Hindawi
Journal of Mathematics
Volume 2022, Article ID 9419427, 8 pages
https://doi.org/10.1155/2022/9419427

Research Article

A Note on *n*-Divisible Positive Definite Functions

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Received 10 June 2022; Accepted 14 September 2022; Published 30 November 2022

Academic Editor: Firdous A. Shah

Let $PD(\mathbb{R})$ be the family of continuous positive definite functions on $\mathbb{R}$. For an integer $n > 1$, a $f \in PD(\mathbb{R})$ is called n-divisible if there is $g \in PD(\mathbb{R})$ such that $g^n = f$. Some properties of infinite-divisible and n-divisible functions may differ in essence. Indeed, if f is infinite-divisible, then for each integer $n > 1$, there is an unique g such that $g^n = f$, but there is a n-divisible f such that the factor g in $g^n = f$ is generally not unique. In this paper, we discuss about how rich can be the class $\{g \in PD(\mathbb{R}): g^n = f\}$ for n-divisible $f \in PD(\mathbb{R})$ and obtain precise estimate for the cardinality of this class.

1. Introduction

We start with some notations and definitions. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the families of integers, non-negative integers, real, and complex numbers, respectively. In the sequel, $M(\mathbb{R})$ denotes the Banach algebra of bounded regular complex-valued Borel measures on $\mathbb{R}$ with the convolution as multiplication. $M(\mathbb{R})$ is equipped with the usual total variation norm $\|\mu\|$ of $\mu \in M(\mathbb{R})$. The Fourier-Stieltjes transform of $\mu \in M(\mathbb{R})$ is given by

$$\widehat{\mu}(x) = \int_{\mathbb{R}} e^{-ixt} d\mu(t). \tag{1}$$

A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be positive definite if

$$\sum_{j,k=1}^{m} f(x_j - x_k) c_j \bar{c}_k \geq 0, \tag{2}$$

for each $m \in \mathbb{N}$ and all $c_1, \cdots, c_m \in \mathbb{C}$, $x_1, \cdots, x_m \in \mathbb{R}$. Any such a function satisfies

$$f(-x) = f(\bar{x}), \tag{3}$$

for all $x \in \mathbb{R}$.

Positive definite functions on groups have a long history and have many applications in probability theory and areas such as stochastic processes [1], harmonic analysis [2], potential theory [3], and spectral theory [4]. See [5] for other applications and details. The analysis of the properties of positive definite functions has vast literature, and the above list is only a small sample.

We will denote by $PD(\mathbb{R})$ the family of continuous nontrivial ($\equiv 0$) positive definite functions on $\mathbb{R}$. Note that if positive definite f is continuous in a neighborhood of the origin, then it is uniformly continuous in $\mathbb{R}$ (see, e.g., [4, Corollary 1.4.10]). Bochner's theorem gives a description of $f \in PD(\mathbb{R})$ in terms of the Fourier transform. Namely, according to this theorem (see, e.g., [6], p. 71]), a continuous function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is positive definite if and only if there exists a nonnegative $\mu_f \in M(\mathbb{R})$ such that $f(x) = \widehat{\mu_f}(x)$, $x \in \mathbb{R}$. This statement implies, in particular, that

$$|f(x)| \leq f(0) = \left\|\mu_f\right\|, \tag{4}$$

for all $x \in \mathbb{R}$. If, in addition, positive $\mu_f \in M(\mathbb{R})$ satisfies $\|\mu_f\| = 1$, then on the language of probability theory, such a μ_f and the function $f(x) = \widehat{\mu_f}(-x)$, $x \in \mathbb{R}$, are called a probability measure and its characteristic function, respectively (see, e.g., [7, 8], p.p. 8-9]).

Recall that in the probability theory a random variable ξ is called n-divisible for certain $n \in \mathbb{N}$, $n > 1$, if there exist independent and identically distributed random variables

$\xi_1, \cdots, \xi_n$ such that $\xi_1 + \cdots + \xi_n$ has the sane distribution as ξ. In terms of the characteristic function f of a real-valued random variable, this means that f is n-divisible if there exists $g \in PD(\mathbb{R})$ such that

$$f(x) = g^n(x), \tag{5}$$

for all $x \in \mathbb{R}$. Next, $f \in PD(\mathbb{R})$ is said to be infinite-divisible if it is n-divisible for each $n \in \mathbb{N}$, $n \geq 2$.

In the sequel, $PD_n(\mathbb{R})$ and $PD_\infty(\mathbb{R})$ denotes the families of n-divisible and infinite divisible functions in $PD(\mathbb{R})$, respectively. An early overview over divisibility of distributions is given in [9]. Until very recently, the vast majority of divisible positive definite functions or divisible distributions considered in the literature are also infinite-divisible. Important applications of n-divisibility is in modelling, for example, bug populations in entomology [10] or in financial aspects of various insurance models [11, 12].

The motivation for our investigation are the following: (i) partly from the fact that properties of functions in $PD_\infty(\mathbb{R})$ has rich literature (see, e.g., [9, 13, 14]), but the n-divisible functions have been studied much less; and (ii) partly from the fact that some properties of functions from $PD_\infty(\mathbb{R})$ and from $PD_n(\mathbb{R})$ may differ in essence. One of those properties is the following: if $f \in PD_\infty(\mathbb{R})$, then for each $n \in \mathbb{N}$, $n > 1$, there is an unique $g \in PD(\mathbb{R})$ such that $g^n = f$, but there is n-divisible f such that the factor g in $g^n = f$ is generally not unique. In this paper, we study the following problems: (i) how rich can be the class $\{g \in PD(\mathbb{R}): g^n = f\}$; and (ii) what properties of f determine the size of $\{g \in PD(\mathbb{R}): g^n = f\}$. We present several precise estimates for the cardinality of this class. Also, the main results are validated via illustrative examples.

