Journal of Mathematical Inequalities
Volume 17, Number 1 (2023), 141–151 doi:10.7153/jmi-2023-17-10

# ALGEBRAIC INTEGERS WITH SMALL POTENTIAL ENERGY

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(*Communicated by A. Filipin*)

*Abstract.* In this paper we give a lower bound on the mean of squares of distances between the points in two sets in terms of the products of distances between the points in each of those two sets. These results imply lower bounds on the $p$th power ($p \geqslant 2$) of the average distance between the conjugates of an algebraic integer and of the same quantity between the conjugates of a totally real algebraic integer.

## 1. Introduction

The weighted arithmetic and geometric mean inequality asserts that for any real numbers $a_1,\dots,a_m \geqslant 0$ and any nonnegative weights $w_1,\dots,w_m$ satisfying

$$w_1+\dots+w_m=1$$

we have

$$w_1a_1+\dots+w_ma_m \geqslant a_1^{w_1}\dots a_m^{w_m} \tag{1}$$

(see, e.g., [12]). In particular, for $n \geqslant 2$ and any $x_1,\dots,x_n \in \mathbb{C}$ applying the standard arithmetic and geometric mean inequality (when the weights $w_i$ in (1) are all equal $w_1=\dots=w_m=1/m$) to $m=n(n-1)/2$ numbers of the form $|x_i-x_j|^2$, where $1 \leqslant i<j \leqslant n$, we have

$$\frac{2}{n(n-1)}\sum_{1\leqslant i<j\leqslant n}|x_i-x_j|^2 \geqslant \Big(\prod_{1\leqslant i<j\leqslant n}|x_i-x_j|\Big)^{\frac{4}{n(n-1)}}.$$

For $n \geqslant 9$ this inequality can be improved as follows:

THEOREM 1. *For $n \geqslant 9$ and any $x_1,\dots,x_n \in \mathbb{C}$ we have*

$$\frac{2}{n(n-1)}\sum_{1\leqslant i<j\leqslant n}|x_i-x_j|^2 \geqslant C(n)\Big(\prod_{1\leqslant i<j\leqslant n}|x_i-x_j|\Big)^{\frac{4}{n(n-1)}}, \tag{2}$$

*where*

$$C(n):=\frac{n}{\big(2^23^3\cdots n^n\big)^{\frac{2}{n(n-1)}}}. \tag{3}$$

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*Mathematics subject classification* (2020): 11R04, 11R80, 30A10.

*Keywords and phrases*: Algebraic integer, totally real algebraic integer, potential energy, average distance between conjugates.

*Moreover, if $n \geqslant 2$ and the numbers $x_1, \ldots, x_n$ are all real then*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i<j \leqslant n} |x_i - x_j|^2 \geqslant 2C(n) \Big( \prod_{1 \leqslant i<j \leqslant n} |x_i - x_j| \Big)^{\frac{4}{n(n-1)}}, \tag{4}$$

*with equality if and only if $x_i = a\zeta_i + b$, where $a, b \in \mathbb{R}$ and $\zeta_1, \ldots, \zeta_n$ are the zeros of the Hermite polynomial*

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{2^k k!(n-2k)!} x^{n-2k} = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}. \tag{5}$$

It is easy to verify that $C(n) > 1$ for $n \geqslant 9$ and

$$C(n) \sim \sqrt{e} = 1.64872\ldots \quad \text{as} \quad n \to \infty. \tag{6}$$

Inequality (4) (without describing the cases when equality holds) has been recently established by Cherubini and Yatsyna [3, Theorem 1.5] by the method of Lagrange multipliers. We will derive it directly from an old result of Schur [16, Satz 2], which is stated as Lemma 7. Inequality (2) in a slightly different form has been obtained in [5] by the method of Lagrange multipliers too (see also [6]). Here we give a much more simple and straightforward proof. In [3], the sum $\sum_{1 \leqslant i<j \leqslant n} |x_i - x_j|^2$ is called the *potential energy* of the set $\{x_1, \ldots, x_n\}$.

