mathematics

Article

Truncated Moments for Heavy-Tailed and Related Distribution Classes

Saulius Paukštys [1], Jonas Šiaulys [1,*] and Remigijus Leipus [2]

1 Institute of Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
2 Institute of Applied Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
* Correspondence: jonas.siaulys@mif.vu.lt

Abstract: Suppose that ξ^+ is the positive part of a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the distribution function F_ξ. When the moment $\mathbb{E}(\xi^+)^p$ of order $p > 0$ is finite, then the truncated moment $\overline{F}_{\xi,p}(x) = \min\left\{1, \mathbb{E}(\xi^p \mathbb{1}_{\{\xi>x\}})\right\}$, defined for all $x \geqslant 0$, is the survival function or, in other words, the distribution tail of the distribution function $F_{\xi,p}$. In this paper, we examine which regularity properties transfer from the distribution function F_ξ to the distribution function $F_{\xi,p}$ and which properties transfer from the function $F_{\xi,p}$ to the function F_ξ. The construction of the distribution function $F_{\xi,p}$ describes the truncated moment transformation of the initial distribution function F_ξ. Our results show that the subclasses of heavy-tailed distributions, such as regularly varying, dominatedly varying, consistently varying and long-tailed distribution classes, are closed under this truncated moment transformation. We also show that exponential-like-tailed and generalized long-tailed distribution classes, which contain both heavy- and light-tailed distributions, are also closed under the truncated moment transformation. On the other hand, we demonstrate that regularly varying and exponential-like-tailed distribution classes also admit inverse transformation closures, i.e., from the condition that $F_{\xi,p}$ belongs to one of these classes, it follows that F_ξ also belongs to the corresponding class. In general, the obtained results complement the known closure properties of distribution regularity classes.

Keywords: truncated moment; alternative moment formula; heavy-tailed distribution; dominated variation; consistent variation; regular variation; long-tailed distribution; exponential-like-tailed distribution

MSC: 60G50; 60J80; 91G05

check for updates

Citation: Paukštys, S.; Šiaulys, J.; Leipus, R. Truncated Moments for Heavy-Tailed and Related Distribution Classes. *Mathematics* **2023**, *11*, 2172. https://doi.org/10.3390/math11092172

Academic Editor: Frederico Caeiro

Received: 27 March 2023
Revised: 1 May 2023
Accepted: 3 May 2023
Published: 5 May 2023

1. Introduction

Let ξ be a real-valued random variable (r.v.) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the distribution function (d.f.) F_ξ and the distribution tail (d.t.) $\overline{F}_\xi = 1 - F_\xi$. Suppose that the moment

$$\mathbb{E}(\xi^+)^p = \int_{[0,\infty)} x^p \mathrm{d}F_\xi(x)$$

is finite for positive p values, where ξ^+ denotes the positive part of the r.v. ξ, i.e., $\xi^+ = \max\{\xi, 0\}$. In such cases, the function

$$F_{\xi,p}(x) = \begin{cases} 0 \text{ if } x < 0, \\ \max\left\{0, 1 - \mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi>x\}}\right)\right\} \text{ if } x \geqslant 0, \end{cases}$$

is a new d.f. with the d.t.

$$\overline{F}_{\xi,p}(x) = \min\left\{1, \mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi>x\}}\right)\right\},\ x \geqslant 0.$$

In this paper, we discuss which regularity properties transfer from the distribution F_ξ to the distribution $F_{\xi,p}$ and which properties transfer from the function $F_{\xi,p}$ to the function F_ξ. The construction of the distribution function $F_{\xi,p}$ describes the truncated moment transformation of the initial distribution function F_ξ. It follows from our Theorem 1 that the distribution classes $\mathcal{R}, \mathcal{C}, \mathcal{D}, \mathcal{L}, \mathcal{L}_\gamma (\gamma > 0)$ and $\mathcal{OL}$, whose definitions are presented in Section 2, are closed under this truncated moment transformation. On the other hand, in Theorem 2, we show that the class of regularly varying distributions $\mathcal{R}$ and the class of exponential-like-tailed distributions $\mathcal{L}_\gamma$ are characterized by inverse closures, i.e., from the condition that $F_{\xi,p} \in \mathcal{R}$, it follows that $F_\xi \in \mathcal{R}$ and from the condition $F_{\xi,p} \in \mathcal{L}_\gamma$, it follows that $F_\xi \in \mathcal{L}_\gamma$. The obtained results complement the known closure properties, some of which are discussed in Section 2, where the regularity classes are introduced. The main results are formulated in Section 3. Some auxiliary lemmas are given in Section 4. The proofs of main results are presented in Section 5. The paper ends with our conclusions in Section 6.

2. Regularity Classes

In this paper, we limit ourselves to the following heavy-tailed and related classes: $\mathcal{R}_\alpha$ with $\alpha \geqslant 0$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{L}$, $\mathcal{OL}$ and $\mathcal{L}_\gamma$ with $\gamma > 0$. In this section, we briefly describe these regularity classes. Throughout the paper, we say that a distribution or d.f. F is on $\mathbb{R}^+$ when $F(-0) := \lim_{x \to -0} F(x) = 0$ and we say that a distribution F is on $\mathbb{R}$ when the condition $F(-0) = 0$ is not satisfied. Obviously, a d.f. concentrated on $\mathbb{R}$ describes a real-valued r.v. and a d.f. on $\mathbb{R}^+$ describes a non-negative r.v.

• *A d.f. F on $\mathbb{R}$ is said to be regularly varying with index $\alpha \geqslant 0$, denoted as $F \in \mathcal{R}_\alpha$ if, for any $y > 0$, it holds that*

$$\lim_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

We denote the set of all regularly varying distributions as $\mathcal{R}$:

$$\mathcal{R} := \bigcup_{\alpha \geqslant 0} \mathcal{R}_\alpha.$$

The Pareto, Burr, loggamma, Cauchy, t- and stable distributions, with any exponent not exceeding two, belong to the class $\mathcal{R}$. Information on the properties of regularly varying functions, together with some historical notes and various applications, can be found, for instance, in [1–12]. In [1], Karamata studied the general regularity property for functions on $\mathbb{R}^+$, obtained the main representative expression of regularly varying functions and determined the asymptotic behavior of the integrals of regular functions. All of the main properties of general regularly varying functions and regularly varying distributions were presented in [2–4,6]. Various applications of regularly varying distributions in risk models were provided in [7–12]. Regular distributions admit a number of closure properties. For instance, the results of Foss et al. [5] (see also the proposition on page 278 in [13] or Lemma 1.3.1. in [14]) implied that the class $\mathcal{R}$ of regularly varying d.f.s is closed under strong tail equivalence and convolution. The closure under strong tail equivalence means that

$$F_1 \in \mathcal{R}_\alpha, \alpha \geqslant 0, \text{ and } \overline{F}_2(x) \underset{x \to \infty}{\sim} c\overline{F}_1(x),\ c > 0, \Rightarrow F_2 \in \mathcal{R}_\alpha,$$

whereas closure under convolution means that

$$F_1, F_2 \in \mathcal{R}_\alpha \Rightarrow F_1 * F_2 \in \mathcal{R}_\alpha,$$

where $F_1 * F_2$ denotes the standard convolution of d.f.s:

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y),\ x \in \mathbb{R}.$$

• *A d.f. F on $\mathbb{R}$ is said to be consistently varying, denoted by $F \in \mathcal{C}$ if*

$$\lim_{y \uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

The class of consistently varying d.f.s was introduced in [15,16] as a generalization of regularly varying d.f.s. By definition, $\mathcal{R} \subset \mathcal{C}$. Examples showing that $\mathcal{C}$ is strictly larger than $\mathcal{R}$ were provided in [16,17].