More precise, for $n \in \mathbb{N}$, $n > 1$ and $f \in PD_n(\mathbb{R})$, we wish to study the family

$$D_n(f) = \{g \in PD(\mathbb{R}): g^n = f\} \tag{6}$$

and the quantity card $(D_n(f))$, i.e., the cardinality of $D_n(f)$. Note that there exists $f \in PD_n(\mathbb{R}) \setminus PD_\infty(\mathbb{R})$ such that the factor g in Equation (5) will generally be not unique. It turns out that card $(D_n(f))$ depends on n and in some way also depends on the geometric structure of the zeros set $N_f = \{x \in \mathbb{R} : f(x) = 0\}$ of $f \in PD_n(\mathbb{R})$ (see our Theorem 1 below). The essential support S_f of $f \in PD(\mathbb{R})$ is defined by $S_f = \mathbb{R} \setminus N_f$. Combining Equation (3) with Equation (4), gives

$$0 \in S_f \text{ and } -S_f = S_f. \tag{7}$$

Since functions $f \in PD(\mathbb{R})$ are continuous on $\mathbb{R}$, it follows that S_f is an open subset of $\mathbb{R}$. Therefore, S_f can be represented as a finite or infinite but countable $S_f = \bigcup_{j\in\Sigma} E_j$, where $\{E_j\}_{j\in\Sigma}$ is the family of all open connected components of S_f. In the sequel, comp (S_f) denotes the cardinality of Σ. According to relations (7), we see that either there is an $k \in \mathbb{N}$ such that comp $(S_f) = 2k - 1$ or comp $(S_f) = \infty$.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $f \in PD_n(\mathbb{R})$. Assume that*

$$comp\,(S_f) = 2k - 1, \tag{8}$$

for some $k \in \mathbb{N}$. Then

$$\text{card}\,(D_n(f)) \leq n^{k-1}. \tag{9}$$

The following theorem shows that the estimate (9) is accurate.

Theorem 2. *For each $n \in \mathbb{N}$, $n \geq 2$, each $k \in \mathbb{N}$, and any open subset E of $\mathbb{R}$ which satisfies*

$$0 \in E, -E = E \text{ and } comp\,(E) = 2k - 1, \tag{10}$$

there exists $f \in PD_n(\mathbb{R})$, such that $S_f = E$ and

$$\text{card}\,(D_n(f)) = n^{k-1}. \tag{11}$$

We will present two examples of $f \in PD_n(\mathbb{R})$ such that Equation (11) is satisfied. In order to make the examples easier to understand, we will consider only small values of k. For $\alpha > 0$, set

$$\Lambda_\alpha(x) = (\max\{1 - |x|\,; 0\})^\alpha, \tag{12}$$

$x \in \mathbb{R}$. Note that $\Lambda_\alpha \in PD(\mathbb{R})$ if and only if $\alpha \geq 1$ (see, e.g., [15, 16, p. 282]). We start with the case where S_f is a bounded subset of $\mathbb{R}$.

Example 3. For $n \in \mathbb{N}$, $n > 1$, let

$$\begin{aligned} f_1(x) = \Lambda_n(x) &+ \frac{1}{2^{3n}}[(\Lambda_1(x - \pi - 1) + \Lambda_1(x - 2\pi + 1))^n \\ &+ (\Lambda_1(x + \pi + 1) + \Lambda_1(x + 2\pi - 1))^n] \\ &+ \frac{1}{2^{4n}}[((\Lambda_1(x - 11) + \Lambda_1(x - 12) + \Lambda_1(x - 13) \\ &+ \Lambda_1(x - 14))^n + (\Lambda_1(x + 11) + \Lambda_1(x + 12) \\ &+ \Lambda_1(x + 13) + \Lambda_1(x + 14))^n]. \end{aligned} \tag{13}$$

Then comp $(S_{f_1}) = 5$, since

$$S_{f_1} = (-15, -10) \cup (-2\pi, -\pi) \cup (-1, 1) \cup (\pi, 2\pi) \cup (10, 15). \tag{14}$$

Moreover, $f_1 \in PD_n(\mathbb{R})$ and

$$\text{card}\,(D_n(f_1)) = n^5. \tag{15}$$

Now let us give an example of $f \in PD_n(\mathbb{R})$ such that S_f has unbounded components.