We will also prove the following more general inequality about the mean of squares of distances between the points in two sets:

THEOREM 2. *For $n, m \geqslant 2$ and any $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathbb{C}$ we have*

$$\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i - y_j|^2 \geqslant \frac{(n-1)C(n)}{2n} \Big( \prod_{1 \leqslant i<j \leqslant n} |x_i - x_j| \Big)^{\frac{4}{n(n-1)}} + \frac{(m-1)C(m)}{2m} \Big( \prod_{1 \leqslant i<j \leqslant m} |y_i - y_j| \Big)^{\frac{4}{m(m-1)}}.$$

*Moreover, if $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathbb{R}$ then*

$$\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i - y_j|^2 \geqslant \frac{(n-1)C(n)}{n} \Big( \prod_{1 \leqslant i<j \leqslant n} |x_i - x_j| \Big)^{\frac{4}{n(n-1)}} + \frac{(m-1)C(m)}{m} \Big( \prod_{1 \leqslant i<j \leqslant m} |y_i - y_j| \Big)^{\frac{4}{m(m-1)}}.$$

In particular, (2) and (4) follow from the corresponding inequalities of Theorem 2 by choosing $n = m$ and $x_i = y_i$ for $i = 1, \ldots, n$.

Applying Theorem 1 to the set of conjugates of an algebraic integer and using the power mean inequality

$$\Big( \frac{1}{m} \sum_{k=1}^{m} a_k^p \Big)^{1/p} \geqslant \Big( \frac{1}{m} \sum_{k=1}^{m} a_k^q \Big)^{1/q},$$

where $a_1, \ldots, a_m \geqslant 0$ and $p \geqslant q > 0$ (see, e.g., [12]), in view of (6) we get the following:

COROLLARY 3. *For any real numbers $p \geqslant 2$ and $\varepsilon > 0$ there is the constant $n_1$ such that for every algebraic integer $\alpha$ of degree $n \geqslant n_1$ with conjugates $\alpha_1, \dots, \alpha_n$ we have*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^p > e^{p/4} - \varepsilon, \tag{7}$$

*and the constant $n_2$ such that for every totally real algebraic integer $\alpha$ of degree $n \geqslant n_2$ with conjugates $\alpha_1, \dots, \alpha_n$ we have*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^p > (4e)^{p/4} - \varepsilon. \tag{8}$$

Similarly, by Theorem 2 and (6), we derive the following:

COROLLARY 4. *For any real numbers $p \geqslant 2$ and $\varepsilon > 0$ there is the constant $n_3$ such that for any algebraic integers $\alpha$ of degree $n \geqslant n_3$ with conjugates $\alpha_1, \dots, \alpha_n$ and $\beta$ of degree $m \geqslant n_3$ with conjugates $\beta_1, \dots, \beta_m$ we have*

$$\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |\alpha_i - \beta_j|^p > e^{p/4} - \varepsilon, \tag{9}$$

*and the constant $n_4$ such that for any totally real algebraic integers $\alpha$ of degree $n \geqslant n_4$ with conjugates $\alpha_1, \dots, \alpha_n$ and $\beta$ of degree $m \geqslant n_4$ with conjugates $\beta_1, \dots, \beta_m$ we have*

$$\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |\alpha_i - \beta_j|^p > (4e)^{p/4} - \varepsilon. \tag{10}$$

Inequality (7) has been established in [5], while (8) in [16] (see also [9]). Inequalities (9) and (10) are new. For $p = 2$ the constants in the lower bounds (7), (9) and (8), (10) are $\sqrt{e} = 1.64872\dots$ and $2\sqrt{e} = 3.29744\dots$, respectively. The next statement shows that the latter constant $2\sqrt{e}$ (for $p = 2$ in (8), (10)) cannot be replaced by one greater than $3.79661$.

THEOREM 5. *There are infinitely many totally real algebraic integers $\alpha$ with conjugates $\alpha_1, \dots, \alpha_n$ for which*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^2 < 3.79661.$$

Theorem 5 is a consequence of a recent result of Smith [18], who showed that there are infinitely many totally positive algebraic integers whose trace divided by degree is less than $1.898303$. For some time, by the initial conjecture related to the so-called Schur-Siegel-Smyth trace problem, it was expected that for any $\varepsilon > 0$ there are only finitely many totally positive algebraic integers whose trace divided by degree is less than $2 - \varepsilon$. See, e.g., [1], [8], [13], [17], [19] for some lower bounds in the Schur-Siegel-Smyth trace problem. Currently, the best result in this direction is given in [20], where it was shown that, with finitely many exceptions, the trace of a totally positive algebraic integer divided by its degree is at least $1.793145$.