The class of consistently varying d.f.s also admits many closure properties, such as closures under strong tail equivalence, convolution and minimum. Closure under strong tail equivalence follows immediately from the definition. Closure under convolution is proved in Theorem 2.2 in [17] (see also Lemma 3 in [18]) and closure under minimum follows from Theorem 2.47 in [5]. Closure under minimum means that for two independent r.v.s ξ_1 and ξ_2 with d.f.s from the class $\mathcal{C}$, it holds that $F_{\xi_1 \wedge \xi_2} \in \mathcal{C}$.

• *A d.f. F on $\mathbb{R}$ is said to be dominatedly varying, denoted by $F \in \mathcal{D}$ if, for any (or some)* $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Dominatedly varying distributions were introduced in [19] as a generalization of regularly varying distributions. The various properties of d.f.s from the class $\mathcal{D}$, including closure properties, were established in [3,17,20–27]. We recall only the relationship between a d.f. from the class $\mathcal{D}$ and two special indices. For a d.f. F on $\mathbb{R}$, we denote the upper Matuszewska index (see [28]) as J_F^+ and the L-index (see [29]) as L_F, as follows:

$$J_F^+ := \lim_{y\to\infty} \frac{1}{\log y} \log\left\{ \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right\},\ \ L_F := \lim_{y \searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

It follows from the above definitions that

$$F \in \mathcal{D} \ \Leftrightarrow\ J_F^+ < \infty \ \Leftrightarrow\ L_F > 0.$$

• *A d.f. F on $\mathbb{R}$ is said to be long-tailed, denoted by $F \in \mathcal{L}$ if, for any (or some) $y > 0$,*

$$\overline{F}(x-y) \underset{x\to\infty}{\sim} \overline{F}(x).$$

The class of long-tailed distributions was introduced in [30] within the context of branching processes and became one of the most important subclasses of heavy-tailed distributions. Recall that a d.f. F on $\mathbb{R}$ is said to be *heavy-tailed* when $\int_{-\infty}^{\infty} \mathrm{e}^{\delta x}\mathrm{d}F(x) = \infty$ for any $\delta > 0$. The class $\mathcal{L}$, either alone or in intersection with the class $\mathcal{D}$, was considered in [17,31–36]. A detailed analysis of distributions from the class $\mathcal{L}$ was presented in Chapter 2 in [5]. For instance, in Lemma 2.23 in [5], closure under mixture and closure under maximum were established. Closure under mixture means that when the d.f. $F_1 \in \mathcal{L}$ and either $F_2 \in \mathcal{L}$ or $\overline{F}_2(x) = o(\overline{F}_1(x))$, then $pF_1 + (1-p)F_2 \in \mathcal{L}$ for any $p \in (0,1)$. Closure under maximum means that for two independent r.v.s ξ_1 and ξ_2 with d.f.s from the class $\mathcal{L}$, it holds that $F_{\xi_1 \vee \xi_2} \in \mathcal{L}$. Closure under minimum is similarly defined.

• *A d.f. F on $\mathbb{R}$ is said to be exponential-like-tailed, denoted by $F \in \mathcal{L}_\gamma$ with $\gamma > 0$ if, for any $y > 0$,*

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \mathrm{e}^{-\gamma y}.$$

The class of exponential-like-tailed distributions is a natural generalization of long-tailed distributions. It was introduced in [37,38] and later also investigated in [23,31,33,39–45]. We note only that unlike the class $\mathcal{L}$, the class $\mathcal{L}_\gamma$ with $\gamma > 0$ is a subclass of light-tailed d.f.s. Recall that a d.f. F is *light-tailed* when $\int_{-\infty}^{\infty} \mathrm{e}^{\delta x}\mathrm{d}F(x) < \infty$ for some $\delta > 0$.

An O-generalization of the classes $\mathcal{L}$ and $\mathcal{L}_\gamma$ with $\gamma > 0$ was proposed in [34]. The new class $\mathcal{OL}$ includes both heavy- and light-tailed d.f.s that satisfy the requirement below.

• *A d.f. F on $\mathbb{R}$ is said to belong to the class of generalized long-tailed d.f.s $\mathcal{OL}$ if, for any (or some) $y > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

The full range of the properties of the class $\mathcal{OL}$ was presented in [46–50]. For instance, in Theorem 1 in [50], closure under product convolution was established for d.f.s on $\mathbb{R}^+$. This means that for two non-negative independent r.v.s ξ_1 and ξ_2 with d.f.s from the class $\mathcal{OL}$, it holds that $F_{\xi_1} \otimes F_{\xi_2} := F_{\xi_1\xi_2} \in \mathcal{OL}$.

At the end of this section, we note that

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{D}, \ \ \mathcal{L} \cup \Big\{ \bigcup_{\gamma>0} \mathcal{L}_\gamma \Big\} \subset \mathcal{OL}.$$

All of these relationships follow from the definitions above. Moreover, all of these inclusions are strict.

3. Main Results

In this section, we formulate the main results of the paper. The results are presented in two theorems. In the first theorem, we present the relationships between the inclusions of F_ξ and $F_{\xi,p}$ in the classes under consideration.

Theorem 1. *Let ξ be a real-valued r.v. with the d.f. F_ξ and the finite moment $\mathbb{E}(\xi^+)^p$ for some $p > 0$. Then, the following relationships hold:*

(i) $F_\xi \in \mathcal{R}_\alpha,\ p < \alpha,\ \Rightarrow\ F_{\xi,p} \in \mathcal{R}_{\alpha-p}$;
(ii) $F_\xi \in \mathcal{C}\ \Rightarrow\ F_{\xi,p} \in \mathcal{C}$;
(iii) $F_\xi \in \mathcal{D}\ \Rightarrow\ F_{\xi,p} \in \mathcal{D}$;
(iv) $F_\xi \in \mathcal{L}\ \Rightarrow\ F_{\xi,p} \in \mathcal{L}$;
(v) $F_\xi \in \mathcal{OL}\ \Rightarrow\ F_{\xi,p} \in \mathcal{OL}$;
(vi) $F_\xi \in \mathcal{L}_\gamma,\ \gamma > 0,\ \Rightarrow\ F_{\xi,p} \in \mathcal{L}_\gamma$.

In the second theorem, we present the inverse relationships to those in Theorem 1. In the second theorem, we describe for which classes $\mathcal{K}$ the inclusion of $F_{\xi,p} \in \mathcal{K}$ implies $F_\xi \in \mathcal{K}$ or $F_\xi \notin \mathcal{K}$.