Example 4. For any $a > 15$ and $n \in \mathbb{N}$, $n > 1$, let

$$f_2(x) = f_1(x) + \frac{1}{2^{2n}}\left(\sum_{r=1}^{\infty}\frac{1}{2^r}\Lambda_1(x-a-r)\right)^n + \left(\sum_{r=1}^{\infty}\frac{1}{2^r}\Lambda_1(x+a+r)\right)^n, \quad (16)$$

where f_1 was defined by Equation (13). Then comp $(S_{f_2}) = 7$, since

$$S_{f_2} = (-\infty,-a) \cup (-15,-10) \cup (-2\pi,-\pi) \cup (-1,1) \cup (\pi,2\pi) \cup (10,15) \cup (a,\infty). \quad (17)$$

Moreover, $f_2 \in PD_n(\mathbb{R})$ and

$$\text{card}\,(D_n(f_2)) = n^7. \quad (18)$$

Theorem 5. *Suppose that an open subset E of $\mathbb{R}$ satisfies*

$$0 \in E, -E = E \text{ and comp}\,(E) = \infty. \quad (19)$$

Let $\{E_j\}_{j\in\Sigma}$ be the family of open connected components of E. Assume that there is $\sigma > 0$ such that

$$\inf_{j\in\Sigma}\left(\sup_{a,b\in E_j}|a-b|\right) = 2\sigma. \quad (20)$$

Then, for any $n \in \mathbb{N}$, $n \geq 2$, there exists $f \in PD_n(\mathbb{R})$ such that $S_f = E$ and

$$\text{card}\,(D_n(f)) = \infty. \quad (21)$$

2. Preliminaries and Proofs

If $\nu, \mu \in M(\mathbb{R})$, then the convolution $\nu * \mu$ is defined by

$$\nu * \mu(E) = \int_{\mathbb{R}} \nu(E-x)\, d\mu(x) \quad (22)$$

for each Borel subset E of $\mathbb{R}$. Note that

$$\widehat{\nu * \mu} = \widehat{\nu}\cdot\widehat{\mu}. \quad (23)$$

In particular, for any $n \in \mathbb{N}$,

$$\widehat{(\mu^{*n})} = (\widehat{\mu})^n, \quad (24)$$

where the convolution power μ^{*n} is defined as the n-fold iteration of the convolution of μ with itself.

The Lebesgue space $L^1(\mathbb{R})$ can be identified with the closed ideal in $M(\mathbb{R})$ of measures absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}$. Namely, if $\varphi \in L^1(\mathbb{R})$, then φ is associated with the measure

$$\mu_\varphi(E) = \int_E \varphi(t)\, dt \quad (25)$$

for each Borel subset E of $\mathbb{R}$. Hence $\widehat{\varphi}(x) = \int_{\mathbb{R}} e^{-itx}\varphi(t)\, dt$. In particular, if $\mu = \varphi(t)dt$, where $\varphi \in L^1(\mathbb{R})$ and φ is such that $\|\varphi\|_{L^1(\mathbb{R})} = 1$ and $\varphi \geq 0$ on $\mathbb{R}$, then φ is called the probability density function of μ, or the probability density for short.

We define the inverse Fourier transform by

$$\check{\psi}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\,x}\psi(x)\, dx, \quad (26)$$

$t \in \mathbb{R}$. Then the inversion formula $\widehat{(\check{\psi})} = \psi$ holds for suitable $\psi \in L^1(\mathbb{R})$.

Proof of Theorem 1. The conditions (7) and (8) imply that there exists a sequence of real numbers

$$0 < b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{k-1} < b_{k-1}, \quad (27)$$

such that

$$S_f = \left(\bigcup_{j=-1}^{-(k-1)} E_j\right) \bigcup E_0 \bigcup \left(\bigcup_{j=1}^{k-1} E_j\right), \quad (28)$$

where $E_0 = (-b_0, b_0)$, $E_j = (a_j, b_j)$, and $E_{-j} = -E_j = (-b_j, -a_j)$ for $j = 1, 2, \cdots, k-1$. Note that in Equation (27) also might be $b_{k-1} = +\infty$. Let $g \in D_n(f)$. Then it is immediate that $S_f = S_g$ and

$$|g(x)|^n = |f(x)| \quad (29)$$

for all $x \in S_f$. Fix any $E_j \subset S_f$ in Equation (28). Since E_j is an open connected component of S_f, we have there are two continuous functions $u_{f,j}, u_{g,j} : E_j \longrightarrow (-\pi, \pi]$ such that

$$f(x) = |f(x)|e^{iu_{f,j}(x)} \text{ and } g(x) = |g(x)|e^{iu_{g,j}(x)} \quad (30)$$

for all $x \in E_j$. Using the identity $g^n = f$, it follows from Equations (29) and (30) that, for each for $j \in \{-(k-1), \cdots, k-1\}$, there exists some integer m_j in $\{0, \cdots, n-1\}$ such that

$$n \cdot u_{g,j}(x) = u_{f,j}(x) + 2\pi m_j \quad (31)$$

for all $x \in E_j$. Therefore, for any $x \in S_f$, we have

$$\begin{aligned} g(x) &= \sum_{j=-(k-1)}^{k-1} |g(x)|\chi_{E_j}(x)e^{iu_{g,j}} \\ &= \sum_{j=-(k-1)}^{k-1} |f(x)|_+^{1/n}\chi_{E_j}(x)e^{\left(iu_{f,j}+2\pi m_j\right)/n}, \end{aligned} \tag{32}$$

where χ_{E_j} is the indicator function of the set E_j and $|f(x)|_+^{1/n}$ denotes the positive nth root of positive number $|f(x)|$, $x \in S_f$. We claim that $m_0 = 0$. Indeed, Equation (4) implies that $f(0)$ and $g(0)$ are positive numbers. Then Equation (30) implies that $u_{f,0}(0) = u_{g,0}(0) = 0$. Combining this with Equation (31), yields the claim. Next, applying the property Equation ((3)()), we get that $m_{-j} = -m_j$ for all $j \in \{0, 1, \cdots, k-1\}$. Therefore, we conclude from Equation (32) that

$$\begin{aligned} g(x) = |f(x)|_+^{1/n}\chi_{E_0}(x) + \sum_{j=1}^{k-1} |f(x)|_+^{1/n} \\ \cdot \left(\chi_{E_j}(-x)e^{-\left(iu_{f,j}(x)+2\pi m_j\right)/n} + \chi_{E_j}(x)e^{\left(iu_{f,j}(x)+2\pi m_j\right)/n}\right), \end{aligned} \tag{33}$$

for all $x \in S_g$. Finally, keeping in mind that each m_j, $j = 1, 2, \cdots, k-1$, may take any value in $\{0, 1, \cdots, n-1\}$, we obtain from Equation (33) the estimate Equation (9). Theorem 1 is proved. □