The optimal bounds in (7) and (8) are known only for $p = \infty$. Then,

$$\Big(\frac{2}{n(n-1)} \sum_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j|^p\Big)^{1/p} \sim \max_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j| \quad \text{as} \quad p \to \infty.$$

In [10], Langevin proved that for any $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that for each algebraic integer $\alpha$ of degree $n \geqslant n_0$ with conjugates $\alpha_1,\dots,\alpha_n$ we have

$$\max_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j| > 2 - \varepsilon. \tag{11}$$

Likewise, for a totally real algebraic integer $\alpha$ of sufficiently large degree the corresponding bound is

$$\max_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j| > 4 - \varepsilon$$

(see [16]). Of course, the latter inequality immediately implies that for any totally real algebraic integers $\alpha$ with conjugates $\alpha_1,\dots,\alpha_n$ and $\beta$ with conjugates $\beta_1,\dots,\beta_m$, where $n,m$ are large enough, we have

$$\max_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant m} |\alpha_i - \beta_j| > 4 - \varepsilon.$$

It would be of interest to find out if inequality (11) (which gives an optimal bound in an old problem of Favard [7]) can be generalized to the maximal difference between the conjugates of two arbitrary algebraic integers of sufficiently large degree. Namely, whether for any $\varepsilon > 0$ there is a constant $N(\varepsilon)$ such that for any algebraic integers $\alpha$ of degree $n \geqslant N(\varepsilon)$ with conjugates $\alpha_1,\dots,\alpha_n$ and $\beta$ of degree $m \geqslant N(\varepsilon)$ with conjugates $\beta_1,\dots,\beta_m$ the inequality

$$\max_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant m} |\alpha_i - \beta_j| > 2 - \varepsilon$$

is true or not.

In our final theorem we improve the inequalities (7) and (8) for $p = 4$.

THEOREM 6. *For any $\varepsilon > 0$ there is a constant $n_5$ such that for every algebraic integer $\alpha$ of degree $n \geqslant n_5$ with conjugates $\alpha_1,\dots,\alpha_n$ we have*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j|^4 > e + \sqrt{e} - \varepsilon. \tag{12}$$

*Similarly, for any $\varepsilon > 0$ there is a constant $n_6$ such that for every totally real algebraic integer $\alpha$ of degree $n \geqslant n_6$ with conjugates $\alpha_1,\dots,\alpha_n$ we have*

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i<j \leqslant n} |\alpha_i - \alpha_j|^4 > 8e - \varepsilon. \tag{13}$$

Note that the constants in (12) and (13) are $e+\sqrt{e} = 4.36700\dots$ (vs. $e = 2.71828\dots$ in (7) for $p = 4$) and $8e = 21.74625\dots$ (vs. $4e = 10.87312\dots$ in (8) for $p = 4$).

In the next section we will state and prove some auxiliary lemmas. In Section 3 we will prove Theorems 1, 2, 5 and 6.

## 2. Auxiliary lemmas

The next lemma is [16, Satz 2].

LEMMA 7. *For $n \geqslant 2$ and any real numbers $y_1,\dots,y_n$ satisfying*

$$y_1^2+\dots+y_n^2 \leqslant 1$$

*we have*

$$\prod_{1\leqslant i<j\leqslant n}(y_i-y_j)^2 \leqslant 2^2 3^3 \dots n^n (n^2-n)^{-\frac{n(n-1)}{2}} = \big((n-1)C(n)\big)^{-\frac{n(n-1)}{2}},$$

*with equality if and only if*

$$\{y_1\sqrt{n^2-n}, y_2\sqrt{n^2-n},\dots,y_n\sqrt{n^2-1}\} = \{\zeta_1,\zeta_2,\dots,\zeta_n\},$$

*where $\zeta_1,\zeta_2,\dots,\zeta_n$ is the set of roots of the Hermite polynomial* (5).

Note that the sum of squares of the roots of $H_n$ defined in (5) is $n^2-n$. This corresponds to the fact that the maximum is attained for $y_1,\dots,y_n$ satisfying $y_1^2+\dots+y_n^2=1$.

The next lemma has been proved by Remak in [15]. See also [11] for another proof and [4] for a more general inequality.