Theorem 2. *Let ξ be a real-valued r.v. with the d.f. F_ξ and let the moment $\mathbb{E}(\xi^+)^p$ be finite for some fixed $p > 0$. Then, in general, the following relationships hold:*

(i) $F_{\xi,p} \in \mathcal{R}_{\alpha-p},\ p < \alpha,\ \Rightarrow\ F_\xi \in \mathcal{R}_\alpha$;
(ii) $F_{\xi,p} \in \mathcal{C}\ \not\Rightarrow\ F_\xi \in \mathcal{C}$;
(iii) $F_{\xi,p} \in \mathcal{D}\ \not\Rightarrow\ F_\xi \in \mathcal{D}$;

(iv) $F_{\xi,p} \in \mathcal{L} \ \not\Rightarrow\ F_\xi \in \mathcal{L}$;
(v) $F_{\xi,p} \in \mathcal{OL} \ \not\Rightarrow\ F_\xi \in \mathcal{OL}$;
(vi) $F_{\xi,p} \in \mathcal{L}_\gamma,\ \gamma > 0,\ \Rightarrow\ F_\xi \in \mathcal{L}_\gamma$.

Here, the assertion "$F_{\xi,p} \in \mathcal{K} \ \not\Rightarrow\ F_\xi \in \mathcal{K}$" for some classes $\mathcal{K}$ means that $p > 0$ and the random variable ξ exist , for which $F_{\xi,p} \in \mathcal{K}$ but $F_\xi \notin \mathcal{K}$.

4. Auxiliary Lemmas

In this section, we present a collection of auxiliary results that we use in the proofs of the main results.

Lemma 1. *Let ξ be a real-valued r.v., such that $\mathbb{E}(\xi^+)^p < \infty$ for some $p > 0$. Then, for any $x \geqslant 0$, we have*

$$\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x\}}\big) = x^p \mathbb{P}(\xi > x) + p\int_x^\infty u^{p-1}\mathbb{P}(\xi > u)\,\mathrm{d}u.$$

Proof. The equality of the lemma follows directly from the following well-known formula:

$$\mathbb{E}\eta^p = p\int_0^\infty u^{p-1}\mathbb{P}(\eta > u)\,\mathrm{d}u$$

provided that $p > 0$ and η is a non-negative r.v. (see, for instance, Corollary 2 on page 208 in [51]). Details of the derivation can be found in [52]. For more information on alternative expectation formulae, see [53–55]. □

Lemma 2. *Let ξ be a real-valued r.v. with the d.f. F_ξ, such that $\mathbb{E}(\xi^+)^p < \infty$ for some $p > 0$, and let $\overline{F}_\xi(x) > 0$ for all $x \in \mathbb{R}$. Then, the inequalities*

$$\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)} \leqslant \max\left\{\left(1-\frac{y}{x}\right)^p \frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)},\ \sup_{z\geqslant x}\left(1-\frac{y}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}\right\}, \tag{1}$$

$$\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)} \geqslant \min\left\{\left(1-\frac{y}{x}\right)^p \frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)},\ \inf_{z\geqslant x}\left(1-\frac{y}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}\right\} \tag{2}$$

hold when $x > 0$, $0 < y < x$ and $\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x-y\}}\big) < 1$. In addition,

$$y^p \inf_{z\geqslant x}\frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} \leqslant \frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} \leqslant y^p \sup_{z\geqslant x}\frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} \tag{3}$$

when $x > 0$, $y > 0$, $\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x\}}\big) < 1$ and $\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > xy\}}\big) < 1$.

Proof. Let us begin with inequality (1). The conditions of the lemma imply that

$$\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)} = \frac{\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x-y\}}\big)}{\mathbb{E}\big(\xi^p \mathbb{1}_{\{\xi > x\}}\big)}.$$

Therefore, using Lemma 1, we obtain

$$\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)} = \frac{(x-y)^p\overline{F}_\xi(x-y) + p\int\limits_x^\infty u^{p-1}\left(1-\frac{y}{u}\right)^{p-1}\overline{F}_\xi(u-y)\,\mathrm{d}u}{x^p\overline{F}_\xi(x) + p\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}$$
$$\leqslant \frac{(x-y)^p\overline{F}_\xi(x-y) + p\sup\limits_{z\geqslant x}\left\{\left(1-\frac{y}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}\right\}\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}{x^p\overline{F}_\xi(x) + p\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}.$$

The upper estimate (1) follows from the min–max inequality

$$\min\left\{\frac{a_1}{b_1},\frac{a_2}{b_2}\right\} \leqslant \frac{a_1+a_2}{b_1+b_2} \leqslant \max\left\{\frac{a_1}{b_1},\frac{a_2}{b_2}\right\}, \tag{4}$$

where a_1, a_2 are non-negative and b_1, b_2 are positive. The lower inequality (2) can be derived in a similar way.

Now, let us consider the upper bound in (3). Condition $\mathbb{E}(\xi^p\mathbb{1}_{\{\xi>xy\}}) < 1$ implies that

$$\frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} = \frac{\mathbb{E}(\xi^p\mathbb{1}_{\{\xi>xy\}})}{\mathbb{E}(\xi^p\mathbb{1}_{\{\xi>x\}})}.$$

Hence, using Lemma 1 again, we obtain

$$\frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} = \frac{(xy)^p\overline{F}_\xi(xy) + p\int\limits_{xy}^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}{x^p\overline{F}_\xi(x) + p\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}$$
$$= y^p\,\frac{x^p\overline{F}_\xi(xy) + p\int\limits_x^\infty u^{p-1}\frac{\overline{F}_\xi(uy)}{\overline{F}_\xi(u)}\overline{F}_\xi(u)\,\mathrm{d}u}{x^p\overline{F}_\xi(x) + p\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}$$
$$\leqslant y^p\,\frac{x^p\overline{F}_\xi(xy) + p\sup\limits_{z\geqslant x}\frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)}\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}{x^p\overline{F}_\xi(x) + p\int\limits_x^\infty u^{p-1}\overline{F}_\xi(u)\,\mathrm{d}u}.$$

We obtain the upper bound in (3) by using inequality (4). The lower bound in (3) can be derived in the same manner. Therefore, the lemma is proved. □

Lemma 3. *Let the d.f. H belong to the class $\mathcal{R}_\alpha$ for some $\alpha > 0$. Then, for an arbitrary $\sigma > -\alpha$, it holds that*

$$\lim_{x\to\infty}\frac{\int_{(x,\infty)}u^{-\sigma}\mathrm{d}\overline{H}(u)}{x^{-\sigma}\overline{H}(x)} = -\frac{\alpha}{\sigma+\alpha}. \tag{5}$$

Conversely, when H is a d.f. and the equation (5) holds for some α and σ, such that $\alpha > 0, \sigma+\alpha > 0$, then $H \in \mathcal{R}_\alpha$.

Proof. The formulated lemma is a version of the well-known Karamata's theorem [1,56]. The presented assertion follows directly from Theorem 1.6.5 in [3]. Further generalizations of Karamata's statements can be found in [57–60], among others. □

5. Proofs of the Main Results

In this section, we present the proofs of the main theorems. Both of the presented proofs consist of several parts.