Remark 6. Of course, we are not claiming that each of n^{k-1}th possible functions in Equation (33) belongs to $PD(\mathbb{R})$.

Remark 7. In the proof of Theorem 1 we concerned with the so-called problem of phase retrieval (see the equality Equation (29)), i.e., the problem of the recovery of a measure μ given the amplitude $|f|$ of its Fourier transform $f = \widehat{\mu}$. This problem is well known in various fields of science and engineering, including crystallography, nuclear magnetic resonance and optics (see, for example, survey [17]).

Proof of Theorem 2. Note that as in the proof of Theorem 1, in light of Equation (10), we see that there exists a sequence of real numbers Equation (27) such that

$$E = \left(\bigcup_{j=-1}^{-(k-1)} E_j\right) \bigcup E_0 \bigcup \left(\bigcup_{j=1}^{k-1} E_j\right). \tag{34}$$

Let us split our proof into two cases. First we consider the case when all E_j in Equation (34) are finite intervals. Denote by 2σ the minimal length of E_j, $j = -(k-1), \cdots, k-1$, i.e.,

$$2\sigma = \min\left\{2b_0\,;\, \min_{1\le j\le k-1}\left(b_j - a_j\right)\right\}. \tag{35}$$

□

Let $\varphi \in PD(\mathbb{R})$. Assume, in addition, that φ is real-valued on $\mathbb{R}$ and

$$S_\varphi = (-\sigma, \sigma). \tag{36}$$

For example, we can take $\varphi(x) = \Lambda_\alpha(x/\sigma)$, where the truncated power function Λ_α was defined by Equation (11). Next, for each $j \in \{1, 2, \cdots, k-1\}$, we take any sequence of real numbers $\{\tau_{j,s}\}_{s=1}^{m(j)}$ such that

$$a_j + \sigma = \tau_{j,1} < \tau_{j,2} < \cdots < \tau_{j,m} = b_j - \sigma. \tag{37}$$

In addition, we assume that

$$\tau_{j,(s+1)} - \tau_{j,s} < 2\sigma \tag{38}$$

for all $s = 1, \cdots, m(j)$. Then we define the function

$$u_j(x) = \sum_{s=1}^{m(j)} \varphi\left(x - \tau_{j,s}\right), \tag{39}$$

$x \in \mathbb{R}$. Now Equations (37) and (38) imply that u_j is supported on $[a_j, b_j] = \bar{E}_j$ and $S_{u_j} = E_j$. Since $\varphi \in PD(\mathbb{R})$ is real-valued on $\mathbb{R}$ and satisfies Equation (36), we conclude that φ is even and positive on S_φ. Therefore, the function Equation (39) and the function

$$u_j(-x) = \sum_{s=1}^{m(j)} \varphi\left(x + \tau_{j,s}\right) \tag{40}$$

are strictly positive on E_j and on $E_{-j} = -E_j$, respectively.

Let us define the function u_0 such that it is supported on $[-b_0, b_0] = \bar{E}_0$ and $S_{u_0} = E_0$. To this end, we take any sequence $\{\theta_i\}_{i=0}^{r}$ of real numbers such that

$$-b_0 + \sigma = \theta_r < \cdots \theta_1 < \theta_0 = 0, \tag{41}$$

$$\theta_i - \theta_{i+1} < 2\sigma, \tag{42}$$

for $i = 0, \cdots, l-1$. Next, for an arbitrary sequence of positive numbers $\{\omega_i\}_{i=0}^{l}$, we define

$$u_0(x) = \sum_{i=0}^{l} \omega_i(\varphi(x - \theta_i) + \varphi(x + \theta_i)), \tag{43}$$

$x \in \mathbb{R}$. Of course, Equations (41) and (42) imply that u_0 is supported on $[-b_0, b_0] = \bar{E}_0$ and $S_{u_0} = E_0$.

Finally, given any fixed sequence of positive numbers $\{\alpha_j\}_{j=1}^{k-1}$, we set

$$u(x) = u_0(x) + \sum_{j=1}^{k-1} \alpha_j\left(u_j(x) + u_j(-x)\right), \tag{44}$$

$$x \in \mathbb{R}. \tag{45}$$

We claim that $u \in PD(\mathbb{R})$ and $S_u = E$. First, as real-valued function $\varphi \in PD(\mathbb{R})$ satisfies Equation (36), it follows from Equations (34)–(38) and from Equations (41)–(42) that u is continuous on $\mathbb{R}$ and

$$S_u = \bigcup_{j=-k}^{k} E_j = E. \tag{46}$$

Second, since u is continuous and compactly supported, it follows that $u \in L^1(\mathbb{R})$. Therefore, the inverse Fourier transform of u is well-defined. Hence