LEMMA 8. *For $n \geqslant 2$ and any complex numbers $z_1,\dots,z_n$ satisfying*

$$|z_1| \geqslant |z_2| \geqslant \dots \geqslant |z_n|$$

*we have*

$$\prod_{1\leqslant i<j\leqslant n}|z_i-z_j| \leqslant n^{n/2}|z_1|^{n-1}|z_2|^{n-2}\dots|z_{n-1}|.$$

We remark that in the case when the numbers $z_1,\dots,z_n$ are all real the inequality of Lemma 8 holds with the constant $2^{\lfloor n/2\rfloor}$ instead of $n^n$. This was conjectured by Pohst in [14] and recently proved by Battistoni and Molteni [2].

Applying (1) with appropriate weights we will derive the following inequality:

LEMMA 9. *For $n \geqslant 2$, any positive number $t$ and any complex numbers $z_1,\dots,z_n$ we have*

$$\sum_{k=1}^{n}|z_k|^t \geqslant \frac{(n-1)C(n)}{2n^{\frac{t-2}{n-1}}}\Big(\prod_{1\leqslant i<j\leqslant n}|z_i-z_j|\Big)^{\frac{2t}{n(n-1)}}, \tag{14}$$

*where $C(n)$ is defined in* (3).

*Proof.* We will apply (1) with weights $w_k = \frac{2(n-k)}{n(n-1)}$ for $k = 1, \dots, n$. Note that the weights sum to 1 and $w_n = 0$, so (1) with $m = n-1$ and $a_k^t / w_k$ in place of $a_k$ for $k = 1, \dots, n-1$ yields

$$\prod_{k=1}^{n-1} a_k^{\frac{2(n-k)t}{n(n-1)}} \leqslant \prod_{k=1}^{n-1} \left( \frac{2(n-k)}{n(n-1)} \right)^{\frac{2(n-k)}{n(n-1)}} \sum_{k=1}^{n-1} a_k^t.$$

In view of (3) we have

$$\prod_{k=1}^{n-1} \left( \frac{2(n-k)}{n(n-1)} \right)^{\frac{2(n-k)}{n(n-1)}} = \frac{2}{n(n-1)} \cdot \frac{n}{C(n)} \cdot n^{-\frac{2}{n-1}} = \frac{2n^{-\frac{2}{n-1}}}{(n-1)C(n)}.$$

Therefore, for any $a_n \geqslant 0$

$$\left( \prod_{k=1}^{n-1} a_k^{t(n-k)} \right)^{\frac{2}{n(n-1)}} \leqslant \frac{2n^{-\frac{2}{n-1}}}{(n-1)C(n)} \sum_{k=1}^{n-1} a_k^t \leqslant \frac{2n^{-\frac{2}{n-1}}}{(n-1)C(n)} \sum_{k=1}^{n} a_k^t.$$

Combining this inequality with Lemma 8, where $a_k = |z_k|$, $k = 1, \dots, n$, are labeled so that $|z_1| \geqslant \dots \geqslant |z_{n-1}| \geqslant |z_n|$, we deduce

$$\left( \prod_{1 \leqslant i < j \leqslant n} |z_i - z_j| \right)^{\frac{2t}{n(n-1)}} \leqslant \frac{2n^{\frac{t-2}{n-1}}}{(n-1)C(n)} \sum_{k=1}^{n} |z_k|^t,$$

which implies (14). $\square$

Finally, we will record two simple identities.

LEMMA 10. *For $n \geqslant 2$ and any $x_1, \dots, x_n \in \mathbb{C}$ we have*

$$\sum_{1 \leqslant i < j \leqslant n} |x_i - x_j|^2 = n \sum_{k=1}^{n} |x_k|^2 - \Big| \sum_{k=1}^{n} x_k \Big|^2 \tag{15}$$

*and*

$$\sum_{1 \leqslant i < j \leqslant n} |x_i - x_j|^4 = nS_{4,0} + |S_{0,2}|^2 + 2S_{2,0}^2 - 2(S_{2,1}\overline{S_{0,1}} + \overline{S_{2,1}}S_{0,1}), \tag{16}$$

*where*

$$S_{a,b} := \sum_{k=1}^{n} |x_k|^a x_k^b \tag{17}$$

*for $a, b \in \mathbb{N} \cup \{0\}$.*

*Proof.* From

$$2 \sum_{1 \leqslant i < j \leqslant n} |x_i - x_j|^2 = \sum_{i,j=1}^{n} |x_i - x_j|^2 = \sum_{i,j=1}^{n} (|x_i|^2 + |x_j|^2 - x_i\overline{x_j} - \overline{x_i}x_j)$$
$$= 2n \sum_{k=1}^{n} |x_k|^2 - 2 \Big| \sum_{k=1}^{n} x_k \Big|^2$$

we get (15).