Proof of Theorem 1. Part (i). As $F_\xi \in R_\alpha$, for an arbitrary $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = y^{-\alpha}.$$

Therefore, using inequality (3) from Lemma 2, we obtain

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} \leqslant y^p \limsup_{x\to\infty} \sup_{z\geqslant x} \frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} = y^p \limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = y^{p-\alpha}, \tag{6}$$

and similarly,

$$\liminf_{x\to\infty} \frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} \geqslant y^p \liminf_{x\to\infty} \inf_{z\geqslant x} \frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} = y^p \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = y^{p-\alpha}.$$

The last two derived inequalities show that $F_{\xi,p} \in \mathcal{R}_{\alpha-p}$. □

Part (ii). According to the definition, $F_\xi \in \mathcal{C}$ if and only if

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} \leqslant 1.$$

Therefore, using inequality (6), we obtain

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}_{\xi,p}(xy)}{\overline{F}_{\xi,p}(x)} \leqslant \lim_{y\uparrow 1} y^p \limsup_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)} = 1,$$

which implies that $F_{\xi,p} \in \mathcal{C}$.

Part (iii). According to the definition, $F_\xi \in \mathcal{D}$ if and only if

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x/2)}{\overline{F}_\xi(x)} < \infty.$$

Meanwhile, inequality (6) implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{\xi,p}(x/2)}{\overline{F}_{\xi,p}(x)} \leqslant \frac{1}{2^p} \limsup_{x\to\infty} \frac{\overline{F}_\xi(x/2)}{\overline{F}_\xi(x)},$$

and, therefore, $F_{\xi,p} \in \mathcal{D}$ as well.

Part (iv). Let us suppose that $F_\xi \in \mathcal{L}$ and $\mathbb{E}(\xi^+)^p < \infty$. Since

$$\lim_{x\to\infty} \sup_{z\geqslant x} \frac{\overline{F}_\xi(z-1)}{\overline{F}_\xi(z)} = \limsup_{x\to\infty} \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} = 1,$$

according to inequality (1) from Lemma 2, we have

$$\begin{aligned}&\limsup_{x\to\infty} \frac{\overline{F}_{\xi,p}(x-1)}{\overline{F}_{\xi,p}(x)} \\ &\leqslant \max\left\{ \limsup_{x\to\infty} \left(1-\frac{1}{x}\right)^p \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}, \limsup_{x\to\infty} \sup_{z\geqslant x} \left(1-\frac{1}{z}\right)^{p-1} \frac{\overline{F}_\xi(z-1)}{\overline{F}_\xi(z)} \right\} = 1.\end{aligned}$$

Similarly, by the condition

$$\lim_{x\to\infty}\inf_{z\geqslant x}\frac{\overline{F}_\xi(z-1)}{\overline{F}_\xi(z)}=\liminf_{x\to\infty}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}=1,$$

and inequality (2) from Lemma 2, we obtain

$$\begin{aligned}&\liminf_{x\to\infty}\frac{\overline{F}_{\xi,p}(x-1)}{\overline{F}_{\xi,p}(x)}\\ &\geqslant\ \min\left\{\liminf_{x\to\infty}\left(1-\frac{1}{x}\right)^p\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)},\ \liminf_{x\to\infty}\inf_{z\geqslant x}\left(1-\frac{1}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-1)}{\overline{F}_\xi(z)}\right\}=1.\end{aligned}$$

The derived inequalities imply that $F_{\xi,p}\in\mathcal{L}$.

Part (v). According to the definition, $F_\xi\in\mathcal{OL}$ if and only if

$$\limsup_{x\to\infty}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}<\infty,$$

or equivalently,

$$\sup_{x\in\mathbb{R}}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}<\infty.$$

According to (1) from Lemma 2, we have

$$\sup_{x\geqslant M}\frac{\overline{F}_{\xi,p}(x-1)}{\overline{F}_{\xi,p}(x)}\leqslant c_p\sup_{x\geqslant M}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}$$

with sufficiently large $M\geqslant 2$ and

$$c_p=\begin{cases}1 & \text{if } p\geqslant 1,\\ 2^{1-p} & \text{if } p<1.\end{cases}$$

The last inequality implies that $F_{\xi,p}\in\mathcal{OL}$.

Part (vi). For $F_\xi\in L_\gamma$, according to the definition, we have that

$$\lim_{x\to\infty}\sup_{z\geqslant x}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}=\lim_{x\to\infty}\inf_{z\geqslant x}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}=\lim_{x\to\infty}\frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)}=\mathrm{e}^{\gamma y}$$

for fixed $y>0$. For such y, according to inequalities (1) and (2), we have

$$\begin{aligned}&\limsup_{x\to\infty}\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)}\\ &\leqslant\ \max\left\{\limsup_{x\to\infty}\left(1-\frac{y}{x}\right)^p\frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)},\ \limsup_{x\to\infty}\sup_{z\geqslant x}\left(1-\frac{y}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}\right\}=\mathrm{e}^{\gamma y},\end{aligned}$$

and

$$\begin{aligned}&\liminf_{x\to\infty}\frac{\overline{F}_{\xi,p}(x-y)}{\overline{F}_{\xi,p}(x)}\\ &\geqslant\ \min\left\{\liminf_{x\to\infty}\left(1-\frac{y}{x}\right)^p\frac{\overline{F}_\xi(x-y)}{\overline{F}_\xi(x)},\ \liminf_{x\to\infty}\inf_{z\geqslant x}\left(1-\frac{y}{z}\right)^{p-1}\frac{\overline{F}_\xi(z-y)}{\overline{F}_\xi(z)}\right\}=\mathrm{e}^{\gamma y},\end{aligned}$$

implying that $F_{\xi,p}\in\mathcal{L}_\gamma$.

Proof of Theorem 2. Part (i). Suppose that $\mathbb{E}(\xi^+)^p < \infty$ for some p under the condition $0 < p < \alpha$. For sufficiently large x, we have

$$\overline{F}_{\xi,p}(x) = \mathbb{E}(\xi^p \mathbb{1}_{\{\xi > x\}}) = -\int_{(x,\infty)} u^p \mathrm{d}\overline{F}_\xi(u),$$
$$\int_{(x,\infty)} u^{-p} \mathrm{d}\overline{F}_{\xi,p}(u) = \int_{(x,\infty)} u^{-p} \mathrm{d}\left(\int_{(u,\infty)} z^p \mathrm{d}F_\xi(z)\right) = -\overline{F}_\xi(x).$$

When $F_{\xi,p} \in \mathcal{R}_{\alpha-p}$, then according to Lemma 3, we obtain

$$\lim_{x\to\infty} \frac{\int_{(x,\infty)} u^{-p} \mathrm{d}\overline{F}_{\xi,p}(u)}{x^{-p}\overline{F}_{\xi,p}(x)} = -\frac{\alpha - p}{\alpha},$$

implying that

$$\lim_{x\to\infty} \frac{\int_{(x,\infty)} u^{p} \mathrm{d}\overline{F}_{\xi}(u)}{x^{p}\overline{F}_{\xi}(x)} = -\frac{\alpha}{(-p)+\alpha}.$$

In this case, we have that $\alpha > 0$ and $(-p) + \alpha > 0$. Hence, using the converse part of Lemma 3, we can derive that $F_\xi \in \mathcal{R}_\alpha$, which finishes the proof of Part (i).