$$\begin{aligned}\check{u}(t) &= \check{u}_0(t) + \sum_{j=1}^{k-1}\left(\alpha_j \check{\varphi}(t) \sum_{s=1}^{m(j)} 2\cos\left(\tau_{j,s}\cdot t\right)\right)\\ &= 2\check{\varphi}(t)\left[\sum_{i=0}^{l}\omega_i \cos\left(\theta_i\cdot t\right) + \sum_{j=1}^{k-1}\left(\alpha_j \sum_{s=1}^{m(j)} \cos\left(\tau_{j,s}\cdot t\right)\right)\right]\\ &= 2\check{\varphi}(t)\left[\omega_0 + \sum_{i=1}^{l}\omega_i \cos\left(\theta_i\cdot t\right) + \sum_{j=1}^{k-1}\left(\alpha_j \sum_{s=1}^{m(j)} \cos\left(\tau_{j,s}\cdot t\right)\right)\right].\end{aligned} \tag{47}$$

Let us fix the previously chosen positive numbers $\omega_1, \cdots, \omega_k$ and $\alpha_1, \cdots, \alpha_{k-1}$. Then we increase if necessary, the value of ω_0 in such a way that

$$\omega_0 > \sum_{i=1}^{l}\omega_i + \sum_{j=1}^{k-1}\alpha_j m_j. \tag{48}$$

Bochner's theorem shows that $\check{\varphi}(t) \geq 0$ for all $t \in \mathbb{R}$, since $\varphi \in PD(\mathbb{R})$. Combining Equation (36) with (48), we see that $\check{u}$ is nonnegative on $\mathbb{R}$ and $\check{u} \not\equiv 0$. In addition, we conclude (see, e.g., [15, p. 409]) that $\check{u} \in L^1(\mathbb{R})$. Thus, applying the Fourier transform to $\check{u}$ and using again Bohner's theorem, we see that u is continuous nontrivial positive definite, i.e., $u \in PD(\mathbb{R})$. This proves our claim.

Define

$$f(x) = u_0^n(x) + \sum_{j=1}^{k-1}\alpha_j^n\left(u_j^n(x) + u_j^n(-x)\right), \tag{49}$$

$x \in \mathbb{R}$. We claim that f satisfies the hypotheses of Theorem 2. Let us first prove that $f \in PD(\mathbb{R})$. Indeed, from Equation (27) we see that the essential support S_u of u, defined by Equation (44) can be represented as the union $\bigcup_{j=-(k-1)}^{k-1}E_j$ of a family of pairwise disjoint sets $E_j = S_{u_j}$, $j = -(k-1) \cdots$, $k-1$. Therefore,

$$u^n(x) = u_0^n(x) + \sum_{j=1}^{k-1}\alpha_j^n\left(u_j^n(x) + u_j^n(-x)\right), \tag{50}$$

$x \in \mathbb{R}$. We have already proven that $u \in PD(\mathbb{R})$. On the other hand, it is well known that for each $n \in \mathbb{N}$ and any $\zeta \in PD(\mathbb{R})$, it follows that $\zeta^n \in PD(\mathbb{R})$. Thus, $u^n \in PD(\mathbb{R})$. Combining this fact with Equation (41), we conclude that $f \in PD(\mathbb{R})$ and $S_f = S_u = \bigcup_{j=-(k-1)}^{k-1}E_j = E$.

Second, we will prove that the function f defined by Equation (50) has the property Equation (11). To this end, let $\mathbb{Z}_n$ denote the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \cong \{0, 1, \cdots, n-1\}$. Given $\Lambda = (p_1, p_2, \cdots, p_{k-1}) \in \mathbb{Z}_n^{k-1}$, define

$$g_\Lambda(x) = u_0(x) + \sum_{j=1}^{k-1}\alpha_j\left(e^{i2\pi p_j/n}u_j(x) + e^{-i2\pi p_j/n}u_j(-x)\right), \tag{51}$$

$x \in \mathbb{R}$. We claim that $g_\Lambda \in D_n(f)$. By the same argument as before for the function u defined by Equation (44), we see that $g_\Lambda \in L^1(\mathbb{R})$. Therefore, $\check{g}_\Lambda$ is well-defined and

$$\begin{aligned}\check{g}_\Lambda(t) = 2\check{\varphi}(t)\Bigg[\omega_0 &+ \sum_{i=1}^{l}\omega_i\cos\left(\theta_i\cdot t\right)\\ &+ \sum_{j=1}^{k-1}\left(\alpha_j\sum_{s=1}^{m(j)}\cos\left(\tau_{j,s}\cdot t + \frac{2\pi p_j}{n}\right)\right)\Bigg],\end{aligned} \tag{52}$$

$t \in \mathbb{R}$. Now using Equation (48), we get that $\check{g}_\Lambda(t) \geq 0$ for all $t \in \mathbb{R}$ and $\check{g}_\Lambda \not\equiv 0$. Hence, by Bochner's theorem it follows that $g_\Lambda \in PD(\mathbb{R})$. Next,

$$\begin{aligned}g_\Lambda^n(x) &= u_0^n(x) + \sum_{j=1}^{k-1}\alpha_j^n\left(\left(e^{i2\pi p_j/n}u_j(x)\right)^n + \left(e^{-i2\pi p_j/n}u_j(-x)\right)^n\right)\\ &= u_0^n(x) + \sum_{j=1}^{k-1}\alpha_j^n\left(u_j(x)^n + u_j(-x)^n\right),\end{aligned} \tag{53}$$

$x \in \mathbb{R}$. Combining this representation with Equations (49) and (50), we see that $g_\Lambda \in D_n(f)$, which yields our claim.