To prove (16) observe that $|x_i - x_j|^4 = (|x_i|^2 + |x_j|^2 - x_i\overline{x_j} - \overline{x_i}x_j)^2$ equals

$$|x_i|^4 + |x_j|^4 + x_i^2\overline{x_j}^2 + \overline{x_i}^2 x_j^2 + 4|x_i|^2|x_j|^2 - 2(|x_i|^2 + |x_j|^2)(x_i\overline{x_j} + \overline{x_i}x_j).$$

So, using (17), we find that

$$2\sum_{1\leqslant i<j\leqslant n} |x_i - x_j|^4 = \sum_{i,j=1}^{n} |x_i - x_j|^4$$

equals

$$2nS_{4,0} + 2|S_{0,2}|^2 + 4S_{2,0}^2 - 4(S_{2,1}\overline{S_{0,1}} + \overline{S_{2,1}}S_{0,1}).$$

This implies (16). □

## 3. Proofs of the theorems

*Proof of Theorem* 1. Note that in inequality (2) we can replace each $x_i$ by $x_i^* = x_i + s$ with arbitrary $s \in \mathbb{C}$. In particular we can choose $s$ so that the sum of all $x_i^*$ is zero. Then, $\sum_{1\leqslant i<j\leqslant n} |x_i^* - x_j^*|^2 = n\sum_{k=1}^{n} |x_k^*|^2$ by (15). So, by Lemma 9 with $t = 2$, we derive (2).

To prove (4) we can replace each $x_i$ by $ax_i + b$ with arbitrary $a, b \in \mathbb{R}$. The inequality clearly holds and becomes equality if $a = 0$. If not all $x_i$ are equal then we replace each $x_i$ by $x_i + b$, where $b$ is chosen so that sum over all $x_i$ is zero. Then, we can replace each new $x_i$ by $x_i^* = ax_i$ so that the sum of squares of all $x_i^*$ is 1. Then, as for $x_i^*$ we have $\sum_{i=1}^{n} x_i^* = 0$ and $\sum_{i=1}^{n} x_i^{*2} = 1$, by Lemma 7, we find that

$$\Big(\prod_{1\leqslant i<j\leqslant n} |x_i^* - x_j^*|\Big)^{\frac{4}{n(n-1)}} \leqslant \frac{1}{(n-1)C(n)} = \frac{1}{(n-1)C(n)} \sum_{k=1}^{n} x_k^{*2}.$$

This implies (4) in view of $\sum_{1\leqslant i<j\leqslant n}(x_i^* - x_j^*)^2 = n\sum_{k=1}^{n} x_k^{*2}$, which holds by (15) and $\sum_{k=1}^{n} x_k^* = 0$.

Furthermore, by Lemma 7, under assumptions $\sum_{k=1}^{n} x_k^* = 0$ and $\sum_{k=1}^{n} x_k^{*2} = 1$, equality in (4) holds if and only if

$$\{x_1^*, \dots, x_n^*\} = \{\zeta_1/\sqrt{n^2-n}, \dots, \zeta_n/\sqrt{n^2-n}\},$$

where $\zeta_1, \dots, \zeta_n$ is the set of roots of $H_n$ defined in (5). This completes the proof, because the original $x_i$ are linear transformations of those $x_i^*$. □