Part (ii). To prove this part of the theorem, we present a counterexample. We introduce an r.v. ξ such that $F_{\xi,1} \in \mathcal{C}$ but $F_\xi \notin \mathcal{C}$. Let us define the tail function of the r.v. ξ as follows:

$$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,2)}(x) + \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \mathbb{1}_{[2^n,2^{n+1})}(x).$$

This tail function describes the distribution of the r.v. ξ, for which

$$\mathbb{P}(\xi = 2) = \frac{1}{2},\ \mathbb{P}(\xi = 2^{n+1}) = \frac{1}{n^2 2^n} - \frac{1}{(n+1)^2 2^{n+1}},\ n \in \{1, 2, \ldots\}.$$

The moment of order p

$$\mathbb{E}(\xi^+)^p = \mathbb{E}(\xi)^p = 2^{p-1} + 2^p \sum_{n=1}^{\infty} 2^{n(p-1)} \left(\frac{1}{n^2} - \frac{1}{2(n+1)^2}\right)$$

is finite for each $p \in [0, 1]$.

For $p = 1$ and large x,

$$\mathbb{E}(\xi \mathbb{1}_{\{\xi > x\}}) = 2 \sum_{n \geqslant \lfloor \log_2 x \rfloor} \left(\frac{1}{n^2} - \frac{1}{2(n+1)^2}\right).$$

Therefore,

$$\overline{F}_{\xi,1}(x) \underset{x\to\infty}{\sim} \frac{1}{\log_2 x}, \tag{7}$$

implying that $F_{\xi,1} \in \mathcal{R}_0 \subset \mathcal{C}$.

However, for the sequence $x_n = 2^n + 1/2$, $n \in \mathbb{N}$,

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)} = \lim_{n\to\infty} \frac{\overline{F}_\xi(x_n - 1)}{\overline{F}_\xi(x_n)} = 2. \tag{8}$$

This shows that $F_\xi \notin \mathcal{L} \supset \mathcal{C}$, which finishes the proof of Part (ii) of the theorem.

Part (iii). To prove this part of the theorem, we again present a counterexample. We introduce an r.v. ξ such that $F_{\xi,1} \in \mathcal{D}$ but $F_\xi \notin \mathcal{D}$. Let us take the following two sequences $\{a_n\}$ and $\{b_n\}$:

$$a_n = 2^{n(n+1)/2},\ n = 1, 2, 3, \ldots,$$
$$b_1 = 1,\ b_2 = 1/2,\ b_n = 2^{-1-(n+7)(n-2)/2},\ n = 3, 4, \ldots.$$

For such sequences, we define the distribution of the r.v. ξ using the distribution tail

$$\overline{F}_\xi(x) = b_1 \mathbb{1}_{(-\infty, a_1)}(x) + \sum_{n=2}^{\infty} b_n \mathbb{1}_{[a_{n-1}, a_n)}(x). \tag{9}$$

When $a_n \leqslant x < 2a_n$, $n \geqslant 2$, then

$$\frac{\overline{F}_\xi(x/2)}{\overline{F}_\xi(x)} = \frac{b_n}{b_{n+1}} = 2^{n+3},$$

implying that $F_\xi \notin \mathcal{D}$.

Using the formula from Lemma 1, we find that the moment of order p

$$\begin{aligned}\mathbb{E}(\xi^+)^p &= b_1 a_1^p + \sum_{n=2}^{\infty} b_n \left(a_n^p - a_{n-1}^p\right) \\ &= 2^p + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{2^{(n+7)(n-2)/2}} \left(\left(2^{n(n+1)/2}\right)^p - \left(2^{(n-1)n/2}\right)^p\right)\end{aligned}$$

is finite for each $p \in (0,1]$. In addition,

$$\max_{p \in (0,1]} \mathbb{E}(\xi^+)^p = \mathbb{E}(\xi^+) = \frac{130}{21}.$$

Using Lemma 1 again, for $p \in (0,1]$ and $x \geqslant 0$, we obtain

$$\begin{aligned}\mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi > x\}}\right) &= \left(b_1 a_1^p + \sum_{n=2}^{\infty} b_n \left(a_n^p - a_{n-1}^p\right)\right) \mathbb{1}_{[0, a_1)}(x) \\ &\quad + \sum_{k=2}^{\infty} \left(b_k a_k^p + \sum_{n=k+1}^{\infty} b_n \left(a_n^p - a_{n-1}^p\right)\right) \mathbb{1}_{[a_{k-1}, a_k)}(x).\end{aligned}$$

Therefore, for $x \geqslant 0$,

$$\begin{aligned}\frac{\mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi > x/2\}}\right)}{\mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi > x\}}\right)} &= \mathbb{1}_{[0, a_1)}(x) + \sum_{k=2}^{\infty} \mathbb{1}_{[2a_{k-1}, a_k)}(x) \\ &\quad + \sum_{k=1}^{\infty} \frac{b_k a_k^p + \sum_{n=k+1}^{\infty} b_n \left(a_n^p - a_{n-1}^p\right)}{b_{k+1} a_{k+1}^p + \sum_{n=k+2}^{\infty} b_n \left(a_n^p - a_{n-1}^p\right)} \mathbb{1}_{[a_k, 2a_k)}(x),\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{\overline{F}_{\xi,1}(x/2)}{\overline{F}_{\xi,1}(x)}\\
&=\lim_{k\to\infty}\frac{2^{(k(k+1)-(k+7)(k-2))/2}+\sum\limits_{n=k+1}^{\infty}2^{-(n+7)(n-2)/2}\Big(2^{n(n+1)/2}-2^{(n-1)n/2}\Big)}{2^{\big((k+1)(k+2)-(k+8)(k-1)\big)/2}+\sum\limits_{n=k+2}^{\infty}2^{-(n+7)(n-2)/2}\Big(2^{n(n+1)/2}-2^{(n-1)n/2}\Big)}\\
&=\lim_{k\to\infty}\frac{\frac{1}{3}\,2^{-2(k-4)}-\frac{1}{7}\,2^{-3(k-2)}}{\frac{1}{3}\,2^{-2(k-3)}-\frac{1}{7}\,2^{-3(k-1)}}=4.
\end{aligned}$$

We conclude that $F_{\xi,1}\in\mathcal{D}$ (but $F_\xi\notin\mathcal{D}$), which finishes the proof of Part (iii).

Part (iv). To prove this part of the theorem, we use the same counterexample as in Part (ii). According to relationship (7), the d.f. $F_{\xi,1}$ belongs to the class $\mathcal{L}$. On the other hand, the relationship (8) gives that $F_\xi\notin\mathcal{L}$. Therefore, Part (iv) is proved.