Finally, again using the fact that S_{g_Λ} is the union of a family of pairwise disjoint sets S_{u_j}, $j = -(k-1), \cdots, k-1$, we conclude from Equation (51) that $g_\Lambda \equiv g_{\Lambda_1}$ for some $\Lambda, \Lambda_1 \in \mathbb{Z}_n^{k-1}$, if and only if $\Lambda = \Lambda_1$. This proves Equation (11) in the case where each E_j in Equation (33) is a finite interval.

Now we consider the second case with $b_{k-1} = \infty$, i.e., if in Equation (33) we have $E_{k-1} = (a_{k-1}, \infty)$. Using the same $\varphi \in PD$ satisfying Equation (36), we define the functions u_0 and u_j, $j = -(k-2), \cdots, -1, 1, \cdots, k-2$ by Equations (43) and (39)–(40), respectively. For $j = k-1$, let us take an arbitrary sequence of positive numbers $\{\gamma_r\}_{r=1}^{\infty}$ such that

$$\sum_{r=1}^{\infty}\gamma_r = 1. \tag{54}$$

Then we define

$$u_{k-1}(x)=\sum_{r=1}^{\infty}\gamma_r\varphi(x-a_k-r\sigma), \tag{55}$$

$x\in\mathbb{R}$. Obviously, supp $(u_{k-1})=\bar{E_{k1}}$ and $S_{u_{k-1}}=E_{k-1}$. Next, for the function u, defined by Equation (44), it follows from Equation (47) that

$$\check{u}(t)=2\check{\varphi}(t)\left[\omega_0+\sum_{i=1}^{l}\omega_i\cos(\theta_i\cdot t)+\sum_{j=1}^{k-2}\left(\alpha_j\sum_{s=1}^{m(j)}\cos\left(\tau_{j,s}\cdot t\right)\right)+\alpha_{k-1}\sum_{r=1}^{\infty}\gamma_r\cos((a_{k-1}+r\sigma)t)\right]. \tag{56}$$

Again, for fixed positive numbers $\omega_1,\cdots\omega_k$, $\alpha_1,\cdots,\alpha_{k-1}$, and $\gamma_1,\gamma_2,\cdots$, we take the value of ω_0 in such a way that

$$\omega_0>\sum_{i=1}^{l}\omega_i+\sum_{j=1}^{k-2}\alpha_jm_j+\alpha_{k-1}. \tag{57}$$

Combining Equation (36) with Equation (57), we conclude from Equation (56) that u is nonnegative on $\mathbb{R}$ and $u\equiv 0$. Thus, $u\in PD(\mathbb{R})$. Finally, we claim that the function f defined by Equation (49) also satisfies the hypotheses of Theorem 2 in our case with $E_{k-1}=(a_{k-1},\infty)$. The proof of this claim is exactly the same as that of the first case. Therefore, we skip the details of this proof. Theorem 2 is proved.

Proof of Example 3. We claim that there are u_0, u_1, u_2 and α_1, α_2 such that the function in Equation (13) coincides with the function defined by Equation (50). Indeed, set $\varphi=\Lambda_1$, $j=3$, $m_1=2$, $m_2=4$ and

$$\tau_{11}=\pi+1,\tau_{12}=2\pi-1,\tau_{21}=11,\tau_{22}=12,\tau_{23}=13,\tau_{24}=14. \tag{58}$$

□

Since σ defined by Equation (35) is equal now to 1, then τ_{js} defined above satisfies Equations (37) and (38). Next, set

$$\begin{aligned}v_1(x)&=\Lambda_1(x-\pi-1)+\Lambda_1(x-2\pi+1),\\ v_2(x)&=\Lambda_1(x-11)+\Lambda_1(x-12)+\Lambda_1(x-13)+\Lambda_1(x-14).\end{aligned} \tag{59}$$

For $v_0=\Lambda_1$, $\alpha_1=1/8$ and $\alpha_2=1/16$, let v be given by

$$v(x)=\Lambda_1(x)+\frac{1}{8}(v_1(x)+v_1(-x))+\frac{1}{16}(v_2(x)+v_2(-x)). \tag{60}$$

It is easily seen that Λ_1, v_1 and v_2 are supported on a family of pairwise disjoint sets. Therefore,

$$v^n(x)=\Lambda_n(x)+\frac{1}{8}(v_1^n(x)+v_1^n(-x))+\frac{1}{16}(v_2^n(x)+v_2^n(-x)), \tag{61}$$

since $\Lambda_1^n=\Lambda_n$. Next

$$S_{v^n}=S_v=(-15,-10)\cup(-2\pi,-\pi)\cup(-1,1)\cup(\pi,2\pi)\cup(10,15). \tag{62}$$

The function v^n coincides with the function in Equation (13) and is defined by the same rules as in Equation (50). Our claim is proved.

