}$$

Since homogeneous boundary conditions are specified for $W^{L,R}$, i.e.,:

$$W^L_{0j} = 0, \quad W^L_{J_{x1},j} = 0, \quad W^R_{J_{x2},j} = 0, \quad W^R_{J_x,j} = 0, \quad 0 \leqslant j \leqslant J_y,$$

then both problems are solved efficiently by using the FFT algorithm. We note that problems (45) are independent and can be solved in parallel.

$U^{L,R}$ are solutions of the problems

$$\begin{aligned} &(A_h - dI_h) U^L = 0, \quad (x_i, y_j) \in \omega_{x1} \times \bar{\omega}_y, \\ &U^L_{0j} = 0, \quad U^L_{J_{x1},j} = 1, \quad 0 \leqslant j \leqslant J_y, \end{aligned}$$

and

$$\begin{aligned} &(A_h - dI_h) U^R = 0, \quad (x_i, y_j) \in \omega_{x2} \times \bar{\omega}_y, \\ &U^R_{J_{x2},j} = 1, \quad U^R_{J_x,j} = 0, \quad 0 \leqslant j \leqslant J_y. \end{aligned}$$

It is easy to see that due to special boundary conditions, the functions $U^{L,R}$ satisfy the following properties

$$\begin{aligned} &U^L_{ij} = u^L_i, \quad (x_i, y_j) \in \bar{\omega}_{x1} \times \bar{\omega}_y, \\ &U^R_{ij} = u^R_i, \quad (x_i, y_j) \in \bar{\omega}_{x2} \times \bar{\omega}_y, \end{aligned}$$

where $u^{L,R}$ are one-dimensional vectors in x dimension. These functions are solutions of the following problems

$$\begin{aligned} &(A_{hx} - dI_h) u^L = 0, \quad x_i \in \omega_{x1} \\ &u^L_0 = 0, \quad u^L_{J_{x1}} = 1, \end{aligned}$$

and

$$\begin{aligned} &(A_{hx} - dI_h) u^R = 0, \quad x_i \in \omega_{x2}, \\ &u^R_{J_{x2}} = 1, \quad u^R_{J_x} = 0. \end{aligned}$$

The solution on the mesh $\bar{\omega}_{x3}$ is defined by the discrete problem

$$\begin{aligned} &(A_{hx} - dI_h) V = F, \quad J_{x1} < i < J_{x2}, \\ &\left(\frac{1}{h_x^2} \left(2 - u^L_{i-1} \right) - d \right) V_{i0} - \frac{1}{h_x^2} V_{i+1,0} = F_{i0} + \frac{1}{h_x^2} S_h \left(W^L_{i-1} \right), \quad i = J_{x1}, \\ &\left(\frac{1}{h_x^2} \left(2 - u^L_{i+1} \right) - d \right) V_{i0} - \frac{1}{h_x^2} V_{i-1,0} = F_{i0} + \frac{1}{h_x^2} S_h \left(W^L_{i+1} \right), \quad i = J_{x2}. \end{aligned} \tag{46}$$

This system of linear equations can be solved efficiently by using the classical factorization algorithm. When $V_{J_{x1},0}$, $V_{J_{x2},0}$ are known, the remaining part of the solution V is computed explicitly.

In Table 8, we present the error with which the solution of the partially dimension-reduced scheme approximates the solution of the Crank–Nicolson scheme (28). Results are given for a sequence of truncation parameters δ, fractional power parameter $\alpha = 0.75$, and mesh steps $\tau = 0.004$, $h_x = h_y = 0.004$.

Table 8. Errors $\tilde{e}(\delta)$ for the solution of the BURA-BRASIL-type partially dimension-reduced scheme for a sequence of truncation parameters δ and $\alpha = \frac{3}{4}$.

	$\delta = 0.8$	$\delta = 0.6$	$\delta = 0.4$	$\delta = 0.3$
$\tilde{e}(\delta)$	6.151×10^{-3}	1.909×10^{-2}	6.889×10^{-2}	1.362×10^{-1}

Again, we see that the solution of the hybrid partially dimension-reduced scheme approximates quite accurately approximates a solution of the Crank–Nicolson scheme (28). This scheme solves a parabolic problem with the fractional power elliptic operator.

8. Conclusions

A general approach is proposed regarding how to use the partial dimension reduction technique in order to solve parabolic problems with fractional power elliptic operators. The new technique is based on approximation of the differential problem by the BURA-BRASIL-type Crank–Nicolson scheme. Then, the obtained discrete problems with classical elliptic operators are solved by using the efficient partial dimension reduction techniques.

Non-iterative solvers of linear equations are proposed and implemented. A stability and convergence analysis is given for all steps of the proposed discrete scheme. The results of computational experiments are reported and they illustrate the accuracy of the discrete algorithms.

It is noted that efficient parallel solvers can be used to implement the given scheme. These results will be presented in a separate paper.