Part (v). To prove this part of the theorem, we use the same counterexample as in Part (iii). For the distribution tail (9), we obtain that $F_{\xi,1}\in\mathcal{OL}$ because $\mathcal{D}\subset\mathcal{OL}$, according to the definitions of the classes $\mathcal{D}$ and $\mathcal{OL}$. On the other hand, for the distribution tail (9), we have

$$\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}=\mathbb{1}_{(-\infty,a_1)}(x)+2\mathbb{1}_{[a_1,a_1+1)}(x)+\sum_{n=2}^{\infty}\mathbb{1}_{[a_{n-1}+1,a_n)}(x)+\sum_{n=2}^{\infty}2^{n+3}\mathbb{1}_{[a_n,a_n+1)}(x),$$

where $a_n=2^{n(n+1)/2}$, $n\in\mathbb{N}$. Therefore,

$$\limsup_{x\to\infty}\frac{\overline{F}_\xi(x-1)}{\overline{F}_\xi(x)}=\lim_{n\to\infty}\frac{\overline{F}_\xi(a_n-1)}{\overline{F}_\xi(a_n)}=\infty,$$

implying that $F_\xi\notin\mathcal{OL}$.

Part (vi). The condition $F_{\xi,p}\in\mathcal{L}_\gamma$ implies that the function $\overline{F}_{\xi,p}(\log x)$ with $x>1$ is regularly varying with the index $-\gamma$. The Karamata characterization theorem (see, for instance, Theorem 1.4.1 in [3]) implies that

$$\overline{F}_{\xi,p}(\log x)=x^{-\gamma}L(x),\ x>1,$$

or

$$\overline{F}_{\xi,p}(x)=\mathrm{e}^{-\gamma x}L(\mathrm{e}^x),\ x>0,$$

with some slowly varying functions L.

When x is sufficiently large, then

$$\overline{F}_{\xi,p}(x)=\mathbb{E}\big(\xi^p\mathbb{1}_{\{\xi>x\}}\big)=\int_{(x,\infty)}u^p\mathrm{d}F_\xi(u)\geqslant x^p\overline{F}_\xi(x). \tag{10}$$

Therefore, for large x,

$$\overline{F}_\xi(x)\leqslant x^{-p}\mathrm{e}^{-\gamma x}L(\mathrm{e}^x).$$

Now, suppose that $\varepsilon\in(0,1/2)$ and $\delta\in(0,1)$. According to the last inequality and Lemma 1 for large x, we obtain

$$\overline{F}_{\xi,p}(x)\leqslant x^p\overline{F}_\xi(x)+p\int\limits_x^{(1+\varepsilon)x}u^{p-1}\overline{F}_\xi(u)\mathrm{d}u+p\int\limits_{(1+\varepsilon)x}^{\infty}u^{-1}\mathrm{e}^{-\gamma u}L(\mathrm{e}^u)\mathrm{d}u.$$

Using the basic properties of slowly varying functions (see Theorem 1.3.1 or Proposition 1.3.6 in [3]), we obtain

$$\begin{aligned}\overline{F}_{\xi,p}(x) &\leqslant (1+\varepsilon)^p x^p \overline{F}_\xi(x) + \frac{p}{(1+\varepsilon)x} \int\limits_{(1+\varepsilon)x}^{\infty} \mathrm{e}^{-\gamma(1-\varepsilon/2)u} \mathrm{d}u \\ &\leqslant (1+\varepsilon)^p x^p \overline{F}_\xi(x) + \frac{p}{\gamma(1+\varepsilon)(1-\varepsilon/2)x} \mathrm{e}^{-\gamma(1+\varepsilon/4)x} \end{aligned} \tag{11}$$

because $(1-\varepsilon/2)(1+\varepsilon) > (1+\varepsilon/4)$ for $\varepsilon \in (0, 1/2)$. Using the same basic properties of slowly varying functions again, we obtain

$$\frac{\mathrm{e}^{-\gamma(1+\varepsilon/4)x}}{\overline{F}_{\xi,p}(x)} = \frac{\mathrm{e}^{-\gamma\varepsilon x/4}}{L(\mathrm{e}^x)} \underset{x\to\infty}{\to} 0.$$

Therefore, for large x,

$$\overline{F}_{\xi,p}(x) \leqslant (1+\varepsilon)^p x^p \overline{F}_\xi(x) + \delta \overline{F}_{\xi,p}(x),$$

or

$$\overline{F}_{\xi,p}(x) \leqslant \frac{(1+\varepsilon)^p}{1-\delta} x^p \overline{F}_\xi(x). \tag{12}$$

The derived estimates (10) and (12) imply the following double inequality:

$$\frac{(1-\delta)}{(1+\varepsilon)^p} x^{-p} \overline{F}_{\xi,p}(x) \leqslant \overline{F}_\xi(x) \leqslant x^{-p} \overline{F}_{\xi,p}(x) \tag{13}$$

provided $\varepsilon \in (0, 1/2)$, $\delta \in (0, 1)$ and x is sufficiently large.

Let us now suppose that $y > 0$. From (13), we have

$$\frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} \leqslant \frac{\overline{F}_{\xi,p}(x+y)}{\overline{F}_{\xi,p}(x)} \frac{x^p}{(x+y)^p} \frac{(1+\varepsilon)^p}{1-\delta}.$$

Condition $F_{\xi,p} \in \mathcal{L}_\gamma$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} \leqslant \mathrm{e}^{-\gamma y} \frac{(1+\varepsilon)^p}{1-\delta}.$$

Since $\varepsilon \in (0, 1/2)$ and $\delta \in (0, 1)$ are arbitrarily chosen, we have

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} \leqslant \mathrm{e}^{-\gamma y}.$$

In a similar way, from (13), the opposite inequality follows

$$\liminf_{x\to\infty} \frac{\overline{F}_\xi(x+y)}{\overline{F}_\xi(x)} \geqslant \mathrm{e}^{-\gamma y}.$$

The last two inequalities imply that $F_\xi \in \mathcal{L}_\gamma$. Therefore, the theorem is proved. □

6. Conclusions

In this paper, we studied the closure of regularity classes under the truncated moment transformation. We proved that the classes $\mathcal{R}, \mathcal{C}, \mathcal{D}, \mathcal{L}, \mathcal{OL}$ and $\mathcal{L}_\gamma$ with $\gamma > 0$ are closed under the truncated moment transformation. This means that the condition $F_\xi \in \mathcal{K}$ for the d.f. F_ξ implies that the d.f. $F_{\xi,p}(x) = 1 - \min\left\{1, \mathbb{E}\left(\xi^p \mathbb{1}_{\{\xi > x\}}\right)\right\} \in \mathcal{K}$ for the classes $\mathcal{K} \in \{\mathcal{R}, \mathcal{C}, \mathcal{D}, \mathcal{L}, \mathcal{OL}, \mathcal{L}_\gamma\}$, where γ is supposed to be positive. The second theorem of this work showed that the classes $\mathcal{R}$ and $\mathcal{L}_\gamma$ with $\gamma > 0$ also have the inverse closure property, while the classes $\mathcal{C}, \mathcal{D}, \mathcal{L}$ and $\mathcal{OL}$ do not admit the inverse closure property. The following conclusions were drawn from both of the main theorems in this article.