For any set of complex numbers $X = \{x_1, \dots, x_n\}$ we define

$$\Delta(X) := \prod_{1\leqslant i<j\leqslant n} |x_i - x_j|. \tag{18}$$

*Proof of Theorem* 2. Set $s = -\frac{1}{n}\sum_{i=1}^{n} x_i$ and replace each $x_i$ by $x_i^* = x_i + s$, and also each $y_j$ by $y_j^* = y_j + s$. Then, for $X = \{x_1, \dots, x_n\}$ and $X^* = \{x_1^*, \dots, x_n^*\}$ we have $\Delta(X) = \Delta(X^*)$ by (18). Likewise, $\Delta(Y) = \Delta(Y^*)$, where $Y = \{y_1, \dots, y_m\}$ and $Y^* = \{y_1^*, \dots, y_m^*\}$. Using

$$x_1^* + \dots + x_n^* = 0 = \overline{x_1^*} + \dots + \overline{x_n^*},$$

we find that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i - y_j|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{m}|x_i^* - y_j^*|^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(|x_i^*|^2 + |y_j^*|^2 - x_i^*\overline{y_j^*} - \overline{x_i^*}y_j^*)\\ &= m\sum_{i=1}^{n}|x_i^*|^2 + n\sum_{j=1}^{m}|y_j^*|^2.\end{aligned}$$

Applying Lemma 9 with $t = 2$ and $z_k = x_k^*$ for $k = 1, \dots, n$ we obtain

$$\sum_{k=1}^{n}|x_k^*|^2 \geqslant \frac{(n-1)C(n)}{2}\Delta(X^*)^{\frac{4}{n(n-1)}} = \frac{(n-1)C(n)}{2}\Delta(X)^{\frac{4}{n(n-1)}}.$$

Likewise,

$$\sum_{k=1}^{m}|y_k^*|^2 \geqslant \frac{(m-1)C(m)}{2}\Delta(Y)^{\frac{4}{m(m-1)}}.$$

Adding these two inequalities with weights $1/n$ and $1/m$, respectively, we derive the first inequality of Theorem 2.

The proof of the second inequality is similar to that above. This time, for real $x_i^*$, $i = 1, \dots, n$, and $y_j^*$, $j = 1, \dots, m$, by (4) and (15), we have

$$\begin{aligned}\sum_{k=1}^{n}x_k^{*2} &\geqslant \frac{1}{n}\Big(\sum_{k=1}^{n}x_k^*\Big)^2 + (n-1)C(n)\Delta(X^*)^{\frac{4}{n(n-1)}}\\ &\geqslant (n-1)C(n)\Delta(X^*)^{\frac{4}{n(n-1)}} = (n-1)C(n)\Delta(X)^{\frac{4}{n(n-1)}}.\end{aligned}$$

Likewise,

$$\sum_{k=1}^{m}y_k^{*2} \geqslant (m-1)C(m)\Delta(Y)^{\frac{4}{m(m-1)}}.$$

Adding these inequalities with weights $1/n$ and $1/m$, by the above established identity

$$\sum_{i=1}^{n}\sum_{j=1}^{m}|x_i - y_j|^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i^* - y_j^*)^2 = m\sum_{i=1}^{n}x_i^{*2} + n\sum_{j=1}^{m}y_j^{*2},$$

we arrive to the second inequality of Theorem 2. □

*Proof of Theorem* 5. By [18, Corollary 1.1], there are infinitely many totally positive algebraic integers $\alpha$ for which the sum of their conjugates is less than $\sigma n$, where $n = \deg \alpha$ and $\sigma = 1.898303$. Take any of those $\alpha$ with $n = \deg \alpha$ large enough. Consider the totally real algebraic integer $\beta > 0$ defined by $\beta = \sqrt{\alpha}$. Assume that $\beta$ is of degree $m$ over $\mathbb{Q}$. It is clear that $m = n$ or $m = 2n$. In the first case the squares of $n$ conjugates of $\beta$ are $\alpha_1, \ldots, \alpha_n$, while in the second case the squares of $2n$ conjugates of $\beta$ are $\alpha_1, \alpha_1, \ldots, \alpha_n, \alpha_n$. In both cases the sum of squares of the conjugates of $\beta$ is $m(\alpha_1 + \ldots + \alpha_n)/n$. Thus, by (15), we obtain

$$\sum_{1 \leqslant i < j \leqslant m} |\beta_i - \beta_j|^2 \leqslant m(\beta_1^2 + \ldots + \beta_m^2) = \frac{m^2}{n}(\alpha_1 + \ldots + \alpha_n) < m^2\sigma.$$