Author Contributions: R.Č. (Raimondas Čiegis): conceptualization; methodology; writing—original draft preparation; V.S.: numerical algorithms; O.S.: software; computational experiments, R.Č. (Remigijus Čiegis): analysis of results, writing—editing. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Lischke, A.; Pang, G.; Gulian, M.; Song, F.; Glusa, C.; Zheng, X.; Mao, Z.; Cai, W.; Meerschaert, M.; Ainsworth, M.; Karniadakis, G. What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.* **2020**, *404*, 109009. [CrossRef]
2. Harizanov, S.; Margenov, S.; Marinov, P.; Vutov, Y. Volume constrained 2-phase segmentation method utilizing a linear system solver based on the best uniform polynomial approximation of $x^{-1/2}$. *J. Comput. Appl. Math.* **2017**, *310*, 115–128. [CrossRef]
3. Lee, H.G. A second-order operator splitting Fourier spectral method for fractional-in-space reaction—Diffusion equations. *J. Comput. Appl. Math.* **2018**, *333*, 395–403. [CrossRef]
4. Barrera, O. A unified modelling and simulation for coupled anomalous transport in porous media and its finite element implementation. *Comput. Mech.* **2021**, *68*, 1267–1282. [CrossRef]
5. Akagi, G.; Schimperna, G.; Segatti, A. Fractional Cahn-Hilliard, Allen-Cahn and porous medium equations. *J. Differ. Equat.* **2016**, *261*, 2935–2985. [CrossRef]
6. Danczul, T.; Hofreither, C.; Schöberl, J. A unified rational Krylov method for elliptic and parabolic fractional diffusion problems. *arXiv* **2021**, arXiv:2103.13068.
7. Harizanov, S.; Lazarov, R.; Margenov, S.; Marinov, P, Pasciak, J. Analysis of numerical methods for spectral fractional elliptic equations based on the best uniform rational approximation. *J. Comput. Phys.* **2020**, *408*, 109285. [CrossRef]

8. Čiegis, R.; Dapšys, I.; Čiegis, R. A comparison of parallel algorithms for numerical solution of parabolic problems with fractional power elliptic operators. *Axioms* **2022**, *11*, 98. [CrossRef]
9. Bulle, R.; Barrera, O.; Bordas, S.; Chouly, F.; Hale, J. An a posteriori error estimator for the spectral fractional power of the Laplacian. *Comput. Methods Appl. Mech. Eng.* **2023**, *407*, 115943. [CrossRef]
10. Hofreither, C. A unified view of some numerical methods for fractional diffusion. *Comput. Math. Appl.* **2020**, *80*, 332–350. [CrossRef]
11. Panasenko, G. Method of asymptotic partial decomposition of domain. *Math. Model. Methods Appl. Sci.* **1998**, *8*, 139–156. [CrossRef]
12. Amosov, A.; Panasenko, G. Partial dimension reduction for the heat equation in a domain containing thin tubes. *Math. Methods Appl. Sci.* **2018**, *41*, 9529–9545. [CrossRef]
13. Amosov, A.; Panasenko, G. Partial decomposition of a domain containing thin tubes for solving the diffusion equation. *J. Math. Sci.* **2022**, *264*, 25–33. [CrossRef]
14. Čiegis, R.; Panasenko, G.; Pileckas, K.; Šumskas, V. ADI scheme for partially dimension reduced heat conduction models. *Comput. Math. Appl.* **2020**, *80*, 1275–1286. [CrossRef]
15. Viallon, M. Error estimate for a 1D–2D finite volume scheme. comparison with a standard scheme on a 2D non-admissible mesh. *Comptes Rendus Math.* **2013**, *351*, 47–51. [CrossRef]
16. Čiegis, R.; Čiegis, R.; Subovic, O. Parallel 3D ADI scheme for partially dimension reduced heat conduction problem. *Informatica* **2022**, *33*, 477–497. [CrossRef]
17. Bonito, A.; Pascia, J.E. Numerical approximation of fractional powers of elliptic operators. *Math. Comput.* **2015**, *84*, 2083–2110. [CrossRef]
18. Hundsdorfer, W.; Verwer, J. *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*; Springer Series in Computational Mathematics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA; Tokyo, Japan, 2003; Volume 33.
19. Ilic, M.; Liu, F.; Turner, I.W.; Anh, V. Numerical approximation of a fractional-in-space diffusion equation–II-with nonhomogeneous boundary conditions. *Fract. Calc. Appl. Anal.* **2006**, *9*, 333–349.
20. Čiegis, R.; Dapšys, I. On a framework for the stability and convergence analysis of discrete schemes for nonstationary nonlocal problems of parabolic type. *Mathematics* **2022**, *10*, 2155. [CrossRef]
21. Samarskii, A.A. *The Theory of Difference Schemes*; Marcel Dekker: New York, NY, USA, 2001.
22. Hofreither, C. An algorithm for best rational approximation based on barycentric rational interpolation. *Numer. Algorithms* **2021**, *88*, 365–388. [CrossRef]
23. Salzman, A.; Moēs, N. A two-scale solver for linear elasticity problems in the context of parallel message passing. *Comput. Methods Appl. Mech. Eng.* **2023**, *407*, 115914. [CrossRef]