Now, it is enough to show that $v\in PD(\mathbb{R})$. Indeed,

$$\begin{aligned}\check{v}(t)&=2\check{\Lambda}_1(t)\left[1+\sum_{j=1}^{2}\left(\alpha_j\sum_{s=1}^{m(j)}\cos\left(\tau_{js}\cdot t\right)\right)\right]\\ &=2\check{\Lambda}_1(t)\left[1+\frac{1}{8}(\cos(\tau_{11}t)+\cos(\tau_{12}t))\right.\\ &\quad\left.+\frac{1}{16}(\cos(\tau_{21}t)+\cos(\tau_{22}t)+\cos(\tau_{23}t)+\cos(\tau_{24}t))\right]\\ &\geq\check{\Lambda}_1(t)\geq 0\end{aligned} \tag{63}$$

for all $t\in\mathbb{R}$, since $\Lambda_1\in PD(\mathbb{R})$. Therefore, Bochner's theorem shows that $v\in PD(\mathbb{R})$.

By repeating the finally part of the proof of Theorem 2, we complete the proof of Example 3.

Proof of Example 4. This example concerns the case that was considered in the second part of the proof of Theorem 2, i.e., when S_f contains two unbounded components $(-\infty,-a)$ and (a,∞). Also, as in the proof of Theorem 2, is enough to show that Equations (54) and (57) are satisfied. Indeed, Equation (54) is clear, since $\gamma_r=1/2^r$, $r=1,2,\cdots$. We conclude from Equations (13) and (16) that

$$\begin{aligned}&k=4,\ \sigma=1,\ m_1=2,\ m_2=4,\ \omega_0=1,\\ &\omega_1=0,\alpha_1=\frac{1}{8},\ \alpha_1=\frac{1}{16}\text{ and }\alpha_3=\frac{1}{4}.\end{aligned} \tag{64}$$

□

Therefore, a simple calculation shows that Equation (57) is also satisfied. This completes the proof.

Proof of Theorem 5. We will prove this theorem using essential the same techniques as in the proof of Theorem 2. Therefore, we sketch the proof only. From Equations (19) and (20) we have that there exits an infinite sequence

$$0<b_0<a_1<b_1<a_2<b_2<\cdots<\infty \tag{65}$$

such that

$$E = \bigcup_{j\in\mathbb{Z}} E_j, \tag{66}$$

where $E_0 = (-b_0, b_0)$ and $E_j = (a_j, b_j) = -E_{-j}$ for each $j \in \mathbb{N}$. Moreover, from Equation (20) we see that

$$\min\left\{2b_0\,; \inf_{j\in\mathbb{N}} (b_j - a_j)\right\} = 2\sigma > 0. \tag{67}$$

□

Let $\varphi \in PD(\mathbb{R})$ be the same function satisfying Equation (36). For $j = 0$ and for $j \in \mathbb{N}$, we define the functions u_0 and u_j by Equations (39) and (43)., respectively. Note that the sequences $\{\tau_{j_s}\}_{s=1}^{m(j)}$ and $\{\theta_i\}_{i=0}^{l}$ satisfy Equations (37)–(38) and (40)–(42), respectively. For any $j \in \mathbb{N}$, let us define

$$\alpha_j = \frac{1}{2^j m(j)|E_j|}, \tag{68}$$

where $|E_j|$ is the length of E_j, i.e., $|E_j| = \sup\{|a - b|:\ a, b \in E_j\}$. Note that the condition (20) guarantees that $\{\alpha_j\}_{j\in\mathbb{N}}$ is a well-defined sequence of positive numbers. Set

$$u(x) = u_0(x) + \sum_{j\in\mathbb{N}} \alpha_j \left(u_j(x) + u_j(-x)\right), \tag{69}$$

$x \in \mathbb{R}$. Combining Equations (4), (39), and (43) with (68), we get

$$|u(x)| \le 2\varphi(0)\left[\left(\sum_{i=0}^{l}\omega_i\right) I_{E_0}(x) + \sum_{j\in\mathbb{N}}\left(\frac{1}{2^j|E_j|} I_{E_j}(x)\right)\right] \tag{70}$$

for any $x \in \mathbb{R}$. Here, $I_D(x)$ denotes the indicator function of a subset $D \subset \mathbb{R}$. Hence,

$$\|\varphi\|_{L^1(\mathbb{R})} \le 2\varphi(0)\left[2b_0 \sum_{i=0}^{l}\omega_i + 1\right]. \tag{71}$$

Therefore, $\varphi \in L^1(\mathbb{R})$ and

$$\begin{aligned} u(t) = 2\varphi(t)\Bigg[\omega_0 &+ \sum_{i=1}^{l}\omega_i \cos(\theta_i \cdot t) \\ &+ \sum_{j\in\mathbb{N}}\left(\alpha_j \sum_{s=1}^{m(j)} \cos\left(\tau_{j_s}\cdot t\right)\right)\Bigg]. \end{aligned} \tag{72}$$

Again, for fixed positive numbers $\omega_1, \cdots \omega_k$, we take the value of ω_0 in such a way that

$$\omega_0 > \sum_{i=1}^{l}\omega_i + \frac{1}{2\sigma}. \tag{73}$$

Combining this condition with Equation (67) and Equation (68), we conclude that the function $\check{u}$ in Equation (61) is nonnegative on $\mathbb{R}$ and $\check{u} \not\equiv 0$. Thus, $u \in PD(\mathbb{R})$. Define

$$f(x) = u_0^n(x) + \sum_{j\in\mathbb{N}} \alpha_j^n\left(u_j^n(x) + u_j^n(-x)\right), \tag{74}$$

$x \in \mathbb{R}$. Next, for $\Lambda = (p_1, p_2, \cdots) \in \mathbb{Z}^\infty$, set

$$g_\Lambda(x) = u_0(x) + \sum_{j\in\mathbb{N}} \alpha_j\left(e^{i2\pi p_j/n} u_j(x) + e^{-i2\pi p_j/n} u_j(-x)\right), \tag{75}$$

$x \in \mathbb{R}$. We claim that: (i) $f \in PD(\mathbb{R})$, (ii) E defined by Equation (66) satisfies Equations (19) and (20), and (iii) $g_\Lambda \in D_n(f)$. The proofs of these claims is exactly the same as in the case of functions f and g_Λ defined by Equations (49) and (51), respectively.