Consequently,

$$\frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} |\beta_i - \beta_j|^2 < \frac{2\sigma m}{m-1},$$

which implies the result, since $m \in \{n, 2n\}$ is large enough. $\square$

*Proof of Theorem* 6. Let $\alpha$ be an algebraic integer of degree $n \geqslant 2$ with conjugates $\alpha_1, \ldots, \alpha_n$. Set

$$E(\alpha) = \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^4, \tag{19}$$

and consider the algebraic number $\gamma = \alpha + s$, where $s = -\frac{1}{n}\sum_{j=1}^n \alpha_j$ is in $\mathbb{Q}$. Then, $\gamma_k = \alpha_k + s$ for $k = 1, \ldots, n$ and

$$\gamma_1 + \ldots + \gamma_n = 0. \tag{20}$$

Moreover, by (18) and (19), $E(\alpha) = E(\gamma)$ and $\Delta(\alpha) = \Delta(\gamma)$, where $\alpha$ is understood as the set of its conjugates over $\mathbb{Q}$.

Clearly, $\Delta(\alpha) \geqslant 1$, since $\Delta(\alpha)^2$ is the modulus of the discriminant of $\alpha$. Thus, Lemma 9 with $t = 2$ and $t = 4$ implies that

$$\sum_{k=1}^n |\gamma_k|^2 \geqslant \frac{(n-1)C(n)}{2} \quad \text{and} \quad \sum_{k=1}^n |\gamma_k|^4 \geqslant \frac{(n-1)C(n)}{2n^{\frac{2}{n-1}}}.$$

Now, we will apply (16) to $x_k = \gamma_k$, $k = 1, \ldots, n$. Then, by (20), we have $S_{0,1} = 0$. Also, $|S_{0,2}|^2 \geqslant 0$, so from (16) and the above inequalities it follows that

$$\begin{aligned}
\frac{n(n-1)}{2} E(\gamma) = \sum_{1 \leqslant i < j \leqslant n} |\gamma_i - \gamma_j|^4 &\geqslant n \sum_{k=1}^n |\gamma_k|^4 + 2\Big(\sum_{k=1}^n |\gamma_k|^2\Big)^2 \\
&\geqslant \frac{n(n-1)C(n)}{2n^{\frac{2}{n-1}}} + \frac{(n-1)^2 C(n)^2}{2} \\
&\geqslant \frac{(n-1)^2 (C(n) + C(n)^2)}{2n^{\frac{2}{n-1}}}.
\end{aligned}$$

Hence, $E(\gamma) \geqslant n^{-\frac{2}{n-1}}(1-1/n)(C(n)+C(n)^2)$, which implies (12) by (6), (18) and $E(\alpha)=E(\gamma)$.

Now, let $\alpha$ be a totally real algebraic integer of degree $n \geqslant 2$ with conjugates $\alpha_1,\dots,\alpha_n$. As above, we set $\gamma=\alpha+s$, where $s=-\frac{1}{n}\sum_{j=1}^{n}\alpha_j$. Then, $\gamma_k=\alpha_k+s$ for $k=1,\dots,n$ satisfy (20). So, by (15), we have

$$\sum_{1\leqslant i<j\leqslant n}|\gamma_i-\gamma_j|^2 = n\sum_{k=1}^{n}|\gamma_k|^2 = n\sum_{k=1}^{n}\gamma_k^2,$$

and hence (4) combined with $\Delta(\gamma)=\Delta(\alpha)\geqslant 1$ yields

$$\sum_{k=1}^{n}\gamma_k^2 \geqslant (n-1)C(n).$$

Now, we can apply (16) to $x_k=\gamma_k$, $k=1,\dots,n$, again. This time, we have not only $S_{0,1}=0$ by (20), but also $S_{0,2}=S_{2,0}$ by (17). Consequently, using (16), (17), (19) and the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}\frac{n(n-1)}{2}E(\alpha) = \frac{n(n-1)}{2}E(\gamma) &= n\sum_{k=1}^{n}\gamma_k^4 + 3\Big(\sum_{k=1}^{n}\gamma_k^2\Big)^2\\ &\geqslant 4\Big(\sum_{k=1}^{n}\gamma_k^2\Big)^2 \geqslant 4(n-1)^2C(n)^2.\end{aligned}$$

This finishes the proof of (13) by (6). $\square$

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(Received April 21, 2022)

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