Corollary 1. *Let ξ be a real-valued r.v. with the d.f. F_ξ. Then, $F_\xi \in \mathcal{R}_\alpha$ with $\alpha > 0$ if and only if $F_{\xi,p} \in \mathcal{R}_{\alpha-p}$ for some $0 < p < \alpha$.*

Corollary 2. *Let ξ be a real-valued r.v. with the d.f. F_ξ. Then, $F_\xi \in \mathcal{L}_\gamma$ with $\gamma > 0$ if and only if $F_{\xi,p} \in \mathcal{L}_\gamma$ for some $p > 0$.*

As for Corollary 2, we note that the r.v. ξ with a d.f. from the class $\mathcal{L}_\gamma$ has a finite moment of any order for the positive part of ξ^+.

The results of this work could provide a new way to verify whether d.f.s are regularly varying and whether d.f.s are exponential-like-tailed. Instead of checking $F_\xi \in \mathcal{R}_\alpha$ with $\alpha > 0$, we can check $F_{\xi,p} \in \mathcal{R}_{\alpha-p}$ for some $0 < p < \alpha$ and instead of checking $F_\xi \in \mathcal{L}_\gamma$ with $\gamma > 0$, we can check $F_{\xi,p} \in \mathcal{L}_\gamma$ for some $p > 0$.

The results obtained in this paper complement the results presented in [41,42,61–64], which considered how the d.f.s from some regularity class affect the regularity properties of the integrated tail

$$F_\xi^{(I)}(x) = \frac{1}{\int_0^\infty \overline{F}_\xi(u)\mathrm{d}u} \int_0^x \overline{F}_\xi(u)\mathrm{d}u.$$

For instance, the following assertion can be derived by combining Corollary 2 and Lemma 3.1 in [42] (see also Theorem 1.1 in [41]).

Corollary 3. *Let $\gamma > 0$ and let ξ be a real-valued r.v. with the d.f. F_ξ. Then,*

$$F_\xi \in \mathcal{L}_\gamma \Leftrightarrow \lim_{x\to\infty} \frac{\overline{F}_\xi(x)}{\int_x^\infty \overline{F}_\xi(u)\mathrm{d}u} = \gamma \Leftrightarrow F_\xi^{(I)} \in \mathcal{L}_\gamma \Leftrightarrow F_{\xi,p} \in \mathcal{L}_\gamma \text{ for some } p > 0.$$

For future research, we plan to extend the obtained results to other heavy- and light-tailed distribution classes of interest. Finally, the obtained results are not only interesting from a theoretical point of view but could also have some potential applications, e.g., the construction and investigation of various risk measures.

Author Contributions: Conceptualization, J.Š.; methodology, S.P. and J.Š.; software, S.P.; validation, R.L.; formal analysis, R.L.; investigation, S.P. and J.Š.; writing—original draft preparation, S.P.; writing—review and editing, J.Š. and R.L.; visualization, J.Š. and R.L.; supervision, J.Š.; project administration, J.Š.; funding acquisition, S.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Karamata, J. Sur un mode de croissance régulière. Théorèmes fundamentoux. *Bulletin de la Société Mathématique de France* **1933**, *61*, 55–62.
2. Seneta, E. *Regularly Varying Functions*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1976; Volume 508.
3. Bingham, N.H.; Goldie, C.M.; Teugels, J.L. *Regular Variation*; Cambridge University Press: Cambridge, UK, 1987.
4. Resnick, S.I. *Extreme Values, Regular Variation and Point Process*; Springer: New York, NY, USA, 1987.
5. Foss, S.; Korshunov, D.; Zachary, S. *An Introduction to Heavy-Tailed and Subexponential Distributions*, 2nd ed.; Springer: New York, NY, USA, 2013.

6. Samorodnitsky, G. *Stochastic Processes and Long Range Dependence*; Springer: New York, NY, USA, 2016.
7. Feng, M.; Deng, L.J.; Chen, F.; Perc, M.; Kurths, J. The accumulative law and its probability model: An extention of the Pareto distribution and the log-normal distribution. *Proc. R. Soc. A.* **2020**, *476*, 20200019.
8. Alfaer, N.M.; Gemeay, A.M.; Aljohani, H.M.; Afify, A.Z. The extended log-logistic distribution: Inference and actuarial applications. *Mathematics* **2021**, *9*, 1386.
9. Mehta, N.J.; Yang, F. Portfolio optimization for extreme risks with maximum diversification: An empirical analysis. *Risks* **2021**, *10*, 101.
10. Yang, Y.; Wang, X.; Zhang, Z. Finite-time ruin probability of a perturbed risk model with dependent main and delayed claims. *Nonlinear Anal. Model. Control* **2021**, *26*, 801–820.
11. Bourguignon, M.; Gallardo, D.D.; Gómez, H.J. A note on Pareto-type distributions parametrized by its mean and precision parameters. *Mathematics* **2022**, *10*, 528. [CrossRef]
12. Olmos, N.M.; Gómez-Déniz, E.; Venegas, O. The heavy-tailed Gleser model: Properties, estimation, and applications. *Mathematics* **2022**, *10*, 4577.
13. Feller, W. *An Introduction to Probability Theory and Its Applications*; Wiley: New York, NY, USA, 1971; Volume 2.
14. Embrechts, P.; Klüppelberg, C.; Mikosch, T. *Modelling Extremal Events for Insurance and Finance*; Springer: New York, NY, USA, 1997.
15. Cline, D.B.H. Intermediate regular and Π variation. *Proc. London Math. Soc.* **1994**, *68*, 594–611. [CrossRef]
16. Cline, D.B.H.; Samorodnitsky, G. Subexponentiallity of the product of independent random variables. *Stoch. Process. Appl.* **1994**, *49*, 75–98.
17. Cai, J.; Tang, Q. On max-type equivalence and convolution closure of heavy-tailed distributions and their applications. *J. Appl. Probab.* **2004**, *41*, 117–130. [CrossRef]
18. Kizinevič, E.; Sprindys, J.; Šiaulys, J. Randomly stopped sums with consistently varying distributions. *Mod. Stoch. Theory Appl.* **2016**, *3*, 165–179. [CrossRef]
19. Feller, W. One-sided analogues of Karamata's regular variation. *Enseign. Math.* **1969**, *15*, 107–121.
20. Tang, Q.; Yua, J. A sharp inequality for the tail probabilities of sums of i.i.d. r.v.'s with dominatedly varying tails. *Sci. China Ser. A* **2002**, *45*, 1006–1011.
21. Tang, Q.; Tsitsiashvili, G. Precise estimates for the ruin probability in the finite horizon in a discrete-time risk model with heavy-tailed insurance and financial risks. *Stoch. Process. Appl.* **2003**, *108*, 299–325.
22. Konstantinides, D. A class of heavy tailed distributions. *J. Numer. Appl. Math.* **2008**, *96*, 127–138.
23. Watanabe, T.; Yamamuro, K. Ratio of the tail of an infinitely divisible distribution on the line to that of its Lévy measure. *Electron. J. Probab.* **2010**, *15*, 44–74.
24. Gao, M.; Wang, K.; Chen, L. Precise large deviations for widely orthant dependent random variables with different distributions. *J. Inequal. Appl.* **2018**, *2018*, 21.
25. Dirma, M.; Paukštys, S.; Šiaulys, J. Tails of the moments for sums with dominatedly varying random summands. *Mathematics* **2021**, *9*, 824.
26. Zhang, A.; Liu, S.; Yang, Y. Asymptotics for ultimate ruin probability in a by-claim risk model. *Nonlinear Anal. Model. Control* **2021**, *26*, 259–270. [CrossRef]
27. Wang, S.; Gao, Y. Precise large deviations for aggregate claims of a compound renewal risk model with arbitrary dependence between claim sizes and waiting times. *Lith. Math. J.* **2022**, *62*, 542–552.
28. Matuszewska, W. On generalization of regularly increasing functions. *Studia Math.* **1964**, *24*, 271–279. [CrossRef]
29. Cline, D.B.H.; Hsing, T. *Large Deviation Probabilities for Sums and Maxima of Random Variables With Heavy or Subexponential Tails*; Texas A&M University: College Station, TX, USA, 1991.
30. Chistyakov, V.P. A theorem on sums of independent, positive random variables and its applications to branching processes. *Theory Probab. Appl.* **1964**, *9*, 640–648. [CrossRef]
31. Embrechts, P.; Goldie, C.M. On closure and factorization properties of subexponential and related distributions. *J. Aust. Math. Soc. Ser. A* **1980**, *29*, 243–256. [CrossRef]
32. Klüpelberg, C. Subexponential distributions and integrated tails. *J. Appl. Probab.* **1988**, *25*, 132–141. [CrossRef]
33. Pakes, A.G. Convolution equivalence and infinite divisibility. *J. Appl. Probab.* **2004**, *41*, 407–424. [CrossRef]
34. Shimura, T.; Watanabe, T. Infinite divisibility and generalized subexponentiality. *Bernoulli* **2005**, *11*, 445–469. [CrossRef]
35. Foss, S.; Korshunov, D.; Zachary, S. Convolution of long-tailed and subexponential distributions. *J. Appl. Probab.* **2009**, *46*, 756–767. [CrossRef]
36. Qian, H.; Geng, B.; Wang, S. Tail asymptotics of randomly weighted sums of dependent strong subexponential random variables. *Lith. Math. J.* **2022**, *62*, 113–122. [CrossRef]
37. Chover, J.; Ney, P.; Waigner, S. Degeneracy properties of subcritical branching processes. *Ann. Probab.* **1973**, *1*, 663–673. [CrossRef]
38. Chover, J.; Ney, P.; Waigner, S. Functions of probability measures. *J. d'Analyse Math.* **1973**, *26*, 255–302. [CrossRef]
39. Cline, D.B.H. Convolution tails, product tails and domains of attraction. *Probab. Theory Relat. Fields* **1986**, *72*, 529–557. [CrossRef]
40. Bertoin, J.; Doney, R.A. Some asymptotic results for transients random walks. *Adv. Appl. Probab.* **1996**, *28*, 207–226. [CrossRef]
41. Su, C.; Chen, J.; Hu, Z. Some discussions on the class $\mathcal{L}(\gamma)$. *J. Math. Sci.* **2004**, *122*, 3416–3425. [CrossRef]
42. Tang, Q. The overshoot of a random walk with negative drift. *Stat. Probab. Lett.* **2007**, *77*, 158–165. [CrossRef]