Finally, again using the fact that $S_f = S_u$ and therefore, $S_{g_\Lambda} = S_u$ are the unit of a family of pairwise disjoint sets S_{u_j}, $j \in \mathbb{Z}$, where $u_{-j}(x) = u_j(-x)$, $j \in \mathbb{N}$, we see from Equation (75) that $g_\Lambda \equiv g_{\Lambda_1}$ for some $\Lambda, \Lambda_1 \in \mathbb{Z}^\infty$, if and only if $\Lambda = \Lambda_1$. This proves Equation (21) and therefore completes the proof our theorem.

3. Conclusion

We study the n-divisible functions in $PD(\mathbb{R})$, where $PD(\mathbb{R})$ denotes the family of continuous positive definite functions on the real line $\mathbb{R}$. While there is rich literature on infinite-divisible functions in $PD(\mathbb{R})$, for an integer $n > 1$, properties of n-divisible functions from $PD(\mathbb{R})$ have been studied much less. Surprisingly, it appears that some properties of infinite-divisible and n-divisible functions $f \in PD(\mathbb{R})$ may differ in essence. In this paper, we examine one such property, which has not yet been discussed in detail in the literature. More precisely, if $f \in PD(\mathbb{R})$ is infinite-divisible, then it is well-known that, for each integer $n > 1$, there is an unique $g \in PD(\mathbb{R})$, such that $g^n = f$. On the other hand, there is n-divisible f such that the factor g in $g^n = f$ is generally not unique. For n-divisible $f \in PD(\mathbb{R})$, we study the following questions: (i) how rich can be the class $D_n(f) = \{g \in PD(\mathbb{R}): g^n = f\}$; and (ii) what properties of f determine the size of $D_n(f)$. We present several precise estimates for the cardinality of $D_n(f)$. Also, the main results are validated via illustrative examples.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that they have no competing interests.

Acknowledgments

This research was funded by Vilnius University as part of the project: “Analysis and Application of Probabilistic and Deterministic Models”.

References

[1] D. Alpay and P. E. T. Jorgensen, “Stochastic processes induced by singular operators,” *Numerical Functional Analysis and Optimization*, vol. 33, no. 7-9, pp. 708–735, 2012.

[2] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions*, Springer, New York, 1984.

[3] B. Fuglede, “Boundary minimum principles in potential theory,” *Mathematische Annalen*, vol. 210, no. 3, pp. 213–226, 1974.

[4] G. Auchmuty and Q. Han, “Spectral representations of solutions of linear elliptic equations on exterior regions,” *Journal of Mathematical Analysis and Applications*, vol. 398, no. 1, pp. 1–10, 2013.

[5] Z. Sasvári, “Positive Definite and Definitizable Functions,” in *Mathematical Topics*, Akademie Verlag, Berlin, 1994.

[6] E. Lukacs, *Characteristic Functions*, Hafner Publishing Co., New York, 2nd edition, 1970.

[7] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev, “Phase retrieval with application to optical imaging: a contemporary overview,” *IEEE signal processing magazine*, vol. 32, no. 3, pp. 87–109, 2015.

[8] N. G. Ushakov, *Selected Topics in Characteristic Functions*, Modern Probability and Statistics. VSP, Utrecht, 1999.

[9] F. Steutel and J. Kent, “Infinite divisibility in theory and practice,” *Scandinavian Journal of Statistics*, vol. 6, no. 2, pp. 57–64, 1979.

[10] S. K. Katti, *Infinite Divisibility of Discrete Distributions III*, Colloquia Mathematica Societatis Janos Bolyai, Debrecen Hungary, 1977.

[11] O. Thorin, “On the infinite divisbility of the Pareto distribution,” *Scandinavian Actuarial Journal*, vol. 1977, no. 1, pp. 31–40, 1977.

[12] O. Thorin, “On the infinite divisibility of the lognormal distribution,” *Scandinavian Actuarial Journal*, vol. 1977, no. 3, pp. 121–148, 1977.

[13] F. Steutel and K. van Harn, “Infinite Divisibility of Probability Distributions on the Real Line,” in *Monographs and Textbooks in Pure and Applied Mathematics*, p. 259, Marcel Dekker, Inc., New York, 2004.

[14] L. B. Klebanov, A. V. Kakosyan, and I. V. Volchenkova, “Inequalities for m-divisible distributions and testing of infinite divisibility,” pp. 1–8, 2019, https://arxiv.org/abs/1904.07604.

[15] E. Hewitt and K. Stromberg, *Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable*, Springer-Verlag, New York-Heidelberg, 1975.

[16] R. M. Trigub and E. S. Bellinsky, *Fourier Analysis and Approximation of Functions*, Kluwer, Dordrecht, 2004.

[17] T. Bisgaard and Z. Sasvári, *Characteristic Functions and Moment Sequences*, Nova Science Publishing, New York, 2000.