43. Watanabe, T. Convolution equivalence and distribution of random sums. *Probab. Theory Relat. Fields* **2008**, *142*, 367–397. [CrossRef]
44. Cheng, D.; Ni, S.; Pakes, A.G.; Wang, Y. Some properties of the exponential distribution class with applications to risk theory. *J. Korean Stat. Soc.* **2012**, *41*, 515–527. [CrossRef]
45. Xu, H.; Wang, Y.; Cheng, D.; Yu, C. On the closure under infinitely divisible distribution roots. *Lith. Math. J.* **2022**, *62*, 259–287. [CrossRef]
46. Albin, J.M.P.; Sunden, M. On the asymptotic behaviour of Lévy processes, part I: Subexponential and exponential processes. *Stoch. Process. Appl.* **2009**, *119*, 281–304. [CrossRef]
47. Su, C.; Chen, Y. Behaviors of the product of independent random variables. *Int. J. Math. Anal.* **2007**, *1*, 21–35. [CrossRef]
48. Xu, H.; Scheutzow, M.; Wang, Y. On a transformation between distributions obeing the principle of a single big jump. *J. Math. Anal. Appl.* **2015**, *430*, 672–684. [CrossRef]
49. Danilenko, S.; Šiaulys, J.; Stepanauskas, G. Closure properties of O-exponential distributions. *Stat. Probab. Lett.* **2018**, *140*, 63–70. [CrossRef]
50. Cui, Z.; Wang, Y. On the long tail property of product convolution. *Lith. Math. J.* **2020**, *60*, 315–329. [CrossRef]
51. Shiryaev, A.N. *Probability*; Springer: New York, NY, USA, 1995.
52. Sprindys, J.; Šiaulys, J. Asymptotic formulas for the left truncated moments of sums with consistently varying distributed increments. *Nonlinear Anal. Model. Control* **2021**, *26*, 1200–1212. [CrossRef]
53. Liu, Y. A general treatment of alternative expectation formulae. *Stat. Probab. Lett.* **2020**, *166*, 108863. [CrossRef]
54. Ogasawara, H. Alternative expectation formulas for real-valued random vectors. *Commun. Stat. Theory Methods* **2020**, *49*, 454–470. [CrossRef]
55. Song, P.; Wang, S. A further remark on the alternative expectation formula. *Commun. Stat. Theory Methods* **2021**, *50*, 2586–2591. [CrossRef]
56. Karamata, J. Sur un mode de croissance régulière des functions. *Mathematica* **1930**, *4*, 38–53.
57. De Haan, L. Equivalence classes of regularly varying functions. *Stoch. Process. Appl.* **1974**, *2*, 243–259. [CrossRef]
58. Buldygin, V.V.; Pavlenkov, V.V. A generalisation of Karamata's theorem on the asymptotiic behavior of integrals. *Theory Probab. Math. Stat.* **2010**, *81*, 15–26. [CrossRef]
59. Buldygin, V.V.; Pavlenkov, V.V. Karamata theorem for regulary log-periodic functions. *Ukr. Math. J.* **2013**, *64*, 1635–1657. [CrossRef]
60. Omey, E.; Cadena, M. New results on slowly varying functions in the Zygmund sense. *Proc. Jpn. Acad. A Math. Sci.* **2020**, *96*, 45–49. [CrossRef]
61. Tang, Q.; Su, C. Ruin probabilities for large claims in delayed renewal risk model. *Southeast Asian Bull. Math.* **2002**, *25*, 735–743. [CrossRef]
62. Su, C.; Tang, Q. Characterizations on heavy-tailed distributions by means of hazard rate. *Acta Math. Appl. Sin.* **2003**, *19*, 135–142. [CrossRef]
63. Su, C.; Hu, Z. On two broad classes of heavy-tailed distributions. *Far East J. Math.* **2004**, *5*, 195–204.
64. Su, C.; Hu, Z.; Chen, Y.; Liang, H. A wide class of heavy-tailed distributions and its applications. *Front. Math. China* **2007**, *2*, 257–286. [CrossRef]